

## ON NONSTANDARD PRODUCT MEASURE SPACES

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ABSTRACT. The aim of this paper is to investigate systematically the relationship between the two different types of product probability spaces based on the Loeb space construction. For any two atomless Loeb spaces, it is shown that for fixed  $r < s$  in  $[0, 1]$  there exists an increasing sequence  $(A_t)_{r < t < s, t \in [0, 1]}$  of in the new sense product measurable sets such that  $A_t$  has measure  $t$  and, with respect to the usual product, the inner and outer measures are equal to  $r$  and  $s$ , respectively. By constructing a continuum of increasing Loeb product null sets with large gaps, the Loeb product is shown to be much richer than the usual product even on null sets. General results in terms of outer and inner measures with respect to the usual product are also obtained for Loeb product measurable sets that are composed of almost independent events.

### 1. Introduction

Let  $(\Omega, L_\mu(\mathcal{A}), \hat{\mu})$  and  $(T, L_\nu(\mathcal{T}), \hat{\nu})$  be the respective Loeb probability spaces constructed from any internal probability spaces  $(\Omega, \mathcal{A}, \mu)$  and  $(T, \mathcal{T}, \nu)$ , where  $\mathcal{A}$  and  $\mathcal{T}$  are internal algebras of sets, and  $\mu$  and  $\nu$  are finitely additive internal measures. Let  $(\Omega \times T, \mathcal{A} \otimes \mathcal{T}, \mu \otimes \nu)$  be the internal product of the internal probability spaces  $(\Omega, \mathcal{A}, \mu)$  and  $(T, \mathcal{T}, \nu)$ , where  $\mathcal{A} \otimes \mathcal{T}$  is simply the internal algebra of all \*finite disjoint unions of rectangles  $A \times B$  with  $A \in \mathcal{A}$  and  $B \in \mathcal{T}$ . The corresponding Loeb space of  $(\Omega \times T, \mathcal{A} \otimes \mathcal{T}, \mu \otimes \nu)$  is denoted by  $(\Omega \times T, L_{\mu \otimes \nu}(\mathcal{A} \otimes \mathcal{T}), \widehat{\mu \otimes \nu})$  and called the Loeb product space. On the other hand, the completion of the usual product of the two Loeb spaces is denoted by  $(\Omega \times T, L_\mu(\mathcal{A}) \otimes L_\nu(\mathcal{T}), \hat{\mu} \otimes \hat{\nu})$ .

As already noted by Anderson [2], the usual product  $\sigma$ -algebra  $L_\mu(\mathcal{A}) \otimes L_\nu(\mathcal{T})$  is contained in the Loeb product algebra  $L_{\mu \otimes \nu}(\mathcal{A} \otimes \mathcal{T})$ , and  $\widehat{\mu \otimes \nu}$  is an extension of  $\hat{\mu} \otimes \hat{\nu}$ . D. Hoover and D. Norman provided a specific example showing that this inclusion can be proper (see Albeverio et al. [1, p. 74]). This example is based on a hyperfinite set  $T$  and its internal power set  $\Omega$ , where

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both  $T$  and  $\Omega$  are endowed with the counting probability measure. A general result was presented in Proposition 6.6 of [15] by the third named author of this paper. It shows that the stated inclusion is always proper as long as the relevant Loeb probability spaces are atomless. The proof of this result was based on the general incompatibility of independence and joint-measurability in the usual sense as shown in Proposition 6.5 in [15] and Proposition 1.1 in [14]. Such incompatibility was already observed by Doob [5] in the setting of iid processes with a Lebesgue parameter space.

The purpose of this paper is to provide a systematic study of the relationship between the Loeb product space  $(\Omega \times T, L_{\mu \otimes \nu}(\mathcal{A} \otimes \mathcal{T}), \widehat{\mu \otimes \nu})$  and its counterpart, the usual product space  $(\Omega \times T, L_{\mu}(\mathcal{A}) \otimes L_{\nu}(\mathcal{T}), \hat{\mu} \otimes \hat{\nu})$  of the two Loeb spaces. Note that the distinction between these two types of nonstandard product measures is the starting point for the discovery of some completely new phenomena involving independence (see [13]–[16]). Loeb product measures are also useful in the hyperfinite contexts of model theory (see Keisler [7]) and chaos decompositions (see Cutland and Ng [4] and for details in [11]). An important property for the larger Loeb product framework is the Fubini property, as first shown by Keisler [7] (see also [1] and [10]). In this paper, we consider only the interesting case that the arbitrarily given Loeb measures  $\hat{\mu}$  and  $\hat{\nu}$  are atomless. For the convenience of the reader, some general properties of atomless probability spaces and atomless Loeb spaces are discussed in Section 2.

In Section 3, an explicit example is constructed to show the existence of a class of sets  $\{R^s \in L_{\mu \otimes \nu}(\mathcal{A} \otimes \mathcal{T}) : s \in [0, 1]\}$  such that for each  $s \in [0, 1]$ ,  $\widehat{\mu \otimes \nu}(R^s) = 0$  and  $(\hat{\mu} \otimes \hat{\nu})^{\text{outer}}(R^s) = 1$ , and for all  $s_1, s_2 \in [0, 1]$  with  $s_1 < s_2$ ,  $R^{s_1} \subset R^{s_2}$  and  $(\hat{\mu} \otimes \hat{\nu})^{\text{outer}}(R^{s_2} \setminus R^{s_1}) = 1$ . This means that there is a continuum of increasing Loeb product null sets with large gaps in the sense that their set differences have  $\hat{\mu} \otimes \hat{\nu}$ -outer measure one. Thus, the Loeb product space is much richer than the usual product even on null sets. Hence, each product measurable set in  $L_{\mu}(\mathcal{A}) \otimes L_{\nu}(\mathcal{T})$  can be modified by a null set in the Loeb product space such that the modified set is no longer measurable in  $L_{\mu}(\mathcal{A}) \otimes L_{\nu}(\mathcal{T})$ . Note that our argument for constructing all such non-product Loeb measurable sets is novel. In addition, another simple and concrete construction of sets in  $L_{\mu \otimes \nu}(\mathcal{A} \otimes \mathcal{T}) \setminus L_{\mu}(\mathcal{A}) \otimes L_{\nu}(\mathcal{T})$  is given in this section.

In Section 4, general results in terms of outer and inner measures with respect to the usual product measure  $\hat{\mu} \otimes \hat{\nu}$  are also considered for those Loeb product measurable sets that are composed of almost independent nontrivial events. Some further consequences of the results in Section 3 are given in Section 5. In particular, it is shown that for a set in the Loeb product algebra  $L_{\mu \otimes \nu}(\mathcal{A} \otimes \mathcal{T})$  its measure  $s$  and its outer measure  $t$  and inner measure  $r$  with respect to the usual product measure  $\hat{\mu} \otimes \hat{\nu}$  can be completely arbitrary subject to the obvious condition  $0 \leq r \leq s \leq t \leq 1$ . In addition, it is pointed out that

every set  $E$  in  $L_{\mu \otimes \nu}(\mathcal{A} \otimes \mathcal{T})$  has a subset  $A \in L_{\mu \otimes \nu}(\mathcal{A} \otimes \mathcal{T})$  with  $\hat{\mu} \otimes \hat{\nu}$ -inner measure 0 that has the same  $\hat{\mu} \otimes \hat{\nu}$ -outer measure and  $\widehat{\mu \otimes \nu}$ -measure as  $E$ .

**2. Properties of atomless Loeb spaces**

A measurable set  $A$  in a finite measure space  $(\Lambda, \mathcal{F}, P)$  is called an atom if  $P(A) > 0$  and for any measurable subset  $B$  of  $A$ ,  $P(B)$  is  $P(A)$  or 0. The measure space  $(\Lambda, \mathcal{F}, P)$  is called *atomless* if it has no atoms. The well known Lyapunov theorem says that the range of the atomless measure  $P$  is the closed bounded interval  $[0, P(\Lambda)]$ . It is thus obvious that for any non-negative real numbers  $\alpha_1, \alpha_2, \dots, \alpha_n$  with total sum  $P(\Lambda)$ , there exists a partition  $B_1, \dots, B_n \in \mathcal{F}$  of  $\Lambda$  such that  $P(B_i) = \alpha_i$  for  $i = 1, 2, \dots, n$ . It is easy to show that an atomless probability space has the following well known universality property. (For example, a version of this result was quoted in Proposition 2.2 in [8].)

LEMMA 2.1. *Let  $(\Lambda, \mathcal{F}, P)$  be an atomless probability space. Then, for any Borel probability distribution  $\tau$  on a complete separable metric space  $X$ , there is a Borel measurable mapping  $f$  from  $(\Lambda, \mathcal{F}, P)$  to  $X$  such that the distribution of  $f$  is the given distribution  $\tau$ , i.e.,  $Pf^{-1}(B) = \tau(B)$  for any Borel set  $B$  in  $X$ .*

For any Loeb probability space  $(T, L_\nu(\mathcal{T}), \hat{\nu})$ , by saturation, the internal algebra  $\mathcal{T}$  is rich enough to guarantee that each element of  $L_\nu(\mathcal{T})$  is *equivalent* to an element of  $\mathcal{T}$  in the sense that for each  $B \in L_\nu(\mathcal{T})$  there exists an  $A \in \mathcal{T}$  such that  $\hat{\nu}(A \Delta B) = 0$ , where  $A \Delta B$  is the symmetric difference of  $A$  and  $B$ . In this case  $A$  is called a  $\hat{\nu}$ -approximation of  $B$  (see [6]).

Throughout this paper we assume that the given Loeb probability spaces  $(\Omega, L_\mu(\mathcal{A}), \hat{\mu})$  and  $(T, L_\nu(\mathcal{T}), \hat{\nu})$  are atomless. For simplicity, we denote the product  $\sigma$ -algebra  $L_\mu(\mathcal{A}) \otimes L_\nu(\mathcal{T})$  by  $\mathcal{U}$ . Let  $\mathcal{P}$  denote the set of all finite unions  $X_1 \times Y_1 \cup \dots \cup X_n \times Y_n$  of rectangles  $X_i \times Y_i$  such that  $X_i \in \mathcal{A}$  and  $Y_i \in \mathcal{T}$  and  $Y_1, \dots, Y_n$  form a partition of  $T$ . Notice that  $\mathcal{P}$  is an algebra and generates  $\mathcal{U}$ . Using saturation, the following lemma is an obvious consequence of the Lyapunov theorem.

LEMMA 2.2. *Given  $B \in \mathcal{T}$ . Then there exists an unlimited  $k \in {}^*\mathbb{N}$  having the following property: for each  $n \in {}^*\mathbb{N}$  with  $n \leq k$ , there exists an internal partition  $C_1, \dots, C_n \in \mathcal{T}$  of  $B$  such that  $|\nu(C_i) - \hat{\nu}(B)/n| < 1/k$ ; in particular,  $\nu(C_i) \approx \hat{\nu}(B)/n$  and  $\nu(C_i) \approx 0$  if  $n$  is unlimited.*

*Proof.* We restrict our attention to the atomless measure space  $(B, L_\nu(\mathcal{T}) \cap B, \hat{\nu}|_B)$ . Take any  $k \in \mathbb{N}$ . Then for each  $n \leq k$ , the Lyapunov theorem implies that there is a partition  $C_1, \dots, C_n$  of  $B$  such that  $\hat{\nu}(C_i) = \hat{\nu}(B)/n$  for each  $i$ . By taking  $\hat{\nu}$ -approximations of these sets, we can assume that the sets  $C_i$  are

internal. Thus, we have  $|\nu(C_i) - \hat{\nu}(B)/n| < 1/k$ . By saturation, there exists an unlimited  $k \in {}^*\mathbb{N}$  with the desired property.  $\square$

Recall that for any set  $C \subseteq \Omega \times T$ , the *outer* and *inner* measures of  $C$  are defined by setting

$$(\hat{\mu} \otimes \hat{\nu})^{\text{inner}}(C) := \sup \{ \hat{\mu} \otimes \hat{\nu}(D) \mid D \subseteq C \text{ and } D \in \mathcal{U} \}$$

and

$$(\hat{\mu} \otimes \hat{\nu})^{\text{outer}}(C) := \inf \{ \hat{\mu} \otimes \hat{\nu}(D) \mid C \subseteq D \in \mathcal{U} \}.$$

The following lemma shows that for an internal set  $C$  the condition  $D \in \mathcal{U}$  in the above definition can be replaced by  $D \in \mathcal{P}$ .

LEMMA 2.3. *For any internal set  $C \subseteq \Omega \times T$ , we have:*

- (1)  $(\hat{\mu} \otimes \hat{\nu})^{\text{inner}}(C) := \sup \{ \hat{\mu} \otimes \hat{\nu}(D) \mid D \subseteq C \text{ and } D \in \mathcal{P} \}$ ; in particular, if  $C$  is product measurable in the usual sense with  $\hat{\mu} \otimes \hat{\nu}(C) > 0$ , then there exist  $X \in \mathcal{A}$  and  $Y \in \mathcal{T}$  such that  $X \times Y \subseteq C$  and  $0 < \hat{\mu}(X)\hat{\nu}(Y)$ .
- (2)  $(\hat{\mu} \otimes \hat{\nu})^{\text{outer}}(C) := \inf \{ \hat{\mu} \otimes \hat{\nu}(D) \mid C \subseteq D \in \mathcal{P} \}$ .

*Proof.* Let us first prove (1). Fix any  $\epsilon > 0$ . Since  $\mathcal{P}$  forms an algebra and generates  $\mathcal{U}$ , Proposition 32 of Royden [12, p. 320] implies that there is a decreasing sequence  $E_n \in \mathcal{P}, n \in \mathbb{N}$  such that  $\bigcap_{n \in \mathbb{N}} E_n \subseteq C$  and  $\hat{\mu} \otimes \hat{\nu}(\bigcap_{n \in \mathbb{N}} E_n) \geq (\hat{\mu} \otimes \hat{\nu})^{\text{inner}}(C) - \epsilon$ . Since the internal set  $C$  contains  $\bigcap_{n \in \mathbb{N}} E_n$ , the underflow principle implies that there exists an  $n \in \mathbb{N}$  such that  $D := \bigcap_{k=1}^n E_k \subseteq C$ . Obviously,  $(\hat{\mu} \otimes \hat{\nu})^{\text{inner}}(C) - \epsilon < \hat{\mu} \otimes \hat{\nu}(D)$  and  $D \in \mathcal{P}$ . The result then follows from the arbitrary choice of  $\epsilon$ .

(2) can be proven from (1) by using contraposition.  $\square$

We end this section with some definitions.

Let  $C$  be a set in the Loeb product algebra  $L_{\mu \otimes \nu}(\mathcal{A} \otimes \mathcal{T})$ . The *indicator function* of  $C$  is denoted by  $1_C$ . For each  $t \in T$ ,  $C(\cdot, t)$  denotes the *section*  $\{\omega \in \Omega : (\omega, t) \in C\}$ , and for each  $\omega \in \Omega$ ,  $C(\omega, \cdot)$  denotes the *section*  $\{t \in T : (\omega, t) \in C\}$ . We will write  $C_t$  instead of  $C(\cdot, t)$ . If for  $\widehat{\nu \otimes \nu}$ -almost all  $(s, t) \in T \times T$ , the events  $C_t$  and  $C_s$  are independent, then we say the events  $C_t, t \in T$ , are  *$\hat{\nu}$ -almost independent*. Processes with almost independent random variables are studied systematically in [13]–[16].

### 3. Explicit constructions

We consider general atomless Loeb probability spaces  $(\Omega, L_\mu(\mathcal{A}), \hat{\mu})$  and  $(T, L_\nu(\mathcal{T}), \hat{\nu})$ . For any  $p$  in the open unit interval  $(0, 1)$ , Theorem 6.2 and Proposition 6.6 in [15] say that there exists a set  $C$  in  $L_{\mu \otimes \nu}(\mathcal{A} \otimes \mathcal{T})$  whose events  $C_t$  are  $\hat{\nu}$ -almost independent with common probability  $p$ . Then the incompatibility of independence and joint measurability stated in Proposition 1.1 in [14] (see also Proposition 6.5 in [15]) implies that  $C \in L_{\mu \otimes \nu}(\mathcal{A} \otimes \mathcal{T}) \setminus \mathcal{U}$ .

In Example 3.1 below, a simple and concrete construction of such a non-product measurable set  $C$  with  $0 < \widehat{\mu \otimes \nu}(C) < 1$  is given.

EXAMPLE 3.1. Let  $p \in (0, 1)$  be arbitrarily given. By Lemma 2.2, we can find some unlimited  $\eta \in {}^*\mathbb{N}$  and internal partitions  $\{B_j : j = 1, \dots, \eta\} \subseteq \mathcal{T}$  of  $T$  and  $\{A_{j_1 \dots j_\eta} : j_1, \dots, j_\eta = 1, \dots, \eta\} \subseteq \mathcal{A}$  of  $\Omega$  such that  $|\nu(B_j) - \eta^{-1}| < \eta^{-2}$  and  $|\mu(A_{j_1 \dots j_\eta}) - \eta^{-\eta}| < \eta^{-2\eta}$ , respectively. Let us fix such a hyperfinite number  $\eta$ . For each  $k = 1, \dots, \eta$ , define

$$D_k = \bigcup_{j_1, \dots, \widehat{j_k}, \dots, j_\eta=1}^{\eta} \bigcup_{j_k=1}^{[p\eta]} A_{j_1 \dots j_\eta},$$

where  $[p\eta]$  stands for the integer part of  $p\eta$  and  $\widehat{j_k}$  means that  $j_k$  is omitted in the index. Note that

$$\sum_{j_1, \dots, \widehat{j_k}, \dots, j_\eta=1}^{\eta} \sum_{j_k=1}^{[p\eta]} \eta^{-\eta} = \eta^{-1}[p\eta] \text{ and } p - \eta^{-1} < \eta^{-1}[p\eta] \leq p,$$

which implies that

$$|\mu(D_k) - p| \leq \sum_{j_1, \dots, \widehat{j_k}, \dots, j_\eta=1}^{\eta} \sum_{j_k=1}^{[p\eta]} |\mu(A_{j_1 \dots j_\eta}) - \eta^{-\eta}| + \eta^{-1} \leq \eta^{-\eta} + \eta^{-1}.$$

Let  $C = \cup_{k=1}^{\eta} D_k \times B_k$ . Then it is clear that  $C$  is in the internal algebra  $\mathcal{A} \otimes \mathcal{T}$ . By the previous estimation on  $\mu(D_k)$ , we obtain

$$|\mu \otimes \nu(C) - p| = \left| \sum_{k=1}^{\eta} ((\mu(D_k) - p) \cdot \nu(B_k)) \right| \leq (\eta^{-\eta} + \eta^{-1}).$$

Therefore,  $\widehat{\mu \otimes \nu}(C) = p$ .

Now we will show that  $(\widehat{\mu} \otimes \widehat{\nu})^{\text{inner}}(C) = 0$ . Suppose otherwise. Then it follows from Lemma 2.3 (1) that there exist internal sets  $X \in \mathcal{A}$  and  $Y \in \mathcal{T}$  such that  $X \times Y \subseteq C$  with  $\widehat{\mu}(X)\widehat{\nu}(Y) > 0$ .

Let  $K = \{1 \leq k \leq \eta : Y \cap B_k \neq \emptyset\}$  and  $G = \cap_{k \in K} D_k$ . For each  $k \in K$ , there is a  $t \in Y \cap B_k$ . Since  $X \times \{t\} \subseteq \cup_{j=1}^{\eta} D_j \times B_j$  and  $t \notin B_j$  for  $j \neq k$ , we must have  $X \subseteq D_k$ . Hence  $Y \subseteq \cup_{k \in K} B_k$  and  $X \subseteq G$ .

Let  $h_1, h_2, \dots, h_{|K|}$  and  $i_1, i_2, \dots, i_{\eta-|K|}$  be the respective lists of the elements in  $K$  and  $\{1, 2, \dots, \eta\} - K$  in an increasing order. Then it is clear that

$$G = \bigcup_{j_{i_1}, j_{i_2}, \dots, j_{i_{\eta-|K|}}=1}^{\eta} \bigcup_{j_{h_1}, j_{h_2}, \dots, j_{h_{|K|}}=1}^{[p\eta]} A_{j_1, \dots, j_\eta}.$$

Thus  $\mu(G) \leq (\eta^{-\eta} + \eta^{-2\eta})\eta^{\eta-|K|}[p\eta]^{|K|}$ . Therefore,

$$\mu(X)\nu(Y) \leq (\eta^{-1} + \eta^{-2})|K|(1 + \eta^{-\eta})p^{|K|},$$

which is an infinitesimal no matter whether  $K$  is finite or infinite. This contradicts  $\hat{\mu}(X)\hat{\nu}(Y) > 0$ . Therefore,  $(\hat{\mu} \otimes \hat{\nu})^{\text{inner}}(C) = 0$ . It follows that  $C \in L_{\mu \otimes \nu}(\mathcal{A} \otimes \mathcal{T}) \setminus \mathcal{U}$ . Note that the events  $C_t, t \in T$ , are  $\hat{\nu}$ -almost independent with equal probability  $p$ . By Theorem 4.2 in Section 4 below, it follows that  $(\hat{\mu} \otimes \hat{\nu})^{\text{outer}}(C) = 1$ .

In the next example, we construct a class of sets  $\{R^s \in \mathcal{A} \otimes \mathcal{T} \setminus \mathcal{U} : s \in [0, 1]\}$  such that for each  $s \in [0, 1]$ ,  $\widehat{\mu \otimes \nu}(R^s) = 0$  and  $(\hat{\mu} \otimes \hat{\nu})^{\text{outer}}(R^s) = 1$ , and for all  $s_1, s_2 \in [0, 1]$  with  $s_1 < s_2$ ,  $R^{s_1} \subset R^{s_2}$  and  $(\hat{\mu} \otimes \hat{\nu})^{\text{outer}}(R^{s_2} \setminus R^{s_1}) = 1$ . Thus we move beyond the previous results to show that there is a continuum of increasing Loeb product null sets with large gaps. Here a continuum of null rectangles are put together to produce a Loeb product null set. As a by-product, we also obtain a class of sets  $\{C^s \in \mathcal{A} \otimes \mathcal{T} : s \in [0, 1]\}$  such that for each  $s \in [0, 1]$ ,  $\widehat{\mu \otimes \nu}(C^s) = s$  and  $(\hat{\mu} \otimes \hat{\nu})^{\text{inner}}(C^s) = 0$  and  $(\hat{\mu} \otimes \hat{\nu})^{\text{outer}}(C^s) = 1$ , and for all  $s_1, s_2 \in [0, 1]$  with  $s_1 < s_2$ ,  $C^{s_1} \subset C^{s_2}$  and  $(\hat{\mu} \otimes \hat{\nu})^{\text{outer}}(C^{s_2} \setminus C^{s_1}) = 1$ .

EXAMPLE 3.2. For any positive integers  $k, n \in \mathbb{N}$ , consider the space  $\{1, 2, \dots, n\}$  with the counting probability measure  $\tau$  and its  $k$ -fold product probability space  $\{1, 2, \dots, n\}^k$  with probability measure  $\tau^k$ . By the universality result in Lemma 2.1, there is a random variable  $\Phi^{n,k}$  from the atomless Loeb space  $(\Omega, L_\mu(\mathcal{A}), \hat{\mu})$  to  $\{1, 2, \dots, n\}^k$  with distribution  $\tau^k$ . Since the target space is discrete, we can assume without loss of generality that  $\Phi^{n,k}$  is internal. Let the  $i$ -th component of  $\Phi^{n,k}$  be  $\phi_i^{n,k}$  for  $i = 1, 2, \dots, k$ . Then the random variables in the collection  $\{\phi_i^{n,k} : i = 1, 2, \dots, k\}$  are iid with common distribution  $\tau$ .

Next, for any fixed  $1 \leq p \leq l < n$ , let  $A_i^{p,l,n,k} = (\phi_i^{n,k})^{-1}(\{p+1, p+2, \dots, l\})$ . Then the internal events in the collection  $\{A_i^{p,l,n,k} : i = 1, 2, \dots, k\}$  are iid with common probability  $(l-p+1)/n$ .

By saturation, there exists an unlimited number  $Q \in {}^*\mathbb{N}$  such that for each unlimited  $k \in {}^*\mathbb{N}$  with  $k \leq Q$ , and for each  $p, l, n \in {}^*\mathbb{N}$  with  $1 \leq p \leq l < n \leq k$ , there exist sets  $A_1^{p,l,n,k}, \dots, A_k^{p,l,n,k} \in \mathcal{A}$  having the following properties:

- (1)  $\hat{\mu}(A_i^{p,l,n,k}) \approx (l-p+1)/n$  for each  $i = 1, \dots, k$ .
- (2)  $A_i^{p,l,n,k} \subseteq A_i^{p,l+1,n,k}$  for  $p < l+1 < n$  and for each  $i = 1, \dots, k$ .
- (3) For each  $1 \leq m \leq k$  and for each strictly increasing  $m$ -tuple  $i_1 < \dots < i_m$  in  $\{1, \dots, k\}$ ,

$$\mu \left( A_{i_1}^{p,l,n,k} \cup \dots \cup A_{i_m}^{p,l,n,k} \right) \approx \left( 1 - \left( \frac{n+p-l-1}{n} \right)^m \right)$$

and

$$\mu \left( A_{i_1}^{p,l,n,k} \cap \dots \cap A_{i_m}^{p,l,n,k} \right) \approx \frac{(l-p+1)^m}{n^m}.$$

$$(4) \quad A_i^{p+1,l,n,k} = A_i^{1,l,n,k} \setminus A_i^{1,p,n,k} \text{ for } 1 \leq p \leq l < n \text{ and for each } i = 1, \dots, k.$$

Now fix unlimited  $n, k \in {}^*\mathbb{N}$  such that  $k \leq Q$  and  $n^2/k \approx 0$ . By Lemma 2.2, there exists an internal partition  $B_1, \dots, B_k \subseteq \mathcal{T}$  of  $T$  with  $\mu(B_i) < 2/k$ . Set

$$D^{p,l} := \left( A_1^{p,l,n,k} \times B_1 \right) \cup \dots \cup \left( A_k^{p,l,n,k} \times B_k \right).$$

Then  $D^{p,l} \in \mathcal{A} \times \mathcal{T}$  with

$$(1) \quad \mu \otimes \nu (D^{p,l}) \approx \frac{(l-p+1)}{n}.$$

By Property (2), it is clear that  $D^{p,l} \subseteq D^{p,q}$  for  $p \leq l \leq q < n$ . By Property (4), we know that for  $1 \leq p \leq l < n$ ,

$$(2) \quad D^{p+1,l} = D^{1,l} \setminus D^{1,p}.$$

We shall now show that the inner measure  $(\hat{\mu} \otimes \hat{\nu})^{\text{inner}}(D^{p,l})$  is 0. Since we work with fixed  $p, l, n, k$ , we can omit the upper indices  $p, l, n, k$  at  $A_i^{p,l,n,k}$ . Suppose that  $(\hat{\mu} \otimes \hat{\nu})^{\text{inner}}(D^{p,l}) > 0$ . It follows from Lemma 2.3 (1) that there exist  $X \in \mathcal{A}$  and  $Y \in \mathcal{T}$  with  $0 < \hat{\mu}(X), \hat{\nu}(Y)$  such that  $X \times Y \subseteq D^{p,l}$ . There must be more than  $n^2$  many elements  $i_1, \dots, i_m \in \{1, \dots, k\}$  with  $Y \cap B_{i_1} \neq \emptyset, \dots, Y \cap B_{i_m} \neq \emptyset$ . (If this is not true, then  $Y \subseteq B_{i_1} \cup \dots \cup B_{i_m}$  for  $m \leq n^2$ , which implies  $\mu(B_{i_1} \cup \dots \cup B_{i_m}) \leq 2n^2/k \approx 0$ ; this is a contradiction to  $\hat{\mu}(Y) > 0$ .) It follows that  $X \subseteq A_{i_1} \cap \dots \cap A_{i_m}$ . By Property (3) above, we obtain

$$\mu(X) \leq \mu(A_{i_1} \cap \dots \cap A_{i_m}) \approx \frac{(l-p+1)^m}{n^m} \leq \left(1 - \frac{1}{n}\right)^{n^2} \approx 0,$$

since  $\lim_{n \rightarrow \infty} (1 - 1/n)^{n^2} = 0$ . This is a contradiction to  $\hat{\mu}(X) \neq 0$ . Therefore,  $(\hat{\mu} \otimes \hat{\nu})^{\text{inner}}(D^{p,l}) = 0$ .

To show that  $(\hat{\mu} \otimes \hat{\nu})^{\text{outer}}(D^{p,l}) = 1$ , fix  $Z = (X_1 \times Y_1) \cup \dots \cup (X_h \times Y_h) \in \mathcal{P}$  with  $D^{p,l} \subseteq Z$ . By Lemma 2.3 (2) it suffices to show that  $\hat{\mu} \otimes \hat{\nu}(Z) = 1$ . Recall that we can require  $Y_1, \dots, Y_h$  to form a partition of  $T$ . Fix  $j \in \{1, \dots, h\}$  with  $\hat{\nu}(Y_j) > 0$ . Then again there exist more than  $n^2$  many elements  $i_1, \dots, i_m$  in  $\{1, \dots, k\}$  with  $Y_j \cap B_{i_1} \neq \emptyset, \dots, Y_j \cap B_{i_m} \neq \emptyset$ . Therefore,  $A_{i_1} \cup \dots \cup A_{i_m} \subseteq X_j$ . By Property (3) above,

$$\mu(X_j) \geq \mu(A_{i_1} \cup \dots \cup A_{i_m}) \approx 1 - \left(\frac{n+p-l-1}{n}\right)^m \geq 1 - \left(1 - \frac{1}{n}\right)^{n^2} \approx 1.$$

Hence  $\hat{\mu}(X_j) = 1$  and  $\hat{\mu}(X_j)\hat{\nu}(Y_j) = \hat{\nu}(Y_j)$ . When  $\hat{\nu}(Y_j) = 0$ , we still have  $\hat{\mu}(X_j)\hat{\nu}(Y_j) = \hat{\nu}(Y_j)$ . Therefore,

$$\hat{\mu} \otimes \hat{\nu}(Z) = (\hat{\mu}(Y_1) + \dots + \hat{\mu}(Y_h)) = 1.$$

It follows that  $(\hat{\mu} \otimes \hat{\nu})^{\text{outer}}(D^{p,l}) = 1$ .

Take an unlimited  $l_0 \in {}^*\mathbb{N}$  such that  $l_0/n \approx 0$ . Fix any  $s \in [0, 1]$ . Choose  $q$  with  $1 \leq q < n$  such that  $q/n \approx s$ , and let  $C^s = D^{1,q}$ . Choose  $p$  with  $1 \leq p \leq l_0$  such that  $p/l_0 \approx s$ , and let  $R^s = D^{1,p}$ .

By Equation (1) and the above results on  $\hat{\mu} \otimes \hat{\nu}$  outer and inner measures, it is clear that  $\widehat{\mu \otimes \nu}(C^s) = s$ ,  $(\hat{\mu} \otimes \hat{\nu})^{\text{inner}}(C^s) = 0$ ,  $(\hat{\mu} \otimes \hat{\nu})^{\text{outer}}(C^s) = 1$ , and  $\widehat{\mu \otimes \nu}(R^s) = 0$ ,  $(\hat{\mu} \otimes \hat{\nu})^{\text{inner}}(R^s) = 0$ ,  $(\hat{\mu} \otimes \hat{\nu})^{\text{outer}}(R^s) = 1$ .

For all  $s_1, s_2 \in [0, 1]$  with  $s_1 < s_2$  we have  $C^{s_1} \subset C^{s_2}$ ,  $(\hat{\mu} \otimes \hat{\nu})^{\text{outer}}(C^{s_2} \setminus C^{s_1}) = 1$ ,  $R^{s_1} \subset R^{s_2}$ ,  $(\hat{\mu} \otimes \hat{\nu})^{\text{outer}}(R^{s_2} \setminus R^{s_1}) = 1$ . Here Equation (2) and the fact that  $(\hat{\mu} \otimes \hat{\nu})^{\text{outer}}(D^{p,l}) = 1$  are used.

#### 4. Outer and inner product Loeb measures

The non-product-measurable sets in Example 3.2 depend on the notion of  $\hat{\nu}$ -almost independence. These particular non-product-measurable sets are also shown to have  $\hat{\mu} \otimes \hat{\nu}$  outer measure one and inner measure zero. In this section, we consider some general results.

Let  $C$  be a set in  $L_{\mu \otimes \nu}(\mathcal{A} \otimes \mathcal{T})$  whose events  $C_t, t \in T$ , are  $\hat{\nu}$ -almost independent, i.e., for  $\widehat{\nu \otimes \nu}$ -almost all  $(s, t) \in T \times T$ , the events  $C_t$  and  $C_s$  are independent. By Theorems 1 and 2 in [13] (also see Theorem 4.6 in [15]), the conditional expectation  $\mathbb{E}(1_C | \mathcal{U})$  of the indicator function  $1_C$  of  $C$  in  $\Omega \times T$  is simply the mean function  $t \mapsto \hat{\mu}(C_t)$  of  $1_C$ . Take any set  $D$  in the usual product algebra  $\mathcal{U}$ . Then  $\mathbb{E}(1_D 1_C | \mathcal{U}) = 1_D \mathbb{E}(1_C | \mathcal{U}) = 1_D \hat{\mu}(C_t)$ . Hence, by integrating the functions on  $\Omega \times T$ , we obtain that  $\widehat{\mu \otimes \nu}(D \cap C) = \int_{\Omega \times T} 1_D \hat{\mu}(C_t) d\widehat{\mu \otimes \nu}$ , which equals to  $\int_T \hat{\mu}(D_t) \hat{\mu}(C_t) d\hat{\nu}$  by Keisler's Fubini theorem. For easy reference, let us state the following lemma.

LEMMA 4.1. *Let  $C$  be a set in  $L_{\mu \otimes \nu}(\mathcal{A} \otimes \mathcal{T})$  whose events  $C_t, t \in T$ , are  $\hat{\nu}$ -almost independent, and  $D \in \mathcal{U}$ . Then*

$$(3) \quad \widehat{\mu \otimes \nu}(D \cap C) = \int_T \hat{\mu}(D_t) \hat{\mu}(C_t) d\hat{\nu}.$$

THEOREM 4.2. *Let  $C$  be a set in  $L_{\mu \otimes \nu}(\mathcal{A} \otimes \mathcal{T})$  whose events  $C_t, t \in T$ , are  $\hat{\nu}$ -almost independent. If  $0 < \hat{\mu}(C_t) < 1$  for  $\hat{\nu}$ -almost all  $t \in T$ , then  $(\hat{\mu} \otimes \hat{\nu})^{\text{outer}}(C) = 1$  and  $(\hat{\mu} \otimes \hat{\nu})^{\text{inner}}(C) = 0$ .*

*Proof.* Take any  $D \in \mathcal{U}$ . If  $D \subseteq C$ , then Equality (3) in Lemma 4.1 implies that

$$\widehat{\mu \otimes \nu}(D) = \int_T \hat{\mu}(D_t) d\hat{\nu} = \widehat{\mu \otimes \nu}(D \cap C) = \int_T \hat{\mu}(D_t) \hat{\mu}(C_t) d\hat{\nu},$$

which further implies that  $\int_T \hat{\mu}(D_t)(1 - \hat{\mu}(C_t)) d\hat{\nu} = 0$ . By the nonnegativity of the integrand, we obtain  $\hat{\mu}(D_t)(1 - \hat{\mu}(C_t)) = 0$  for  $\hat{\nu}$ -almost all  $t$ . Since  $(1 - \hat{\mu}(C_t)) > 0$  for  $\hat{\nu}$ -almost all  $t$ , we obtain  $\hat{\mu}(D_t) = 0$  for  $\hat{\nu}$ -almost all  $t$ . Hence  $\hat{\mu} \otimes \hat{\nu}(D) = 0$ . Thus  $(\hat{\mu} \otimes \hat{\nu})^{\text{inner}}(C) = 0$ .



If  $C \subseteq D$ , then Equality (3) in Lemma 4.1 implies that

$$\widehat{\mu \otimes \nu}(C) = \int_T \hat{\mu}(C_t) d\hat{\nu} = \int_T \hat{\mu}(D_t) \hat{\mu}(C_t) d\hat{\nu}.$$

Hence  $\int_T (1 - \hat{\mu}(D_t)) \hat{\mu}(C_t) d\hat{\nu} = 0$ . As above, the nonnegativity of the integrand implies that  $(1 - \hat{\mu}(D_t)) \hat{\mu}(C_t) = 0$  for  $\hat{\nu}$ -almost all  $t$ . Since  $\hat{\mu}(C_t) > 0$  for  $\hat{\nu}$ -almost all  $t$ , we obtain  $\hat{\mu}(D_t) = 1$  for  $\hat{\nu}$ -almost all  $t$ , and hence  $\hat{\mu} \otimes \hat{\nu}(D) = 1$ . Therefore,  $(\hat{\mu} \otimes \hat{\nu})^{\text{outer}}(C) = 1$ .  $\square$

Note that the above theorem cannot be applied to compute the  $\hat{\mu} \otimes \hat{\nu}$  outer and inner measures for the sets  $D^{p,l}$  in Example 3.2 since the sections of these sets may be null.

The following lemma and corollary are obvious.

LEMMA 4.3. *Let  $C \in L_{\mu \otimes \nu}(\mathcal{A} \otimes T)$  with  $(\hat{\mu} \otimes \hat{\nu})^{\text{outer}}(C) = 1$ . Then, for any  $F \in \mathcal{U}$ ,  $(\hat{\mu} \otimes \hat{\nu})^{\text{outer}}(F \cap C) = (\hat{\mu} \otimes \hat{\nu})(F)$ .*

COROLLARY 4.4. *Let  $C$  be a set in  $L_{\mu \otimes \nu}(\mathcal{A} \otimes T)$  whose events  $C_t, t \in T$ , are  $\hat{\nu}$ -almost independent, and  $F$  a set in  $\mathcal{U}$ . If  $0 < \hat{\mu}(C_t) < 1$  for  $\hat{\nu}$ -almost all  $t \in T$ , then  $(\hat{\mu} \otimes \hat{\nu})^{\text{outer}}(F \cap C) = (\hat{\mu} \otimes \hat{\nu})(F)$  and  $(\hat{\mu} \otimes \hat{\nu})^{\text{inner}}(F \cap C) = 0$ .*

The following lemma is a version of Lemma 7 in [16].

LEMMA 4.5. *For any positive numbers  $a_1, \dots, a_n$  with sum 1, there is a Loeb product measurable partition  $\{C^1, \dots, C^n\}$  of  $\Omega \times T$  such that for each  $i$  the events  $(C^i)_t, t \in T$ , are  $\hat{\nu}$ -almost independent with equal probability  $a_i$ .*

The following theorem shows that every product Loeb measurable set of positive measure  $\beta$  can be decomposed as any finite union of Loeb product measurable sets with  $\hat{\mu} \otimes \hat{\nu}$  outer measure  $\beta$  and inner measure 0.

THEOREM 4.6. *Let  $F$  be any set in  $\mathcal{U}$  with  $\hat{\mu} \otimes \hat{\nu}(F) = \beta$ . Then for any positive numbers  $a_1, \dots, a_n$  with sum 1, there is a Loeb product measurable partition  $\{A^1, \dots, A^n\}$  of  $F$  such that for all  $i$ ,  $\widehat{\mu \otimes \nu}(A^i) = a_i \beta$ ,  $(\hat{\mu} \otimes \hat{\nu})^{\text{outer}}(A^i) = \beta$  and  $(\hat{\mu} \otimes \hat{\nu})^{\text{inner}}(A^i) = 0$ .*

*Proof.* Take a Loeb product measurable partition  $\{C^1, \dots, C^n\}$  of  $\Omega \times T$  as in Lemma 4.5. Take  $A^i = F \cap C^i$ . Then Lemma 4.1 implies that  $\widehat{\mu \otimes \nu}(A^i) = \int_T \hat{\mu}(F_t) \hat{\mu}(C_t^i) d\hat{\nu} = a_i \beta$ . The rest follows from Theorem 4.2 and Lemma 4.3.  $\square$

### 5. Consequences of the examples

The following theorem shows that for a set in the Loeb product algebra  $L_{\mu \otimes \nu}(\mathcal{A} \otimes T)$ , its  $\widehat{\mu \otimes \nu}$  measure  $s$  and its  $\hat{\mu} \otimes \hat{\nu}$  outer measure  $t$  and inner

measure  $r$  can be completely arbitrary subject to the obvious condition  $0 \leq r \leq s \leq t \leq 1$ .

**THEOREM 5.1.** *For any  $0 \leq r \leq s \leq t \leq 1$  there is a set  $E \in L_{\mu \otimes \nu}(\mathcal{A} \otimes \mathcal{T})$  such that  $\widehat{\mu \otimes \nu}(E) = s$ ,  $\hat{\mu} \otimes \hat{\nu}^{\text{outer}}(E) = t$ , and  $\hat{\mu} \otimes \hat{\nu}^{\text{inner}}(E) = r$ .*

*Proof.* When  $r = t$ , the result is clear by the Lyapunov theorem. Fix any  $r, t \in [0, 1]$  with  $r < t$ . By Example 3.2, there is a class of sets  $\{C^{s'} \in \mathcal{A} \otimes \mathcal{T} : s' \in [0, 1]\}$  such that for each  $s' \in [0, 1]$ ,  $\widehat{\mu \otimes \nu}(C^{s'}) = s'$ ,  $(\hat{\mu} \otimes \hat{\nu})^{\text{inner}}(C^{s'}) = 0$ , and  $(\hat{\mu} \otimes \hat{\nu})^{\text{outer}}(C^{s'}) = 1$ , and for all  $s_1, s_2 \in [0, 1]$  with  $s_1 < s_2$ ,  $C^{s_1} \subset C^{s_2}$ .

By the Lyapunov theorem, we can find disjoint internal sets  $U$  and  $W$  in  $\mathcal{A}$  such that  $\hat{\mu}(U) = r$  and  $\hat{\mu}(W) = t - r$ . Since  $r \leq s \leq t$ , we can define  $s' = (s - r)/(t - r)$  and  $D^s = (U \times T) \cup (C^{s'} \cap (W \times T))$ . Since the events  $C_t^{s'}, t \in T$ , are  $\hat{\nu}$ -almost independent, Lemma 4.1 implies that

$$\widehat{\mu \otimes \nu}((W \times T) \cap C^{s'}) = \int_T \hat{\mu}(W) \hat{\mu}(C_t^{s'}) d\hat{\nu} = (t - r) \widehat{\mu \otimes \nu}(C^{s'}) = s - r.$$

Hence  $\widehat{\mu \otimes \nu}(D^s) = s$ . By Lemma 4.3,  $(\hat{\mu} \otimes \hat{\nu})^{\text{outer}}((W \times T) \cap C^{s'}) = (\hat{\mu} \otimes \hat{\nu})(W \times T) = t - r$ . Since  $U$  and  $W$  are disjoint, it follows from the Caratheodory identity for outer measures (see [12]) that  $\hat{\mu} \otimes \hat{\nu}^{\text{outer}}(D^s) = t$ . Using similar ideas as in Lemma 4.3 and here, we derive that  $\hat{\mu} \otimes \hat{\nu}^{\text{inner}}(D^s) = r$ . We can simply take  $E = D^s$ . Note that when  $s$  is regarded as a parameter, the sets in the class  $\{D^s : s \in [r, t]\}$  are increasing in  $s$ .  $\square$

**REMARK 5.2.** Take a  $\widehat{\mu \otimes \nu}$ -null set  $R^s$  as in Example 3.2. Then for any  $D \in \mathcal{U}$ ,  $D \cup R^s \notin \mathcal{U}$  if  $\hat{\mu} \otimes \hat{\nu}(D) < 1$ , and  $D - R^s \notin \mathcal{U}$  if  $\hat{\mu} \otimes \hat{\nu}(D) > 0$ . On the other hand, let  $\mathcal{N}$  be the collection of all  $\widehat{\mu \otimes \nu}$ -null sets, and  $\mathcal{U} \vee \mathcal{N}$  the  $\sigma$ -algebra generated by the sets in  $\mathcal{U} \cup \mathcal{N}$ . Then for any  $s \in (0, 1)$ , the set  $C^s$  in Example 3.2 is not in  $\mathcal{U} \vee \mathcal{N}$ . We note that Theorems 1 and 2 in [13] (and also Theorem 4.6 in [15]) say that the conditional expectation  $\mathbb{E}(1_{C^s} | \mathcal{U})$  is essentially the constant function  $s$ . It follows that  $\mathbb{E}(1_{C^s} | \mathcal{U} \vee \mathcal{N})(\omega, t) = s$  for  $\widehat{\mu \otimes \nu}$ -almost all  $(\omega, t)$ . Thus, if  $C^s \in \mathcal{U} \vee \mathcal{N}$ , then  $1_{C^s}(\omega, t) = s$  for  $\widehat{\mu \otimes \nu}$ -almost all  $(\omega, t)$ . This is a contradiction with the condition  $s \in (0, 1)$ .

In the next theorem, we shall show that every set  $E$  in  $L_{\mu \otimes \nu}(\mathcal{A} \otimes \mathcal{T})$  has a subset  $A \in L_{\mu \otimes \nu}(\mathcal{A} \otimes \mathcal{T})$  with  $\hat{\mu} \otimes \hat{\nu}$ -inner measure 0 that has the same  $\hat{\mu} \otimes \hat{\nu}$ -outer measure and  $\widehat{\mu \otimes \nu}$ -measure as  $E$ . In order to prove this general result, we first consider the special case  $E \in \mathcal{U}$ .

**LEMMA 5.3.** *For any  $F \in \mathcal{U}$  with  $\hat{\mu} \otimes \hat{\nu}(F) = \beta$ , there are Loeb product measurable subsets  $B^0$  and  $B^1$  of  $F$  such that  $\widehat{\mu \otimes \nu}(B^0) = 0$ ,  $\widehat{\mu \otimes \nu}(B^1) = \beta$ , and for  $i = 0, 1$ ,  $(\hat{\mu} \otimes \hat{\nu})^{\text{outer}}(B^i) = \beta$  and  $(\hat{\mu} \otimes \hat{\nu})^{\text{inner}}(B^i) = 0$ .*

*Proof.* Take  $C^0$  and  $C^1$  as in Example 3.2. Let  $B^i = C^i \cap F$ . Then  $\widehat{\mu \otimes \nu}(B^0) = 0$  and  $\widehat{\mu \otimes \nu}(B^1) = \widehat{\mu \otimes \nu}(F) = \hat{\mu} \otimes \hat{\nu}(F) = \beta$ . It is obvious that  $(\hat{\mu} \otimes \hat{\nu})^{\text{inner}}(B_i) \leq (\hat{\mu} \otimes \hat{\nu})^{\text{inner}}(C^i) = 0$ . Since  $(\hat{\mu} \otimes \hat{\nu})^{\text{outer}}(C^i) = 1$ , Lemma 4.3 implies that  $(\hat{\mu} \otimes \hat{\nu})^{\text{outer}}(B^i) = \beta$ .  $\square$

**THEOREM 5.4.** *For any set  $E \in L_{\mu \otimes \nu}(\mathcal{A} \otimes \mathcal{T})$  with  $\widehat{\mu \otimes \nu}(E) = \alpha$  (where  $E$  need not be in  $\mathcal{U}$ ) there is a Loeb product measurable subset  $A$  of  $E$  such that  $\widehat{\mu \otimes \nu}(A) = \alpha$ ,  $(\hat{\mu} \otimes \hat{\nu})^{\text{outer}}(A) = (\hat{\mu} \otimes \hat{\nu})^{\text{outer}}(E)$ , and  $(\hat{\mu} \otimes \hat{\nu})^{\text{inner}}(A) = 0$ .*

*Proof.* Set  $\beta = (\hat{\mu} \otimes \hat{\nu})^{\text{inner}}(E)$ . Note that there exists an  $F \in \mathcal{U}$  with  $F \subseteq E$  and  $(\hat{\mu} \otimes \hat{\nu})(F) = \beta$ . It follows that  $(\hat{\mu} \otimes \hat{\nu})^{\text{inner}}(E \setminus F) = 0$ .

By Lemma 5.3, there is a Loeb product measurable subset  $B$  of  $F$  such that  $\widehat{\mu \otimes \nu}(B) = \beta = (\hat{\mu} \otimes \hat{\nu})^{\text{outer}}(B)$  and  $(\hat{\mu} \otimes \hat{\nu})^{\text{inner}}(B) = 0$ .

Let  $A = B \cup D$  with  $D = E \setminus F$ . The verification that  $A$  satisfies the requirements is straightforward and thus left to the reader.  $\square$

**REMARK 5.5.** We end this section with a question about the following generalization of Theorem 5.4: Can the  $\widehat{\mu \otimes \nu}$  measure, and the  $\hat{\mu} \otimes \hat{\nu}$  outer and inner measures of subsets of a general set  $E \in L_{\mu \otimes \nu}(\mathcal{A} \otimes \mathcal{T})$  be arbitrary subject to some obvious conditions? That is, for any set  $E \in L_{\mu \otimes \nu}(\mathcal{A} \otimes \mathcal{T})$  with  $\hat{\mu} \otimes \hat{\nu}^{\text{inner}}(E) = r$ ,  $\widehat{\mu \otimes \nu}(E) = s$  and  $\hat{\mu} \otimes \hat{\nu}^{\text{outer}}(E) = t$ , and for any  $0 \leq a \leq b \leq c$  with  $a \leq r$ ,  $b \leq s$ , and  $c \leq t$ , does there exist a subset  $A$  of  $E$  such that  $\hat{\mu} \otimes \hat{\nu}^{\text{inner}}(A) = a$ ,  $\widehat{\mu \otimes \nu}(A) = b$ , and  $\hat{\mu} \otimes \hat{\nu}^{\text{outer}}(A) = c$ ?

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