Illinois Journal of Mathematics
Volume 46, Number 1, Spring 2002, Pages 287–318
S 0019-2082

TOPOLOGICAL VECTOR SPACES OF BOCHNER MEASURABLE FUNCTIONS

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Abstract. The notion of a topological vector space of Bochner measurable functions is introduced and studied. Among the main results obtained are characterizations of completeness and of containment of copies of c_0 or ℓ_∞.

1. Introduction

Recall two classical results characterizing Banach lattices (in the terminology of [AB]) that have both the Lebesgue and the Levi property (also called KB-spaces), or the Lebesgue property alone (which is the same as the order continuity of norm), in terms of containment of copies of c_0 or ℓ_∞.

Theorem 1.0. *Let E be a Banach lattice.*

(i) *E is Lebesgue and Levi iff E contains no lattice (or Riesz homeomorphic) copy of c_0 iff E contains no (linearly homeomorphic) copy of c_0.*
(ii) *Assume that E is σ-Dedekind complete. Then E is Lebesgue iff E contains no lattice copy of ℓ_∞ iff E contains no copy of ℓ_∞.*

In [DL2, Prop. 2.1, Thm. 2.4, Thm. 2.7], we refined the existing extensions of the 'lattice copy part' of (i) and (ii) to general topological Riesz spaces by proving the following result.

Theorem 1.1. *Let L be a topological Riesz space.*

(i) *L is σ-Lebesgue and σ-Levi iff it has the σ-monotone completeness property and contains no lattice copy of c_0.*
(ii) *A disjointly σ-Dedekind complete L is pre-Lebesgue iff it contains no lattice copy of ℓ_∞.*

Received June 1, 2001; received in final form November 19, 2001.
2000 *Mathematics Subject Classification.* 46E40, 46E30.

We then focused on the case of topological Riesz spaces L of measurable functions over a locally finite measure and were able to show that, for such spaces, the other parts of both (i) and (ii) remain also valid (see Thms. 5.5 and 5.8 in [DL2]):

Theorem 1.2. *Let L be a topological Riesz space of λ-measurable functions.*

(i) *L with the monotone completeness property is σ-Lebesgue and σ-Levi iff L contains no lattice copy of c_0 iff L contains no copy of c_0 iff L has the Property (O) of Orlicz.*
(ii) *L having the σ-monotone completeness property is σ-Lebesgue iff L contains no lattice copy of ℓ_∞ iff L contains no copy of ℓ_∞.*

Moreover, with the help of a representation theorem, we translated these results back to general topological Riesz spaces with separating dual.

In the present research, we pursue the natural question to what degree the results mentioned above can be extended to spaces of comparable generality but consisting of *vector-valued* functions.

Accordingly, we develop a *theory of topological vector spaces of Bochner measurable functions*. The theory is sufficiently general to include topological Riesz spaces of measurable functions as a particular case, and to cover – via a representation theorem – at least those abstract topological Riesz spaces that have the Lebesgue property. This allows applications (see [DL3] and [DL1]) and explains why we adopted the setting of submeasure spaces rather than measure spaces.

Topological vector spaces of Bochner measurable functions are introduced in Section 3. We investigate their Lebesgue type and Levi type properties in Sections 4 and 5. Their topological completeness is studied in Section 6, where, in particular, sequentially complete and complete $L_0(\mu, E)$ spaces are characterized.

A major subclass of topological vector spaces of Bochner measurable functions is formed by the 'mixtures' $L(E)$ of a topological Riesz space of scalar measurable functions L with a Banach space E. If E is a Banach lattice, then $L_0(\mu, E)$ as well as the 'mixtures' $L(E)$ are topological Riesz spaces. We examine this case thoroughly in Section 7. There, we are concerned with special kinds of completeness such as the interval or monotone completeness which often can successfully be used in the absence of 'full' completeness. Then we characterize the spaces $L(E)$ that have the (σ-)Lebesgue and/or (σ-)Levi properties, or which contain no lattice copy of c_0 or ℓ_∞.

In Section 8 we prove an Orlicz-Pettis type theorem for our class of spaces. It is worth noting that this result is strong enough to imply the coincidence of subseries convergence for all Hausdorff Lebesgue topologies on a Riesz space [DL1].

In Section 9, using the Orlicz-Pettis theorem as main tool, we study the (isomorphic) embeddings of c_0 and ℓ_∞ into topological vector spaces of Bochner measurable functions. Most of the results about copies of c_0 and ℓ_∞ obtained in [DL2] for topological Riesz spaces of measurable functions carry over to this setting.

Finally, in Section 10, we specialize and somewhat improve these results for the particular case of spaces of type $L(E)$. For example, we show that if E contains no copy of c_0 and $L(E)$ is quasi-complete, then $L(E)$ contains no copy of c_0 iff L is σ-Lebesgue and σ-Levi. We conclude this paper by proving that if L is complete and has separating dual space, then $L(E)$ contains a copy of ℓ_∞ iff either L or E does.

This paper is, to some extent, a continuation of [DL2]. Our terminology and notation is rather standard and mostly follows that of [AB]; all unexplained terms and facts that will occur below can also be found in [DL2].

We use the abbreviations TVS and TRS for Hausdorff *topological vector space* and Hausdorff *locally solid topological Riesz space*, respectively. It is worth noting that for topological Riesz spaces the notions of completeness and quasi-completeness coincide (see [We, Prop. 2.8] and [Wn, Prop. 1.4]).

We denote by $\mathbb{N}$ the set of positive integers, and let $\mathcal{P}(\mathbb{N})$ stand for its power set.

2. Scalar measurable functions

Except in Sections 9 and 10, we will usually deal with spaces of measurable functions over *submeasure spaces*. In what follows, (S, Σ, μ) is a *submeasure space* with a locally order continuous submeasure μ which, for the sake of convenience (though not of necessity), will be assumed to be *null-complete* (i.e., whenever $B \subset A \in \Sigma$ and $\mu(A) = 0$, then $B \in \Sigma$ and $\mu(B) = 0$).

Thus S is a set, Σ a σ-algebra of subsets of S, and $\mu\colon \Sigma \to \overline{\mathbb{R}}_+$ a submeasure (i.e., μ is nondecreasing, subadditive, and $\mu(\emptyset) = 0$). The assumption that μ is *locally order continuous* means that each $A \in \Sigma$ with $\mu(A) > 0$ contains a $B \in \Sigma$ with $\mu(B) > 0$ on which μ is order continuous (o.c.). That is, $\mu(B_n) \to 0$ whenever $B_n \in \Sigma$, $B_n \subset B$ and $B_n \downarrow \emptyset$. From this it follows easily that the class $\mathcal{N}(\mu)$ of μ-null sets is a σ-ideal in Σ.

We set

$$\begin{aligned}
\Sigma_{\mathrm{oc}}(\mu) &= \text{all sets } A \in \Sigma \text{ such that } \mu \text{ is o.c. on } A,\\
\Sigma_{\mathrm{oc}}^{+}(\mu) &= \{A \in \Sigma_{\mathrm{oc}}(\mu) : \mu(A) > 0\},\\
\Sigma_{\mathrm{oc}}^{\sigma}(\mu) &= \text{all countable unions of sets in } \Sigma_{\mathrm{oc}}(\mu).
\end{aligned}$$

Clearly, $\Sigma_{\mathrm{oc}}(\mu)$ is an ideal, and $\Sigma_{\mathrm{oc}}^{\sigma}(\mu)$ is a σ-ideal in Σ; in particular, both are directed upward by inclusion. Also note that if $A \in \Sigma$ and $A \cap B \in \mathcal{N}(\mu)$ for each $B \in \Sigma_{\mathrm{oc}}^{+}(\mu)$, then $A \in \mathcal{N}(\mu)$. If S itself is in $\Sigma_{\mathrm{oc}}^{\sigma}(\mu)$, then μ is said to be *countably o.c.-decomposable*.

If $A \in \Sigma$ and $(A_i : i \in I)$ is a net in Σ, then we write $A_i \uparrow A$ provided that the net (A_i) is increasing and $A = \sup_i A_i$. Strictly speaking, these requirements are to be satisfied for the corresponding elements in the quotient Boolean algebra $\Sigma/\mathcal{N}(\mu)$. Thus, in particular, the equality $A = \sup_i A_i$ means that $\mu(A_i \setminus A) = 0$ for all i, and whenever $B \in \Sigma$ is such that $\mu(A_i \setminus B) = 0$ for all i, then $\mu(A \setminus B) = 0$. Note that if $A \in \Sigma_{\mathrm{oc}}^{\sigma}(\mu)$ and $A_i \uparrow A$, then there exists a subsequence (A_{i_n}) of (A_i) such that $A_{i_n} \uparrow A$; this fact will be used freely throughout the paper without explicitly mentioning it.

We denote by $L_0(\mu) = L_0(S, \Sigma, \mu)$ the vector lattice (or Riesz space) of all (μ-equivalence classes of) measurable scalar functions on S. It is equipped with the locally solid vector topology τ_μ of convergence in submeasure μ on all sets in $\Sigma_{\mathrm{oc}}(\mu)$. A base of solid neighborhoods at zero for τ_μ consists of the sets

$$U(A,\varepsilon) = \big\{ f \in L_0(\mu) : \mu\{s \in A : |f(s)| \geqslant \varepsilon\} < \varepsilon \big\}, \quad A \in \Sigma_{\mathrm{oc}}(\mu),\ \varepsilon > 0.$$

Of course, if (f_i) is a net in $L_0(\mu)$ and $f \in L_0(\mu)$, then

$$f_i \to f \ (\tau_\mu) \quad \text{iff} \quad \lim_i \mu(\{s \in A : |f(s) - f_i(s)| \geqslant \varepsilon\}) = 0 \text{ for all } A \in \Sigma_{\mathrm{oc}}(\mu).$$

Proposition 2.1. *$L_0(\mu)$ is a Hausdorff σ-universally complete TRS having the σ-Lebesgue and σ-Levi properties. It is metrizable iff μ is countably o.c.-decomposable.*

We shall say that the submeasure space (S, Σ, μ), or the submeasure μ itself, is *of type* (C) (resp. (SC)) if the corresponding TRS $L_0(\mu)$ is complete (resp. sequentially complete). Note that a countably o.c.-decomposable submeasure is of type (C).

The following result corresponds to a characterization of measures of type (C) in [F1] (called there Maharam measures) with a slight addition, viz., condition (b).

Theorem 2.2. *For the submeasure space (S, Σ, μ), let Σ_μ denote the associated submeasure Boolean algebra $\Sigma/\mathcal{N}(\mu)$. Then the following statements are equivalent.*

(a) *μ is of type (C).*
(b) *The algebra Σ_μ is complete as a topological abelian group with the topology induced from $L_0(\mu)$ via the map $A \to 1_A$.*
(c) *The algebra Σ_μ is Dedekind complete.*
(d) *$L_0(\mu)$ is a universally complete Lebesgue Levi TRS.*

Extending the terminology from [DL2], by a TRS *of μ-measurable functions* we shall mean an ideal (or solid subspace) L of $L_0(\mu)$ equipped with a Hausdorff locally solid topology $\tau = \tau_L$. Note that all such TRS's are automatically σ-Dedekind complete.

PROPOSITION 2.3. *Let the submeasure μ be countably o.c.-decomposable. Then $L_0(\mu)$ has the countable sup property. Consequently, if a TRS L of μ-measurable functions is σ-Lebesgue, or σ-Levi, or σ-Fatou, then it is Lebesgue, or Levi, or Fatou, respectively.*

The inclusion $L \subset L_0(\mu)$ is often automatically continuous. The following result is given in [D2, Prop. 3.6] or [KA, Thm. IV.3.1] for the metrizable case, and in [DL2, Prop. 3.4] for the σ-Fatou case.

PROPOSITION 2.4. *If a TRS L of μ-measurable functions is metrizable or has the σ-Fatou property, then it is continuously included in $L_0(\mu)$.*

The next result explains our insistence on working with submeasure spaces rather than just measure spaces. It is a form of Theorem 2.7 in [L4].

THEOREM 2.5. *If a TRS X is Lebesgue (and Dedekind complete), then there exists a submeasure space (S, Σ, μ) of type (C) such that X is continuously included as an order dense (and solid) sublattice in $L_0(\mu)$.*

EXAMPLE 2.6. The following distinguishes between type (SC) and (C) measures. Let S be any uncountable set, $\mathcal{C}$ the class of its countable subsets together with their complements to S, and γ the counting measure on $\mathcal{C}$. That is, $\gamma(A)$ is the number of elements of A if A is finite, and $\gamma(A) = \infty$ otherwise. Then $L_0(\gamma)$ consists of all functions $f\colon S \to \mathbb{R}$ that are constant outside a countable subset of S, and τ_γ is simply the topology of pointwise convergence on S. Clearly, $L_0(\gamma)$ is sequentially complete, but not complete: its completion is the product space $\mathbb{R}^S$. Thus $(S, \mathcal{C}, \gamma)$ is of type (SC), but not (C).

3. Spaces of vector valued measurable functions

Let E be an F-space (i.e., a complete metrizable TVS), and thus, in particular, a Banach space.

By a *countably Σ-simple* function from S to E we mean one that assumes at most countably many values, each on a set from Σ. A function $f\colon S \to E$ will be called *Bochner measurable* if it satisfies any of the following equivalent conditions.

(a) There is a sequence of countably Σ-simple functions $f_n\colon S \to E$ such that
$$f(s) = \lim_n f_n(s) \text{ uniformly for a.e. } s \in S;$$
(b) there is a sequence of countably Σ-simple functions $f_n\colon S \to E$ such that
$$f(s) = \lim_n f_n(s) \text{ for a.e. } s \in S;$$

(c) there is a μ-null set A such that $f(S \setminus A)$ is a separable subset of E, and f is Σ-Borel measurable; that is,

$$f^{-1}(B) \in \Sigma \text{ for every Borel set } B \subset E.$$

Below, we refer to such functions f as *μ-measurable.* If the submeasure μ is countably o.c.-decomposable, or if $E = \mathbb{R}$, then the conditions stated above are also equivalent to

($\overline{\text{b}}$) there is a sequence of (finitely) Σ-simple functions $f_n \colon S \to E$ such that

$$f(s) = \lim_n f_n(s) \text{ for a.e. } s \in S,$$

which is the usual definition of Bochner measurability. The main reason for our departure from this common usage is that with the definition adopted here one can easily prove for *general* submeasure (or measure) spaces (S, Σ, μ) that a.e. limits of sequences of μ-measurable functions are μ-measurable. (Note also that, for the modified definition of measurability, the latter fact follows solely from the property that $\mathcal{N}(\mu)$ is a σ-ideal, without any additional requirements on μ.) Evidently, the above definition coincides with the standard (inverse image) definition of measurability in the case of scalar functions.

We denote by $L_0(\mu, E) = L_0(S, \Sigma, \mu; E)$ the space of all (μ-equivalence classes of) E-valued μ-measurable functions on S, with its Hausdorff vector topology τ_μ defined, similarly as in the scalar case, as the topology of convergence in submeasure μ on sets in $\Sigma_{\mathrm{oc}}(\mu)$. Note that the familiar convergence properties of sequences of measurable functions, such as the Egoroff theorem, remain valid in the submeasure setting provided the sets of finite measure are replaced by sets in $\Sigma_{\mathrm{oc}}(\mu)$.

We let τ_{u} stand for the group topology in $L_0(\mu, E)$ of (μ-a.e.) uniform convergence on S. For $A \in \Sigma$, we denote by P_A the linear projection in $L_0(\mu, E)$ defined by $P_A(f) = 1_A f$. We say that a subset V of $L_0(\mu, E)$ is Σ-*solid* if $P_A(V) \subset V$ for every $A \in \Sigma$.

Let X be a Σ-solid subspace of $L_0(\mu, E)$, and let (P) be a property which any particular subspace

$$X_A = P_A(X), \qquad \text{where } A \in \Sigma,$$

may or may not have. Then we shall say that property (P) holds *piecewise*, or that X *is piecewise* (P), if every $A \in \Sigma_{\mathrm{oc}}^+(\mu)$ contains a $B \in \Sigma_{\mathrm{oc}}^+(\mu)$ such that X_B has property (P).

For example, if τ and ρ are two topologies on X, then τ is said to be *piecewise weaker* than ρ if every $A \in \Sigma_{\mathrm{oc}}^+(\mu)$ contains a $B \in \Sigma_{\mathrm{oc}}^+(\mu)$ such that $\tau | X_B \leqslant \rho | X_B$.

Properties (P) encountered below will also be hereditary, i.e., whenever X_A has (P), so does X_B for $B \subset A$. Evidently, if properties (P) and (Q) are hereditary, and X is piecewise (P) and piecewise (Q), then X is piecewise (P) *and* (Q).

PROPOSITION 3.1. *For any property* (P) *as above, if X is piecewise* (P), *then every A in $\Sigma_{\mathrm{oc}}^{\sigma}(\mu)$ admits a countable partition (A_n) such that each X_{A_n} has property* (P).

By a TVS *of E-valued μ-measurable functions* we shall mean a Σ-solid subspace X of $L_0(\mu, E)$ equipped with a Hausdorff vector topology $\tau = \tau_X$ such that

- the projections P_A $(A \in \Sigma)$ are equicontinuous on X; equivalently, the topology τ has a base of Σ-solid neighborhoods of zero;
- the topology τ is piecewise weaker than the topology τ_{u}.

Note that the inclusion $X \subset L_0(\mu, E)$ is not assumed to be continuous. The subspaces of X of the form X_A, where $A \in \Sigma_{\mathrm{oc}}^{\sigma}(\mu)$, will be called *fundamental*.

PROPOSITION 3.2. *A TRS L of μ-measurable functions is also a TVS of scalar-valued μ-measurable functions.*

Proof. The first condition of the definition above is obviously satisfied. To check the second, pick an $A \in \Sigma_{\mathrm{oc}}^{+}(\mu)$ and observe that either $L_A = 0$ or there exists $B \in \Sigma_{\mathrm{oc}}^{+}(\mu)$, $B \subset A$, with $1_B \in L$. Hence, given any solid neighborhood V of zero in L, there is $\varepsilon > 0$ such that $\varepsilon[-1_B, 1_B] \subset V$, and the needed condition follows. $\square$

The most common types of TVS's of Bochner measurable functions arise in the literature as special cases of the following construction (see, e.g., [KPS, Ch. 4] and [FN]).

Let (S, Σ, λ) be a measure space (usually assumed to be finite or σ-finite), $L = (L, \|\cdot\|_L)$ a Banach (or Köthe) function space over (S, Σ, λ), and $E = (E, \|\cdot\|_E)$ an arbitrary Banach space. For every function $f\colon S \to E$, let $\|f\|_E$ denote the scalar function $s \to \|f(s)\|_E$. Then define the subspace $L(E)$ of $L_0(\lambda, E)$ by $L(E) = \{f \in L_0(\lambda, E) : \|f\|_E \in L\}$, and equip it with the norm $\|f\|_{L(E)} := \|\|f\|_E\|_L$. The spaces $L_p(\lambda, E)$ or Orlicz spaces $L_\varphi(\lambda, E)$ are just particular instances of this construction.

In this paper we will consider an analogous 'mixture' $L(E)$ of a *general* TRS $L = (L, \tau_L)$ of μ-measurable functions with a Banach space E. Thus $L(E)$, as a vector space, is again defined as above, that is,

$$L(E) = \{f \in L_0(\mu, E) : \|f\|_E \in L\},$$

and we equip it with the vector topology $\tau_{L(E)}$ determined by the neighborhoods of zero

$$V_E = \{f \in L(E) : \|f\|_E \in V\}, \qquad V \in \mathcal{V},$$

where $\mathcal{V}$ is any base of τ_L-neighborhoods of zero in L (e.g., the base of solid neighborhoods of zero). It is evident that $L(E) = \big(L(E), \tau_{L(E)}\big)$ is a TVS of E-valued μ-measurable functions (which does not change when the norm $\|\cdot\|_E$

of E is replaced by an equivalent norm). Moreover, if E is a Banach lattice, then $L(E)$, with its natural order, is also a TRS, and an ideal (solid subspace) of the TRS $L_0(\mu, E)$.

Note the following two obvious facts.

(a) For any $0 \neq u \in E$, we have $L = \{f \in L_0(\mu) : f \cdot u \in L(E)\}$, and the map $f \to f \cdot u$ is an isomorphic embedding of L into $L(E)$.
(b) For any $A \in \Sigma$ with $0 \neq 1_A \in L$, the map $z \to 1_A \cdot z$ is an isomorphic embedding of E into $L(E)$.

Moreover, if E is a Banach lattice, and $0 < u \in E$, then the maps in (a) and (b) are both Riesz homeomorphic embeddings.

The spaces $L(E)$ will serve as our basic 'testing ground' for the theory of TVS's of Bochner measurable functions. Of course, additional assumptions on L or E will be imposed whenever needed.

Note. In what follows, we often write Σ' for $\Sigma^{\sigma}_{\text{oc}}(\mu)$, and consider Σ' to be directed upward by inclusion. Furthermore, unless stated otherwise, $L = (L, \tau)$ stands for a TRS of scalar μ-measurable functions, E for an F-space, and $X = (X, \tau)$ for a TVS of E-valued μ-measurable functions.

Remark 3.3.

(a) The 'mixtures' $L(E)$ may also be considered when $E = (E, \|\cdot\|_E)$ is an F-space, with a slightly modified definition to ensure continuity of multiplication by scalars: $L(E)$ is defined as the set of all $f \in L_0(\mu, E)$ such that $\|\alpha f\|_E \in L$ for some $\alpha > 0$ and $\tau\text{-}\lim_{\alpha \to 0} \|\alpha f\|_E = 0$. The topology in $L(E)$ is defined as before. This time, however, $L(E)$ depends heavily on which of the equivalent F-norms in E is used in the construction, and fact (a) above is no longer valid.

(b) Let (S, Σ, λ) be a locally finite measure space. Assume that λ is nontrivial, i.e., $\dim L_0(\lambda) = \infty$, and that the Banach space E is infinite dimensional. Then a natural example of a TVS of E-valued λ-measurable functions which is *not* of type $L(E)$ is provided by the space X of Pettis integrable functions $f \in L_0(\lambda, E)$, with its usual norm $\|f\| = \sup\{\int_S |x'f|\, d\lambda : \|x'\| \leqslant 1\}$. For, suppose that $X = L(E)$ for some TRS $L \subset L_0(\lambda)$. Then, by fact (a) above, choosing $u \in E$ with $\|u\| = 1$, we have $L = \{f \in L_0(\lambda) : f \cdot u \in X\} = L_1(\lambda)$. Moreover, the map $f \to f \cdot u$ is an isomorphic embedding of L into X, and since $\|f \cdot u\| = \int_S |f|\, d\lambda$, we conclude that L and $L_1(\lambda)$ are identical as TVS's. It follows that X is equal to $L_1(\lambda)(E) = L_1(\lambda, E)$, the space of Bochner integrable functions. However, as a consequence of the Dvoretzky–Rogers theorem, $L_1(\lambda, E)$ is a proper subspace of X, and we have thus arrived at a contradiction.

It is also worth noting that if, in addition, the measure λ is not purely atomic, then X is not continuously included in $L_0(\lambda, E)$. To justify this statement assume, as we may, that λ is finite and nonatomic, and that E is separable. Then $X = \mathcal{P}(\lambda, E)$ is an incomplete normed space (see [DLi, p. 8] for more information). However, by a result of Heiliö [H, Thm. 4.4.2] (see also [DLi, Lemma 5.3]), if X is considered with the topology γ defined as the supremum of its normed topology and the topology induced from $L_0(\lambda, E)$, then it becomes an F-space (i.e., a complete metrizable TVS). Hence the inclusion $X \subset L_0(\lambda, E)$ cannot be continuous. Actually, it can be shown that (X, γ) is always sequentially complete.

(c) There exist also 'weak' type spaces $L(E)$, which can be defined by (see [DFP, Example 5.2])

$$(wL)(E) = \{f \in L_0(\mu, E) : x'f \in L \text{ for each } x' \in E'\},$$

with the topology determined by the neighborhoods of zero

$$V_E^w = \big\{f \in (wL)(E) : \{x'f : \|x'\| \leqslant 1\} \subset V\big\},$$

where V runs through solid neighborhoods of zero in L.

(d) A still more general definition of Bochner measurability than the one adopted here, in which a function $f\colon S \to E$ is declared measurable if its restriction to each set in $\Sigma_{\mathrm{oc}}(\mu)$ is measurable in the usual sense, is also used quite frequently when the underlying (sub)measure space (S, Σ, μ) is decomposable (see [F1], [F2], and [L4]). In that case the new space $L_0(\mu, E)$ can be identified with a product of L_0-spaces of E-valued functions over order continuous (sub)measure spaces (and thus, in particular, is complete).

4. Lebesgue type properties

Recall that a general TRS X, or its topology τ, is said to be Lebesgue (resp. σ-Lebesgue) if $f_i \to 0$ (τ) for every net (resp. sequence) $f_i \downarrow 0$ in X.

We shall say that a TVS X of E-valued μ-measurable functions, or its topology τ, is

- *μ-continuous* (resp. *sequentially μ-continuous*) if $f = \tau\text{-}\lim_i 1_{A_i} f$ for every $f \in X$ and every net (resp. sequence) (A_i) in Σ such that $A_i \uparrow S$.

As is easily seen, a fundamental subspace of X is μ-continuous iff it is sequentially μ-continuous (see Proposition 2.3).

Proposition 4.1. *Let L be a TRS of μ-measurable functions. Then L is Lebesgue* (*resp. σ-Lebesgue*) *if and only if it is μ-continuous* (*resp. sequentially μ-continuous*).

Proof. The 'only if' part is obvious.

'If': Consider a net (resp. sequence) $f_i \downarrow 0$ in L. We may assume that $f_i \leqslant f \in L$ for all i. Let V be a solid neighborhood of zero in L, and choose $\varepsilon > 0$ so that $\varepsilon f \in V$. For each i let $A_i = \{s \in S : f_i(s) \leqslant \varepsilon f(s)\}$. Then $A_i \uparrow$ and, in fact, $A_i \uparrow S$. For otherwise there would exist $B \in \Sigma_{\mathrm{oc}}^+(\mu)$ such that for each i, $A_i \subset S \setminus B$ μ-a.e., or $\varepsilon f(s) < f_i(s)$ for μ-a.e. $s \in B$. As

$f \geqslant f_i$, $\varepsilon f(s) > 0$ μ-a.e. on B, and we would have a contradiction with the assumption that $f_i \downarrow 0$. By the μ-continuity (resp. sequential μ-continuity) of L, there is k such that $1_{S \setminus A_k} f \in V$. Then $f_k \leqslant 1_{S \setminus A_k} f + \varepsilon 1_A f \in V + V$ so that $f_i \in V + V$ for all $i \geqslant k$. □

By a *projective net* in X we shall mean a net $(f_A : A \in \Sigma')$ in X such that $f_A = 1_A f_B$ whenever $A, B \in \Sigma'$ and $A \subset B$.

Proposition 4.2.

(a) *If a projective net (f_A) has a limit f in X, then $f_A = 1_A f$ for every $A \in \Sigma'$.*

(b) *Every projective net in X is bounded.*

(c) *If every fundamental subspace of X is sequentially μ-continuous, then every projective net in X is Cauchy.*

Proof. (a) follows from the continuity of the projections P_A, while (b) and (c) are easily justified using the fact that Σ' is a σ-ideal. □

Proposition 4.3. *Let (f_n) be a sequence in X satisfying the condition*

$$(*) \qquad \lim_{A \in \Sigma'} 1_A f_n = f_n \quad \text{for } n = 1, 2, \ldots$$

(a) *If, for every $A \in \Sigma'$, the sequence $(1_A f_n)$ is Cauchy, then the sequence (f_n) is Cauchy in X.*

(b) *If, for some $f \in X$ and every $A \in \Sigma'$, the sequence $(1_A f_n)$ converges to $1_A f$, then the sequence (f_n) converges to f.*

(c) *If, for every $A \in \Sigma'$, the sequence $(1_A f_n)$ is bounded, then the sequence (f_n) itself is bounded.*

Proof. (a): Since Σ' is a σ-ideal, it is easy to see that the sequences $(1_A f_n)$ are equi-Cauchy for $A \in \Sigma'$. That is, given a neighborhood V of zero in X, there is n_0 such that $1_A f_n - 1_A f_m \in V$ for all $A \in \Sigma'$ and $n, m \geqslant n_0$. From this, assuming as we may that V is closed and using $(*)$, we get $f_n - f_m \in V$ for all $n, m \geqslant n_0$.

(b) and (c) can be verified by a similar argument. □

We shall say that X is

- *fundamentally μ-continuous* if every fundamental subspace of X is sequentially μ-continuous;
- *projectively μ-continuous* if for every $f \in X$ the associated projective net $(1_A f)$ converges to f;
- *projectively complete* if every Cauchy projective net in X is convergent.

Proposition 4.4. *The space X is μ-continuous iff it is fundamentally and projectively μ-continuous.*

Proof. Let $f \in X$ and $A_i \uparrow S$. Let V be a Σ-solid neighborhood of zero in X. Since X is projectively μ-continuous, there is $A \in \Sigma'$ such that $f - 1_A f \in V$. Next, since $A \in \Sigma'$, we can select an increasing sequence (i_n) of indices so that $A \cap A_{i_n} \uparrow A$. Hence, as X_A is sequentially μ-continuous, there is an index j such that $1_A f - 1_{A\cap A_i} f \in V$ for all $i \geqslant j$. Finally, if $i \geqslant j$, then

$$f - 1_{A_i} f = (f - 1_A f) + (1_A f - 1_{A\cap A_i} f) + 1_{A_i}(1_A f - f) \in V + V + V,$$

which completes the argument. □

Corollary 4.5. *If X is fundamentally μ-continuous and projectively complete, then X is μ-continuous.*

Proposition 4.6. *Let X be μ-continuous. Then every closed Σ-solid neighborhood V of zero in X is also closed for the topology induced from $L_0(\mu, E)$.*

Proof. Let $(f_i) \subset V$, $f \in X$, and $f_i \to f$ (τ_μ). Let $A \in \Sigma'$. Since the subspace $P_A\big(L_0(\mu, E)\big)$ is metrizable, there exists a sequence $i_1 < i_2 < \dots$ of indices such that $1_A f_{i_n} \to 1_A f$ (τ_μ), and we may also assume that $1_A f_{i_n} \to 1_A f$ μ-a.e. Then, by the Egoroff theorem and Proposition 3.1, we find $A_k \uparrow A$, such that, for every k, $\tau|X_{A_k} \leqslant \tau_{\mathrm{u}}|X_{A_k}$ and $1_{A_k} f_{i_n} \to 1_{A_k} f$ uniformly. Then also $1_{A_k} f_{i_n} \to 1_{A_k} f$ (τ). Since V is Σ-solid, all the functions $1_{A_k} f_{i_n}$ are in V, and since V is also τ-closed, we see that $1_{A_k} f \in V$ for every k. Now, as τ is μ-continuous, we have $1_{A_k} f \to 1_A f$ (τ) whence $1_A f \in V$, and next $1_A f \to f$ (τ), and we conclude that $f \in V$. □

Corollary 4.7. *Let $X = (X, \tau)$ be μ-continuous. If a net (f_i) in X is Cauchy, $f \in X$ and $f_i \to f$ (τ_μ), then also $f_i \to f$ (τ).*

5. Levi type properties

Recall that a general TRS $X = (X, \tau)$, or its topology, is said to be Levi (resp. σ-Levi) if every increasing τ-bounded net (resp. sequence) in X has a supremum in X.

We now reformulate these Levi properties in a purely topological form that is adequate for spaces of vector-valued functions. We shall say that a TVS X of E-valued μ-measurable functions, or its topology τ, is

- *boundedly closed* (resp. *boundedly sequentially closed*) *in* $L_0(\mu, E)$ if, for every bounded subset of X, its closure (resp. sequential closure) in $L_0(\mu, E)$ is a subset of X.

Note that a fundamental subspace of X is boundedly closed iff it is boundedly sequentially closed.

The same proof as for Proposition 3.6 in [DL2] gives the following.

Proposition 5.1. *Let L be a TRS of μ-measurable functions.*

(a) *If L is σ-Levi, then every fundamental band of L is boundedly closed in $L_0(\mu)$.*
(b) *If $L \subset L_0(\mu)$ continuously and L is boundedly sequentially closed in $L_0(\mu)$, then L is σ-Levi.*
(c) *If L is Levi, then L is boundedly closed in $L_0(\mu)$.*
(d) *If μ is of type (C), $X \subset L_0(\mu)$ continuously, and L is boundedly closed in $L_0(\mu)$, then L is Levi.*

A series in X is said to be *disjoint* if its terms have pairwise disjoint supports. Clearly, such a series converges in $L_0(\mu, E)$ to its pointwise sum. In general, a series in a TVS is called *bounded* if the sequence of its partial sums is bounded.

We shall say that X is

- *disjointly boundedly closed in* $L_0(\mu, E)$ if, for each bounded disjoint series in X, its $L_0(\mu, E)$-sum belongs to X;
- *projectively closed in* $L_0(\mu, E)$ if, whenever a projective net in X has a limit in $L_0(\mu, E)$, the limit belongs to X;
- *piecewise uniformly closed in* $L_0(\mu, E)$ if every $A \in \Sigma^+_{\mathrm{oc}}(\mu)$ contains a $B \in \Sigma^+_{\mathrm{oc}}(\mu)$ such that X_B is closed in $(L_0(\mu, E), \tau_{\mathrm{u}})$; that is, whenever (f_n) is a sequence in X_B and $f_n \to f$ uniformly, then $f \in X_B$.

The qualifier 'in $L_0(\mu, E)$' will sometimes be omitted.

Our main purpose below is to show that X is boundedly closed iff each of its fundamental subspaces is piecewise uniformly closed and disjointly boundedly closed, and X itself is projectively closed (Corollary 5.7).

We shall say that X has

- the *disjoint Property* (O) if every bounded disjoint series in X is (subseries) convergent in X.

Note that $L_0(\mu, E)$ has always the disjoint Property (O). We omit the easy proofs of the next two propositions.

Proposition 5.2. *The following conditions are equivalent.*

(a) *X is boundedly closed in $L_0(\mu, E)$.*
(b) *Every fundamental subspace of X is boundedly closed in $L_0(\mu, E)$, and X is projectively closed in $L_0(\mu, E)$.*

Proposition 5.3. *The following conditions are equivalent.*

(a) *X is both sequentially μ-continuous and disjointly boundedly closed in $L_0(\mu, E)$.*
(b) *X has the disjoint property (O).*

Proposition 5.4. *If $X = (X, \tau) \subset L_0(\mu, E)$ continuously and X is piecewise sequentially complete, then it is piecewise uniformly closed in $L_0(\mu, E)$.*

Proof. Given $A \in \Sigma_{\mathrm{oc}}^{+}(\mu)$, choose $B \subset A$ with $\mu(B) > 0$ so that $\tau|X_B \leqslant \tau_{\mathrm{u}}|X_B$ and X_B is τ-sequentially complete. Let (f_n) be a sequence in X_B which converges uniformly to a function f. Then (f_n) is τ-Cauchy whence, by the τ-sequential completeness, it has a τ-limit $g \in X_B$. Since $X \subset L_0(\mu, E)$ continuously, $g = f$. $\square$

Let us say that a function $f \in L_0(\mu, E)$ *is piecewise* in X if every $A \in \Sigma_{\mathrm{oc}}^{+}(\mu)$ contains a $B \in \Sigma_{\mathrm{oc}}^{+}(\mu)$ such that $1_B f \in X$.

PROPOSITION 5.5. *Let X be piecewise uniformly closed in $L_0(\mu, E)$.*

(a) *If a function $f \in L_0(\mu, E)$ is the τ_μ-limit of a sequence from X, then f is piecewise in X.*
(b) *For every $f \in X$ and every $A \in \Sigma_{\mathrm{oc}}^{+}(\mu)$ there exists a $B \in \Sigma_{\mathrm{oc}}^{+}(\mu)$ contained in A and such that $1_B \varphi f \in X$ for all $\varphi \in L_\infty(\mu)$.*

Proof. (a): Let (f_n) be a sequence in X converging in $L_0(\mu, E)$ to f, and let $A \in \Sigma_{\mathrm{oc}}^{+}(\mu)$. By passing to a subsequence, we may assume that $f_n \to f$ μ-a.e.

Combining the assumption that X is piecewise uniformly closed with the Egoroff theorem, we find $B \subset A$ with $\mu(B) > 0$ such that X_B is uniformly closed in $L_0(\mu, E)$ and $\lim_n 1_B f_n = 1_B f$ uniformly. Since each $1_B f_n$ is in X, so is $1_B f$.

(b): Choose a $B \subset A$ with $B \in \Sigma_{\mathrm{oc}}^{+}(\mu)$ so that X_B is uniformly closed in $L_0(\mu, E)$ and the range of $1_B f$ is bounded in E. Let $\varphi \in L_\infty(\mu)$, and pick a sequence (φ_n) of simple functions such that $\varphi_n \to \varphi$ uniformly. Then all $1_B \varphi_n f$ are in X, and $1_B \varphi_n f \to 1_B \varphi f$ uniformly. Consequently, $1_B \varphi f \in X$. $\square$

REMARK. A TRS $L \subset L_0(\mu)$ is always piecewise uniformly closed in $L_0(\mu)$ (and even relatively uniformly complete; see the Remark at the end of Section 2 of [DL2]). It is so because, by definition, L is solid in $L_0(\mu)$.

PROPOSITION 5.6. *The following conditions are equivalent.*

(a) *X is piecewise uniformly closed and every fundamental subspace of X is disjointly boundedly closed in $L_0(\mu, E)$.*
(b) *Every fundamental subspace of X is boundedly closed in $L_0(\mu, E)$.*

Proof. (a) $\Longrightarrow$ (b): Let $A \in \Sigma_{\mathrm{oc}}^{\sigma}(\mu)$, and let (f_n) be a bounded sequence in X_A converging in $L_0(\mu, E)$ to some f. By passing to a subsequence, we may assume that $f_n \to f$ μ-a.e.

Next, by combining Proposition 5.5 (a) with the Egoroff theorem, and taking also Proposition 3.1 into account, we find a $\Sigma_{\mathrm{oc}}(\mu)$-partition $\{A_k : k \in \mathbb{N}\}$ of A such that, for every k,

$$(1)\ \tau_{\mathrm{u}}\text{-}\lim_n 1_{A_k} f_n = 1_{A_k} f =: g_k; \qquad (2)\ \tau|X_{A_k} \leqslant \tau_{\mathrm{u}}|X_{A_k}; \qquad (3)\ g_k \in X.$$

Consequently, (g_k) is a disjoint sequence in X_A. As X_A is Σ-solid, each $1_{A_k} f_n$ is in X_A, and from (1) and (2) it follows that $\tau\text{-}\lim_n 1_{A_k} f_n = g_k$. Now, it is easily seen that the sequence $h_n = \sum_{k=1}^{n} g_k$ is bounded in X and $h_n \to f$ in $L_0(\mu, E)$. Since X_A is disjointly boundedly closed, we conclude that f is in X_A.

(b) $\Longrightarrow$ (a): It is enough to show that if $B \in \Sigma_{\mathrm{oc}}(\mu)$ and $\tau | X_B \leqslant \tau_{\mathrm{u}} | X_B$, then X_B is uniformly closed in $L_0(\mu, E)$. Let a sequence (f_n) from X_B converge uniformly to a function f. Then it is τ-Cauchy and a fortiori τ-bounded. Since, obviously, $f_n \to f$ (τ_μ), it follows that $f \in X$. □

Remark. It is also not hard to verify that X is piecewise uniformly closed provided that (i) every fundamental subspace of X is disjointly boundedly closed, and (ii) whenever a function f is the limit in $L_0(\mu, E)$ of a bounded sequence from X, then f is piecewise in X.

Combining Propositions 5.2 and 5.6, we derive the following.

Corollary 5.7. *The space X is boundedly closed in $L_0(\mu, E)$ iff X is piecewise uniformly closed, each of its fundamental subspaces is disjointly boundedly closed, and X is projectively closed in $L_0(\mu, E)$.*

If L is a TRS of μ-measurable functions then, clearly, L is disjointly σ-Levi iff L is disjointly boundedly closed in $L_0(\mu)$. This along with the Remark made after Proposition 5.5 yield the following consequence of the above corollary.

Corollary 5.8. *If a TRS L of μ-measurable functions has the disjoint σ-Levi property and is projectively closed in $L_0(\mu)$, then it is boundedly closed in $L_0(\mu)$.*

Remark. The corollary can also be obtained in a more direct way: In the proof of Proposition 5.1 (a) (modeled on that of [DL2, Prop. 3.6]) it is, in fact, the disjoint σ-Levi property that is used to show that fundamental bands of L are boundedly closed in $L_0(\mu)$. To finish, proceed as in the proof of Proposition 5.1 (c) (loc. cit.), or apply Proposition 5.2.

Recall that a general TRS X is said to have the (σ-)*monotone completeness property*, (σ-)MCP, if every increasing positive Cauchy net (sequence) in X converges (see [AB, Def. 7.4]). If X is metrizable, then the σ-MCP is equivalent to the completeness of X (see [AB, Thm. 16.1]).

Proposition 5.9. *Let a TRS L of μ-measurable functions be Lebesgue and have the σ-MCP. Then L is σ-Levi iff every fundamental band of L is σ-Levi.*

Proof. 'If': Let (f_n) be a bounded increasing positive sequence in L. For each $A \in \Sigma'$, as L_A is both σ-Lebesgue and σ-Levi, the sequence $(1_A f_n)$

converges in $L_A \subset L$. Hence, by Proposition 4.3 (a), the sequence (f_n) is Cauchy. By the σ-MCP, it converges in L. □

6. Completeness

It is well known that if μ is countably o.c.-decomposable, then $L_0(\mu)$ and, more generally, by the same argument, $L_0(\mu, E)$ is metrizable and complete, i.e., is an F-space. It is, however, not clear that if we merely know $L_0(\mu)$ to be sequentially complete or complete, then so must be $L_0(\mu, E)$. In this respect there is a sharp distinction between these two kinds of completeness. We assume that the F-space $E = (E, \|\cdot\|)$ is nonzero.

THEOREM 6.1. *The space $L_0(\mu, E)$ is sequentially complete iff μ is of type (SC).*

Proof. The 'only if' direction is obvious.

'If': Let (f_n) be a Cauchy sequence in $L_0(\mu, E)$. Then there is a closed separable subspace F of E such that each f_n is μ-a.e. F-valued. Clearly, we expect our sequence to have a limit in the space $L_0(\mu, F)$. Now, as every separable metric space is isometric with a subset of $G = C[0, 1]$, we may assume that $F \subset G$. Since the set of μ-measurable F-valued functions is easily seen to be sequentially closed in $L_0(\mu, G)$, it suffices to show that $L_0(\mu, G)$ is sequentially complete. Let (b_k) be a Schauder basis of G and (b_k^*) the associated coefficient functionals.

For every k, the sequence $(b_k^* f_n)_{n=1}^{\infty}$ in $L_0(\mu)$ is Cauchy and, by assumption, has a limit $\varphi_k \in L_0(\mu)$.

If $A \in \Sigma_{\mathrm{oc}}(\mu)$, then $L_0(A, \mu, G)$ is complete. Hence there exists a function f_A in $L_0(\mu, G)$ such that $1_A f_n \to f_A$ (τ_μ). Choose $n_j \uparrow \infty$ so that $f_{n_j} \to f_A$ μ-a.e. on A. Then, for each k, we have on the one hand $\lim_j b_k^* f_{n_j} = b_k^* f_A$ μ-a.e. on A, while on the other hand $\lim_j b_k^* f_{n_j} = \varphi_k$ in submeasure μ on A. Therefore, $b_k^* f_A = \varphi_k$ μ-a.e. on A. In consequence, $f_A = \sum_k \varphi_k \, b_k$ μ-a.e. on A. Now, since the series $\sum_k \varphi_k \, b_k$ converges μ-a.e. on every set $A \in \Sigma_{\mathrm{oc}}(\mu)$, it has to converge μ-a.e. on S to a function $f \in L_0(\mu, G)$. By what we have seen above, $f_A = 1_A f$ μ-a.e. for every $A \in \Sigma_{\mathrm{oc}}(\mu)$. Thus $f_n \to f$ in $L_0(\mu, G)$. □

THEOREM 6.2.

(a) *If E is nonseparable, then $L_0(\mu, E)$ is complete iff μ is countably o.c.-decomposable.*
(b) *If E is separable, then $L_0(\mu, E)$ is complete iff μ is of type (C).*

Proof. (a): The 'if' part is obvious.

'Only if': Suppose $\Sigma_{\mathrm{oc}}^+(\mu)$ contains an uncountable disjoint subfamily $\{A_i : i \in I\}$. Since E is nonseparable, for some $\varepsilon > 0$ there exists an uncountable family $\{x_j : j \in J\}$ such that $\|x_j - x_k\| \geqslant \varepsilon$ if $j, k \in J$ and $j \neq k$. We may assume that $I = J$. For every finite subset K of I, let $f_K = \sum_{i \in K} 1_{A_i} x_i$.

Clearly, the net (f_K) is Cauchy in $L_0(\mu, E)$. If $L_0(\mu, E)$ were complete, this net would converge to some $f \in L_0(\mu, E)$. However, since $1_{A_i} f = 1_{A_i} x_i$ μ-a.e., f would not be μ-almost separably valued, contradicting its Bochner measurability.

Thus if $L_0(\mu, E)$ is complete, and $\mathcal{A}$ is a maximal disjoint subfamily in $\Sigma_{\mathrm{oc}}^+(\mu)$, then $\mathcal{A}$ is countable. Since then $\mu(S \smallsetminus \bigcup \mathcal{A}) = 0$, the submeasure μ has to be countably o.c.-decomposable.

(b): We have to prove that if μ is of type (C) and E is separable, then $L_0(\mu, E)$ is complete. Let $\{x_i : i \in \mathbb{N}\}$ be a countable dense subset of E. For $i, n \in \mathbb{N}$, let $K_n(x_i)$ denote the open ball centered at x_i and having radius $1/n$.

Let (f_α) be a Cauchy net in $L_0(\mu, E)$. For every $A \in \Sigma_{\mathrm{oc}}^+(\mu)$, the net $(1_A f_\alpha)$ has a limit f_A in $L_0(\mu, E)$. We must show that there exists $g \in L_0(\mu, E)$ such that $1_A g = f_A$ for every $A \in \Sigma_{\mathrm{oc}}^+(\mu)$.

Fix $i, n \in \mathbb{N}$. For every $A \in \Sigma_{\mathrm{oc}}^+(\mu)$, let

$$A_{n,i} := A \cap f_A^{-1}\big(K_n(x_i)\big).$$

Since μ is of type (C), the (increasing) net $\big(A_{n,i} : A \in \Sigma_{\mathrm{oc}}^+(\mu)\big)$ has a supremum $S_{n,i}$ in Σ (see Theorem 2.2 (c)). Set

$$R_{n,1} = S_{n,1} \quad \text{and} \quad R_{n,i} = S_{n,i} \smallsetminus (S_{n,1} \cup \cdots \cup S_{n,i-1}) \quad \text{for } i > 1,$$

and define a function $g_n \colon S \to E$ by

$$g_n = \sum_{i=1}^{\infty} 1_{R_{n,i}} x_i \qquad \text{(pointwise sum)}.$$

Also, for every $A \in \Sigma_{\mathrm{oc}}^+(\mu)$, let

$$A'_{n,1} = A_{n,1} \quad \text{and} \quad A'_{n,i} = A_{n,i} \smallsetminus (A_{n,1} \cup \cdots \cup A_{n,i-1}) \quad \text{for } i > 1,$$

Note that $A'_{n,i}$ and $A \cap R_{n,i}$ are μ-equal for all $A \in \Sigma_{\mathrm{oc}}^+(\mu)$ and $i, n \in \mathbb{N}$.

Let $m < n$ and $A \in \Sigma_{\mathrm{oc}}^+(\mu)$. Let $i, j \in \mathbb{N}$ and suppose the set $A \cap R_{m,i} \cap R_{n,j}$ is of positive μ submeasure. Since this set is μ-equal to $A'_{m,i} \cap A'_{n,j}$, taking any point t in the latter set, we have

$$\|g_m(s) - g_n(s)\| = \|x_i - x_j\| \leqslant \|x_i - f_A(t)\| + \|f_A(t) - x_j\| \leqslant \frac{1}{m} + \frac{1}{n}$$
$$\text{for } s \in A \cap R_{m,i} \cap R_{n,j}.$$

Hence

$$\|g_m(s) - g_n(s)\| \leqslant \frac{1}{m} + \frac{1}{n} \qquad \text{for } \mu\text{-a.e. } s \in A.$$

Consequently, as μ is locally o.c., the above estimate holds μ-a.e. on S.

Thus the sequence (g_n) satisfies the Cauchy condition for μ-a.e. uniform convergence on S and hence converges in this sense to a Bochner measurable function g. It is not hard to see that g is as required. $\square$

REMARK. In both parts of Theorem 6.2, the condition '$L_0(\mu, E)$ is quasi-complete' could be included as a third condition that is equivalent to each of the other two conditions.

PROPOSITION 6.3. *Let X be μ-continuous and projectively complete. If each fundamental subspace of X is complete, or quasi-complete, or sequentially complete, then so is, respectively, X.*

Proof. Consider, for instance, the case of quasi-completeness. Let (f_i) be a bounded Cauchy net in X. Let $A \in \Sigma_{\mathrm{oc}}^{\sigma}(\mu)$. Since X_A is quasi-complete, there is $f_A \in X_A$ such that $1_A f_i \to f_A$ (τ). Clearly, the net (f_A) is projective and, by Proposition 4.2, it is Cauchy in X. Since X is projectively complete, it converges to some $f \in X$.

We now show that $f_i \to f$ (τ). Let V be a Σ-solid balanced neighborhood of zero in X. Choose an index k so that $f_i - f_k \in V$ for $i \geqslant k$. Next, as X is μ-continuous, there is a set $A \in \Sigma'$ such that $1_{S \smallsetminus A} f_k \in V$ and $1_{S \smallsetminus A} f \in V$. Since $1_A f_i \to 1_A f$ (τ), there is $j \geqslant k$ such that $1_A f - 1_A f_i \in V$ for $i \geqslant j$. Finally, if $i \geqslant j$, then

$$f - f_i = 1_{S \smallsetminus A} f + (1_A f - 1_A f_i) - 1_{S \smallsetminus A}(f_i - f_k) - 1_{S \smallsetminus A} f_k \in V + V + V + V,$$

which concludes the proof. □

PROPOSITION 6.4. *Let X be μ-continuous, and assume that $L_0(\mu, E)$ is (sequentially) complete. If $X \subset L_0(\mu, E)$ continuously and X is boundedly (sequentially) closed in $L_0(\mu, E)$, then X is (sequentially complete) quasi-complete.*

Proof. Let (f_i) be a bounded Cauchy net (sequence) in X. It is also Cauchy in $L_0(\mu, E)$, and since the latter space is (sequentially) complete, it τ_μ-converges to some f. As X is boundedly (sequentially) closed, $f \in X$. By Corollary 4.7, we have $f_i \to f$ in X. □

Questions concerning the (topological) completeness of spaces $L_0(\mu, E)$, as defined in this paper, are answered in a rather satisfactory manner by Theorems 6.1 and 6.2. However, condition (SC) itself is not yet well understood. Furthermore, as we pointed out in Remark 3.3 (d), also other notions of measurability of vector valued functions could conceivably be considered. The next theorem stresses the importance of a 'good' choice of $L_0(\mu, E)$ in this type of research.

THEOREM 6.5. *Let $L = (L, \tau)$ be a TRS of μ-measurable functions continuously included in $L_0(\mu)$, and $E = (E, \|\cdot\|_E)$ a Banach space. Assume that both L and $L_0(\mu, E)$ are sequentially complete, or quasi-complete, or complete. Then so is, respectively, $L(E)$.*

Proof. As is easily seen, $L(E)$ is continuously included in $L_0(\mu, E)$.

We give a proof for the case of quasi-completeness. Let (f_i) be a bounded Cauchy net in $L(E)$. Denote $\varphi_i = \|f_i\|_E$. Then the net (φ_i) in L is bounded, and since $|\varphi_i(s) - \varphi_j(s)| \leqslant \|f_i(s) - f_j(s)\|_E$, it is also Cauchy. Hence it has a τ-limit $\varphi \in L$. As $L \subset L_0(\mu)$ continuously, $\varphi_i \to \varphi$ in $L_0(\mu)$.

Since $L(E) \subset L_0(\mu, E)$ continuously, and the latter is quasi-complete, the net (f_i) is Cauchy in $L_0(\mu, E)$ and so converges there to a function f. But then $\varphi_i = \|f_i\|_E \to \|f\|_E$ in $L_0(\mu)$ and, consequently, $\|f\|_E = \varphi \in L$. Therefore, $f \in L(E)$.

Take any closed zero-neighborhood V in L. As (f_i) is Cauchy in $L(E)$, there is k such that $\|f_i - f_j\|_E \in V$ for all $i, j \geqslant k$. Fix $i \geqslant k$ and consider the net $\big(\|f_i - f_j\|_E : j \geqslant k\big)$ in $V \subset L$. It is bounded and Cauchy, and hence has a τ-limit in L. Evidently, that limit is nothing but $\|f_i - f\|_E$. Since V is τ-closed, $\|f_i - f\|_E \in V$, and this holds for all $i \geqslant k$. Thus $f_i \to f$ in $L(E)$. $\square$

7. The topological Riesz spaces $L(E)$

Throughout this section $L = (L, \tau)$ is a TRS of μ-measurable functions *continuously* included in $L_0(\mu)$ and $E = (E, \|\cdot\|_E)$ is a Banach lattice. Note, however, that Propositions 7.1–7.4 and 7.6 below are also valid when E is an F-lattice.

Recall that a general TRS X is said to be (*sequentially*) *intervally complete* if each of its order intervals is (sequentially) complete. We shall say that X has the (σ-)MCP *for intervals* if every Cauchy monotone, or just increasing, net (sequence) contained in an order interval in X is convergent. It is known (see [V] and [L3, Prop. 6.2]) that a metrizable Riesz space is intervally complete iff it has the σ-MCP for intervals. Also note that the (σ-)MCP for intervals implies the pseudo (σ-)Lebesgue property (see [AB, Def. 17.1]): $x_i \to x$ whenever the net (sequence) (x_i) is Cauchy and $0 \leqslant x_i \uparrow x$. The last two properties obviously coincide for a (σ-)Dedekind complete TRS.

Proposition 7.1. *$L_0(\mu, E)$ has the σ-MCP.*

Proof. Let an increasing and positive sequence (f_n) in $L_0(\mu, E)$ be Cauchy. Then it is τ_μ-convergent, and hence also μ-a.e. convergent, on each set from $\Sigma_{\mathrm{oc}}(\mu)$. As is well known, the set C of all points $s \in S$, where the sequence $\big(f_n(s)\big)$ is convergent, is in Σ. In fact,

$$C = \bigcap_r \bigcup_k \bigcap_{n \geqslant k} \{s \in S : \|f_n(s) - f_k(s)\| \leqslant r^{-1}\}.$$

From what was said above it follows that $S \setminus C$ is a μ-null set. Hence the sequence (f_n) is μ-a.e. convergent, and a fortiori τ_μ-convergent, to a function $f \in L_0(\mu, E)$. $\square$

The σ-MCP seems to be the only property of completeness that $L_0(\mu, E)$ has without any additional assumption on (S, Σ, μ). Other types of completeness may or may not hold and, as we show below, fall into three categories.

PROPOSITION 7.2. *The following statements are equivalent.*

(a) $L_0(\mu, E)$ *is sequentially complete.*
(b) $L_0(\mu, E)$ *is sequentially intervally complete.*
(c) (S, Σ, μ) *is of type* (SC).

Proof. In view of Theorem 6.1, and since (a) implies (b) is trivial, we only have to verify that (b) implies (c). Evidently, (b) implies that $L_0(\mu)$ is also sequentially intervally complete.

Now, to prove that $L_0(\mu)$ is actually sequentially complete, it suffices to show that every positive Cauchy sequence (f_n) in $L_0(\mu)$ is convergent. In view of Proposition 7.1, this is a consequence of [AB, Sec. 7, Exerc. 9]. For the reader's convenience, we provide a direct argument. For each $k \in \mathbb{N}$ consider the sequence $f_n \wedge k = f_n \wedge k1_S$ $(n = 1, 2, \dots)$. It is Cauchy and contained in the order interval $[0, k1_S]$, and hence $(\tau_\mu$-)convergent to some $g_k \in [0, k1_S]$. Note that if $k < m$, then $g_k = g_m \wedge k \leqslant g_m$.

We verify that the sequence (g_k) is Cauchy. Fix $A \in \Sigma_{\mathrm{oc}}(\mu)$ and $\varepsilon > 0$. Since the sequence (f_n) is Cauchy, and hence bounded, there is $N > 0$ such that

$$\alpha_n = \mu(\{s \in A : f_n(s) > N\}) < \varepsilon/4.$$

Let $m > k \geqslant N$ and choose n so that

$$\beta_j = \mu(\{s \in A : |g_j - f_n \wedge j|(s) \geqslant \varepsilon/4\}) < \varepsilon/4 \quad \text{for } j = k, m.$$

Since $g_m - g_k = (g_m - f_n \wedge m) + (f_n \wedge m - f_n) + (f_n - f_n \wedge k) + (f_n \wedge k - g_k)$, from the above it follows easily that $\mu(\{s \in A : |g_m - g_k|(s) \geqslant \varepsilon\}) \leqslant \beta_m + \alpha_n + \alpha_n + \beta_k < \varepsilon$. Thus (g_k) is Cauchy.

By Proposition 7.1, the sequence (g_k) converges to some $f \in L_0(\mu)$. Note that $f - f_n = (f - g_k) + (g_k - f_n \wedge k) + (f_n \wedge k - f_n)$ for any n and k. Using this representation and a similar reasoning as above, it is not hard to see that $f_n \to f(\tau_\mu)$. $\square$

PROPOSITION 7.3. *Suppose E has a nonseparable order interval. Then the following statements are equivalent.*

(a) $L_0(\mu, E)$ *is complete.*
(b) $L_0(\mu, E)$ *has the MCP for intervals,*
(c) (S, Σ, μ) *is countably o.c.-decomposable.*

Consequently, in this case quasi-completeness, MCP, and interval completeness are all equivalent to completeness.

Proof. Only the implication (b) $\Longrightarrow$ (c) needs a proof. Let $u \in E_+$ be such that the interval $[0, u]$ is nonseparable. We proceed as in the first part of the proof of Theorem 6.2 (a) choosing the x_i's in $[0,u]$. Note that the net (f_K) is increasing, Cauchy, and contained in the interval $[0, 1_S u]$. By (b), it converges, and we conclude as in 6.2 that μ has to be countably o.c.-decomposable. $\square$

Proposition 7.4. *Suppose E has separable order intervals. Then the following statements are equivalent.*

(a) *The closure of every ideal in $L_0(\mu, E)$ generated by a singleton is complete.*
(b) *$L_0(\mu, E)$ is intervally complete.*
(c) *$L_0(\mu, E)$ has the MCP for intervals.*
(d) *(S, Σ, μ) is of type (C).*

Proof. The implications (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) are obvious.

(c) $\Longrightarrow$ (d): If (c) holds, then $L_0(\mu)$ also has the MCP for intervals, and in particular for the interval $[0, 1_S]$. From this it follows that μ is of type (C) (cf. Theorem 2.2).

(d) $\Longrightarrow$ (a): For $z \in E$, denote by E_z the ideal in E generated by z. Since the interval $[-|z|, |z|]$ is separable, so is E_z, and also its closure $\overline{E_z}$. By Theorem 6.2 (b), the space $L_0(\mu, \overline{E_z})$ is complete. Now, take any $g \in L_0(\mu, E)$, and consider the closure $\overline{I_g}$ of the ideal $I_g \subset L_0(\mu, E)$ generated by g. We may assume that the range of g is separable. Let (z_n) be a sequence dense in $g(S)$. As is well known, one can find a $z \in E$ such that $(z_n) \subset E_z$. (For example, $z = \sum_n \alpha_n |z_n|$, where $\alpha_n > 0$ and $\sum_n \|\alpha_n z_n\| < \infty$, will work.) Then, obviously, $g(S) \subset \overline{E_z}$. Hence g is in $L_0(\mu, \overline{E_z})$, and since the latter is an ideal in $L_0(\mu, E)$, it follows that $\overline{I_g} \subset L_0(\mu, \overline{E_z})$. But we already know that $L_0(\mu, \overline{E_z})$ is complete. Hence so must be $\overline{I_g}$. $\square$

Remark. There are here two possibilities. If E is nonseparable and μ is of type (C) but not countably o.c.-decomposable then, in view of Theorem 6.2 (a) and Proposition 7.4, the interval completeness of $L_0(\mu, E)$ is all we can expect. If E happens to be separable, then $L_0(\mu, E)$ is actually complete by Theorem 6.2 (b).

Proposition 7.5. *Suppose L has the σ-MCP (resp. the σ-MCP for intervals). Then the TRS $L(E)$ has the σ-MCP (resp. the σ-MCP for intervals).*

Proof. The proof is, with obvious changes, the same as that of Theorem 6.5 above, noting that the σ-MCP of $L_0(\mu, E)$ is automatic by Proposition 7.1. $\square$

Also, by the same type of reasoning, we have:

Proposition 7.6.

(a) *Suppose $L_0(\mu, E)$ is sequentially complete. If L is sequentially intervally complete or sequentially complete, then these properties are inherited by the space $L(E)$.*

(b) *Suppose $L_0(\mu, E)$ is intervally complete. If L is intervally complete or has the MCP for intervals, then these properties are inherited by the space $L(E)$.*

(c) *Suppose $L_0(\mu, E)$ is complete. If L is complete or has the MCP, then these properties are inherited by the space $L(E)$.*

PROPOSITION 7.7. *The TRS $L(E)$ is σ-Lebesgue and σ-Dedekind complete iff both L and E are σ-Lebesgue, and E is σ-Dedekind complete.*

Proof. The 'only if' part is obvious.

'If': Let $0 \leqslant f_n \uparrow \leqslant g$ in $L(E)$. We have to show that there is $f \in L(E)$ such that $f_n \to f$. Obviously, $0 \leqslant f_n(s) \uparrow \leqslant g(s)$ a.e. in S. Hence, as E is σ-Dedekind complete and σ-Lebesgue, there is a function $f\colon S \to E$ such that $f_n(s) \uparrow f(s)$ a.e. in S and $\|f(s) - f_n(s)\|_E \to 0$ a.e. Hence $f \in L_0(\mu, E)$, and as $0 \leqslant f \leqslant g \in L(E)$, we conclude that $f \in L(E)$. Moreover, $\|f - f_n\|_E \downarrow 0$ in L so that, by the σ-Lebesgue property of L, $\|f - f_n\|_E \to 0$ (τ). That is, $f_n \to f$ in $L(E)$. □

PROPOSITION 7.8. *The TRS $L(E)$ is Lebesgue if and only if both L and E are Lebesgue.*

Proof. The 'only if' part is obvious.

'If': Let $0 \leqslant f_i \uparrow f$ in $L(E)$. Since E, being Lebesgue, is also Dedekind complete, it follows from Proposition 7.7 that every (increasing) subsequence of the net (f_i) converges in $L(E)$. Hence the net (f_i) is Cauchy. We want to show that $f_i \to f$ in $L(E)$. Take any τ-neighborhood V of zero in L. By Proposition 4.1, there is $A \in \Sigma_{\mathrm{oc}}(\mu)$ such that $1_{S\smallsetminus A} f \in V_E$, and hence also $1_{S\smallsetminus A}(f - f_i) \in V_E$ for all i. Now, the net $(1_A f_i)$ in the TRS $L_A(E)$ is Cauchy and contained in the order interval $[0, 1_A f]$. Since L_A is Lebesgue and Dedekind complete, it trivially has the MCP for intervals. Hence $L_A(E)$ also has this property, by Proposition 7.6 (b). [Alternatively, since L_A is Lebesgue (hence Fatou) and Dedekind complete, it is intervally complete (see [AB, Thms. 11.6 and 13.1]). Then $L_A(E)$ is also intervally complete, by Proposition 7.6 (b).] Therefore, the net $(1_A f_i)$ converges, and it should be clear that its limit in $L_A(E)$ is $1_A f$. Hence there is i_0 such that for $i \geqslant i_0$ we have $1_A(f - f_i) \in V_E$ and, consequently, $f - f_i \in V_E + V_E$. □

PROPOSITION 7.9. *Let $L = (L, \tau)$ be pseudo σ-Lebesgue and E σ-Dedekind complete. Then $L(E)$ contains a lattice copy of ℓ_∞ iff either L or E does.*

Proof. Suppose that neither L nor E contains a lattice copy of ℓ_∞. Then E is σ-Lebesgue by Theorem 1.0 (ii). It follows that L is also σ-Lebesgue. Indeed, let $0 \leqslant f_n \uparrow f$ in L. By Theorem 1.2, (f_n) is a Cauchy sequence and so $f_n \to f$ (τ) by the pseudo σ-Lebesgue property of L. In view of

Proposition 7.7, $L(E)$ is σ-Lebesgue and σ-Dedekind complete, and hence cannot contain a lattice copy of ℓ_∞. $\square$

Proposition 7.10. *The TRS $L(E)$ is σ-Lebesgue and σ-Levi iff both L and E are.*

Proof. The 'only if' part is clear.

'If': Take a bounded sequence $0 \leqslant f_n \uparrow$ in $L(E)$. We have to show that (f_n) converges in $L(E)$. As the sequence $(\|f_n\|_E)$ in L is increasing and bounded, by the σ-Levi property of L it has a supremum φ in L. It follows that for a.e. $s \in S$ the sequence $\big(f_n(s)\big)$ is bounded in E. Therefore, by the σ-Lebesgue σ-Levi property of E, $f(s) := \lim_n f_n(s)$ exists in E for a.e. s. The function f thus obtained is Bochner measurable and $\varphi = \|f\|_E$. Thus $f \in L(E)$ and $f = \sup_n f_n$. Then $\|f - f_n\|_E \downarrow 0$ in L and, therefore, $\|f - f_n\|_E \to 0$ (τ) by the σ-Lebesgue property of L. This means that $f_n \to f$ in $L(E)$. $\square$

Corollary 7.11. *Let L have the σ-MCP. Then $L(E)$ contains a lattice copy of c_0 iff either L or E does.*

Proof. 'Only if': Suppose neither L nor E contains a lattice copy of c_0. Then, by Theorem 1.1, both L and E are σ-Lebesgue and σ-Levi. Hence so is $L(E)$, by Proposition 7.10. It follows that $L(E)$ cannot contain a lattice copy of c_0. $\square$

Proposition 7.12. *The following are equivalent.*

(a) *$L(E)$ is Lebesgue Levi.*
(b) *Both L and E are Lebesgue Levi and $L(E)$ is projectively complete.*

Proof. (b) $\Longrightarrow$ (a): We first show that $L(E)$ is complete. As L is Lebesgue Levi, it is complete (see [AB, Thms. 11.6 and 13.9]). Therefore, by Theorem 6.5, for each A in Σ' the (fundamental) band $L_A(E)$ of $L(E)$ is complete. Moreover, by Proposition 7.8, $L(E)$ is Lebesgue. Therefore, by Proposition 6.3, $L(E)$ is complete.

Now, consider a bounded net $0 \leqslant f_i \uparrow$ in $L(E)$. As in the proof of Proposition 7.10 above, there is $\varphi \in L$ such that $\|f_i\|_E \leqslant \varphi$ for all i. Hence, again as above, every increasing subsequence of the net converges in $L(E)$. It follows that the net (f_i) is Cauchy, and hence convergent, in $L(E)$.

(a) $\Longrightarrow$ (b): Obvious. $\square$

8. An Orlicz-Pettis theorem

Let $\mathcal{R}$ be a ring of sets. Given a finitely additive measure $\mathbf{m}\colon \mathcal{R} \to X \subset L_0(\mu, E)$, we define $\Sigma_{\mathrm{ca}}(\mathbf{m})$ and $\Sigma_{\mathrm{exh}}(\mathbf{m})$ to be the classes of sets $A \in \Sigma$ such that the measure $P_A \circ \mathbf{m}\colon \mathcal{R} \to X$ is countably additive or exhaustive, respectively.

PROPOSITION 8.1. *Let* $\mathbf{m}\colon \mathcal{R} \to X$ *be a finitely additive measure.*

(a) *If* $\mathcal{R}$ *is a* σ*-ring and* X *is sequentially* μ*-continuous, then* $\Sigma_{\mathrm{ca}}(\mathbf{m})$ *and* $\Sigma_{\mathrm{exh}}(\mathbf{m})$ *are* σ*-ideals in* Σ.

(b) *If the net* $(P_A\circ\mathbf{m} : A \in \Sigma')$ *converges to* $\mathbf{m}$ *pointwise on* $\mathcal{R}$ (*in particular, if* X *is* μ*-continuous*), *then* $\mathbf{m}$ *is countably additive* (*resp. exhaustive, or bounded*) *if* (*and only if*) *for each* $A \in \Sigma'$ *the measure* $P_A\circ\mathbf{m}$ *is countably additive* (*resp. exhaustive, or bounded*).

Proof. (a): Clearly, both $\Sigma_{\mathrm{ca}}(\mathbf{m})$ and $\Sigma_{\mathrm{exh}}(\mathbf{m})$ are ideals. Let (A_n) be an increasing sequence in $\Sigma_{\mathrm{ca}}(\mathbf{m})$ (resp. $\Sigma_{\mathrm{exh}}(\mathbf{m})$) with union A. Then for each $N \in \mathcal{R}$, $P_{A_n}\circ\mathbf{m}(N) \to P_A\circ\mathbf{m}(N)$ in X. By the Nikodym theorem (resp. the Brooks-Jewett theorem; see [D1]), $P_A\circ\mathbf{m}$ is countably additive (resp. exhaustive). Thus $A \in \Sigma_{\mathrm{ca}}(\mathbf{m})$ (resp. $A \in \Sigma_{\mathrm{exh}}(\mathbf{m})$).

(b): This follows easily from Proposition 4.3 (b). □

COROLLARY 8.2. *Let* X *be* μ*-continuous and projectively complete. Suppose the net* $(\mathbf{m}_A : A \in \Sigma')$ *of finitely additive measures from* $\mathcal{R}$ *to* X *is projective pointwise on* $\mathcal{R}$*; that is, for each* $N \in \mathcal{R}$, $(\mathbf{m}_A(N) : A \in \Sigma')$ *is a projective net. Then the pointwise limit measure* $\mathbf{m}\colon \mathcal{R} \to X$ *of the net* $(\mathbf{m}_A : A \in \Sigma')$ *exists and is countably additive* (*resp. exhaustive, or bounded*) *provided the measures* $\mathbf{m}_A$, $A \in \Sigma'$, *have these properties.*

Proof. As X is μ-continuous, the nets $(\mathbf{m}_A(N) : A \in \Sigma')$ are Cauchy by Proposition 4.2 (c). Further, the limits $\mathbf{m}(N) = \lim_{A\in\Sigma'} \mathbf{m}_A(N)$ exist in X by its projective completeness. Clearly, $P_A\circ\mathbf{m} = \mathbf{m}_A$ for each $A \in \Sigma'$. To finish, apply Proposition 8.1 (b). □

The next result is a direct consequence of Proposition 4.3 (a).

PROPOSITION 8.3. *Suppose* X *is sequentially complete and projectively* μ*-continuous. If* $\sum_n f_n$ *is a series in* X *such that for every* $A \in \Sigma'$ *the series* $\sum_n 1_A f_n$ *is subseries convergent, then* $\sum_n f_n$ *is subseries convergent in* X.

The following result is a common generalization of [D2, Thm. 2.2] and the theorem proved in [DL1]. However, it does not cover the Orlicz-Pettis theorem in [DL2] which was established for σ-*Lebesgue* TRS's of measurable functions over *measure* spaces.

THEOREM 8.4. *Let* (X,τ) *be a* μ*-continuous TVS of* E*-valued* μ*-measurable functions,* $\mathcal{R}$ *a* σ*-ring of sets, and* $\mathbf{m}\colon \mathcal{R} \to X$ *a finitely additive measure.*

(a) *If* $\mathbf{m}$ *is* τ_μ*-countably additive, then it is* τ*-countably additive.*

(b) *If* $\mathbf{m}$ *is* τ_μ*-exhaustive, then it is* τ*-exhaustive.*

Proof of (a). The proof will be split into three cases.

Case 1. μ is o.c., τ is weaker than the topology of uniform convergence on S.

In this case the assertion is a particular case of Thm. 2.2 in [D2] noting that, as is easily seen by inspecting the arguments, the assumption of metrizability of $\tau = \mathfrak{t}$ in [D2] was unnecessary.

Case 2. μ is countably o.c.-decomposable.

Using Proposition 3.1, we can find an increasing sequence (S_k) in $\Sigma_{\mathrm{oc}}(\mu)$ such that $S_k \uparrow S$ and $\tau|X_{S_k} \leqslant \tau_{\mathrm{u}}|X_{S_k}$ for every k. Then, by Case 1, $S_k \in \Sigma_{\mathrm{ca}}(\mathbf{m})$ for every k. By Proposition 8.1 (a), $S \in \Sigma_{\mathrm{ca}}(\mathbf{m})$; that is, $\mathbf{m}$ is countably additive.

Case 3. μ is locally o.c.

From Case 2 it follows that $\Sigma' \subset \Sigma_{\mathrm{ca}}(\mathbf{m})$ so that each of the measures $P_A \circ \mathbf{m}$ $(A \in \Sigma')$ is countably additive. As X is μ-continuous, $\mathbf{m}$ is the pointwise limit in X of the net $(P_A \circ \mathbf{m} : A \in \Sigma')$. Now apply Proposition 8.1 (b). □

Proof of (b). We consider here the same three cases as in the proof of (a).

Case 1. As τ_μ is metrizable, the τ-exhaustivity of $\mathbf{m}$ follows easily from Case 1 in (a) using Proposition 1.1 of [DL2].

Case 2. The proof is the same as in (a) replacing $\Sigma_{\mathrm{ca}}(\mathbf{m})$ by $\Sigma_{\mathrm{exh}}(\mathbf{m})$, and appealing to the 'exhaustive' part of Proposition 8.1 (a).

Case 3. Again, the proof is the same as in part (a). □

Remark. Note that if part (b) is proved first, then one can deduce (a) from (b) using Corollary 4.7.

Corollary 8.5. *Let τ_1 and τ_2 be two Hausdorff Lebesgue topologies on a Riesz space L, $\mathcal{R}$ a σ-ring of sets, and $\mathbf{m} \colon \mathcal{R} \to L$ a finitely additive measure.*

(a) $\mathbf{m}$ *is τ_1-countably additive iff it is τ_2-countably additive.*

(b) $\mathbf{m}$ *is τ_1-exhaustive iff it is τ_2-exhaustive.*

In particular, subseries convergence coincides for all Hausdorff Lebesgue topologies on a Riesz space.

Proof. In view of [AB, Thm. 11.10] (or [L4, Prop. 8]), both τ_1 and τ_2 admit a Hausdorff Lebesgue extension to the Dedekind completion of L. Thus we may assume that L is Dedekind complete. Furthermore, the infimum τ of τ_1 and τ_2 is a Hausdorff Lebesgue topology on L (see, e.g., [L4, Lemma 3.4]). Now, by Theorem 2.5, we may embed (L, τ) continuously as a solid subspace in $L_0(\mu)$ for some submeasure μ of type (C). It now suffices to apply Theorem 8.4. □

9. Property (O) and copies of c_0 and ℓ_∞

We first recall a few definitions from [DL2, Sec. 1].

A series in a TVS X is *perfectly bounded* if the set of all its finite sums is bounded, and *convexly bounded* if the convex hull of the set of all its finite sums is bounded. Likewise, a finitely additive measure with values in X is *convexly bounded* if the convex hull of its range is bounded.

A TVS X has *Property* (O) if every perfectly bounded series in X is (subseries) convergent, and *contains a copy* of c_0 (resp. ℓ_∞) if there exists an isomorphism from the Banach space c_0 (resp. ℓ_∞) onto a subspace of X. Evidently, if X has Property (O), then X contains no copy of c_0.

In all of the present (and most of the next) section, (S, Σ, λ) is a measure space, with a locally order continuous σ*-additive measure* λ, and E is a *Banach* space. Furthermore, X is a TVS of E-valued λ-measurable functions continuously included in $L_0(\lambda, E)$.

Remark. For the sake of conformity with the definitions and results of the previous sections, we impose on the measure λ the condition of local order continuity instead of the more familiar (though slightly stronger) condition that λ be *locally finite* (or semi-finite). Recall that the latter means that each $A \in \Sigma$ with $\lambda(A) > 0$ contains a $B \in \Sigma$ such that $0 < \lambda(B) < \infty$. If one prefers to work with locally finite measures λ, the results of the preceding sections still hold if one replaces $\Sigma_{\mathrm{oc}}(\lambda)$ and $\Sigma^\sigma_{\mathrm{oc}}(\lambda)$ by the families of sets of finite or σ-finite λ measure, respectively.

In [DL2], where we considered TRS's of scalar λ-measurable functions, we used a theorem of Orlicz on perfectly bounded series in $L_0(\lambda)$. Here, we will need a vector-valued extension of this result. In the theorem below, the implication (a) $\Longrightarrow$ (c) is implicit in Hoffmann-Jørgensen [HJ] and Kwapień [K], and is stated and proven explicitly in [L2, Thm. 2.11]. Other implications can also be found in [L2, loc. cit.]; the proof is the same as that of Proposition 9.5 below using [L1, Cor. B of Thm. 1].

Theorem 9.1. *The following conditions are equivalent.*

(a) *E contains no copy of c_0.*
(b) *$L_0(\lambda, E)$ contains no copy of c_0.*
(c) *Every perfectly bounded series in $L_0(\lambda, E)$ is unconditionally Cauchy.*

Moreover, if λ is of type (SC), the above conditions are also equivalent to

(d) *$L_0(\lambda, E)$ has Property (O).*

We now generalize Theorem 9.1 to some other spaces of measurable functions.

Theorem 9.2. *Assume E contains no copy of c_0, X is μ-continuous, and either*

(a) *X is sequentially complete and each fundamental subspace of X is boundedly closed in $L_0(\lambda, E)$, or*
(b) *λ is of type (SC) and X is boundedly sequentially closed in $L_0(\lambda, E)$.*

Then X has Property (O).

Proof. (a): Let $\sum_n f_n$ be a perfectly bounded series in X. Applying Theorem 9.1, we see that for every $A \in \Sigma^{\sigma}_{\mathrm{oc}}(\lambda)$ the series $\sum_n 1_A f_n$ is subseries convergent in $L_0(\lambda, E)$. Since X_A is boundedly closed in $L_0(\lambda, E)$, the series $\sum_n 1_A f_n$ is in fact subseries convergent in X for the topology inherited from $L_0(\lambda, E)$. It is subseries convergent in the original topology of X by Theorem 8.4. To complete the proof, apply Proposition 8.3.

(b): This follows from (a) using Proposition 6.4; alternatively, apply Theorem 9.1 and Theorem 8.4. □

Remark. Applying 8.2 and 8.3, one can easily verify the following: *Let X be μ-continuous and sequentially or projectively complete. If each fundamental subspace of X has Property (O), so does X.*

Corollary 9.3. *Let E contain no copy of c_0, and let X be piecewise uniformly closed in $L_0(\lambda, E)$ and projectively complete. If X has the disjoint Property (O), then it has Property (O).*

Proof. By Proposition 5.3 and Corollary 5.7, X is sequentially μ-continuous and boundedly closed in $L_0(\lambda, E)$. Next, by Corollary 4.5, X is μ-continuous which, in view of Proposition 6.3, implies that X is quasi-complete. To conclude, apply Theorem 9.2 (a). □

We shall say that X contains a *disjointly supported copy* of c_0 (resp. ℓ_∞) if there exists an isomorphic embedding $J\colon c_0 \to X$ (resp. $J\colon \ell_\infty \to X$) which is disjointness preserving. That is, whenever elements x, y in c_0 (or ℓ_∞) have disjoint supports, so do their images $J(x)$, $J(y)$ in X.

The next result corresponds to Theorem 1.3 (i.e., Theorem 5.5 in [DL2]). It is a culmination of a long line of research including, e.g., [MO, Thm. 3], [S], [C, Thm. 5].

Theorem 9.4. *Assume E contains no copy of c_0, and let X be a quasi-complete TVS of E-valued λ-measurable functions. Also assume that*

(s) *every bounded disjoint series in X is convexly bounded.*

Then the following conditions are equivalent.

(a) *X contains no copy of c_0.*
(b) *X contains no disjointly supported copy of c_0.*
(c) *X has the disjoint Property (O).*
(d) *X has Property (O).*

Proof. (c) $\Longrightarrow$ (d): In view of Proposition 5.4, X is piecewise uniformly closed in $L_0(\lambda, E)$; apply Corollary 9.3. Since the implications (d) $\Longrightarrow$ (a) and (a) $\Longrightarrow$ (b) are trivial, it remains to show that

(b) $\Longrightarrow$ (c): If not then, as X is sequentially complete, there exists a bounded disjoint series $\sum_n f_n$ in X with $f_n \nrightarrow 0$. By condition (s), the series $\sum_n f_n$ is convexly bounded. Then, as in [DL2, proof of Prop. 1.3 or Remark after Thm. 2.4], one arrives at a contradiction with (b) by producing an isomorphism $c_0 \to X$ the usual way out of the operator $J(a_n) = \sum_n a_n f_n$. $\square$

Remark. Condition (s) is often satisfied automatically. It is so, e.g., when X is locally convex (or, more generally, locally pseudoconvex) or is a TRS of measurable functions (cf. Lemma 2.3 in [DL2]); see also Section 10.

Proposition 9.5. *The following conditions are equivalent.*

(a) *$L_0(\lambda, E)$ contains no copy of ℓ_∞.*
(b) *Every bounded finitely additive measure $\mathbf{m}\colon \mathcal{P}(\mathbb{N}) \to L_0(\lambda, E)$ is exhaustive.*

Proof. Only (a) $\Longrightarrow$ (b) is nontrivial. Suppose $\mathbf{m}$ is not exhaustive. Then, for a set $A \in \Sigma^+_{\mathrm{oc}}(\lambda)$, the measure $P_A \circ \mathbf{m}$ is not exhaustive. By passing to a subset of A, we may assume that either A is an atom of infinite λ measure, or $\lambda(A) < \infty$. In the first case it is easy to see that E, and a fortiori $L_0(\lambda, E)$, has to contain a copy of ℓ_∞, contradicting (a). In the second case it can be assumed that λ is finite. Then, as $L_0(\lambda, E)$ has the bounded multiplier property (see [P, Cor. 3], [RW]), a contradiction with (a) arises using [L1, Cor. A of Thm. 1]. $\square$

Remark. It is a natural conjecture (already made in [L2, 2.11$'$, p. 235]) that the two conditions above are equivalent to the noncontainment of ℓ_∞ in E. Only partial results are known. For instance, if E is WCG, then $L_0(\lambda, E)$ cannot contain ℓ_∞.

The following result corresponds to Theorem 1.4 (i.e., Theorem 5.8 in [DL2]) and generalizes [L2, Thm. 2.12 A]; it can be considered as a vector-valued Lozanovskii type theorem.

Theorem 9.6. *Let $L_0(\lambda, E)$ contain no copy of ℓ_∞ and let X be a quasi-complete TVS of E-valued λ-measurable functions. Also assume that*

(t) *for every $f \in X$ and disjoint sequence of sets (A_n) in Σ, the measure $\mathbf{m}\colon \mathcal{P}(\mathbb{N}) \to X$ defined by $\mathbf{m}(N) = 1_{A(N)} f$, $A(N) = \bigcup_{n \in N} A_n$, is convexly bounded.*

Then the following conditions are equivalent.

(a) *X contains no copy of ℓ_∞.*
(b) *X contains no disjointly supported copy of ℓ_∞.*
(c) *X is sequentially μ-continuous.*
(d) *Every bounded finitely additive measure $\mathbf{m}\colon \mathcal{P}(\mathbb{N}) \to X$ is exhaustive.*

Proof. The implications (d) $\Longrightarrow$ (a) $\Longrightarrow$ (b) are obvious.

(b) $\Longrightarrow$ (c): Suppose X is not sequentially μ-continuous. Then there exist $f \in X$ and a disjoint sequence $(A_n) \subset \Sigma$ with union S such that the series $\sum_n 1_{A_n} f$ is not convergent to f. Since X is sequentially complete and continuously included in $L_0(\lambda, E)$, this series is not Cauchy. Clearly, we may assume that all the functions $f_n = 1_{A_n} f$ are outside a neighborhood V of zero in X. Now, using the sequential completeness of X and assumption (t), we may define a continuous linear operator $T\colon \ell_\infty \to X$ by $T(a) = \int_{\mathbb{N}} a\, d\mathbf{m}$, $a = (a_n) \in \ell_\infty$. By applying the projections P_{A_n} it is easily seen that $T(a) = \sum_n a_n f_n$ (pointwise sum). Obviously, T is disjointness preserving. Since $T(e_n) = f_n \notin V$ for every n, we get a contradiction with (b) by applying the generalized Rosenthal ℓ_∞-theorem proved in [D3] (see [DL2, Thm. 1.2]).

(c) $\Longrightarrow$ (d): By Proposition 9.5, $\mathbf{m}$ is exhaustive in the topology induced from $L_0(\lambda, E)$. Since X is quasi-complete and sequentially μ-continuous, it is μ-continuous. By Theorem 8.4, $\mathbf{m}$ is exhaustive for the original topology of X. □

Remark. A somewhat stronger form of condition (t) is the following condition:

(t′) *For every $f \in X$ and every disjoint sequence (A_n) in Σ, the set of all pointwise sums $\sum_n c_n 1_{A_n} f$, where $|c_n| \leqslant 1$, is contained in X and bounded there.*

For sequentially complete spaces, both forms are equivalent. Also note that, by the closed graph theorem, if X is an F-space, then the phrase '*and bounded there*' can be omitted in (t′).

As in the case of (s), condition (t) is automatically satisfied in the presence of local convexity or in TRS's. In the latter case, it is the property of solidness of the space and local solidness of its topology that causes (t) to be satisfied. For a general TVS $X \subset L_0(\mu, E)$, condition (t) can be achieved by imposing the following property on X: For each $\varphi \in L_\infty(\mu)$, the operator M_φ defined by $M_\varphi(f) = \varphi f$ maps X into itself, and the family M_φ, where $\|\varphi\|_\infty \leqslant 1$, is equicontinuous. Of course, this property is much stronger than that of being Σ-solid (cf. Proposition 5.5 (b)).

10. Property (O) and copies of c_0 and ℓ_∞ in the TVS $L(E)$

Below $E = (E, \|\cdot\|_E)$ is a Banach space and $L = (L, \tau)$ is a TRS of λ-measurable functions continuously included in $L_0(\lambda)$ with the measure space (S, Σ, λ) as in Section 9.

We first gather together a few facts about spaces $L(E)$ that are easy consequences of the way they were defined and topologized.

Proposition 10.1. *Let L be a TRS of μ-measurable functions.*

(a) *If L is sequentially μ-continuous, or μ-continuous, or projectively μ-continuous, so is, respectively, $L(E)$.*
(b) *If L is (sequentially) boundedly closed, so is $L(E)$.*
(c) *If L has the disjoint Property (O), so does $L(E)$.*
(d) *$L(E)$ is piecewise uniformly closed in $L_0(\mu, E)$.*
(e) *$L(E)$ satisfies conditions* (s) *and* (t).

THEOREM 10.2. *Assume E contains no copy of c_0, L is Lebesgue, and either*

(a) *L is σ-Levi and $L(E)$ is sequentially complete, or*
(b) *λ is of type (SC) and L is boundedly sequentially closed in $L_0(\lambda)$.*

Then $L(E)$ has Property (O).

Proof. (a): As L is Lebesgue and sequentially complete (because so is $L(E)$), by Propositions 5.1 and 5.9 the conditions 'every fundamental band in L is boundedly closed in $L_0(\lambda)$' and 'L is σ-Levi' are equivalent. Now apply Theorem 9.2 (a).

(b): This is a direct consequence of Theorem 9.2 (b). □

In view of Proposition 10.1 (c) and (d), our next result is immediate from Corollary 9.3.

COROLLARY 10.3. *Let E contain no copy of c_0, and $L(E)$ be projectively complete. If L has the disjoint Property (O), then $L(E)$ has Property (O).*

Finally, note that if $L(E)$ is quasi-complete, so is L; in particular, L has the MCP. Combining this with Theorems 1.3 and 1.4, and taking also Proposition 10.1 (e) into account, we derive the following two results from Theorems 9.4 and 9.6.

THEOREM 10.4. *Assume that E contains no copy of c_0, and let $L(E)$ be quasi-complete. Then the following conditions are equivalent.*

(a) *$L(E)$ contains no copy of c_0.*
(b) *$L(E)$ contains no disjointly supported copy of c_0.*
(c) *L has the disjoint Property (O).*
($\underline{\mathrm{c}}$) *L is σ-Lebesgue and σ-Levi.*
($\overline{\mathrm{c}}$) *L contains no lattice copy of c_0.*
(d) *$L(E)$ has Property (O).*

THEOREM 10.5. *Assume that $L_0(\lambda, E)$ contains no copy of ℓ_∞, and let $L(E)$ be quasi-complete. Then the following conditions are equivalent.*

(a) *$L(E)$ contains no copy of ℓ_∞.*
(b) *$L(E)$ contains no disjointly supported copy of ℓ_∞.*
(c) *L is σ-Lebesgue.*

($\overline{\mathrm{c}}$) *L contains no lattice copy of ℓ_∞.*
(d) *Every bounded finitely additive measure $\mathbf{m}\colon \mathcal{P}(\mathbb{N}) \to L(E)$ is exhaustive.*

The following result is a significant generalization of a result of Emmanuele [E] for Banach spaces $L(E)$. It should also be stressed that the Köthe (or Banach) function spaces L used in [E] are far more restrictive than their 'counterparts' here. Emmanuele's theorem was, in turn, an extension of a result of Mendoza [M] (see also [CM, Thm. 4.2.1]) for spaces $L_p(\lambda, E)$, $1 \leqslant p < \infty$. In fact, Emmanuele's proof was a reduction to Mendoza's result. A similar reduction is achieved in the proof below.

THEOREM 10.6. *Assume that a TRS L of μ-measurable functions is complete and has a separating continuous dual L'. If $L(E)$ contains a copy of ℓ_∞, then either L or E contains a copy of ℓ_∞.*

Proof. Let $T\colon \ell_\infty \to L(E)$ be an isomorphic embedding, and $\mathbf{m}\colon \mathcal{P}(\mathbb{N}) \to L(E)$ the associated measure; that is, $\mathbf{m}(N) = T(1_N)$ for $N \subset \mathbb{N}$. Clearly, $\mathbf{m}$ is not exhaustive. Suppose L does not contain a copy of ℓ_∞. Then, by Theorem 1.2, L is Lebesgue, or μ-continuous. Consequently, by Proposition 10.1 (a), $L(E)$ is also μ-continuous. From this, using Proposition 8.1 (b), it follows that there is $A \in \Sigma'$ such that the measure $P_A{\circ}\mathbf{m}$ is not exhaustive. Next, choose a sequence (A_n) in $\Sigma_{\mathrm{oc}}(\mu)$ so that $A_n \uparrow A$. Then, applying the Brooks-Jewett theorem (see, e.g., [D1]) to the measures $P_{A_n}{\circ}\mathbf{m}$, we find $B = A_k$ such that the measure $P_B{\circ}\mathbf{m}$ is not exhaustive. Clearly, we may assume that the support of the band L_B is equal to B.

Consider any $0 < x' \in L'_B$. Clearly, the representing measure $\varphi\colon A \to x'(1_A)$ is (σ-additive and) μ-continuous ($\varphi \ll \mu$) on $\Sigma_B = \Sigma \cap B$. By the Lebesgue decomposition, there is $C \in \Sigma_B$ such that $\mu(C) > 0$ and $\varphi(D) > 0$ iff $\mu(D) > 0$ for all $D \in \Sigma_C$. Note that then $x'(f) > 0$ for all $0 < f \in L_C$. Now, let $\mathcal{C}$ be a maximal disjoint family of non-μ-null sets C in Σ_B such that there is $0 < x'_C \in L'_C$ with the property that $x'_C(f) > 0$ whenever $0 < f \in L_C$. Since μ is o.c. on Σ_B, $\mathcal{C}$ is countable, say $\mathcal{C} = \{C_1, C_2, \dots\}$. Moreover, $C_0 = \bigcup \mathcal{C}$ equals B μ-a.e. Otherwise, the band $L_{B \smallsetminus C_0}$ would be nonzero with a nontrivial dual space, and the construction in the first part of this paragraph would lead to a contradiction with the maximality of $\mathcal{C}$. By a similar application of the Brooks-Jewett theorem as above, it can be shown that for some $D = C_k$ the measure $P_D{\circ}\mathbf{m}$ is not exhaustive.

Without loss of generality it can be assumed that $f_n = P_D\big(\mathbf{m}(\{n\})\big) \nrightarrow 0$. Consider the continuous linear operator $P_D T\colon \ell_\infty \to L_D(E)$. Since $P_B T(e_n) = f_n \nrightarrow 0$, by [D3] there is an infinite subset M of $\mathbb{N}$ such that $P_D T|\ell_\infty(M)$ is an isomorphic embedding of $\ell_\infty(M) \cong \ell_\infty$ into $L_D(E)$.

As a result of the above reduction process, we may, therefore, assume that the submeasure μ is o.c. and that there is $0 < x' \in L'$ such that $x'(f) > 0$ whenever $0 < f \in L$. Then the measure λ on Σ defined by $\lambda(A) = x'(1_A)$ is equivalent to μ. Consequently, $L_0(\mu, E) = L_0(\lambda, E)$. Moreover, the formula $\|f\|_1 = x'(|f|) = \int_S |f| \, d\lambda$ defines a continuous L_1-norm on L. This means that we have a continuous inclusion $L \subset L_1(\lambda)$. Hence we also have $L(E) \subset L_1(\lambda, E)$ continuously. Now observe that the measure $\mathbf{m}\colon \mathcal{P}(\mathbb{N}) \to L_1(\lambda, E)$ is not exhaustive; otherwise, by Corollary 8.5 (b), $\mathbf{m}\colon \mathcal{P}(\mathbb{N}) \to L(E)$ would be exhaustive, which is not the case. Therefore, by applying [D3] to the operator $T\colon \ell_\infty \to L_1(\lambda, E)$, we can now produce an isomorphic embedding of ℓ_∞ into $L_1(\lambda, E)$. To finish, apply a result of Mendoza [M] to conclude that E contains an isomorphic copy of ℓ_∞. $\square$

Remark. It would be desirable to replace '$L_0(\lambda, E)$ contains no copy of ℓ_∞' with 'E contains no copy of ℓ_∞' in Theorem 10.5. However, as was already mentioned, whether these two conditions are equivalent is still an open question. Nonetheless, such an improvement is possible if the dual space L' is separating. This should be clear by inspecting the proof of Theorem 9.6 for the case $X = L(E)$, and changing the proof of the implication (c) $\Longrightarrow$ (d) therein by an almost verbatim repetition of the proof given above. Only after arriving at a nonexhaustive bounded measure $\mathbf{m}\colon \mathcal{P}(\mathbb{N}) \to L_1(\lambda, E)$ a slight change of argument would be needed: At that point, define an operator $T\colon \ell_\infty \to L_1(E)$ by $T(a) = \int_{\mathbb{N}} a \, d\mathbf{m}$, and next apply [D3] and [M] to get a contradiction.

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