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DIFFERENTIAL GEOMETRY OF PARTIAL ISOMETRIES AND PARTIAL UNITARIES

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ABSTRACT. Let $\mathcal{A}$ be a C*-algebra. In this paper the sets $\mathcal{I}$ of partial isometries and $\mathcal{I}_\Delta \subset \mathcal{I}$ of partial unitaries (i.e., partial isometries which commute with their adjoints) are studied from a differential geometric point of view. These sets are complemented submanifolds of $\mathcal{A}$. Special attention is paid to geodesic curves. The space $\mathcal{I}$ is a homogeneous reductive space of the group $U_\mathcal{A} \times U_\mathcal{A}$, where $U_\mathcal{A}$ denotes the unitary group of $\mathcal{A}$, and geodesics are computed in a standard fashion. Here we study the problem of the existence and uniqueness of geodesics joining two given endpoints. The space $\mathcal{I}_\Delta$ is *not* homogeneous, and therefore a completely different treatment is given. A principal bundle with base space $\mathcal{I}_\Delta$ is introduced, and a natural connection in it defined. Additional data, namely certain translating maps, enable one to produce a *linear* connection in $\mathcal{I}_\Delta$, whose geodesics are characterized.

1. Introduction

In their study of the problem of unitary equivalence of operators on a Hilbert space H, Halmos and McLaughlin [22] proved that the problem can be reduced to that of the unitary equivalence of partial isometries. In doing so, they characterized the connected components of the set $\mathcal{I}$ of all partial isometries on H: the partial isometries x and y belong to the same component if and only if they have the same nullity (i.e., dimension of the null-space), the same rank (dimension of the image) and the same corank (dimension of the orthogonal complement of the image). They also proved that if $\|x - y\| < 1$ then there exist unitary operators u and v on H such that $y = uxv^*$. This paper is devoted to the study of the differential geometry of the set $\mathcal{I}$. In order to describe the results, we fix a unital C*-algebra $\mathcal{A}$, and denote by $G_\mathcal{A}$ the group of invertible elements of $\mathcal{A}$, by $U_\mathcal{A}$ the subgroup of unitary elements of $\mathcal{A}$ and by $\mathcal{P}$ the set of all hermitian projections of $\mathcal{A}$: $\mathcal{P} = \{p \in \mathcal{A} : p^2 = p = p^*\}$. The set $\mathcal{I}$ of partial isometries of $\mathcal{A}$ is defined by $\mathcal{I} = \{x \in \mathcal{A} : x^*x, xx^* \in \mathcal{P}\}$. The differential geometry of $\mathcal{P}$ is well known by now, and we often use this

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knowledge in order to obtain results on $\mathcal{I}$. The main link between $\mathcal{I}$ and $\mathcal{P}$ is provided by the mapping

$$\mathcal{I} \to \mathcal{P} \times \mathcal{P}, \quad x \mapsto (xx^*, x^*x).$$

Recall that xx^* (resp. x^*x) is called the final (resp. initial) projection of x.

We shall now reformulate the result of Halmos and McLaughlin in geometrical terms. Observe that the map

$$U_{\mathcal{A}} \times U_{\mathcal{A}} \times \mathcal{I} \to \mathcal{I}, \quad (u, v, x) \mapsto uxv^*,$$

defines a left action of $U_{\mathcal{A}} \times U_{\mathcal{A}}$ on $\mathcal{I}$. Their result says that the action is locally transitive, i.e., two partial isometries which are close enough are conjugate by a pair of unitaries. As a corollary, the connected component of x in $\mathcal{I}$ is the orbit of x by the action of $U_{\mathcal{A}} \times U_{\mathcal{A}}$. Moreover, it is a homogeneous space of $U_{\mathcal{A}} \times U_{\mathcal{A}}$ and a C^∞ submanifold of $\mathcal{A} = \mathcal{B}(H)$. For a general C*-algebra $\mathcal{A}$, for which $U_{\mathcal{A}}$ is not necessarily connected, the same argument can be carried out with $U_{\mathcal{A}}$ replaced by U_0, the connected component of 1 in $U_{\mathcal{A}}$. The fact that the connected components of $\mathcal{I}$ are homogeneous spaces of $U_{\mathcal{A}} \times U_{\mathcal{A}}$ but not of $U_{\mathcal{A}}$ depends on the map $\mathcal{I} \to \mathcal{P} \times \mathcal{P}$ mentioned above. In fact, $U_{\mathcal{A}}$ has a left action on $\mathcal{P}$ by $(u, p) \mapsto upu^*$, and the action is locally transitive (see B. Sz.-Nagy [38]): If $p, q \in \mathcal{P}$ and $\|p - q\| < 1$ there exists $u \in U_{\mathcal{A}}$ such that $q = upu^*$. (Originally proved for $\mathcal{A} = \mathcal{B}(H)$, this holds for any C*-algebra, where u can be chosen in U_0.) Hence the orbit of p by the action on U_0 is the connected component of p in $\mathcal{P}$. Now, the motion of $x \in \mathcal{I}$ is determined by the motions of its initial and final projections (spaces) x^*x and xx^* in $\mathcal{P}$, and these projections are moved independently. Therefore it must be the group $U_{\mathcal{A}} \times U_{\mathcal{A}}$ which provides the motions of $x \in \mathcal{I}$ (if one wants the action to be locally transitive and to fill connected components).

For a fixed $p \in \mathcal{P}$, an important role is played by the space $\mathcal{I}^p$ of all partial isometries with initial projection p: $\mathcal{I}^p = \{v \in \mathcal{A} : v^*v = p\}$. $\mathcal{I}^p$ is a submanifold of $\mathcal{I}$ and naturally carries a left action of $U_{\mathcal{A}}$ which makes it a homogeneous space of $U_{\mathcal{A}}$.

The present paper is an addition to the existing literature on the differential geometry of different sets and maps of operators (and their corresponding abstract analogues). The reader is referred to [37], [13], [3], [28], [39], [11], [23], [24] for projections; [12], [28], [29] for n-tuples of projections; [7] for spectral measures; [8], [9] for nilpotent operators; [15] for selfadjoint invertible operators; [16], [17], [19], [20] for positive (hermitian) operators; [2], [18] for representations of groups and algebras, [10] for states; [34], [35] for partial isometries; and [21], [14], [12] for elements which admit generalized inverses.

Let us describe now the contents of this paper. In Section 2 we introduce a linear connection in $\mathcal{I}$, of which we compute the geodesic curves, and investigate how the geometry of $\mathcal{P}$ and $\mathcal{I}^p$ plays a role in the geometric properties of $\mathcal{I}$. The general theory of reductive spaces guarantees the existence of a (uniform) radius R, with the property that if two elements lie at distance (measured with the norm of $\mathcal{A}$) less than R, then they can be joined by a unique geodesic of this connection. For instance, for the space $\mathcal{P}$ of projections of $\mathcal{A}$ this radius $R_{\mathcal{P}}$ is 1 [37]. In Section 3 we estimate (very roughly) the geodesic radius for the spaces $\mathcal{I}^p$ and for $\mathcal{I}$. This estimate relates the three numbers $R_{\mathcal{P}} = 1$, $R_{\mathcal{I}^p}$ and $R_{\mathcal{I}}$.

The second half of this paper (sections 4 and 5) is devoted to the study of what we call *partial unitaries* (there might be another name for them in the literature that we are unaware of): Partial unitaries are partial isometries such that the initial and final spaces coincide, or, equivalently, elements $v \in \mathcal{I}$ such that v commutes with v^*. We denote the set of all such elements by $\mathcal{I}_\Delta$. We show that $\mathcal{I}_\Delta$ is a complemented submanifold of $\mathcal{I}$ (and of $\mathcal{A}$). However, $\mathcal{I}_\Delta$ does not admit a (locally transitive) action of a group of unitaries, as $\mathcal{I}$ does, and it is not a homogeneous space. Therefore, the differential geometric study of $\mathcal{I}_\Delta$ is more complicated. First, note that if $w_1,\ w_2 \in \mathcal{I}_\Delta$ lie at a distance less than 1, their initial (equivalently, final) projections are unitarily equivalent. This motivates the study of the following map. For each fixed $p \in \mathcal{P}$,

$$\begin{aligned} &\pi_p^\Delta : \Delta^p = \Delta := \{(\alpha,\beta) \in U_{\mathcal{A}} \times U_{\mathcal{A}} : \alpha p \alpha^* = \beta p \beta^*\} \to \mathcal{I}_\Delta, \\ &\pi_p^\Delta(\alpha,\beta) = \alpha p \beta^*. \end{aligned}$$

If w_1 is in the range of this map and $w_2 \in \mathcal{I}_\Delta$ satisfies $\|w_2 - w_1\| < 1$, then w_2 also lies in the range of π_p^Δ. In other words, the range of this map fills connected components.

We show that this map is a smooth principal bundle, with structure group

$$G_p = \{(g_1, g_2) \in U_{\mathcal{A}} \times U_{\mathcal{A}} : g_i p = p g_i,\ i = 1,2 \text{ and } g_1 p = g_2 p\}.$$

We introduce a connection on this principal bundle, and compute the horizontal lifting differential equations. These enable one to perform the parallel transport of elements in the fibres. However, our interest is in a connection in the tangent bundle. Unfortunately, this cannot be established in the usual manner in differential geometry because the structure group G_p does not act on the tangent spaces of $\mathcal{I}_\Delta$. We therefore introduce a linear connection by means of a distribution of isomorphisms between the horizontal spaces. Namely, if $(\alpha,\beta),(\delta,\epsilon) \in \Delta$, and $\mathcal{K}_{(\alpha,\beta)}$, $\mathcal{K}_{(\delta,\epsilon)}$ denote the corresponding horizontal subspaces, we define a smooth distribution of (real) linear isomorphisms

$$T_{\alpha,\beta}^{\delta,\epsilon} : \mathcal{K}_{(\alpha,\beta)} \to \mathcal{K}_{(\delta,\epsilon)}$$

with the following properties:

(1) $T_{\alpha,\beta}^{\alpha,\beta} = \text{id}$.
(2) $(T_{\alpha,\beta}^{\delta,\epsilon})^{-1} = T_{\delta,\epsilon}^{\alpha,\beta}$.

(3) The distribution is equivariant with respect to the action of G_p: If $(g_1, g_2) \in G_p$ and $(x_1, x_2) \in \mathcal{K}_{(\alpha,\beta)}$, then

$$\left(T^{\delta,\epsilon}_{\alpha,\beta}(x_1, x_2)\right) \cdot (g_1, g_2) = T^{\delta g_1, \epsilon g_2}_{\alpha g_1, \beta g_2}(x_1 g_1, x_2 g_2).$$

These maps, combined with the horizontal liftings, provide a parallel transport of tangent vectors in $T\mathcal{I}_\Delta$, and therefore a linear connection. The geodesics of this connection are not explicitly computed, but we show that their (horizontal) liftings satisfy a linear differential equation. This implies, in particular, that geodesics of this connection in $\mathcal{I}_\Delta$ exist for all $t \in \mathbb{R}$.

Throughout this paper, if p is a projection, we write $\bar{p} = 1 - p$. If $g \in G_\mathcal{A}$ and $x \in \mathcal{A}$, we set $x^g = gxg^{-1}$.

2. The reductive structure of $\mathcal{I}$

The group $U_\mathcal{A} \times U_\mathcal{A}$ acts on $\mathcal{I}$ by means of

$$(u, w) \cdot v = uvw^*, \ u, w \in U_\mathcal{A}, \ v \in \mathcal{I}.$$

The action is locally transitive, two partial isometries at distance less than $1/2$ are conjugate by this action, with a pair of unitaries which can be chosen by an explicit (and smooth) formula that gives local cross sections for the action. Indeed, note that if $v_0, v \in \mathcal{I}$ are such that $\|v_0 - v\| < 1/2$, then $\|v_0 v_0^* - vv^*\| < 1$ and $\|v_0^* v_0 - v^* v\| < 1$. Then [37],[36] there exist unitaries $\nu, \sigma \in U_\mathcal{A}$, which are smooth functions of v_0, v, such that $\nu v_0^* v_0 \nu^* = v^* v$ and $\sigma v_0 v_0^* \sigma^* = vv^*$. Put $\gamma = v\nu^* v_0^* + \sigma(1 - v_0 v_0^*)$; then the pair of unitaries (γ, ν) satisfies

$$\gamma v_0 \nu^* = v.$$

In other words, the map

$$\mu_{v_0} : \{v \in \mathcal{I} : \|v - v_0\| < 1/2\} \to U_\mathcal{A} \times U_\mathcal{A}, \ \mu_{v_0}(v) = (\gamma, \nu),$$

is a C^∞ cross section for the action.

Let us fix a partial isometry v_0. We will describe the isotropy subgroup and the tangent spaces based on v_0. The isotropy subgroup $V_{v_0} \subset U_\mathcal{A} \times U_\mathcal{A}$ is

$$V_{v_0} = \{(f, g) \in U_\mathcal{A} \times U_\mathcal{A} : fv_0 = v_0 g\}.$$

Note that if $(f, g) \in V_{v_0}$ then f commutes with the final projection $p_0 = v_0 v_0^*$, and g commutes with the initial projection $q_0 = v_0^* v_0$.

Let π_{v_0} be the map (in fact, the C^∞ fibre bundle [1])

$$\pi_{v_0} : U_\mathcal{A} \times U_\mathcal{A} \to \mathcal{I}, \ \pi_{v_0}(u, w) = uv_0 w^*.$$

The tangent space of $U_\mathcal{A} \times U_\mathcal{A}$ at $(1, 1)$ can be identified with $\mathcal{A}_{ah} \times \mathcal{A}_{ah}$, where $\mathcal{A}_{ah}$ is the real Banach space of antihermitian elements of $\mathcal{A}$. Let

$\delta_{v_0} = d(\pi_{v_0})_{(1,1)}$, i.e.,

$$\delta_{v_0} : \mathcal{A}_{ah} \times \mathcal{A}_{ah} \to T\mathcal{I}_{v_0},\ \delta_{v_0}(x, y) = xv_0 - v_0 y.$$

In particular, this implies that

$$T\mathcal{I}_{v_0} = \{xv_0 - v_0 y : x, y \in \mathcal{A}_{ah}\}.$$

The tangent space (and Lie algebra) $(TV_{v_0})_{(1,1)} = \mathcal{V}_{v_0}$ is equal to the kernel of δ_{v_0}.

We shall introduce a reductive structure on $\mathcal{I}$, that is, a (real) closed linear subspace $\mathcal{H}_{v_0}$ of $\mathcal{A}_{ah} \times \mathcal{A}_{ah}$ with the following properties:

(1) $\mathcal{H}_{v_0} \oplus \mathcal{V}_{v_0} = \mathcal{A}_{ah} \times \mathcal{A}_{ah}$.
(2) $\mathrm{ad}(f, g)(\mathcal{H}_{v_0}) = \mathcal{H}_{v_0}$.

Here $\mathrm{ad}(f, g)(x, y) = (x^f, y^g)$.

Consider the linear map

$$\Sigma_{v_0} : \mathcal{A} \to \mathcal{A} \times \mathcal{A},$$
$$\Sigma_{v_0}(a) = (av_0^* - v_0 a^* + v_0 a^* v_0 v_0^*, a^* v_0 - v_0^* a + 2v_0^* a v_0^* v_0).$$

The following result is a straightforward computation.

Lemma 2.1. $\delta_{v_0} \circ \Sigma_{v_0} \circ \delta_{v_0} = \delta_{v_0}$.

Note that, in particular, the range of $\Sigma_{v_0} \circ \delta_{v_0}$ lies in $\mathcal{A}_{ah} \times \mathcal{A}_{ah}$. The map $\Sigma_{v_0} \circ \delta_{v_0}$ is therefore an idempotent in the Banach algebra $B_{\mathbb{R}}(\mathcal{A}_{ah} \times \mathcal{A}_{ah})$ of real linear bounded operators on the space $\mathcal{A}_{ah} \times \mathcal{A}_{ah}$, whose kernel is equal to the kernel of $\delta_{v_0} = \mathcal{V}_{v_0}$. Therefore, its range is a complement of $\mathcal{V}_{v_0}$ in $\mathcal{A}_{ah} \times \mathcal{A}_{ah}$. Let us define

$$\mathcal{H}_{v_0} := R(\Sigma_{v_0} \circ \delta_{v_0}) = R(\Sigma_{v_0}|_{(T\mathcal{I})_{v_0}}).$$

Explicitly, $\mathcal{H}_{v_0} \subset \mathcal{A}_{ah} \times \mathcal{A}_{ah}$ is given by

$$\mathcal{H}_{v_0} = \{(xp_0 + p_0 x - p_0 x p_0 - v_0 y v_0^*, yq_0 + q_0 y - 2q_0 y q_0) : x, y \in \mathcal{A}_{ah}\}.$$

We claim that this space is an invariant complement of $\mathcal{V}_{v_0}$.

Lemma 2.2. *If $(f, g) \in V_{v_0}$, and $(x, y) \in \mathcal{H}_{v_0}$, then*

$$\mathrm{ad}(f, g)(x, y) = (x^f, y^g) \in \mathcal{H}_{v_0}.$$

Proof. The element (x, y) is of the form $\Sigma_{v_0}(av_0 - v_0 b)$ for $a, b \in \mathcal{A}_{ah}$, i.e.,

$$x = ap_0 + p_0 a - p_0 b p_0 - v_0 a v_0^*, \quad \text{and } y = q_0 b + b q_0 - 2q_0 b q_0.$$

Then

$$\begin{aligned} x^f &= fap_0 f^* + fp_0 a f^* - fp_0 a p_0 f^* - fv_0 b v_0^* f^* \\ &= a^f p_0 + p_0 a^f - p_0 a^f p_0 - v_0 b^g v_0^*, \end{aligned}$$

where in the last term we use the relation $fv_0 = v_0 g$. Analogously,

$$y^g = q_0 b^g + b^g q_0 - 2q_0 b^g q_0.$$

Then $(x^f, y^g) = \Sigma_{v_0} \circ \delta_{v_0}(a^f, b^g)$, which lies in $\mathcal{H}_{v_0}$ because $a^f, b^g \in \mathcal{A}_{ah}$. $\square$

This reductive structure induces a linear connection in $\mathcal{I}$. We are interested in the exponential map and the geodesic curves of this connection. These can be computed in a standard fashion [26] (see [33] for a C*-algebraic framework). For instance, if $x \in T\mathcal{I}_{v_0}$, then the unique geodesic $\gamma(t) \in \mathcal{I}$ with $\gamma(0) = v_0$ and $\dot\gamma(0) = x$ is

$$\gamma(t) = e^{t\xi_1} v_0 e^{-t\xi_2},$$

where $\xi = (\xi_1, \xi_2) = \Sigma_{v_0}(x)$. Geodesics starting at an arbitrary point $v = uv_0w^* \in \mathcal{I}$ are transports of geodesics starting at v_0: $\nu(t) = u\gamma(t)w^*$ with γ as above. Note that the action defines linear isomorphisms between the corresponding tangent spaces. As usual in the case of homogeneous spaces, it follows that the local structure of $\mathcal{I}$ can be studied on the neighbourhoods of v_0.

The projections p_0 and q_0 enable one to regard the elements of $\mathcal{A}$ as 2×2 matrices. Let us describe the matrices of horizontal elements $\xi \in \mathcal{H}_{v_0}$. If $\xi = (\xi_1, \xi_2) = (xp_0 + p_0x - p_0xp_0 - v_0yv_0^*, q_0y + yq_0 - 2q_0yq_0),\ y^* = -y,\ x^* = -x$, then it is straightforward that ξ is of the form

$$\xi_1 = \begin{pmatrix} x_{11} & x_{12} \\ -x_{12}^* & 0 \end{pmatrix}_{p_0}, \ \xi_2 = \begin{pmatrix} 0 & y_{12} \\ -y_{12}^* & 0 \end{pmatrix}_{q_0}$$

with $x_{11}^* = -x_{11}$. The subscripts p_0 and q_0 here indicate that the matrices are regarded with respect to these projections.

We conclude this section by recalling the connections of the homogeneous spaces $\mathcal{P}$ [13] and $\mathcal{I}^{q_0}$ (denoted by $\mathcal{S}_{q_0}(\mathcal{A})$ in [6]). This will make apparent the close relationship between these geometries.

Theorem 2.3. *The geodesics of $\mathcal{P}$, starting at q_0, are of the form $\rho(t) = e^{t\xi_2} q_0 e^{-t\xi_2}$, where ξ_2 is the second coordinate of ξ above.*

The geodesics of $\mathcal{I}^{q_0}$, starting at v_0, are of the form $\sigma(t) = e^{t\xi_1} v_0$. where ξ_1 is the first coordinate of ξ above.

3. Existence of geodesics joining two given endpoints

In [13] (see also [36]) it is shown that two projections $p, q_0 \in \mathcal{P}$ such that $\|p - q_0\| < 1$ can be joined by a unique geodesic. In [4] there is no estimation of the geodesic radius of $\mathcal{I}^{q_0}$. The general theory shows the existence of a number $1 \geq R > 0$ with the property that two elements $v', v'' \in \mathcal{I}^{q_0}$ such that $\|v' - v''\| < R$ can be joined by a unique geodesic. In this section we relate this constant R with the geodesic radius of $\mathcal{I}$, and also give a rough estimate for R.

Let $v, v_0 \in \mathcal{I}$ be two partial isometries. We want to establish the existence of a geodesic curve joining v and v_0.

Let us denote by $\mathcal{H}'_{v_0}, \mathcal{H}''_{v_0}$ the subspaces of the first and second coordinates of elements of $\mathcal{H}_{v_0}$, i.e.,

$$\mathcal{H}_{v_0} = \mathcal{H}'_{v_0} \times \mathcal{H}''_{v_0}.$$

PROPOSITION 3.1. *Suppose that $\|v - v_0\| < \min\{1/2, R/4\}$. Then there exists a unique $\xi = (\xi_1, \xi_2) \in \mathcal{H}_{v_0}$ such that $v = e^{\xi_1} v_0 e^{-\xi_2}$, with $\|\xi_1\| < R$ and $\|\xi_2\| < \pi$.*

Proof. First, note that $\|v^*v - q_0\| \leq \|v^*v - v^*q_0\| + \|v^*q_0 - v_0^*v_0\| \leq \|v^*\|\|v - v_0\| + \|v^* - v_0^*\| \leq 2\|v - v_0\| < 1$. It is known [13] that two projections at (norm) distance strictly less that 1 are joined by a unique geodesic of the space of projections. Therefore there exists a unique $\xi_2 \in \mathcal{A}_{ah}$, with $\|\xi_2\| < \pi$, $\xi_2 \in \mathcal{H}''_{v_0}$, such that $v^*v = e^{\xi_2} q_0 e^{-\xi_2}$. Let $\hat{v} = ve^{\xi_2}$. Then it is clear that $\hat{v}^*\hat{v} = q_0$, i.e., $\hat{v} \in \mathcal{I}^{q_0}$. We compute

$$\|\hat{v} - v_0\| = \|ve^{\xi_2} - v_0\| \leq \|ve^{\xi_2} - v\| + \|v - v_0\| < \|e^{\xi_2} - 1\| + \|v - v_0\|.$$

Let us estimate $\|e^{\xi_2} - 1\|$. This norm is equal to $r(e^{\xi_2} - 1)$ (where r is the spectral radius), which is bounded by $\sqrt{2(1 - \cos(\|\xi_2\|))}$. In [36] (see also [5]) the norm $\|\xi_2\|$ is computed in terms of the projections q_0 and v^*v, namely,

$$\|\xi_2\| = \arcsin(\|q_0 - v^*v\|) \leq \arcsin(2\|v - v_0\|).$$

Therefore,

$$\|e^{\xi_2} - 1\| < 2\sin\left(\frac{1}{2}\arcsin(2\|v - v_0\|)\right).$$

Note that if $0 \leq t < 1/2$, then $\arcsin(t) \leq \frac{3}{2}t$. It follows that $\|e^{\xi_2} - 1\| < 3\|v - v_0\|$, and therefore

$$\|\hat{v} - v_0\| < 4\|v - v_0\| \leq R.$$

Hence there exists a unique $\xi_1 \in \mathcal{H}'_{v_0}$ with $\|\xi_1\| < R$ such that $e^{-\xi_1} v_0 = \hat{v}$. In other words, we have found $\xi = (\xi_1, \xi_2) \in \mathcal{H}_{v_0}$ such that

$$e^{\xi_1} v_0 e^{-\xi_2} = \hat{v} e^{-\xi_2} = v. \qquad \square$$

Let us now give a rough estimate for R.

Let $\mathcal{V} = \{x \in \mathcal{A}_{ah} : xp_0 = 0\}$. Note that $\mathcal{V}$ is a complement for $\mathcal{H}'_{v_0}$ in $\mathcal{A}_{ah}$. (In fact, it is the Lie algebra of vertical elements of the homogeneous space $\mathcal{I}^{q_0}$ [4].)

Our estimate will be based on the map

$$\nu : \mathcal{A}_{ah} \to \mathcal{A}_{ah}, \quad \nu(x) = \log(e^{x_{\mathcal{H}}} e^{x_{\mathcal{V}}}),$$

where

$$x = x_{\mathcal{H}} + x_{\mathcal{V}}$$

is the decomposition of x with $x_{\mathcal{H}} \in \mathcal{H}'_{v_0}$ and $x_{\mathcal{V}} \in \mathcal{V}$, and log is the analytic inverse of the usual exponential of antihermitian elements, on a neighbourhood of $1 \in U_{\mathcal{A}}$, i.e., log is defined on $\{u \in U_{\mathcal{A}} : \|u-1\| < 1\}$ as the series

$$\log(u) = \sum_{n\geq 1} \frac{(-1)^{n+1}}{n}(u-1)^n.$$

Note that since $x_{\mathcal{V}} = \bar{q}_0 x_{ah} \bar{q}_0$, (where x_{ah} is the antihermitian part of x), we have $\|x_{\mathcal{V}}\| \leq \|x_{ah}\| \leq \|x\|$. On the other hand, we have the obvious estimate $\|x_{\mathcal{H}}\| \leq 2\|x\|$. Clearly $d\nu_0 = I$. Therefore ν is a local diffeomorphism around the origin. Let $B_r(0)$ denote the ball with centre 0 and radius r in $\mathcal{A}_{ah}$.

LEMMA 3.2. *$\nu : B_r(0) \to B_{r/2}(0)$ is a diffeomorphism for $r = 0.036$.*

Proof. Recall the usual proof of the Inverse Function Theorem in the context of Banach spaces (see, e.g., [30]). Let $\gamma = \mathrm{id} - \nu$. We need to estimate the norm of the differential of γ at $a \in \mathcal{A}_{ah}$,

$$d\gamma_a(x) = x - d\nu_a(x).$$

Let $b, y \in \mathcal{A}_{ah}$ and let E be the usual exponential, $E(y) = e^y$. Then $dE_b(y) = \frac{d}{dt} e^{b+ty}\big|_{t=0}$. This derivative is equal to the series

$$y + \frac{1}{2}(yb + by) + \frac{1}{6}(yb^2 + byb + b^2y) + \cdots$$

Analogously, if $u \in U_{\mathcal{A}}$ with $\|u-1\| < 1$, $d\log_u(x)$ is given by

$$x - \frac{1}{2}((u-1)x + x(u-1)) + \frac{1}{3}((u-1)^2x + (u-1)x(u-1) + x(u-1)^2) - \cdots$$

A straightforward estimate yields

$$\|d\log_u\| \leq \frac{1}{1 - \|u-1\|}.$$

Let us set $\exp(a) = e^{a_{\mathcal{H}}} e^{a_{\mathcal{V}}}$. Then $d\gamma_a(x) = x - d\log_{\exp(a)}(d\exp_a(x))$, and

$$\begin{aligned}\|d\gamma_a(x)\| &\leq \|x - d\log_{\exp(a)}(x)\| + \|d\log_{\exp(a)}[x - d\exp_a(x)]\| \\ &\leq \|x\| \frac{\|\exp(a) - 1\|}{1 - \|\exp(a) - 1\|} + \frac{1}{1 - \|\exp(a) - 1\|}\|x - d\exp_a(x)\|.\end{aligned}$$

Let us estimate $\|x - d\exp_a(x)\|$. As above,

$$d\exp_a(x) = \frac{d}{dt} e^{a_{\mathcal{H}} + tx_{\mathcal{H}}}\big|_{t=0} e^{a_{\mathcal{V}}} + e^{a_{\mathcal{H}}} \frac{d}{dt} e^{a_{\mathcal{V}} + tx_{\mathcal{V}}}\big|_{t=0}.$$

Note that $\frac{d}{dt}e^{a_{\mathcal{H}}+tx_{\mathcal{H}}}\big|_{t=0}e^{a_{\mathcal{V}}}$ is equal to

$$\left(x_{\mathcal{H}}+\frac{1}{2}(a_{\mathcal{H}}x_{\mathcal{H}}+x_{\mathcal{H}}a_{\mathcal{H}})++\frac{1}{6}(a_{\mathcal{H}}^2x_{\mathcal{H}}+a_{\mathcal{H}}x_{\mathcal{H}}a_{\mathcal{H}}+x_{\mathcal{H}}a_{\mathcal{H}}^2)+\dots\right)e^{a_{\mathcal{V}}}$$
$$=(x_{\mathcal{H}}+R_{2,a_{\mathcal{H}}}(x_{\mathcal{H}}))(1+(e^{a_{\mathcal{V}}}-1))$$
$$=x_{\mathcal{H}}+x_{\mathcal{H}}(e^{a_{\mathcal{V}}}-1)+R_{2,a_{\mathcal{H}}}(x_{\mathcal{H}})e^{a_{\mathcal{V}}}.$$

Analogously,

$$e^{a_{\mathcal{H}}}\frac{d}{dt}e^{a_{\mathcal{V}}+tx_{\mathcal{V}}}|_{t=0}=x_{\mathcal{V}}+(e^{a_{\mathcal{H}}}-1)x_{\mathcal{V}}+e^{a_{\mathcal{H}}}R_{2,a_{\mathcal{V}}}(x_{\mathcal{V}}).$$

Therefore $\|x-d\exp_a(x)\|$ is bounded by

$$\|x_{\mathcal{H}}\|\|e^{a_{\mathcal{V}}}-1\|+\|R_{2,a_{\mathcal{H}}}(x_{\mathcal{H}})\|+\|x_{\mathcal{V}}\|\|e^{a_h}-1\|+\|R_{2,a_{\mathcal{V}}}(x_{\mathcal{V}})\|.$$

Note that $\|R_{2,b}(y)\|\le\|y\|(e^{\|b\|}-1)$. Recall the estimates $\|e^z-1\|\le 2\sin(\|z\|/2)$, $\|z_{\mathcal{V}}\|\le\|z\|$ and $\|z_{\mathcal{H}}\|\le 2\|z\|$ for $z\in\mathcal{A}_{ah}$. Therefore

$$\|x-d\exp_a(x)\|\le 2\|x_{\mathcal{H}}\|\sin(\|a_{\mathcal{V}}\|/2)+2\|x_{\mathcal{V}}\|\sin(\|a_{\mathcal{H}}\|/2)$$
$$+\|x_{\mathcal{H}}\|(e^{\|a_{\mathcal{H}}\|}-1)+\|x_{\mathcal{V}}\|(e^{\|a_{\mathcal{V}}\|}-1)$$
$$\le\|x\|\{4\sin(\|a\|/2)+2\sin(\|a\|)+2e^{2\|a\|}+e^{\|a\|}-3\}.$$

We also need to estimate $\|\exp(a)-1\|$ in terms of $\|a\|$:

$$\|\exp(a)-1\|=\|e^{a_{\mathcal{H}}}e^{a_{\mathcal{V}}}-1\|\le\|e^{a_{\mathcal{H}}}e^{a_{\mathcal{V}}}-e^{a_{\mathcal{V}}}\|+\|e^{a_{\mathcal{V}}}-1\|$$
$$\le 2\sin(\|a\|)+2\sin(\|a\|/2).$$

It follows that $\|d\gamma_a\|$ is bounded by

$$\frac{1}{1-2\sin(\|a\|)-2\sin(\|a\|/2)}\left\{6\sin(\|a\|/2)+4\sin(\|a\|)+2e^{2\|a\|}+e^{\|a\|}-3\right\}.$$

Therefore, if $\|a\|<0.036$, then $\|d\gamma(a)\|<1/2$. As in the proof of the Inverse Function Theorem [30], this implies that $\nu:B_r(0)\to B_{r/2}(0)$ is a diffeomorphism. □

For $z\in\mathcal{A}_{ah}$ with small norm, as in our case, we have in fact the equality

$$\|e^z-1\|=2\sin(\|z\|/2).$$

Therefore $z\in B_{r/2}(0)$ if and only if $\|e^z-1\|<2\sin(r/4)$. In other words, the above lemma states that the map

$$\exp:B_r(0)\to\{u\in U_{\mathcal{A}}:\|u-1\|<2\sin(r/4)\}$$

is an analytic diffeomorphism for $r=0.036$.

For the uniqueness part we shall need the following result:

Lemma 3.3. *Let $\xi,\xi'\in\mathcal{H}'_{v_0}$ be such that $e^{\xi}v_0=e^{\xi'}v_0$ with $\|\xi\|,\|\xi'\|<d=\ln(5/4)$. Then $\xi=\xi'$.*

Proof. Let $\mu : \mathcal{A} \to \mathcal{A}$ be the (real) linear map

$$\mu(x) = xv_0^* - v_0 x^* \bar{p_0}.$$

Clearly $\|\mu(x)\| \leq 2\|x\|$. If $x \in \mathcal{A}$, let $x = x_h + x_{ah}$ be the decomposition of x into its hermitian and antihermitian parts. Let

$$\begin{aligned}&\theta : \mathcal{A} \to \mathcal{A},\\ &\theta(x) = \mu\big(E\left(x_{ah} - \bar{p_0} x_{ah} \bar{p_0}\right) v_0\big) + \bar{p_0} x_{ah} \bar{p_0} + x_h - v_0,\end{aligned}$$

where E is the usual exponential. Clearly this map is C^∞. The differential of θ at the origin is the identity. Indeed,

$$\begin{aligned}d\theta_0(x) &= \mu\big(dE_0(x_{ah} - \bar{p_0} x_{ah} \bar{p_0}) v_0\big) + \bar{p_0} x_{ah} \bar{p_0} + x_h\\ &= \mu(x_{ah} v_0) + \bar{p_0} x_{ah} \bar{p_0} + x_h\\ &= x_{ah} p_0 - p_0 x_{ah}^* \bar{p_0} + \bar{p_0} x_{ah} \bar{p_0} + x_h\\ &= x_{ah} p_0 + x_{ah} \bar{p_0} + x_h = x.\end{aligned}$$

It follows that θ is a local diffeomorphism with $\theta(0) = 0$. We proceed as in the previous lemma, considering the auxiliary map $\gamma(x) = x - \theta(x)$ and estimating $\|d\gamma_a\|$. Now

$$\begin{aligned}d\gamma_a(x) = x - d\theta_a(x) &= x_{ah} - \bar{p_0} x_{ah} \bar{p_0} - \mu(dE_{a_{ah}}(x_{ah}) v_0)\\ &= \mu\big(x_{ah} v_0 - dE_{a_{ah}}(x_{ah}) v_0\big).\end{aligned}$$

Therefore

$$\|d\gamma_a(x)\| \leq 2\|x_{ah} - dE_{a_{ah}}(x_{ah})\| \leq 2\|x\|(e^{\|a\|} - 1).$$

Thus $\|d\gamma_a\| < 1/2$ if $\|a\| < \ln(5/4)$. As above, this implies that $\theta : B_d(0) \to B_{d/2}(0)$ is a diffeomorphism for $d = \ln(5/4)$. Note that the map $\theta + v_0$ restricted to $\mathcal{H}'_{v_0} \cap B_d(0)$,

$$\theta + v_0 : \mathcal{H}'_{v_0} \cap B_d(0) \to \mu(E(\mathcal{H}'_{v_0} v_0)),\ \xi \mapsto e^{\xi} v_0,$$

is bijective. Therefore, if $\xi, \xi' \in \mathcal{H}'_{v_0}$ with $\|\xi\|, \|\xi'\| < d = \ln(5/4)$ satisfy $e^{\xi} v_0 = e^{\xi'} v_0$, then $\mu(e^{\xi} v_0) = \mu(e^{\xi'} v_0)$, that is, $\theta(\xi) = \theta(\xi')$, and therefore $\xi = \xi'$. $\square$

Note that $\ln(5/4) \simeq 0.223$, which is larger than the value of r that we estimated. We can summarize both lemmas as follows:

Corollary 3.4. *Let $r > 0$ be such that* exp *is a diffeomorphism on $B_r(0)$ (with $r \leq d$). If a partial isometry $v \in \mathcal{I}^{q_0}$ satisfies that $\|v - v_0\| < \epsilon = \frac{2}{\sqrt{3}} \sin(r/4)$, then there exists a unique $\xi \in \mathcal{H}'_{v_0}$ such that $v = e^{\xi} v_0$ with $\|\xi\| < 2r$.*

Proof. Suppose that $\|v - v_0\| < \epsilon$. This implies that $\|vv^* - v_0v_0^*\| < 2\epsilon \leq 1$. It follows [6] [36] that there exists a unitary element $w \in U_{\mathcal{A}}$ such that $wv_0v_0^*w^* = vv^*$ and $\|w - 1\| \leq \|vv^* - v_0v_0^*\| < 2\epsilon$. Let $s = vv_0^* + (1 - vv^*)w$. It is clear that this element is also a unitary of $\mathcal{A}$ and verifies

$$sv_0 = v.$$

Note that $1 - s = v(v^* - v_0^*) + (1 - vv^*)(1 - w)$. Suppose that $\mathcal{A}$ is faithfully represented in H. Since v and $1 - vv^*$ have orthogonal ranges, if $\eta \in H$,

$$\|(s-1)\eta\|^2 = \|v(v^* - v_0^*)\eta\|^2 + \|(1 - vv^*)(1 - w)\eta\|^2.$$

It follows that

$$\|s - 1\| < \sqrt{3}\epsilon = 2\sin(r/4).$$

Hence there exists a unique $a \in B_r(0)$, with $a = a_{\mathcal{H}} + a_{\mathcal{V}}$ as above, such that $s = \exp(a) = e^{a_{\mathcal{H}}}e^{a_{\mathcal{V}}}$. Note that $a_{\mathcal{V}}v_0 = 0$. Therefore

$$v = sv_0 = e^{a_{\mathcal{H}}}e^{a_{\mathcal{V}}}v_0 = e^{a_{\mathcal{H}}}v_0,$$

with $\|a_{\mathcal{H}}\| \leq 2\|a\| < 2r$. Take $\xi = a_{\mathcal{H}}$; then ξ is clearly unique. $\square$

The above result shows that the value R in Proposition 3.1 satisfies $R \geq 0.0034$.

We combine these facts in our main result, which is as follows:

THEOREM 3.5. *Let $r > 0$ be such that* exp *is a diffeomorphism on $B_r(0)$ ($r \leq d$). If $v, v_0 \in \mathcal{I}$ satisfy $\|v - v_0\| < r' = \min\{1/2, \frac{2}{4\sqrt{3}}\sin(r/4)\}$, then there exists a unique geodesic with velocity vector $\xi = (\xi_1, \xi_2)$ such that $\|\xi_1\| < r'$, $\|\xi_2\| < \pi$, which joins v and v_0.*

4. Partial unitaries

Let us call *partial unitary* a partial isometry w such that the initial and final spaces coincide. Equivalently, $w^*w = ww^* = q$, or w is a unitary element of the reduced algebra $q\mathcal{A}q$. Let

$$\mathcal{I}_\Delta = \{w \in \mathcal{I} : ww^* = w^*w\} = \bigcup_{q \in \mathcal{P}} U_{q\mathcal{A}q}.$$

In [1] it is shown that this set $\mathcal{I}_\Delta$, or rather its connected components, are C^∞ submanifolds of $\mathcal{I}$ (and of $\mathcal{A}$). In this section we study the properties of this set.

First let us describe the tangent space of $\mathcal{I}_\Delta$ at w. Differentiating $ww^* = w^*w$, we get

$$(T\mathcal{I}_\Delta)_w = \{z \in T\mathcal{I}_w : z^*w + w^*z = zw^* + wz^*\}.$$

An alternative description is

$$(T\mathcal{I}_\Delta)_w = \{z \in T\mathcal{I}_w : w^*z\bar{q} = wz^*\bar{q}\},$$

where $q = w^*w = ww^*$. Indeed, suppose first that $z \in T\mathcal{I}_w$ satisfies $w^*z\bar{q} = wz^*\bar{q}$. We have $z = xw - wy$ for some $x, y \in \mathcal{A}_{ah}$. Then $wz^* + zw^* = xq - qx$ and $w^*z + z^*w = yq - qy$, and so $wz^*\bar{q} = -qx\bar{q}$ and $w^*z\bar{q} = -qy\bar{q}$. Our assumption $w^*z\bar{q} = wz^*\bar{q}$ then implies $qx\bar{q} = qy\bar{q}$. Since x and y are antihermitian, this implies that $xq-qx = yq-qy$, i.e., $wz^*+zw^* = w^*z+z^*w$. On the other hand, multiplying this relation on the right by $\bar{q}$ yields $wz^*\bar{q} = w^*z\bar{q}$.

We now construct a natural principal bundle over $\mathcal{I}_\Delta$ (which is not a homogeneous space). Fix a projection p in $\mathcal{A}$. Let

$$\Delta = \Delta^p = \{(\alpha, \beta) \in U_\mathcal{A} \times U_\mathcal{A} : p^\alpha = p^\beta\}.$$

First note that Δ is not a subgroup of $U_\mathcal{A} \times U_\mathcal{A}$. It inherits a right action from the group $U_\mathcal{A} \cap \{p\}' \times U_\mathcal{A} \cap \{p\}'$. In fact, we shall be interested in a subgroup of this group and its right action on Δ. Consider the map

$$\pi_p^\Delta : \Delta \to \mathcal{I}_\Delta, \quad \pi_p^\Delta(\alpha, \beta) = \alpha p \beta^*,$$

given by the restriction to Δ of the map π_p defined in Section 2. Note that this map is indeed well defined, i.e., takes values in $\mathcal{I}_\Delta$: If $w = \alpha p \beta^*$ with $(\alpha, \beta) \in \Delta$, then $ww^* = p^\alpha = p^\beta = w^*w$. Clearly this map is C^∞. Let us examine the fibre of π_p^Δ over p. If $\alpha p \beta^* = p$, then elementary computations show that α and β commute with p and $\alpha p = \beta p$. Let us set

$$G_p = \{(g_1, g_2) \in \Delta : g_1 p = p g_2\}.$$

It is clear that G_p is a (Banach-Lie) subgroup of $U_\mathcal{A} \cap \{p\}' \times U_\mathcal{A} \cap \{p\}'$, and therefore acts on Δ by right multiplication. Also, it is clear that this action is free.

The range of π_p^Δ consists of all partial unitaries w such that their unit $ww^* = w^*w$ is unitarily equivalent to p. This set fills connected components (see [1]).

Theorem 4.1. *The set Δ is a C^∞ submanifold of $\mathcal{A} \times \mathcal{A}$, and the map*

$$\pi_p^\Delta : \Delta \to \mathcal{I}_\Delta, \quad \pi_p^\Delta(\alpha, \beta) = \alpha p \beta^*$$

is a C^∞ principal bundle with structure group G_p.

Proof. The map $\pi : U_\mathcal{A} \to \mathcal{P}$, $\pi(u) = p^u$ is a C^* submersion [13]. Therefore the map $\pi \times \pi : U_\mathcal{A} \times U_\mathcal{A} \to \mathcal{P} \times \mathcal{P}$, $(u, v) \mapsto (p^u, p^v)$ is also a submersion. The subset $D = \{(q, q) : q \in \mathcal{P}\} \subset \mathcal{P} \times \mathcal{P}$ is clearly a submanifold of $\mathcal{P} \times \mathcal{P}$. It follows that $\Delta = (\pi \times \pi)^{-1}(D)$ is a submanifold of $U_\mathcal{A} \times U_\mathcal{A}$.

To prove that the map π_p^Δ is a principal bundle, it will suffice to construct smooth local cross sections. Fix $w_0 = \alpha_0 p \beta_0^*$. Let $w \in \mathcal{I}_\Delta$ be such that $\|w - w_0\| < 1/2$. Then there exists a unitary $u = u(w_0, w) \in U_\mathcal{A}$, which is a C^∞ function on the parameter w such that $u\alpha_0 w_0^* w_0 \alpha_0^* u^* = p$ [1]. Let

$w' = u\alpha_0 w\alpha_0^* u^*$. Clearly w' is a unitary of $p\mathcal{A}p$. Put

$$\alpha' = w'^2 + 1 - p, \quad \beta' = w' + 1 - p.$$

Then α', β' are unitaries in $\mathcal{A}$ which verify $\alpha' p \beta'^* = w'$. Finally, put

$$\alpha = \alpha_0^* u^* \alpha', \quad \beta = \alpha_0^* u^* \beta'.$$

A straightforward computation shows that $\alpha p \beta^* = w$. Also note that

$$p^\alpha = \alpha_0^* u^* \alpha' p \alpha'^* u \alpha_0 = \alpha_0 u^* p u \alpha_0 = w_0^* w_0,$$

and analogously $p^\beta = w_0^* w_0$. It follows that $(\alpha, \beta) \in \Delta$. In other words, we have found a C^∞ cross section $\sigma_{w_0}(w) = (\alpha, \beta) = (\alpha_w, \beta_w)$ for π_p^Δ, defined on the neighbourhood $\mathcal{U}_{w_0} = \{w \in \mathcal{I}_\Delta : \|w - w_0\| < 1/2\}$ of w_0 in $\mathcal{I}_\Delta$. These cross sections provide equivariant trivializations for π_p^Δ in a standard fashion [25]. $\square$

Next, we shall introduce a connection in this principal bundle. To do so, we must first compute the tangent spaces of Δ and the fibres $(\pi_p^\Delta)^{-1}(\{w\})$. Let $(\alpha(t), \beta(t))$ be a smooth curve in Δ with $\alpha(0) = \alpha$, $\beta(0) = \beta$, $\dot\alpha(0) = a$, $\dot\beta(0) = b$. First note that since $\alpha(t), \beta(t)$ are unitaries, $\alpha^* a, a\alpha^*, \beta^* b, b\beta^*$ belong to $\mathcal{A}_{ah}$. Next, differentiating $\alpha(t) p \alpha^*(t) = \beta(t) p \beta^*(t)$ at $t = 0$ shows that $ap\alpha^* + \alpha p a^* = bp\beta^* + \beta p b^*$. Note that

$$ap\alpha^* + \alpha p a^* = (a\alpha^*) p^\alpha + p^\alpha (\alpha a^*) = (a\alpha^*) p^\alpha + p^\alpha(-a\alpha^*) = [a\alpha^*, p^\alpha],$$

and analogously $bp\beta^* + \beta p b^* = [b\beta^*, \beta p \beta^*]$. Since $\alpha p \alpha^* = \beta p \beta^*$, we have

$$(T\Delta)_{(\alpha,\beta)} = \{(a, b) \in \mathcal{A} \times \mathcal{A} : a\alpha^*, b\beta^* \in \mathcal{A}_{ah}, [a\alpha^* - b\beta^*, \alpha p \alpha^*] = 0\}.$$

The tangent space of the fibre of π_p^Δ over $\alpha p \beta^*$ can be computed in a similar way. Let $(\omega_1(t), \omega_2(t))$ be a curve in Δ such that $\omega_1(t) p \omega_2^*(t) = \alpha p \beta^*$, with $\dot\omega_1(0) = x, \dot\omega_2(0) = y$. This implies that $(\alpha^* \omega_1(t), \beta^* \omega_2(t)) \in G_p$. Therefore, denoting by $\Omega_{(\alpha,\beta)}$ the "vertical" space over $\alpha p \beta^*$, that is, the tangent space at (α, β) of the fibre of π_p^Δ, we get

$$\Omega_{(\alpha,\beta)} = \{(x, y) \in \mathcal{A} \times \mathcal{A} : \alpha^* x, \beta^* y \in \mathcal{A}_{ah} \cap \{p\}', \text{ and } \alpha^* x p = \beta^* y p\}.$$

Equivalently, $\Omega_{(\alpha,\beta)} = \{(x, y) : (\alpha^* x, \beta^* y) \in \mathcal{G}_p\}$, where $\mathcal{G}_p$ denotes the Lie algebra of G_p.

A connection in the principal bundle consists of a distribution

$$(\alpha, \beta) \mapsto \mathcal{K}_{(\alpha,\beta)}$$

of subspaces of $(T\Delta)_{(\alpha,\beta)}$ with the following properties:

(1) $\mathcal{K}_{(\alpha,\beta)} \oplus \Omega_{(\alpha,\beta)} = (T\Delta)_{(\alpha,\beta)}$.
(2) The distribution is smooth, i.e., if (A, B) is a smooth tangent vector field on a neighbourhood of (α, β) in Δ, then the vector $(A_\mathcal{K}, B_\mathcal{K})$, which is the (pointwise) projection of (A, B) onto $\mathcal{K}$, is also smooth.

(3) The distribution is equivariant under the action of G_p. Note that because the right action of G_p on Δ is in fact the restriction of a linear action, this property is equivalent to $\mathcal{K}_{(\alpha g_1,\beta g_2)} = \mathcal{K}_{(\alpha,\beta)} \cdot (g_1, g_2)$ for any $(g_1, g_2) \in G_p$.

An element $\alpha^* x$ commutes with p if and only if $x\alpha^*$ commutes with p^α, and analogously for $\beta^* y$. Therefore it is clear that $\Omega_{(\alpha,\beta)} \subset (T\Delta)_{(\alpha,\beta)}$. Moreover,

$$\Omega_{(\alpha,\beta)} \subset \mathcal{Z} \subset (T\Delta)_{(\alpha,\beta)},$$

where

$$\mathcal{Z} := \{(z_1, z_2) : z_1\alpha^*, z_2\beta^* \in \mathcal{A}_{ah} \text{ and commute with } p^\alpha\}.$$

In order to find a complement for $\Omega_{(\alpha,\beta)}$ in $(T\Delta)_{(\alpha,\beta)}$ we shall construct a complement for Ω in $\mathcal{Z}$ and add it to a complement of $\mathcal{Z}$ in $(T\Delta)_{(\alpha,\beta)}$.

Observe that $(z_1, z_2) \in \mathcal{Z}$ if and only if $\alpha^* z_1$ and $\beta^* z_2$ are antihermitian elements of $\mathcal{A}$ which commute with p. Therefore a natural complement for $\Omega_{(\alpha,\beta)}$ in $\mathcal{Z}$ consists of all pairs of matrices (in terms of p) of the form

$$\alpha \begin{pmatrix} c_{11} & 0 \\ 0 & 0 \end{pmatrix}, \ \beta \begin{pmatrix} -c_{11} & 0 \\ 0 & 0 \end{pmatrix}.$$

Next, we find a complement for $\mathcal{Z}$ inside $(T\Delta)_{(\alpha,\beta)}$, or, equivalently, a complement for $\mathcal{Z} \cdot (\alpha^*, \beta^*)$ inside $(T\Delta)_{(\alpha,\beta)} \cdot (\alpha^*, \beta^*)$. These subspaces of $\mathcal{A}_{ah} \times \mathcal{A}_{ah}$ are, respectively,

$$\{(v_1, v_2) : [v_i, p^\alpha] = 0, i = 1, 2\} \text{ and } \{(x_1, x_2) : [x_1 - x_2, p^\alpha] = 0\}.$$

The condition $[x_1 - x_2, p^\alpha] = 0$ means that in the decomposition (in terms of p^α) into diagonal and codiagonal matrices, the elements x_1 and x_2 have the same codiagonal part. A natural complement for $\mathcal{Z}$ in $(T\Delta)_{(\alpha,\beta)}$ is the set

$$\{(a, a) : a \in \mathcal{A}_{ah} \text{ is codiagonal with respect to } p^\alpha\} \cdot (\alpha, \beta).$$

Instead of describing explicitly the complement $\mathcal{K}_{(\alpha,\beta)}$, let us write down the projection $P_{(\alpha,\beta)} = P_{\mathcal{K}_{(\alpha,\beta)}}$ that corresponds to this decomposition of $(T\Delta)_{(\alpha,\beta)}$. After routine calculations we get

$$P_{(\alpha,\beta)}(x, y) = (\tilde{x}, \tilde{y}),$$

where

$$\tilde{x} = \frac{1}{2}(p^\alpha x - \alpha p\beta^* y)p + p^\alpha x(1 - p) + (1 - p^\alpha)xp, \tag{4.1}$$

$$\tilde{y} = \frac{1}{2}(p^\beta y - \beta p\alpha^* x)p + p^\beta y(1 - p) + (1 - p^\beta)yp. \tag{4.2}$$

It is clear that the map $(\alpha, \beta) \mapsto P_{(\alpha,\beta)}$, as a map from Δ to $B_{\mathbb{R}}(\mathcal{A})$, is C^∞. Thus the smoothness requirement is fulfilled. Moreover, if $(g_1, g_2) \in G_p$, then $(\alpha g_1)p(\beta g_2)^* = \alpha p\beta^*$, $\beta g_2 p(\alpha g_1)^* = ((\alpha g_1 p(\beta g_2)^*)^* = (\alpha p\beta^*)^* = \beta p\alpha^*$

and $\alpha g_1 p(\alpha g_1)^* = \beta g_2 p(\beta g_2)^* = \alpha p \alpha^*$. Therefore, if $x, y \in (T\Delta)_{(\alpha,\beta)}$ and $P_{\alpha,\beta}(x,y) = (\tilde{x}, \tilde{y})$ are as above, then

$$(\tilde{x}, \tilde{y}) \cdot (g_1, g_2) = P_{\alpha g_1, \beta g_2}(x g_1, y g_2)$$

(where $(xg_1, yg_2) \in (T\Delta)_{(\alpha g_1, \beta g_2)}$), i.e., the distribution $(\alpha, \beta) \mapsto \mathcal{K}_{(\alpha,\beta)}$ is G_p-invariant. Therefore we have the following result:

Proposition 4.2. *Let $\mathcal{K}_{(\alpha,\beta)} = R(P_{(\alpha,\beta)})$. Then the distribution*

$$(\alpha, \beta) \mapsto \mathcal{K}_{(\alpha,\beta)}$$

defines a connection in the principal bundle π_p^Δ.

The differential of π_p^Δ at (α, β) is the map

$$\delta_{\alpha,\beta} : (T\Delta)_{(\alpha,\beta)} \to (T\mathcal{I}_\Delta)_w, \ \delta_{\alpha,\beta}(a,b) = ap\beta^* + \alpha p b^*,$$

where $w = \alpha p \beta^*$. Note that $\Omega_{(\alpha,\beta)}$ is the kernel of $\delta_{\alpha,\beta}$. It follows that

$$\delta_{\alpha,\beta}\big|_{\mathcal{K}_{\alpha,\beta}} : \mathcal{K}_{\alpha,\beta} \to (T\mathcal{I}_\Delta)_w$$

is a (real) linear isomorphism. A useful description for the connection is the distribution of the inverses $\kappa_{\alpha,\beta} := \left(\delta_{\alpha,\beta}\big|_{\mathcal{K}_{\alpha,\beta}}\right)^{-1}$, given by

$$\kappa_{\alpha,\beta} : (T\mathcal{I}_\Delta)_w \to \mathcal{K}_{\alpha,\beta}, \ \kappa_{\alpha,\beta}(z) = (\hat{x}, \hat{y}),$$

where

$$\begin{aligned} \hat{x} &= \frac{1}{2}\, p^\alpha z \beta p + (1 - p^\alpha) z \beta p - \beta p \alpha^* z \alpha (1-p), \\ \hat{y} &= -\frac{1}{2} \beta p \alpha^* z \beta p - \beta p \alpha^* z \beta (1-p) + (1-p^\alpha) z \alpha p. \end{aligned}$$

Let us conclude this section by computing the horizontal lifting differential equation of this connection. Fix $w = \alpha p \beta^*$ and a piecewise C^1 curve $\gamma(t) \in \mathcal{I}_\Delta$, $t \in [0,1]$, with $\gamma(0) = w$. We seek a piecewise C^1 curve $\Gamma(t) = (\Gamma_1(t), \Gamma_2(t)) \in \Delta$ such that Γ lifts γ and $\dot{\Gamma}$ is horizontal, i.e.,

$$\pi_p^\Delta(\Gamma(t)) = \Gamma_1(t) p \Gamma_2^*(t) = \gamma(t)$$

and

$$\dot{\Gamma}(t) \in \mathcal{K}_{\Gamma(t)}, t \in [0,1].$$

Differentiating the first condition, we get $\delta_{\Gamma(t)}(\dot{\Gamma}(t)) = \dot{\gamma}(t)$, and since $\dot{\Gamma}(t) \in \mathcal{K}_{\Gamma(t)}$, applying $\kappa_{\Gamma(t)}$ we obtain the differential equation

$$\dot{\Gamma}(t) = \kappa_{\Gamma(t)}(\dot{\gamma}(t)), \tag{4.3}$$

or explicitly (omitting the parameter t)

$$\dot{\Gamma_1} = \frac{1}{2}\Gamma_1 p\Gamma_1^*\dot{\gamma}\Gamma_2 p + (1-\Gamma_1 p\Gamma_1^*)\dot{\gamma}\Gamma_2 p - \Gamma_2 p\Gamma_1^*\dot{\gamma}\Gamma_1(1-p),$$
$$\dot{\Gamma_2} = -\frac{1}{2}\Gamma_2 p\Gamma_1^*\dot{\gamma}\Gamma_2 p + (1-\Gamma_1 p\Gamma_1^*)\dot{\gamma}\Gamma_1 p - \Gamma_2 p\Gamma_1^*\dot{\gamma}\Gamma_2(1-p).$$

If we make the (a posteriori) assumption that Γ lifts γ, these equations may be rewritten, using the relations $\Gamma_1 p\Gamma_2^* = \gamma$ and $\Gamma_1 p\Gamma_1^* = \gamma^*\gamma = \gamma\gamma^* = \Gamma_2 p\Gamma_2^*$, in the form

$$\dot{\Gamma_1} = \left\{\frac{1}{2}\gamma^*\gamma\dot{\gamma}\gamma^* + (1-\gamma^*\gamma)\dot{\gamma}\gamma^* - \gamma^*\dot{\gamma}(1-\gamma^*\gamma)\right\}\Gamma_1, \tag{4.4}$$

$$\dot{\Gamma_2} = \left\{-\frac{1}{2}\gamma^*\dot{\gamma}\gamma^*\gamma + (1-\gamma^*\gamma)\dot{\gamma}\gamma - \gamma^*\dot{\gamma}(1-\gamma^*\gamma)\right\}\Gamma_2. \tag{4.5}$$

We shall use the latter equations, and prove that their solutions lift γ horizontally. Note that these equations are linear, and therefore existence and uniqueness under initial conditions are guaranteed.

We need the following result (see [32]): If $\dot{\Omega} = \Sigma\Omega$ is a linear differential equation in $\mathcal{A}$ such that $\Omega(t_0) \in U_{\mathcal{A}}$ and $\Sigma \in \mathcal{A}_{ah}$, then $\Omega(t) \in U_{\mathcal{A}}$ for all t.

To apply this result, we require the following lemma.

Lemma 4.3. *Let γ be a piecewise C^1 curve in $\mathcal{I}_\Delta$. Then both*

$$\frac{1}{2}\gamma^*\gamma\dot{\gamma}\gamma^* + (1-\gamma^*\gamma)\dot{\gamma}\gamma^* - \gamma^*\dot{\gamma}(1-\gamma^*\gamma)$$

and

$$-\frac{1}{2}\gamma^*\dot{\gamma}\gamma^*\gamma + (1-\gamma^*\gamma)\dot{\gamma}\gamma^* - \gamma^*\dot{\gamma}(1-\gamma^*\gamma)$$

lie in $\mathcal{A}_{ah}$.

Proof. Note that both terms share the last two summands. To prove the lemma, it suffices to show that

$$\frac{1}{2}\gamma^*\gamma\dot{\gamma}\gamma^*,\ -\frac{1}{2}\gamma^*\dot{\gamma}\gamma^*\gamma \text{ and } (1-\gamma^*\gamma)\dot{\gamma}\gamma^* - \gamma^*\dot{\gamma}(1-\gamma^*\gamma)$$

lie in $\mathcal{A}_{ah}$. We first deal with the last term. Differentiating $(1-\gamma^*\gamma)\gamma = 0$, we get

$$(1-\gamma^*\gamma)^{\cdot}\gamma + (1-\gamma^*\gamma)\dot{\gamma} = 0.$$

Multiplying by γ^* on the right (note that γ and γ^* commute), we obtain

$$(\gamma^*\gamma)^{\cdot}\gamma^*\gamma = -(1-\gamma^*\gamma)^{\cdot}\gamma\gamma^* = (1-\gamma^*\gamma)\dot{\gamma}\gamma^*.$$

Analogously, we have

$$-\gamma^*\gamma(\gamma^*\gamma)^{\cdot} = -\gamma^*\dot{\gamma}(1-\gamma^*\gamma).$$

Therefore,

$$(1-\gamma^*\gamma)\dot{\gamma}\gamma^* - \gamma^*\dot{\gamma}(1-\gamma^*\gamma) = (\gamma^*\gamma)^{\cdot}\gamma^*\gamma - \gamma^*\gamma(\gamma^*\gamma)^{\cdot},$$

which is antihermitian because $(\gamma^*\gamma)^{\cdot}$ is selfadjoint, since it is the derivative of a curve of selfadjoints. Next, note that $\gamma = \gamma\gamma^*\gamma$ implies

$$\dot{\gamma} = \dot{\gamma}\gamma^*\gamma + \gamma\dot{\gamma}^*\gamma + \gamma\gamma^*\dot{\gamma}. \tag{4.6}$$

Multiplying this relation by γ^* on the right yields

$$\dot{\gamma}\gamma^* = \dot{\gamma}\gamma^* + \gamma\dot{\gamma}^*\gamma\gamma^* + \gamma\gamma^*\dot{\gamma}\gamma^*,$$

i.e.,

$$\gamma\gamma^*\dot{\gamma}\gamma^* = -\gamma\dot{\gamma}^*\gamma\gamma^* = -(\gamma\gamma^*\dot{\gamma}\gamma^*)^*.$$

Analogously, we obtain $\gamma^*\dot{\gamma}\gamma^*\gamma \in \mathcal{A}_{ah}$. □

THEOREM 4.4. *Let γ be a piecewise C^1 curve in $\mathcal{I}_\Delta$ with $\gamma(t_0) = w = \alpha p\beta$. Let Γ_1, Γ_2 be the unique solutions of, respectively,*

$$\dot{\Gamma_1} = \left\{\frac{1}{2}\gamma^*\gamma\dot{\gamma}\gamma^* + (1-\gamma^*\gamma)\dot{\gamma}\gamma^* - \gamma^*\dot{\gamma}(1-\gamma^*\gamma)\right\}\Gamma_1$$

and

$$\dot{\Gamma_2} = \left\{-\frac{1}{2}\gamma^*\dot{\gamma}\gamma^*\gamma + (1-\gamma^*\gamma)\dot{\gamma}\gamma - \gamma^*\dot{\gamma}(1-\gamma^*\gamma)\right\}\Gamma_2,$$

with initial conditions $\Gamma_1(t_0) = \alpha$ and $\Gamma_2(t_0) = \beta$. Then $\Gamma = (\Gamma_1, \Gamma_2)$ is the horizontal lifting of γ with $\Gamma(t_0) = (\alpha, \beta)$.

Proof. Set

$$\Sigma_1 = \frac{1}{2}\gamma^*\gamma\dot{\gamma}\gamma^* + (1-\gamma^*\gamma)\dot{\gamma}\gamma^* - \gamma^*\dot{\gamma}(1-\gamma^*\gamma)$$

and

$$\Sigma_2 = -\frac{1}{2}\gamma^*\dot{\gamma}\gamma^*\gamma + (1-\gamma^*\gamma)\dot{\gamma}\gamma - \gamma^*\dot{\gamma}(1-\gamma^*\gamma).$$

The above lemmas show that Γ_1, Γ_2 lie in $U_{\mathcal{A}}$. Let us prove that Γ lifts γ, i.e., that $\Gamma_1 p\Gamma_2^* = \gamma$, or, equivalently, $\Gamma_1^*\gamma\Gamma_2 = p$. Differentiating the right hand side of this relation, we obtain

$$(\Gamma_1^*\dot{\gamma}\Gamma_2) = \dot{\Gamma}_1^*\gamma\Gamma_2 + \Gamma_1^*\dot{\gamma}\Gamma_1 + \Gamma_1^*\gamma\dot{\Gamma_2} = \Gamma_1^*(-\Sigma_1\gamma + \dot{\gamma} + \gamma\Sigma_2)\Gamma_2.$$

If we set $q = \gamma^*\gamma = \gamma\gamma^*$, then

$$-\Sigma_1\gamma + \dot{\gamma} + \gamma\Sigma_2 = -\frac{1}{2}q\dot{\gamma}q - \bar{q}\dot{\gamma}q + \dot{\gamma} - \frac{1}{2}q\dot{\gamma}q - q\dot{\gamma}\bar{q} = \bar{q}\dot{\gamma}\bar{q}.$$

Note that $\gamma\bar{q} = \gamma(1-\gamma^*\gamma) = 0$, and similarly $\bar{q}\gamma = 0$. Therefore

$$0 = \frac{d}{dt}\{\bar{q}\gamma\bar{q}\} = -\dot{q}\gamma\bar{q} + \bar{q}\dot{\gamma}\bar{q} - \bar{q}\gamma\dot{q} = \bar{q}\dot{\gamma}\bar{q}.$$

Hence $(\Gamma_1^*\dot{\gamma}\Gamma_2) = 0$ and $\Gamma_1^*(t_0)\gamma(t_0)\Gamma_2(t_0) = p$, and thus $\Gamma_1 p\Gamma_2^* = \gamma$. Note that this implies, in particular, that $\Gamma \in \Delta$: Indeed, we have

$$\Gamma_1 p\Gamma_1^* = \Gamma_1 p\Gamma_2^*(\Gamma_2 p\Gamma_1^*) = \gamma\gamma^* = \gamma^*\gamma = \Gamma_2 p\Gamma_2^*.$$

Finally, we check that Γ is horizontal. Since we know that Γ lifts γ, we can reverse the argument leading to the equations (4.4) and (4.5), and obtain that these equations are equivalent to the condition

$$\dot{\Gamma} = \kappa_\Gamma(\dot{\gamma}) \in \mathcal{K}_\Gamma,$$

i.e., that Γ is horizontal. □

5. A linear connection in $\mathcal{I}_\Delta$

In this section we introduce a linear connection in $\mathcal{I}_\Delta$. We shall use the horizontal lifting equation in order to define a parallel transport in the tangent bundle of $\mathcal{I}_\Delta$. To do this, one piece of data is still missing: We need a way to move elements from one horizontal space to another, i.e., we need a map $\mathcal{K}_{(\alpha,\beta)} \to \mathcal{K}_{(\delta,\epsilon)}$, defined for any $(\alpha, \beta), (\delta, \epsilon) \in \Delta$, which is equivariant under the action of G_p. We construct this map through an intermediate coordinate space. Consider the set

$$\mathcal{C} = \{(a, b) \in \mathcal{A}_{ah} \times \mathcal{A}_{ah} : \bar{p}a\bar{p} = \bar{p}b\bar{p} = 0, pap + pbp = 0 \text{ and } [a - b, p] = 0\}.$$

Lemma 5.1. *Let $(\alpha, \beta) \in \Delta$. Then the map*

$$c_{\alpha,\beta} : \mathcal{C} \to (\alpha^*, \beta^*) \cdot \mathcal{K}_{(\alpha,\beta)},$$
$$c_{\alpha,\beta}(a, b) = (a, pbp + (pb\bar{p} + \bar{p}bp)^{\beta^*\alpha})$$

is an isomorphism with inverse

$$c_{\alpha,\beta}^{-1}(x, y) = (x, pyp + (py\bar{p} + \bar{p}yp)^{\alpha^*\beta}).$$

Proof. That the maps above are inverses of each other is clear, in light of the fact that $\alpha^*\beta$ commutes with p. The fact that $c_{\alpha,\beta}$ maps $\mathcal{C}$ onto $(\alpha^*, \beta^*)\cdot\mathcal{K}_{(\alpha,\beta)}$ requires a proof. Note that, by (4.1) and (4.2), $(\alpha^*, \beta^*) \cdot \mathcal{K}_{(\alpha,\beta)}$ consists of pairs $(\alpha^*\tilde{x}, \beta^*\tilde{y})$ with

$$\tilde{x} = \frac{1}{2}(p^\alpha x - \alpha p\beta^* y)p + p^\alpha x(1 - p) + (1 - p^\alpha)xp$$

and

$$\tilde{y} = \frac{1}{2}(p^\beta y - \beta p\alpha^* x)p + p\beta y(1 - p) + (1 - p^\beta)yp.$$

Then we have

$$\alpha^*\tilde{x} = \frac{1}{2}p(\alpha^* x)p - \frac{1}{2}p(\beta^* y)p + p(\alpha^* x)(1 - p) + (1 - p)(\alpha^* x)p,$$

where $\alpha^* x, \beta^* y \in \mathcal{A}_{ah}$, and an analogous relation holds for $\beta^* \tilde{y}$. Given this description, observe that $(\alpha^*, \beta^*) \cdot \mathcal{K}_{(\alpha,\beta)}$ consists of pairs (r, s) of antihermitian matrices (in terms of p) satisfying the following relations:

(1) $r_{11} + s_{11} = 0$.
(2) $r_{22} = s_{22} = 0$.
(3) $r - \alpha^* \beta s \beta^* \alpha$ commutes with p.

Consider $c_{\alpha,\beta}^{-1}(r, s) = (a, b)$. Clearly the first two relations are preserved. There is an alternate description for $c_{\alpha,\beta}$ (and for $c_{\alpha,\beta}^{-1}$), namely

$$c_{\alpha,\beta}^{-1}(r, s) = (r, s^{\alpha^*\beta} + psp - ps^{\alpha^*\beta} p).$$

Hence $a - b = r - s^{\alpha^*\beta} + psp - ps^{\alpha^*\beta} p$, where $r - s^{\alpha^*\beta}$ commutes with p by the above description, and the other summands lie in $p\mathcal{A}p$. This proves that $c_{\alpha,\beta}^{-1}((\alpha^*, \beta^*) \cdot \mathcal{K}_{(\alpha,\beta)}) \subset \mathcal{C}$. An analogous argument shows that $c_{\alpha,\beta}(\mathcal{C}) \subset (\alpha^*, \beta^*) \cdot \mathcal{K}_{(\alpha,\beta)}$. □

We can now introduce the transport map

$$T_{\alpha,\beta}^{\delta,\epsilon} : \mathcal{K}_{\alpha,\beta} \to \mathcal{K}_{\delta,\epsilon} \tag{5.1}$$
$$\mathcal{K}_{(\alpha,\beta)} \overset{l_{(\alpha^*,\beta^*)}}{\longrightarrow} (\alpha^*, \beta^*) \cdot \mathcal{K}_{(\alpha,\beta)} \overset{c_{\alpha,\beta}}{\longrightarrow} \mathcal{C} \overset{c_{\delta,\epsilon}^{-1}}{\longrightarrow} (\delta^*, \epsilon^*) \cdot \mathcal{K}_{(\delta,\epsilon)} \overset{l_{(\delta,\epsilon)}}{\longrightarrow} \mathcal{K}_{(\delta,\epsilon)}.$$

Explicitly, we have

$$T_{\alpha,\beta}^{\delta,\epsilon}(x, y) = (\delta\alpha^* x, \delta\alpha^* y \beta^* \alpha \delta^* \epsilon - \delta p \alpha^* y \beta^* \alpha \delta^* \epsilon p + \epsilon p \beta^* y p).$$

Note that this map has the following properties:

$$T_{\alpha,\beta}^{\alpha,\beta} = \mathrm{id} \quad \text{and} \quad \left(T_{\alpha,\beta}^{\delta,\epsilon}\right)^{-1} = T_{\delta,\epsilon}^{\alpha,\beta}.$$

We now show that it is equivariant:

Proposition 5.2. *Let $(\alpha, \beta), (\delta, \epsilon) \in \Delta$ and $(g_1, g_2) \in G_p$. Then, if $v \in (T\mathcal{I}_\Delta)_w$, where $w = \alpha p \beta^*$, we have*

$$\left(T_{\alpha,\beta}^{\delta,\epsilon}(\kappa_{\alpha,\beta}(v))\right) \cdot (g_1, g_2) = T_{\alpha g_1, \beta g_2}^{\delta g_1, \epsilon g_2}(\kappa_{\alpha g_1, \beta g_2}(v)).$$

Proof. Let $(x, y) = \kappa_{\alpha,\beta}(v)$. Then $\kappa_{\alpha g_1, \beta g_2}(v) = (x g_1, y g_2)$ by Proposition 4.2. The proof consists of checking what happens if, in the above explicit version of T, $(\alpha, \beta), (\delta, \epsilon)$ and (x, y) are replaced by $(\alpha g_1, \beta g_2), (\delta g_1, \epsilon g_2)$ and $(x g_1, y g_2)$, respectively. It is clear that

$$T_{\alpha g_1, \beta g_2}^{\delta g_1, \epsilon g_2}(x g_1, y g_2) = \left(T_{\alpha,\beta}^{\delta,\epsilon}(x, y)\right) \cdot (g_1, g_2).$$ □

This property enables one to define the parallel transport of tangent vectors along piecewise smooth curves of $\mathcal{I}_\Delta$. Let $\gamma(t)$ be a piecewise C^1 curve of $\mathcal{I}_\Delta$,

$t \in [0,1]$, with $\gamma(0) = w = \alpha p \beta^*$. Let $\Gamma(t)$ be the horizontal lifting of $\gamma(t)$, with $\Gamma(0) = (\alpha, \beta)$. Then we define

$$\tau_{\gamma(t)} : (T\mathcal{I}_\Delta)_w \to (T\mathcal{I}_\Delta)_{\gamma(t)}, \ \tau_{\gamma(t)}(v) = \delta_{\Gamma(t)}(T^{\Gamma(t)}_{\alpha,\beta}(\kappa_{\alpha,\beta}(v))). \tag{5.2}$$

THEOREM 5.3. *The map τ above is well defined (i.e., does not depend on the choice of (α, β) in the fibre of w), and is a linear isomorphism.*

Proof. It is clear that τ is an isomorphism. Let us prove that it is well defined. Let $(\alpha g_1, \beta g_2)$ be another element in the fibre of w (where $(g_1, g_2) \in G_p$). Then it is clear that $\Gamma(t) \cdot (g_1, g_2)$ is the solution of the horizontal lifting equations (4.4) and (4.5) with initial conditions $\Gamma(0) \cdot (g_1, g_2) = (\alpha g_1, \beta g_2)$. If we compute $\tau_{\gamma(t)}$ using these data, we get

$$\delta_{\Gamma(t)\cdot(g_1,g_2)}(T^{\Gamma(t)\cdot(g_1,g_2)}_{\alpha g_1,\beta g_2}(\kappa_{\alpha g_1,\beta g_2}(v))),$$

which, by Proposition 5.2, is equal to

$$\delta_{\Gamma(t)\cdot(g_1,g_2)}(T^{\Gamma(t)}_{\alpha,\beta}(\kappa_{\alpha,\beta}(v)) \cdot (g_1, g_2)).$$

Recall that $\delta_{(u_1,u_2)}(x_1, x_2) = x_1 p u_2^* + u_1 p x_2^*$, and therefore

$$\delta_{(u_1 g_1, u_2 g_2)}(x_1 g_1, x_2 g_2) = x_1 g_1 p g_2^* u_2^* + u_1 g_1 p g_2^* x_2^* = \delta_{(u_1,u_2)}(x_1, x_2).$$

Then the above expression is equal to

$$\delta_{\Gamma(t)}(T^{\Gamma(t)}_{\alpha,\beta}(\kappa_{\alpha,\beta}(v))). \qquad \square$$

The covariant derivative of a vector field $X = X_{\gamma(t)}$ that is tangent along a curve $\gamma(t) \in \mathcal{I}_\Delta$, $t \in [0,1]$, with $\gamma(0) = w = \alpha p \beta^*$, is given by

$$\frac{DX}{dt}\Big|_{t=t_0} = \frac{d}{dt}\tau^{-1}_{\gamma(t)}(X_{\gamma(t)})\Big|_{t=t_0}.$$

If Γ is the horizontal lifting of γ with $\Gamma(0) = (\alpha, \beta)$, then

$$\tau_\gamma^{-1}(X_\gamma) = \delta_{(\alpha,\beta)}(T^{\alpha,\beta}_{\Gamma}(\kappa_\Gamma(X_\gamma))).$$

In particular, we have the following result.

PROPOSITION 5.4. *Let $v \in (T\mathcal{I})_w$, with $w = \alpha p \beta^*$. Then the unique geodesic $\omega(t)$, $t \in \mathbb{R}$, of this connection, with $\omega(0) = w$ and $\dot\omega(0) = v$, is given by*

$$\omega(t) = \Omega_1(t) p \Omega_2^*(t), \ t \in \mathbb{R},$$

where $\Omega = (\Omega_1, \Omega_2)$ is characterized by

$$\dot\Omega(t) = T^{\Omega(t)}_{\alpha,\beta}(\kappa_{\alpha,\beta}(v)), \ \Omega(0) = (\alpha, \beta),$$

where $\alpha p \beta^* = w$.

Proof. The geodesic ω satisfies $D\dot{\omega}/dt = 0$, i.e.,

$$\frac{d}{dt}\big(\delta_{(\alpha,\beta)}(T_{\Omega}^{\alpha,\beta}(\kappa_{\Omega}(\dot{\omega})))\big) = 0,$$

where Ω is the horizontal lifting of ω with initial condition $\Omega(0) = (\alpha,\beta)$. Recall (see (4.3)) that $\kappa_{\Omega}(\dot{\omega}) = \dot{\Omega}$, so

$$0 = \frac{d}{dt}\big(\delta_{(\alpha,\beta)} \circ T_{\Omega}^{\alpha,\beta}(\dot{\Omega})\big) = \delta_{(\alpha,\beta)}\left(\frac{d}{dt}(T_{\Omega}^{\alpha,\beta}(\dot{\Omega}))\right).$$

This derivative is taken in the Banach space $\mathcal{K}_{(\alpha,\beta)}$, on which $\delta_{(\alpha,\beta)}$ is an isomorphism. It follows that

$$\frac{d}{dt}T_{\Omega}^{\alpha,\beta}(\dot{\Omega}) = 0,$$

i.e., $T_{\Omega}^{\alpha,\beta}(\dot{\Omega})$ is constant and equals

$$T_{\Omega(0)}^{\alpha,\beta}(\dot{\Omega}(0)) = T_{\alpha,\beta}^{\alpha,\beta}(\kappa_{(\alpha,\beta)}(\dot{\omega}(0))) = \kappa_{(\alpha,\beta)}(v).$$

Hence, using the fact that $\big(T_{\Omega}^{\alpha,\beta}\big)^{-1} = T_{\alpha,\beta}^{\Omega}$, it follows that

$$\dot{\Omega} = T_{\alpha,\beta}^{\Omega}(\kappa_{(\alpha,\beta)}(v)). \qquad \square$$

Remark 5.5. The above proposition asserts that geodesics exist for all times $t \in \mathbb{R}$. This is clear if one makes the above equations explicit. Set $(x_1, x_2) = \kappa_{\alpha,\beta}(v)$. Then $\dot{\Omega} = T_{\alpha,\beta}^{\Omega}(x_1,x_2)$ gives

$$\dot{\Omega}_1 = \Omega_1\alpha^* x_1 \text{ with } \Omega(0) = \alpha,$$

and

$$\dot{\Omega}_2 = \Omega_1\alpha^* x_2\beta^*\alpha\Omega_1^*\Omega_2 - \Omega_1 p\alpha^* x_2\beta^*\alpha\Omega_1^*\Omega_2 p + \Omega_2 p\beta^* x_2 p \text{ with } \Omega_2(0) = \beta.$$

Note that $\Omega_1^*\Omega_2$ commutes with p, because $\Omega = (\Omega_1,\Omega_2) \in \Delta$. Therefore the second summand on the right hand side can be written as

$$-\,\Omega_1 p\alpha^* x_2\beta^*\alpha p\Omega_1^*\Omega_2.$$

The first equation has solution

$$\Omega_1(t) = \alpha e^{t\alpha^* x_1}.$$

Substituting this into the second equation gives (with the modification pointed out above)

$$\begin{aligned}\dot{\Omega}_2 = {} & \alpha e^{t\alpha^* x_1}\alpha^* x_2\beta^*\alpha e^{-t\alpha^* x_1}\alpha^*\Omega_2 \\ & - \alpha e^{t\alpha^* x_1} p\alpha^* x_2\beta^*\alpha p e^{-t\alpha^* x_1}\alpha^*\Omega_2 + \Omega_2 p\beta^* x_2 p.\end{aligned}$$

Using the fact that $\alpha e^{t\alpha^* x_1}\alpha^* = e^{tx_1\alpha^*}$, the first two terms on the right hand side are equal to

$$e^{t\alpha^* x_1} x_2\beta^* e^{-tx_1\alpha^*}\Omega_2 - e^{t\alpha^* x_1} p^\alpha x_2\beta^* p^\alpha e^{-tx_1\alpha^*}\Omega_2$$
$$= e^{t\alpha^* x_1}\{x_2\beta^* - p^\alpha x_2\beta^* p^\alpha\} e^{-tx_1\alpha^*}\Omega_2.$$

Since $(x_1, x_2) = \kappa_{\alpha,\beta}(v) \in \mathcal{K}_{(\alpha,\beta)}$, it follows (see Lemma 4.3) that $x_1\alpha^*$ and $x_2\beta^*$ have the same codiagonal entries in their matrices in terms of p^α. Therefore

$$x_2\beta^* - p^\alpha x_2\beta^* p^\alpha = x_1\alpha^* - p^\alpha x_1\alpha^* p^\alpha.$$

Hence the second equation is

$$\dot{\Omega}_2 = x_1\alpha^*\Omega_2 - e^{tx_1\alpha^*} p^\alpha x_1 p\alpha^* e^{-tx_1\alpha^*}\Omega_2 + \Omega_2 p\beta^* x_2 p.$$

This is a linear differential equation, with solutions defined for all $t \in \mathbb{R}$.

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