Illinois Journal of Mathematics
Volume 48, Number 1, Spring 2004, Pages 89–96
S 0019-2082

ON THE GEOMETRY OF POSITIVELY CURVED MANIFOLDS WITH LARGE RADIUS

QIAOLING WANG

Abstract. Let M be an n-dimensional complete connected Riemannian manifold with sectional curvature $K_M \geq 1$ and radius $\text{rad}(M) > \pi/2$. For any $x \in M$, denote by $\text{rad}(x)$ and $\rho(x)$ the radius and conjugate radius of M at x, respectively. In this paper we show that if $\text{rad}(x) \leq \rho(x)$ for all $x \in M$, then M is isometric to a Euclidean n-sphere. We also show that the radius of any connected nontrivial (i.e., not reduced to a point) closed totally geodesic submanifold of M is greater than or equal to that of M.

1. Introduction

Let M be an n-dimensional complete connected Riemannian manifold with sectional curvature $K_M \geq 1$. Many interesting results about M have been proven during the past years. It was shown by Grove and Shiohama [GS] that M is homeomorphic to S^n, the n-dimensional sphere, if $\text{diam}(M)$, the diameter of M, is greater than $\pi/2$. In the case $\text{diam}(M) = \pi/2$ (where the theorem is false, as shown by the example of the real projective space) a classification was given by Gromoll and Grove [GG]. It should be mentioned that in the proof of their result Grove and Shiohama established a critical point theory of distance functions on complete Riemannian manifolds, which serves as an important tool in Riemannian geometry (cf. [C]). In 1989, Shiohama and Yamaguchi [SY] proved that if the radius of M is close to π, then M is diffeomorphic to S^n. Recall that for a compact metric space (X, d), the radius of X at a point $x \in X$ is defined as $\text{rad}(x) = \max_{y \in X} d(x, y)$, and the radius of X is given by $\text{rad}(X) = \min_{x \in X} \text{rad}(x)$ (cf. [SY]).

Colding [C1], [C2] extended the result of Shiohama and Yamaguchi as follows: An n-dimensional complete connected Riemannian manifold with Ricci curvature larger than or equal to $n-1$ and radius close to π is diffeomorphic to S^n (cf. [C1], [C2]). A classical result due to Toponogov [T] states that if $n = 2$ and M contains a closed geodesic without self-intersections of length 2π, then

Received March 2, 2003; received in final form May 12, 2003.
2000 *Mathematics Subject Classification.* 53C20.

M is isometric to a 2-dimensional unit sphere. Recently, Xia [X] partially extended Toponogov's theorem to higher dimensional Riemannian manifolds. In the case when the radius of M is greater than $\pi/2$, Grove and Petersen [GP] showed that the volume of M satisfies $C(n) \le \mathrm{vol}(M) \le \{\mathrm{rad}(M)/\pi\} \cdot \omega_n$, where ω_n is the volume of a unit Euclidean n-sphere and $C(n)$ is a positive constant depending only on n.

In this article, we study complete manifolds with sectional curvature bounded below by 1 and radius greater than $\pi/2$. In order to state our first result we fix some notation.

Let x be a point in a complete Riemannian manifold M and let γ be a unit speed geodesic with $\gamma'(0) = v \in T_xM$. The conjugate value c_v of v is defined to be the first number $r > 0$ such that there is a Jacobi field J along γ satisfying $J(0) = J(r) = 0$. Set

$$\rho(x) := \inf_{v \in S_xM} c_v,$$

where S_xM is the unit tangent sphere of M at x. We call $\rho(x)$ the *conjugate radius* of M at x. The conjugate radius of M is defined as $\rho(M) = \inf_{p\in M} \rho(p)$.

Our first theorem is motivated by the simple fact that the radius and the conjugate radius at any point on a Euclidean sphere are the same. Theorem 1 below shows that in the set of closed manifolds with sectional curvature larger than or equal to 1 and radius greater than $\pi/2$ this phenomenon can only happen for the spheres.

THEOREM 1. *Let M be an n-dimensional complete connected Riemannian manifold with $K_M \ge 1$ and $\mathrm{rad}(M) > \pi/2$. If for any $x \in M$ we have $\rho(x) \ge \mathrm{rad}(x)$, then M is isometric to an n-sphere.*

We next prove the following result.

THEOREM 2. *Let M be an $n(\ge 3)$-dimensional complete connected Riemannian manifold with $K_M \ge 1$ and $\mathrm{rad}(M) > \pi/2$. Then the radius of any connected nontrivial (i.e., not reduced to a point) closed totally geodesic submanifold of M is greater than or equal to that of M.*

As a direct consequence of Theorem 2 and the diameter sphere theorem of Grove and Shiohama, we have the following corollary, first obtained by Xia [X].

COROLLARY 3. *Let M be an $n(\ge 3)$-dimensional complete Riemannian manifold with sectional curvature $K_M \ge 1$ and radius $\mathrm{rad}\ M > \pi/2$. Suppose that N is a $k(\ge 2)$-dimensional complete connected totally geodesic submanifold. Then N is homeomorphic to a k-sphere.*

Combining Theorem 2 and the above-mentioned theorem of Grove and Petersen, we obtain the following result.

COROLLARY 4. *Let M be an $n(\geq 3)$-dimensional complete Riemannian manifold with sectional curvature $K_M \geq 1$ and radius* rad $M > \pi/2$. *Suppose that N is a $k(\geq 2)$-dimensional closed connected totally geodesic submanifold. Then there exists a positive constant $C(k)$ such that* $\mathrm{vol}(N) \geq C(k)$.

2. Proof of the theorems

Before proving our results, we list some known facts that we will need. Let M be a complete connected Riemannian n-manifold satisfying $K_M \geq 1$ and $\mathrm{rad}(M) > \pi/2$. By using the Toponogov comparison theorem one can show that for any $x \in M$ there exists a unique point $A(x)$ which is at maximal distance from x. The map $A : M \to M$ is easily seen to be continuous (cf. [GP], [X]). Since M is homeomorphic to S^n, the Brouwer fixed point theorem implies that A is surjective.

We shall assume throughout this paper that all geodesics are parametrized by arc-length.

A connected simply connected compact Riemannian n-manifold M without boundary such that for any $m \in M$ the cut locus of m in M is a single point is called a wiedersehen manifold (cf. [Gn]). From the work of Green [Gn], Berger [B], Weinstein [W] and Yang [Y1], [Y2] we know that a wiedersehen manifold is isometric to a Euclidean sphere.

Now we are ready to prove our main theorems.

Proof of Theorem 1. The Bonnet-Myers Theorem implies that M is compact. Since the diameter of M is greater than or equal to $\mathrm{rad}(M) > \pi/2$, M is homeomorphic to S^n and, in particular, M is simply connected. For any $x \in M$, let $D(x)$ be the cut locus of x. It is well known that the function $g : M \to R^+$ given by $f(x) = d(x, D(x))$ is continuous. We shall show that our M is a wiedersehen manifold and therefore is isometric to an n-sphere. It then suffices to show that $D(x) = \{A(x)\}$ for all $x \in M$, where $A : M \to M$ is the map defined at the beginning of this section. To do this, we fix a point $p \in M$. Since $D(p)$ is closed and hence is compact, there exists $q \in D(p)$ such that $d(p, q) = \inf_{x \in D(p)} d(p, x)$. We claim that $q = A(p)$. In fact, set $s = d(p, q)$; from well known results in Riemannian geometry (cf. [Ca, p. 274]) we conclude that either

(a) there exists a minimizing geodesic σ from p to q along which q is conjugate to p, or
(b) there exist exactly two minimizing geodesics σ_1 and σ_2 from p to q with $\sigma_1'(s) = -\sigma_2'(s)$.

If (a) holds, then we have $s \geq \rho(p) \geq \mathrm{rad}(p)$. Thus $s = \mathrm{rad}(p)$ and so $q = A(p)$ since $A(p)$ is the unique point which is at maximal distance from p.

Suppose that (b) holds and $q \neq A(p)$. Set $t = d(q, A(p)), r = d(p, A(p))$ and consider first the case when $s > \pi/2$. Take a minimal geodesic σ_3 from q

to $A(p)$; then either

$$\angle\left(\sigma_3'(0), -\sigma_1'(s)\right) \leq \frac{\pi}{2},$$

or

$$\angle\left(\sigma_3'(0), -\sigma_2'(s)\right) \leq \frac{\pi}{2}.$$

We assume without loss of generality that $\angle(\sigma_3'(0), -\sigma_1'(s)) \leq \pi/2$.

Applying the Toponogov inequality to the hinge (σ_1, σ_3), we obtain

$$0 > \cos r \geq \cos s \cos t + \sin s \sin t \cos \angle\left(\sigma_3'(0), -\sigma_1'(s)\right) \geq \cos s \cos t. \tag{2.1}$$

On the other hand, since $A(p)$ is at maximal distance from p, by the well known Berger Lemma (cf. [CE]) there exists a minimal geodesic γ from $A(p)$ to p such that $\angle(-\sigma_3'(t), \gamma'(0)) \leq \pi/2$. Applying the Toponogov comparison theorem to the hinge (γ, σ_3), we obtain

$$\cos s \geq \cos r \cos t + \sin r \sin t \cos \angle\left(-\sigma_3'(t), \gamma'(0)\right) \geq \cos r \cos t. \tag{2.2}$$

Since $s > \pi/2$, (2.1) and (2.2) imply that

$$\cos r \sin^2 t \geq 0, \tag{2.3}$$

which is a contradiction.

Suppose now that $s \leq \pi/2$. We suppose that $p = A(z)$ is the unique point which is at maximal distance from some point $z \in M$. Then $z \neq q$ since $d(p, z) > \pi/2 \geq d(p, q)$. Set $t_1 = d(p, z)$ and $t_2 = d(q, z)$; then $t_1 > t_2$. Take a minimal geodesic c from q to z. Since we have either

$$\angle\left(c'(0), -\sigma_1'(s)\right) \leq \frac{\pi}{2},$$

or

$$\angle\left(c'(0), -\sigma_2'(s)\right) \leq \frac{\pi}{2},$$

one can use the Toponogov comparison theorem to the hinge (c, σ_1) or (c, σ_2) to get

$$0 > \cos t_1 \geq \cos s \cos t_2. \tag{2.4}$$

This implies that $s \neq \pi/2$, and so we obtain from

$$\cos t_1 < \cos t_2$$

and (2.4) that

$$\cos t_1 > \cos s \cos t_1. \tag{2.5}$$

Thus,

$$\cos t_1 (1 - \cos s) > 0,$$

which clearly contradicts the fact that $t_1 > \pi/2$. Thus our claim is true. For any $x \in D(p)$, we then conclude from

$$d(p, q) = d(p, A(p)) \geq d(p, x) \geq d(p, D(p)) = d(p, q) \tag{2.6}$$

that $x = A(p)$. Consequently, we have $D(p) = \{A(p)\}$. Hence, our M is a wiedersehen manifold and so is isometric to an n-sphere. This completes the proof of Theorem 1. □

Proof of Theorem 2. Let N be a closed totally geodesic submanifold of M. We consider two cases:

Case 1. $\dim N \geq 2$. Denote by d and d^N the distance functions on M and N, respectively. Let $\mathrm{rad}_N : N \to R$ be the radius function on N, i.e., $\mathrm{rad}_N(x) = \max_{y \in N} d^N(x, y)$ for all $x \in N$, and define rad_M similarly. It then suffices to prove that $\mathrm{rad}_N(x) \geq \mathrm{rad}_M(x)$ for all $x \in N$. In order to prove this, we fix a point $p \in N$ and take $q \in N$ satisfying

$$\mathrm{rad}_N(p) = d^N(p, q). \tag{2.7}$$

Let Γ_{qp} be the set of unit vectors in T_qN corresponding to the set of normal minimal geodesics of N from q to p. Then, by Berger's Lemma, Γ_{qp} is $\pi/2$-dense in S_qN, that is,

$$\Gamma_{qp}(\pi/2) := \{u \in S_qN \mid \angle(u, \Gamma_{qp}) \leq \pi/2\} = S_qN, \tag{2.8}$$

where S_xN denotes the unit tangent sphere of N at x. Since a $\pi/2$-dense subset of a great sphere S^l in a unit sphere S^m, $l < m$, is also $\pi/2$-dense in S^m, Γ_{qp} is $\pi/2$-dense in S_qM.

Let $A : M \to M$ be the map defined above. Set $s = d^N(p, q)$ and $r = d(p, A(p))$. We claim that $s > \pi/2$. Suppose on the contrary that $s \leq \pi/2$. Take a point $z \in M$ so that $p = A(z)$. It follows from

$$d(p, z) > \frac{\pi}{2} \geq d^N(p, q) \geq d(p, q)$$

that $q \neq z$. Set $l = d(p, z)$ and $t = d(q, z)$; then $l > t$. Take a minimal geodesic c of M from q to z. Since Γ_{qp} is $\pi/2$-dense in S_qM, we can find $v \in \Gamma_{qp}$ such that $\angle(v, c'(0)) \leq \pi/2$. Thus, by the definition of Γ_{qp}, there exists a minimal geodesic c_1 of N from q to p such that $c_1'(0) = v$. Since N is totally geodesic, c_1 is also a geodesic of M. We apply the Toponogov comparison theorem to the hinge (c, c_1) to get

$$0 > \cos l \geq \cos s \cos t + \sin s \sin t \cos \angle(c'(0), c_1'(0)) \geq \cos s \cos t. \tag{2.9}$$

Since $\cos l < \cos t$ and $s \leq \pi/2$, we get from (2.9) that

$$\cos l (1 - \cos s) > 0. \tag{2.10}$$

This is a contradiction. Hence $s > \pi/2$.

Now we are ready to show that $s \geq r$. Assume by contradiction that $s < r$. Since

$$d(p, q) \leq d^N(p, q) < d(p, A(p)),$$

we have $A(p) \neq q$. Let $w = d(q, A(p))$ and take a minimal geodesic γ of M from q to $A(p)$. We can find a minimal geodesic γ_1 of N from q to p such that $\angle(\gamma'(0), \gamma_1'(0)) \leq \pi/2$. Since γ_1 is also a geodesic of M, applying the Toponogov inequality to the hinge (γ, γ_1), we conclude that

$$0 > \cos r \geq \cos s \cos w, \tag{2.11}$$

which gives $\cos w > 0$ since $s > \pi/2$. Hence, since $s < r$, we have

$$\cos w \cos s > \cos w \cos r. \tag{2.12}$$

Combining (2.11) and (2.12), we conclude

$$\cos r(1 - \cos w) > 0. \tag{2.13}$$

This is a contradiction.

Case 2. N is a closed geodesic. The proof in this case is similar to that in Case 1; for the sake of completeness, we give the argument. Denote by $c : [0, a] \to M$ the closed geodesic N. Set $p = c(0)$ and $q = c(a/2)$. Let us first show that $a > \pi$. Take $z \in M$ so that $p = A(z)$ and assume that $a \leq \pi$. Then we have $q \neq z$ since

$$d(p, z) > \frac{\pi}{2} \geq \frac{a}{2} = L[c|_{[0,a/2]}] \geq d(p, q),$$

where d is as before the distance function on M. Set $l = d(p, z)$ and $t = d(q, z)$; then $l > t$. Let γ be a minimal geodesic of M from q to z; then we have either

$$\angle\left(\gamma'(0), c'\left(\frac{a}{2}\right)\right) \leq \frac{\pi}{2},$$

or

$$\angle\left(\gamma'(0), -c'\left(\frac{a}{2}\right)\right) \leq \frac{\pi}{2}.$$

Thus, we can apply the Toponogov inequality to the hinges $(\gamma, c|_{[0,a/2]})$ or $(\gamma, c|_{[a/2,a]})$ to get

$$0 > \cos l \geq \cos \frac{a}{2} \cos t,$$

which contradicts the fact that $\cos l < \cos t$ and $a \leq \pi$. Thus we have $a > \pi$.

Set $r = d(p, A(p))$. Then we need only show that $a \geq 2r$, since the (intrinsic) radius of c is equal to its intrinsic diameter, which in turn is equal to half of its length, i.e., $a/2$. Suppose on the contrary that $a < 2r$. Then $A(p) \neq q$ since

$$d(p, q) \leq \frac{a}{2} < r.$$

Take a minimal geodesic σ of M from q to $A(p)$ and let $w = d(q, A(p))$. Applying the Toponogov inequality to $(\sigma, c|_{[0,a/2]})$ or $(\sigma, c|_{[a/2,a]})$, we have

$$0 > \cos r \geq \cos \frac{a}{2} \cos w, \tag{2.14}$$

and so $\cos w > 0$ since $a/2 > \pi/2$. Since $a/2 < r$, we conclude that

$$\cos w \cos \frac{a}{2} > \cos w \cos r. \tag{2.15}$$

From (2.14) and (2.15) it follows that $\cos r(1 - \cos w) > 0$, which is a contradiction. The proof of Theorem 2 is complete. □

In view of Theorem 2, it is interesting to study the following problem.

PROBLEM. Let M be a complete Riemannian manifold with $K_M \geq 1$ and $\mathrm{rad}(M) > \pi/2$. Does the "antipodal" map A of M restricted to a totally geodesic submanifold agree with the "antipodal" map of the submanifold?

Acknowledgment. The author would like to thank the referee for the encouragement and valuable remarks.

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Departamento de Matemática-IE, Fundação Universidade de Brasília, Campus Universitário, 70910-900-Brasília-DF, Brasil
E-mail address: wang@mat.unb.br