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PRESCRIBING MEAN CURVATURE VECTORS FOR FOLIATIONS

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ABSTRACT. Given a foliation $\mathcal{F}$ of a compact manifold M and a vector field X on M, we provide some necessary and sufficient conditions for X to become the mean curvature vector of (the leaves of) $\mathcal{F}$ with respect to some Riemannian metric on M.

1. Introduction

Sullivan [13], using his work on foliation cycles [12] and a formula due to Rummler [10], provided a homological characterization of p-dimensional foliations $\mathcal{F}$ whose leaves become minimal submanifolds under an appropriate choice of a Riemannian metric on the ambient manifold. Roughly speaking, the condition is that there exists a volume p-form ω along the leaves that is relatively closed, i.e., $d\omega(v_1, \dots, v_{p+1}) = 0$ whenever $v_1, \dots, v_p$ are tangent to $\mathcal{F}$. Haefliger [1] has shown that this criterion depends only on the transverse structure of $\mathcal{F}$.

Motivated by these results, G. Oshikiri [4]–[7] and the second author [14]–[16] considered the following problem: Given a transversely oriented codimension-one foliation $\mathcal{F}$ of a compact oriented manifold M and a function $f \in C^\infty(M)$, decide whether one can find a Riemannian metric g on M such that f coincides with $h = h_{\mathcal{F},g}$, the mean curvature function of (the leaves of) $\mathcal{F}$ on (M, g). The well known formula [9]

$$d\Omega_{\mathcal{F},g} = -h\Omega_{M,g}, \tag{1}$$

$\Omega_{\mathcal{F},g}$ and $\Omega_{M,g}$ being the Riemannian volume forms of (the leaves of) $\mathcal{F}$ and of M, implies immediately that

$$\int_M h = 0, \quad \int_{D_+} h > 0 \quad \text{and} \quad \int_{D_-} h < 0 \tag{2}$$

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for any positive saturated domain D_+ and any negative saturated domain D_-, defined as follows. A saturated domain is *positive* (resp., *negative*) if the positive oriented unit vector field ν orthogonal to $\mathcal{F}$ points outwards (resp., inwards) at all points of its boundary. Therefore, any such domain D_+ (resp., D_-) has to contain a point x_+ (resp., x_-) for which $h(x_+) > 0$ (resp., $h(x_-) < 0$). If such domains do not exist, then $\mathcal{F}$ is said to be topologically taut [13] and either $h \equiv 0$ or $h(x) \cdot h(y) < 0$ for some x and y of M. In [7], Oshikiri proved that any function h satisfying the sign conditions described above can be realized as the mean curvature function of $\mathcal{F}$ with respect to some Riemannian metric g on M.

In this article, we consider the analogous problem for foliations of arbitrary codimension. In this case, the mean curvature $H = H_{\mathcal{F},g}$ of $\mathcal{F}$ (with respect to a Riemannian metric g with its Levi-Cività connection ∇) is defined by

$$H = \sum_{i=1}^{p} (\nabla_{E_i} E_i)^{\perp}, \tag{3}$$

$E_1, \ldots, E_p$ ($p = \dim \mathcal{F}$) being a local orthonormal frame of vector fields tangent to $\mathcal{F}$, and becomes a vector field orthogonal to $\mathcal{F}$. Therefore, our problem can be formulated as follows: Given M, $\mathcal{F}$ and a vector field X on M, find necessary and/or sufficient conditions for X to coincide with $H_{\mathcal{F},g}$ for some Riemannian metric g. (Let us note that even in the case of foliations of codimension one this problem is quite different from the scalar problem considered in the papers mentioned before.)

Our Theorem 1 in Section 3 provides such conditions under the assumption that we can solve the problem in a neighbourhood of Σ, the set of all singularities of X, and leads to Corollary 1, which provides such conditions for nowhere vanishing vector fields. Another result, Theorem 2 in Section 4, provides sufficient conditions for the existence of a Riemannian metric g with $H_{\mathcal{F},g} = X$ in a neighbourhood of Σ. The conditions formulated in Theorems 1 and 2 are expressed in terms of currents and are analogous to those of [13], where the problem of geometrical tautness of foliations was considered. (Recall that a foliation is geometrically taut when there exists a Riemannian metric for which all leaves become minimal (i.e., $H \equiv 0$).) As in [13], the Hahn-Banach Theorem plays an important role in the proofs. Similar methods were proposed in [2] (see Theorem 2.17 there) to get conditions for $\mathcal{F}$ to be either geometrically tense or taut, expressed in terms of currents and Q-valued 1-forms, where $Q = TM/T\mathcal{F}$. This should be closely related to the realization of such forms as mean curvature forms of $\mathcal{F}$. We expect that our conditions can be transformed into more geometric ones like those in [7].

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2. Preliminaries

Let $(M, \mathcal{F})$ be an oriented closed manifold with an oriented foliation, $m = \dim M$, $p = \dim \mathcal{F}$. For any $k \in \{1, \dots, m\}$ and any closed set A of M denote by $D_k(A)$ the space of k-currents supported in A. By definition, $D_k = D_k(M)$ is the dual of D^k, the space of differentiable k-forms on M with the C^∞ topology. (For information concerning currents we refer to Schwartz [11]. In particular, we recall Theorem XIV in [11] which states that the dual of D_k coincides with D^k.) Also let X be a smooth vector field on M and let Σ be the set of all singular points of X. Given A, a closed subset of M contained in $M \setminus \Sigma$, consider the following subsets of D_p determined by $\mathcal{F}$ and X:

- $C_{\mathcal{F}}(A)$, the closed convex cone generated by all Dirac currents of the form $e_1 \wedge \dots \wedge e_p$, where $(e_1, \dots, e_p)$ is a positively oriented frame of $T_x\mathcal{F}$, the space tangent to the leaf through x, $x \in A$;
- $\tilde{C}_{\mathcal{F},X}(A)$, the closed convex cone generated by all boundaries of the form $\partial(-X(x) \wedge e_1 \wedge \dots \wedge e_p)$, where $x \in A$ and $e_1, \dots, e_p$ are as above;
- $C_{\mathcal{F},X}(A)$, the closed convex cone generated by the union $C_{\mathcal{F}}(A) \cup \tilde{C}_{\mathcal{F},X}(A)$;
- $P_{\mathcal{F},X}(A)$, the closed linear space generated by all Dirac currents $X(x) \wedge v_1 \dots \wedge v_{p-1}$, where $v_1, \dots, v_{p-1} \in T_x\mathcal{F}$ and $x \in A$.

We write $C_{\mathcal{F}}$ instead of $C_{\mathcal{F}}(M)$ and, if $\Sigma = \emptyset$, $\tilde{C}_{\mathcal{F},X}$, $C_{\mathcal{F},X}$ and $P_{\mathcal{F},X}$ instead of $\tilde{C}_{\mathcal{F},X}(M)$, $C_{\mathcal{F},X}(M)$ and $P_{X,\mathcal{F}}(M)$, respectively.

In [12], Sullivan proved that the cone $C_{\mathcal{F}}$ has a compact base, that is, that there exists a continuous linear functional $\lambda : D_k \to \mathbb{R}$, positive on $C_{\mathcal{F}} \setminus \{0\}$, and such that the set $\lambda^{-1}(1) \cap C_{\mathcal{F}}$ is compact. Obviously, the same holds for all cones $C_{\mathcal{F}}(A)$ with A closed in M. However, Sullivan's argument cannot be applied to the cones $\tilde{C}_{\mathcal{F},X}(A)$ (and, *a fortiori*, to $C_{\mathcal{F},X}(A)$). In fact, in some situations these cones do not have compact bases.

Example 1. Let M be a closed oriented 3-dimensional manifold equipped with a 2-dimensional foliation $\mathcal{F}$ and a nowhere vanishing vector field X transverse to $\mathcal{F}$. If $(M, \mathcal{F})$ contains a Reeb component R bounded by a 2-dimensional torus T, then both currents $\int_T$ and $-\int_T$ belong to $\tilde{C}_{\mathcal{F},X} \subset C_{\mathcal{F},X}$. Indeed, $\int_T = \partial \int_R$, $-\int_T = \partial \int_{M \setminus R}$ and the integrals $\int_R$ and $\int_{M\setminus R}$ can be expressed as limits of convex combinations of Dirac currents of the form $X(x) \wedge e_1 \wedge e_2$, where $e_i \in T_x\mathcal{F}$ and $x \in M$. Since the cones $\tilde{C}_{\mathcal{F},X}$ and $C_{\mathcal{F},X}$ contain two non-trivial elements $\pm c$, $c \in D_2$, they cannot have compact bases. The same argument applies to any codimension one foliation which admits a domain U of type D_+ mentioned in the Introduction: Again, the currents $\pm \int_{\partial U}$ belong to $\tilde{C}_{\mathcal{F},X}$.

Now assume that $X = H_{\mathcal{F},g}$ for some Riemannian metric g. Let $\Omega = \Omega_{\mathcal{F},g}$ be the volume form of $\mathcal{F}$ with respect to g (i.e., the differential p-form which gives the p-volume on the leaves and vanishes if any argument is a vector orthogonal to the leaf). Rummler [10] proved that

$$d\Omega(Z, E_1, \dots, E_p) = -\langle Z, X\rangle \tag{4}$$

for every vector Z tangent to M. (Here, $E_1, \dots, E_p$ is a positive oriented orthonormal local frame of sections of $T\mathcal{F}$.) If $X \neq 0$ on a closed set A, then setting $Z = X$ in (4) yields

$$\Omega|\tilde{C}_{\mathcal{F},X}(A) \setminus \{0\} > 0. \tag{5}$$

Since obviously $\Omega|C_{\mathcal{F}} \setminus \{0\} > 0$, condition (5) implies that

$$\Omega|C_{\mathcal{F},X}(A) \setminus \{0\} > 0. \tag{6}$$

Moreover, in this case we have the following result.

LEMMA 1. *If $X = H_{\mathcal{F},g}$ and $A \subset M \setminus \Sigma$ is closed in M, then the cone $C_{\mathcal{F},X}(A)$ has a compact base.*

Proof. Let $B = C_{\mathcal{F},X}(A) \cap \Omega^{-1}(1)$. We have to show that $B \subset D_p$ is compact, i.e., that any continuous linear functional $\lambda : D_p \to \mathbb{R}$ is bounded on B.

According to the theorem of Schwartz cited above, any such λ can be represented by a differential p-form ω. Take any current

$$c = \sum_i t_i e_1^i \wedge \cdots \wedge e_p^i - \sum_j t_j \partial(X(x_j) \wedge e_1^j \wedge \cdots \wedge e_p^j),$$

where $x_j \in A$, the $(e_1^k, \dots, e_p^k)$'s are positive oriented orthonormal frames of $T\mathcal{F}$ at certain points of A and all coefficients t_i and t_j are positive. Note that

$$\Omega(\partial(X(x_j) \wedge e_1^j \wedge \cdots \wedge e_p^j) = d\Omega(X(x_j) \wedge e_1^j \wedge \cdots \wedge e_p^j) = -\|X(x_j)\|^2$$

by (4). Hence if $c \in B$, then

$$1 = \Omega(c) = \sum_i t_i + \sum_j \|X(x_j)\|^2 t_j \geq \alpha \left(\sum_i t_i + \sum_j t_j \right),$$

where

$$\alpha = \min\left\{1, \min\{\|X(x)\|^2; x \in A\}\right\} > 0.$$

Therefore,

$$\begin{aligned} |\omega(c)| &\leq \sum_i t_i |\omega(e_1^i \wedge \cdots \wedge e_p^i)| + \sum_j t_j |d\omega(X(x_j) \wedge e_1^j \wedge \cdots \wedge e_p^j)| \\ &\leq \alpha^{-1} \cdot (\|\omega\| + \|d\omega\| \cdot \|X\|), \end{aligned}$$

where

$$\begin{aligned} \|X\| &= \max\{\|X(x)\|; x \in M\}, \\ \|\omega\| &= \max\left\{|\omega(\xi)|; \xi \in \Lambda^p T_x M, \|\xi\| = 1, x \in M\right\}, \end{aligned}$$

and so on. □

Finally, it is obvious that if $X = H_{\mathcal{F},g}$, then

$$\Omega_{\mathcal{F}}|P_{\mathcal{F},X}(A) \equiv 0 \tag{7}$$

for any closed set A contained in $M \setminus \Sigma$. Conditions (6) and (7) imply that in this case the cone $C_{\mathcal{F},X}(A)$ intersects the space $P_{\mathcal{F},X}(A)$ trivially, i.e.,

$$C_{\mathcal{F},X}(A) \cap P_{\mathcal{F},X}(A) = \{0\}. \tag{8}$$

3. Away from singularities

In this section, we shall consider the problem of prescribing mean curvature assuming that it is already solved in a neighbourhood of the singular set Σ of a given vector field X. More precisely, given a closed oriented manifold M with an oriented foliation $\mathcal{F}$ and a vector field X on M which vanishes precisely on Σ, we shall prove the following.

THEOREM 1. *Suppose that g_0 is a Riemannian metric on a neighbourhood U of Σ on which $X = H_{\mathcal{F},g_0}$. Let B be a smooth compact submanifold of M with*

$$\Sigma \subset \mathrm{Int}(B) \subset B \subset U$$

and let A be the closure of $M \setminus B$, so that $\partial A = \partial B = A \cap B$. Then there exists a Riemannian metric g on M extending $g_0|_B$ and such that $X = H_{\mathcal{F},g}$ on M if and only if the cone $C_{\mathcal{F},X}(A)$ has a compact base and intersects the space $P_{\mathcal{F},X}(A)$ trivially.

Proof. In the previous section it was shown that if the metric g exists, then the cone $C_{\mathcal{F},X}(A)$ has a compact base and intersects the space $P_{\mathcal{F},X}(A)$ in $\{0\}$. To prove the converse, let us suppose these conditions on the cone $C_{\mathcal{F},X}(A)$ hold.

In order to obtain the desired metric g, we shall first construct a differential p-form Ω and then construct g so that Ω is the volume form along the leaves of $\mathcal{F}$. Let Ω_0 be the volume form of $\mathcal{F}$ on U with respect to g_0.

CLAIM. *Under our hypotheses, the sum* $\ker(\Omega_0|B) + P_{\mathcal{F},X}(A)$ *also intersects the cone $C_{\mathcal{F},X}(A)$ trivially.*

To prove the Claim, take a small normal neighbourhood $N \subset U$ of $A \cap B$ such that each point x in N has a unique shortest g_0-geodesic γ_x joining x to a point of $A \cap B$ and so that γ_x is entirely contained in N. If we denote the endpoint of γ_x by $\mathrm{pr}(x)$, then the resulting projection $\mathrm{pr} : N \to A \cap B$ is a smooth map. Also equip the bundle $T\mathcal{F}|U$ with a linear connection ∇ (for example, the one induced by the Levi-Cività connection on (U, g_0)) and let τ_x denote parallel transport along γ_x. Finally choose a smooth function $f : M \to \mathbb{R}$ supported in N and equal to 1 at all points of $A \cap B$. With these tools in hand, define a projection map $\pi : W \to D_p(A \cap B)$, W being the closure of the linear subspace of $D_p(A)$ generated by the union $C_{\mathcal{F},X}(A) \cup P_{\mathcal{F},X}(A) \cup D_p(A \cap B)$, in the following way. First, for generators of $C_{\mathcal{F},X}(A)$ and $P_{\mathcal{F},X}(A)$ put

$$\pi(e_1 \wedge \cdots \wedge e_p) = f(x) \cdot \tau_x(e_1) \wedge \cdots \wedge \tau_x(e_p),$$

$$\pi(-\partial(X(x) \wedge e_1 \wedge \cdots \wedge e_p)) = -f(x) \cdot \partial(X(\mathrm{pr}(x)) \wedge \tau_x e_1 \wedge \cdots \wedge \tau_x e_p),$$

and

$$\pi(X(x) \wedge v_1 \wedge \cdots \wedge v_{p-1}) = f(x) \cdot X(\mathrm{pr}(x)) \wedge \tau_x v_1 \wedge \cdots \wedge \tau_x v_{p-1},$$

whenever $e_1 \wedge \cdots \wedge e_p$ is a positively oriented frame of $T_x\mathcal{F}$, $v_j \in T_x\mathcal{F}$ and $x \in N$. Then π is well defined, since the generators of the cone (in the first two formulas) are linearly independent, as can easily be shown by evaluating a linear combination of them on appropriate differential p-forms, and this cone meets the subspace $P_{\mathcal{F},X}(A)$ (whose generators appear in the third formula) trivially. If $x \notin N$, set π equal to 0 in all these cases. Next, put $\pi(c) = c$ for all $c \in D_p(A \cap B)$. From the above formulas it is clear that π is continuous where defined. Therefore, it can be extended over W by linearity and continuity. This projection $\pi : W \to D_p(A \cap B)$ maps the cone $C_{\mathcal{F},X}(A)$ into the cone $C_{\mathcal{F},X}(A \cap B)$ and the subspace $P_{\mathcal{F},X}(A)$ into the subspace $P_{\mathcal{F},X}(A \cap B)$.

Now suppose that a non-zero element c of $C_{\mathcal{F},X}(A)$ decomposes as $c = c_1 + c_2$ with $c_1 \in D_p(B)$ and $c_2 \in P_{\mathcal{F},X}(A)$. Since $c_1 = c - c_2$ is supported in $A \cap B$, $\pi(c_1) = c_1$. Therefore, $\pi(c) \neq 0$, since otherwise $c_1 = -\pi(c_2)$ would be supported in $A \cap B$ while c itself would lie in the intersection $C_{\mathcal{F},X}(A) \cap P_{\mathcal{F},X}(A) = \{0\}$. Consequently, $\Omega_0(c_1) = \Omega_0(\pi(c)) \neq 0$, as claimed.

Now Ω_0 on $D_p(B)$ extends to a continuous linear functional $\lambda_0 : D_p(B) + P_{\mathcal{F},X}(A) \to \mathbb{R}$ by setting $\lambda_0 = 0$ on $P_{\mathcal{F},X}(A)$, for Ω_0 vanishes on the generators of $P_{\mathcal{F},X}(A)$. Furthermore, since Ω_0 is positive on currents in $C_{\mathcal{F},X}(A) \setminus \{0\}$, where it is defined, the same holds for λ_0, so we can apply the Hahn-Banach Theorem in the form stated in [3] (any continuous linear functional on a closed subspace W of a Frechet space V that is positive on the intersection of a compact closed cone C with W extends to a continuous functional on V that is positive on C) to extend λ_0 to a continuous linear functional $\lambda : D_p \to \mathbb{R}$ that is positive on $C_{\mathcal{F},X}(A) \setminus \{0\}$. Clearly $\lambda = \Omega_0$ on $D_p(B)$ and $\lambda \equiv 0$ on $P_{\mathcal{F},X}(A)$. As before, λ is represented by a unique globally defined p-form ω.

Let Ω denote Sullivan's purification of ω with respect to $T\mathcal{F}$ [13]. This means that if $x \in M$ and $P_\omega : T_xM \to T_x\mathcal{F}$ is the projection defined by the condition

$$\iota_{P_\omega(v)}(\omega|\Lambda^p T_x\mathcal{F}) = (\iota_{P_\omega(v)}\omega)|\Lambda^{p-1}T_x\mathcal{F} = (\iota_v\omega)|\Lambda^{p-1}T_x\mathcal{F}, \quad v \in T_xM,$$

then

$$\Omega(v_1, \ldots, v_p) = \omega(P_\omega(v_1), \ldots, P_\omega(v_p))$$

for all $v_1, \dots, v_p \in T_xM$, $x \in M$. From the above it follows directly that

$$\Omega(w, v_1, \dots, v_{p-1}) = \omega(w, v_1, \dots, v_{p-1}) \tag{9}$$

whenever $w \in T_xM$, $v_j \in T_x\mathcal{F}$, $x \in M$. Since the bundle $T\mathcal{F}$ is integrable, for any $w \in T_{x_0}M$, $x_0 \in M$ one can find a $(p+1)$-dimensional submanifold N such that $w \in TN$ and $T_x\mathcal{F} \subset T_xN$ for all $x \in N$. If $\iota_N : N \to M$ is the canonical inclusion, then $\iota_N^*\Omega = \iota_N^*\omega$ and consequently $\iota_N^* d\Omega = \iota_N^* d\omega$. In particular,

$$d\Omega(w, v_1, \dots, v_p) = d\omega(w, v_1, \dots, v_p) \tag{10}$$

for any $v_1, \dots, v_p \in T_{x_0}\mathcal{F}$. Equalities (9) and (10) show that Ω is strictly positive on $C_{\mathcal{F},X}(A) \setminus \{0\}$ and vanishes identically on $P_{\mathcal{F},X}(A)$. Also, since $\omega = \Omega_0$ on $D_p(B)$ and Ω_0 is pure, the forms Ω and Ω_0 coincide on B.

Now decompose the tangent bundle TM over A into the direct sum

$$TM = T\mathcal{F} \oplus \ker\Omega = T\mathcal{F} \oplus \mathrm{Span}(X) \oplus E, \tag{11}$$

where E is the $(m-p-1)$-dimensional subbundle of $\ker\Omega$ defined by the equation

$$\iota_w d\Omega | \Lambda^p T_x\mathcal{F} = 0.$$

Observe that $\ker\Omega$ coincides with the g_0-orthogonal complement of $T\mathcal{F}$ at all points of B. Also, Rummler's formula (4) implies that E is g_0-orthogonal to X on B.

Choose any Riemannian metric $g = \langle\cdot,\cdot\rangle$ on M making all components of the decomposition in (11) orthogonal and such that the volume form $\Omega_{\mathcal{F},g}$ equals Ω while

$$\langle X, Z\rangle = -d\Omega(Z, E_1, \dots, E_p), \tag{12}$$

where $E_1, \dots, E_p$ is any g-orthonormal positive oriented local frame of $T\mathcal{F}$. Note that such a metric exists since $d\Omega(X, E_1, \dots, E_p) < 0$ on $M \setminus \Sigma$. Also, without loss of generality, we may assume that $g|T\mathcal{F}\otimes T\mathcal{F} = g_0|T\mathcal{F}\otimes T\mathcal{F}$ and $g|E\otimes E = g_0|E\otimes E$ at all points of B. Then the Riemannian metrics g and g_0 will coincide on B. Comparing (4) and (12) yields the equality $X = H_{\mathcal{F},g}$. □

The above Theorem (applied to the case $\Sigma = U = \emptyset$, $A = M$) yields immediately the following corollary.

Corollary 1. *A nowhere vanishing vector field X on a closed foliated manifold $(M, \mathcal{F})$ can become the mean curvature of $\mathcal{F}$ (with respect to some Riemannian metric g on M) if and only if the cone $C_{\mathcal{F},X}$ has a compact base and intersects the subspace $P_{\mathcal{F},X}$ trivially.* □

Let us now discuss some consequences of the conditions which appeared in Theorem 1 and Corollary 1. We have already seen that if X is nonsingular and transverse to a codimension one foliation $\mathcal{F}$ and the cone $C_{\mathcal{F},X}$ has a compact base, then $(M, \mathcal{F})$ contains no domains of type $D_\pm$. Triviality of the intersection of the cone $C_{\mathcal{F},X}$ and space $P_{\mathcal{F},X}$ implies transversality of X and $\mathcal{F}$ along $M \setminus \Sigma$. Indeed, if $X(x_0) \neq 0$ belongs to $T_{x_0}\mathcal{F}$ and $v_1, \dots, v_{p-1}$ are such that $(X(x_0), v_1, \dots, v_{p-1})$ is a positive oriented frame of $T_{x_0}\mathcal{F}$, then $X(x_0) \wedge v_1 \wedge \cdots \wedge v_{p-1}$ belongs to the intersection $C_{\mathcal{F}} \cap P_{\mathcal{F},X}(A) \subset C_{\mathcal{F},X}(A) \cap P_{\mathcal{F},X}(A)$ for any closed set $A \subset M \setminus \Sigma$ which contains x_0. Another consequence of triviality of this intersection is described in the following example.

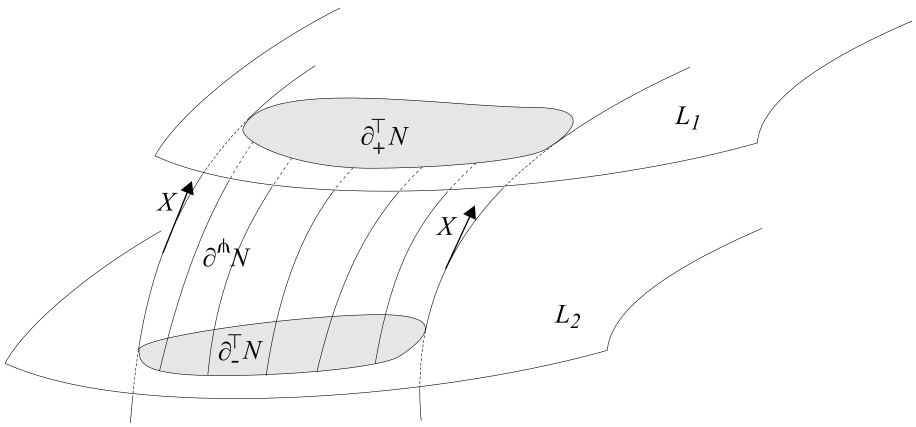

FIGURE 1. A submanifold spanned by X and $\mathcal{F}$

EXAMPLE 2. Let N be a compact $(p+1)$-dimensional submanifold (with boundary and corners) of $(M, \mathcal{F})$ such that TN is spanned by X and $T\mathcal{F}$ with the orientation given by $(X(x), v_1, \dots, v_p)$, where $(v_1, \dots, v_p)$ is a positive oriented frame of $T_x\mathcal{F}$, and suppose that ∂N decomposes into $\partial^\pitchfork N$, a submanifold tangent to X, and a union $\partial_+^\top N \cup \partial_-^\top N$ of pieces of leaves, with X pointing outwards along $\partial_+^\top N$ and inwards along $\partial_-^\top N$ (Figure 1). Then the p-current $\int_{\partial N}$ which takes a p-form α to the integral $\int_{\partial N} \alpha$ (where ∂N has the induced orientation) has a decomposition

$$\int_{\partial N} = \int_{\partial_+^\top N} - \int_{\partial_-^\top N} + \int_{\partial^\pitchfork N}$$

with $-\int_{\partial N} \in \tilde{C}_{\mathcal{F},X}$, $\int_{\partial_+^\top N} \in C_{\mathcal{F}}$, $\int_{\partial_-^\top N} \in C_{\mathcal{F}}$, and $\int_{\partial^\pitchfork N} \in P_{\mathcal{F},X}$.

Now suppose that the flow (ϕ_t) of $-X$ (or of $-fX$, for some positive function f) maps the closed bounded domain $\partial_+^\top N$ on a leaf into itself, i.e., $\partial_-^\top N = \phi_{t_0}(\partial_+^\top N) \subset \partial_+^\top N$ for some $t_0 > 0$ (see Figure 2 below). Then $\int_{\partial_+^\top N} - \int_{\partial_-^\top N} = \int_B \in C_{\mathcal{F}}$, where $B = \partial_+^\top N \setminus \partial_-^\top N$, and consequently

$$-\int_{\partial^\pitchfork N} \in C_{\mathcal{F},X} \cap P_{\mathcal{F},X},$$

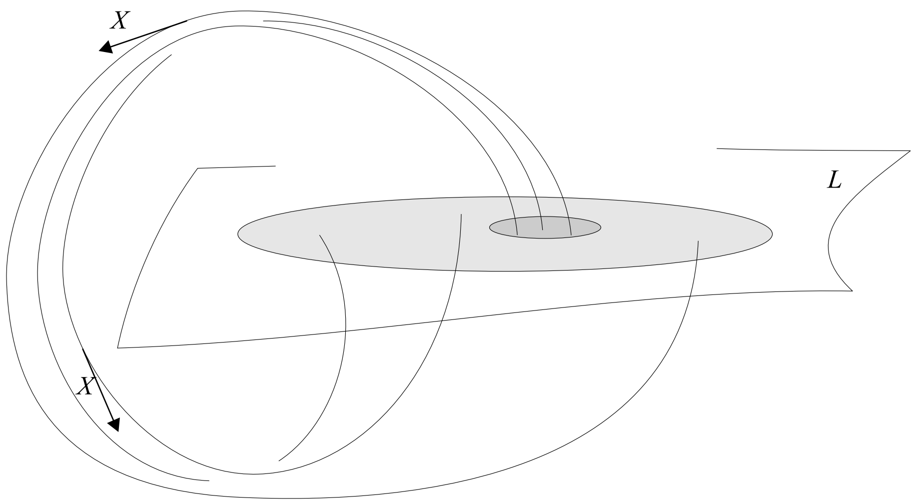

FIGURE 2. A vector field which cannot be mean curvature

so X cannot be realized as the mean curvature of $\mathcal{F}$ with respect to any Riemannian metric on M. In fact, the mean curvature $H = H_{\mathcal{F},g}$ is the negative gradient of the leaf volume, and therefore the volumes of pieces of leaves have to increase when deformed by the flow of $-H$ (or of $-fH$, $f > 0$).

The preceding example can be generalized to a construction involving a submanifold N of dimension $p+k+1$ in the presence of a suitable invariant measure, so that a family of pieces of leaves flows backwards into itself without preserving single leaves.

EXAMPLE 3. Let T and D be compact manifolds (possibly with boundary) of dimensions k and p, respectively, and suppose that an embedding $h_0 : T \times D \to M$ is given such that the induced foliation $h_0^*(\mathcal{F})$ coincides with the vertical foliation $\mathcal{V}$ of $T \times D$ whose leaves are $\{y\} \times D$, $y \in T$. Suppose that the flow (ϕ_t) of $-X$ (or of $-fX$, for some positive function f) generates an immersion $h : T \times D \times I \to M$, defined by setting

$$h(x, y, t) = \phi_t(h_0(x, y)),$$

where I is an interval $[0, t_0]$, such that the induced foliation $h^*(\mathcal{F})$ on $T \times D \times I$ has leaves $\{y\} \times D \times \{t\}$, $(y, t) \in T \times I$. Also suppose that ϕ_{t_0} takes $N_0 = h_0(T \times D)$ into itself, and that this is the only identification of points under h. It follows that there exist a smooth embedding $\phi : T \to T$ and uniquely determined subsets $D_y \subset D$ for each $y \in \phi(T)$ such that

$$h(\{y\} \times D \times \{t_0\}) = \phi_{t_0} \circ h_0(\{y\} \times D) = h_0(\{\phi(y)\} \times D_{\phi(y)}). \tag{13}$$

Finally, suppose that D and $\mathcal{F}$ are oriented and h takes the orientation of D into the orientation of the leaves of $\mathcal{F}$. In this situation we can extend the argument of Example 2 to show that the vector field X cannot be the mean curvature of $\mathcal{F}$ for any Riemannian metric on M. To this end, take any non-trivial positive ϕ-invariant Borel measure μ on T. Such a measure exists by the Krylov-Bogolyubov Theorem ([17, Cor. 6.9.1]). The boundary of $N = h(T \times D \times I)$ admits a decomposition $\partial N = \partial^\top N \cup \partial^\pitchfork N$, where $\partial^\top N$ is the closure of $N_0 \setminus \phi_{t_0}(N_0)$. For every value of the parameter $y \in T$ we have

$$\int_{\{y\}\times\partial D\times I} = -\partial \int_{\{y\}\times D\times I} + \int_{\{y\}\times D\times\{0\}} - \int_{\{y\}\times D\times\{t_0\}}, \tag{14}$$

where these integrals are interpreted as p-currents operating on differential p-forms on $T \times D \times I$. Integration over T yields

$$\begin{aligned}\int_T \int_{\{y\}\times\partial D\times I} d\mu = -\int_T \partial \int_{\{y\}\times D\times I} d\mu + \int_T \int_{\{y\}\times D\times\{0\}} d\mu \\ - \int_T \int_{\{y\}\times D\times\{t_0\}} d\mu.\end{aligned} \tag{15}$$

Then h_* will take the left hand side of (15) into $P_{\mathcal{F},X}$ and the first term on the right hand side into $\tilde{C}_{\mathcal{F},X}$. Next, we shall show that h_* takes the sum of the two remaining terms on the right hand side of (15) into $C_{\mathcal{F}}$. From (13) and the invariance of the measure μ under ϕ we get

$$h_* \int_T \int_{\{y\}\times D\times\{t_0\}} d\mu = h_* \int_T \int_{\{\phi(y)\}\times D_{\phi(y)}\times\{0\}} d\mu = h_* \int_{\phi(T)} \int_{\{y\}\times D_y\times\{0\}} d\mu.$$

Therefore, if we set $D_y = \emptyset$ for $y \in T \setminus \phi(T)$, we have

$$h_* \int_T \int_{\{y\}\times D\times\{0\}} d\mu - h_* \int_T \int_{\{y\}\times D\times\{t_0\}} d\mu = h_* \int_T \int_{\{y\}\times(D\setminus D_y)\times\{0\}} d\mu,$$

which belongs to $C_{\mathcal{F}}$, as desired.

Consequently,

$$h_* \int_T \int_{\{y\}\times\partial D\times I} d\mu \in C_{\mathcal{F},X} \cap P_{\mathcal{F},X},$$

and X cannot be the mean curvature vector field of $\mathcal{F}$ for any metric.

To give a concrete example of this situation, let $M = S^1 \times S^p \times \mathbb{R}/\sim$, where $\sim$ is the equivalence relation generated by setting $(y, x, t+1) \sim (\phi(y), f_y(x), t)$ for diffeomorphisms $\phi : S^1 \to S^1$ and $f_y : S^p \to S^p$ for every $y \in S^1$, with the property that $\psi(y, x) = (\phi(y), f_y(x))$ is a diffeomorphism of $S^1 \times S^p$ to itself. (Thus M is the suspension of ψ.) Let $\mathcal{F}$ be the foliation with the slices $\{x\} \times S^p \times \{t\}$ as leaves, oriented by the standard orientation of S^p. Letting $T = S^1$ and fixing a p-ball $D \subset S^p$, we suppose that for every y, $f_y(D) \subset D$. (For example, ϕ could be a rotation and every f_y could be the identity on S^p, or f_y could contract D into a smaller concentric ball.) Let

$h_0 : S^1 \times D \equiv S^1 \times D \times \{0\} \to M$ be the inclusion and set $X = -\partial/\partial t$. Then the flow (ϕ_t) of $-X$ on M preserves $\mathcal{F}$ and at time $t_0 = 1$ satisfies

$$\phi_1 \circ h_0(T \times D) \subset h_0(T \times D),$$

so the preceding argument shows that there is no Riemannian metric for which X is the mean curvature.

4. At singular sets

A priori, the behaviour of vector fields in neighbourhoods of singular points can be very complicated. In the case of mean curvature vectors of foliations, one can get immediate obstructions. For instance, such vector fields take values in the orthogonal complements of tangent bundles of foliations. Let us look at this situation more closely.

So, again let $(M, \mathcal{F})$ be a closed foliated manifold equipped with a Riemannian structure g. Let $X = H_{\mathcal{F},g}$ be the mean curvature vector and $E = T^{\perp}\mathcal{F}$, the orthogonal complement of $T\mathcal{F}$, the tangent bundle of $\mathcal{F}$. Then X takes values in E and $\Omega = \Omega_{\mathcal{F},g}$, the volume form of $\mathcal{F}$ with respect to g, vanishes on $P_{\mathcal{F},E}$, the closed subspace of D_p generated by all Dirac currents of the form $w \wedge v_1 \cdots \wedge v_{p-1}$, where $w \in E_x$, $v_i \in T_x\mathcal{F}$ and $x \in M$. If Σ is, as before, the set of all singular points of X, then by (4) Ω also vanishes on $B_{\mathcal{F}}(\Sigma)$, the closed subspace of D_p generated by all boundaries of the form $\partial(w \wedge v_1 \wedge \cdots \wedge v_p)$ with $w \in T_xM$, $v_i \in T_x\mathcal{F}$ and $x \in \Sigma$. Consequently, $\Omega \equiv 0$ on the sum $P_{\mathcal{F},E} + B_{\mathcal{F}}(\Sigma)$. Since Ω is positive on the cone $C_{\mathcal{F}} \setminus \{0\}$, it follows that

$$C_{\mathcal{F}} \cap (P_{\mathcal{F},E} + B_{\mathcal{F}}(\Sigma)) = \{0\}. \tag{16}$$

On the other hand, if K is an arbitrary compact subset of M satisfying the condition analogous to (16)

$$C_{\mathcal{F}} \cap (P_{\mathcal{F},E} + B_{\mathcal{F}}(K)) = \{0\}, \tag{17}$$

and $(U_k)_{k=1}^{\infty}$ is a decreasing nested family of open neighbourhoods of K such that $\bigcap_{k=1}^{\infty} \overline{U_k} = K$, then $B_{\mathcal{F}}(K) = \bigcap_{k=1}^{\infty} B_{\mathcal{F}}(\overline{U_k})$ and, in view of the existence of a compact base of $C_{\mathcal{F}}$,

$$C_{\mathcal{F}} \cap (P_{\mathcal{F},E} + B_{\mathcal{F}}(\overline{U_k})) = \{0\}$$

for k large enough. By arguments similar to those of [13] we get the following result.

Proposition 1. *If $K \subset M$ is compact, $E \subset TM$ is a subbundle complementary to $T\mathcal{F}$ and condition* (17) *is satisfied, then there exists a Riemannian metric g on M such that E is orthogonal to $\mathcal{F}$ and $H_{\mathcal{F},g} \equiv 0$ in a neighbourhood of K. Furthermore, given any Riemanian metric g_0 on M, g can be chosen so that $g(v, w)$ coincides with $g_0(v, w)$ whenever v and w belong to E.* □

The following elementary fact can be obtained by easy calculation (or found in the literature, for example in [4] or [14]).

LEMMA 2. *If g and $g' = e^{2\phi}g$ are conformally equivalent Riemannian metrics on a foliated manifold $(M, \mathcal{F})$, then the mean curvature vectors H and H' of $\mathcal{F}$ with respect to g and g' are related by the formula*

$$H' = e^{-2\phi}\left(H - p(\nabla\phi)^{\perp}\right), \tag{18}$$

where $p = \dim \mathcal{F}$ and $(\nabla\phi)^{\perp}$ denotes the component of the g-gradient of ϕ orthogonal to $\mathcal{F}$. □

From (18) it follows that if $H = 0$ on an open subset U of M and $X = (\nabla f)^{\perp}$ for some g and $f \in C^{\infty}(M)$, then $H' = X$ on U when $g' = e^{2\phi}g$ with

$$\phi = \frac{1}{2} \cdot \log \frac{p}{2(f + c)},$$

where c is a positive constant greater than $\max_{x \in M} |f(x)|$. Also, if X is a gradient field (i.e., $X = \nabla^0 f$ is the gradient of a function f with respect to some Riemannian metric g_0), X takes values in E, a subbundle of TM complementary to $T\mathcal{F}$, and the conditions of Proposition 1 are satisfied, then there exists a Riemannian metric g on U for which E is orthogonal to $\mathcal{F}$, the volume form $\Omega_{\mathcal{F}}$ of $\mathcal{F}$ (with respect to g) is $\mathcal{F}$-closed (i.e., vanishes on $B_{\mathcal{F}}(\overline{U})$) and $g(v, w)$ coincides with $g_0(v, w)$ whenever v and $w \in E$. For this g, $H_{\mathcal{F},g} = 0$ and $(\nabla f)^{\perp} = X$ on U. Therefore, Proposition 1 and Theorem 1 imply the following result.

THEOREM 2. *If X is a vector field on $(M, \mathcal{F})$ which takes values in a subbundle $E \subset TM$ complementary to $T\mathcal{F}$, E and $\Sigma = \{x \in M; X(x) = 0\}$ satisfy (16), $X|U$ is a gradient field for some open neighbourhood U of Σ, and the cone $C_{\mathcal{F},X}(A)$ has a compact base and intersects the subspace $P_{\mathcal{F},X}(A)$ trivially for every closed set $A \subset M \setminus \Sigma$, then there exists a Riemannian metric g on M for which $X = H_{\mathcal{F},g}$, the mean curvature of the foliation $\mathcal{F}$ on (M, g).* □

Let us conclude this article with some remarks on gradient vector fields. These fields play an important role in the theory of smooth dynamical systems. For instance (see [8]), Morse-Smale gradient fields are open and dense in the space of all gradients. They have a number of simple properties: they admit no closed orbits and the limit sets of their orbits consist of singularities (infinitely many if more than one). These properties can be expressed in terms of currents in the following way.

Let X be a vector field and Σ, as before, the set of all singular points of X. Consider the closed convex cone $C_X \subset D_1$ generated by all Dirac currents $X(x)$, $x \in M$, and the closed linear subspace $P_{\Sigma} \subset D_1$ generated by all 1-currents c for which there exist currents z supported in Σ such that $\partial c = \partial z$.

If $X = \nabla f$ for some $f \in \mathrm{C}^\infty(M)$ and some Riemannian metric g, then

$$C_X \cap P_\Sigma \subset D_1(\Sigma). \tag{19}$$

Indeed, if $c = \lim_{n\to\infty} c_n$ belongs to P_Σ, where $c_n = \sum_i t_{n,i} X(x_{n,i})$ for some $t_{n,i} > 0$ and $x_{n,i} \in M$, then setting $X_{n,i} = X(x_{n,i})$ we have

$$\|X_{n,i}\|^2 = g(X_{n,i}, X_{n,i}) = g(X_{n,i}, \nabla f) = \langle X_{n,i}, df\rangle$$

and therefore

$$\begin{aligned}\lim_{n\to\infty} \sum_i t_{n,i} \|X_{n,i}\|^2 &= \langle c, df\rangle \\ &= \langle \partial c, f\rangle = \langle \partial z, f\rangle = \langle z, df\rangle = 0\end{aligned}$$

for some $z \in D_1$ supported in Σ. This implies condition (19).

Figure 3 below shows some situations when (19) is not satisfied.

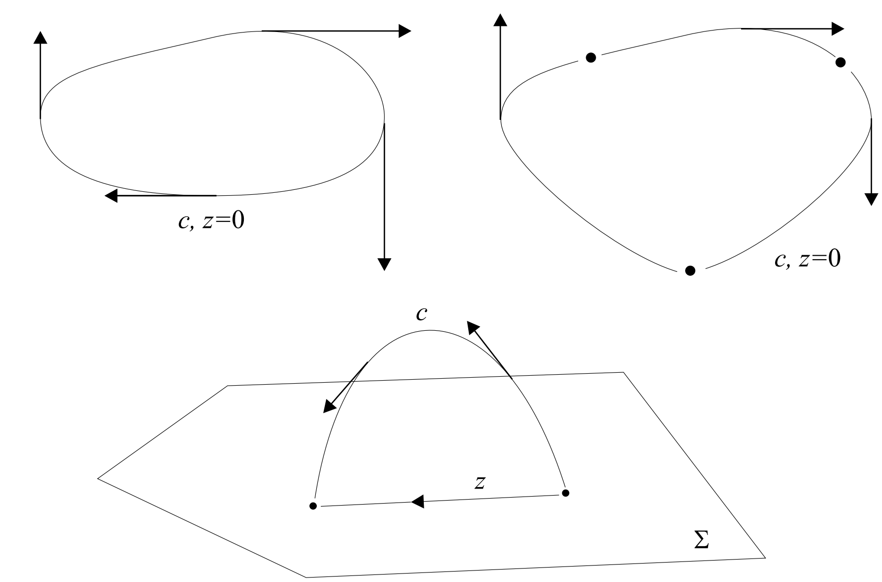

FIGURE 3. Some non-gradient phenomena

Unfortunately, the cone C_X need not have a compact base even if X is a gradient field. For example, if $M = S^1$ is a standard unit circle parametrized by the angle θ, $f : S^1 \to [0,1]$ is a function which has exactly two critical points, x_0 with $f(x_0) = 0$ and x_1 with $f(x_1) = 1$, and is strictly increasing on oriented arcs I_0 and I_1 with initial point x_0 and end point x_1, then the Dirac currents corresponding to the vectors $\pm(\partial/\partial\theta)(x_0)$ and $\pm(\partial/\partial\theta)(x_1)$ belong to $C_{\nabla f}$, and, therefore, $C_{\nabla f}$ has no compact base. However, a slight modification of this simple example provides a gradient field Y for which C_Y

has a compact base. In fact, take $f : S^1 \to [0,1]$ such that $f|J_0 \equiv 0$ and $f|J_1 \equiv 1$ for some disjoint non-trivial arcs J_0 and J_1 contained in S^1, while f is strictly increasing on the complementary arcs I_0 and I_1 whose endpoints coincide with those of J_0 and J_1. Then there exists a 1-form η on S^1 which coincides with $d\theta$ on I_0 and $-d\theta$ on I_1. In this case, η is positive on $C_Y \setminus \{0\}$ and the set $\eta^{-1}(1) \cap C_Y$ is compact.

If (19) is satisfied, the cone C_X is compact, and U is an open subset of M for which $\overline{U} \cap \Sigma = \emptyset$, then, again by the Hahn-Banach Theorem, there exists a 1-form ω such that $\omega > 0$ on $C_X(\overline{U}) \setminus \{0\}$ and $\omega \equiv 0$ on P_Σ. Since P_Σ contains all 1-cycles, ω is exact, i.e., $\omega = df$ for some function f. Since ω is positive on $C_X(\overline{U}) \setminus \{0\}$, $df(X(x)) > 0$ whenever $x \in \overline{U}$. Therefore, one can find a Riemannian metric g on U for which X is orthogonal to $\ker df$ and $\|X\|^2 = df(X)$. For this g, $X = \nabla f$ on U.

The problem of the realization of a vector field as a gradient in a neighbourhood of its singular set seems to be much more delicate.

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