

## DECOMPOSITION THEOREMS FOR HARDY SPACES ON CONVEX DOMAINS OF FINITE TYPE

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ABSTRACT. In this paper we study the holomorphic Hardy space  $\mathcal{H}^p(\Omega)$ , where  $\Omega$  is a smoothly bounded convex domain of finite type in  $\mathbb{C}^n$ . We show that for  $0 < p \leq 1$ ,  $\mathcal{H}^p(\Omega)$  admits an atomic decomposition. More precisely, we prove that each  $f \in \mathcal{H}^p(\Omega)$  can be written as  $f = P_S(\sum_{j=0}^{\infty} \nu_j a_j) = \sum_{j=0}^{\infty} \nu_j P_S(a_j)$ , where  $P_S$  is the Szegő projection, the  $a_j$ 's are real variable  $p$ -atoms on the boundary  $\partial\Omega$ , and the coefficients  $\nu_j$  satisfy the condition  $\sum_{j=0}^{\infty} |\nu_j|^p \lesssim \|f\|_{\mathcal{H}^p(\Omega)}^p$ . Moreover, we prove the following factorization theorem. Each  $f \in \mathcal{H}^p(\Omega)$  can be written as  $f = \sum_{j=0}^{\infty} f_j g_j$ , where  $f_j \in \mathcal{H}^{2p}$ ,  $g_j \in \mathcal{H}^{2p}$ , and  $\sum_{j=0}^{\infty} \|f_j\|_{\mathcal{H}^{2p}} \|g_j\|_{\mathcal{H}^{2p}} \lesssim \|f\|_{\mathcal{H}^p(\Omega)}$ . Finally, we extend these theorems to a class of domains of finite type that includes the strongly pseudoconvex domains and the convex domains of finite type.

### Introduction

Let  $\Omega$  be a smoothly bounded domain in  $\mathbb{C}^n$ . For  $0 < p \leq \infty$  let  $L^p(\Omega)$  denote the Lebesgue space with respect to the volume form and  $L^p(\partial\Omega)$  the Lebesgue space on  $\partial\Omega$  with respect to the induced surface measure  $d\sigma$ . For  $0 < p \leq 1$  let  $H^p(\partial\Omega)$  be the real-variable Hardy space on  $\partial\Omega$ .

We let  $\mathcal{H}^p(\Omega)$  denote the Hardy space of holomorphic functions on  $\Omega$ , with norm given by

$$\|f\|_{\mathcal{H}^p(\Omega)}^p := \sup_{0 < \varepsilon < \varepsilon_0} \int_{\delta(w)=\varepsilon} |f(w)|^p d\sigma_\varepsilon(w),$$

where  $\delta(w)$  is the distance from  $w$  to  $\partial\Omega$  and  $d\sigma_\varepsilon$  denotes the surface measure on the manifold  $\{\delta(w) = \varepsilon\}$ . To any function  $f \in \mathcal{H}^p(\Omega)$  corresponds a unique boundary function in  $L^p(\partial\Omega)$ , which we also denote by  $f$ , obtained as normal almost everywhere limit [St1]. Thus, we may identify  $\mathcal{H}^p(\Omega)$  with a closed subspace of  $L^p(\partial\Omega)$ .

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The Hilbert space orthogonal projection  $P_S$  of  $L^2(\partial\Omega)$  onto  $\mathcal{H}^2(\Omega)$  is given by the Szegő projection

$$P_S f(z) = \int_{\partial\Omega} f(\zeta) S_\Omega(z, \zeta) d\sigma(\zeta),$$

where  $S_\Omega(z, \zeta)$  is the Szegő kernel.

When  $\Omega$  is a smoothly bounded convex domain of finite type, there exists a natural pseudo-distance  $d_b$  on  $\partial\Omega$  that makes  $\partial\Omega$  into a space of homogeneous type (see [Mc]). The real-variable Hardy space  $H^p(\partial\Omega)$ ,  $0 < p \leq 1$ , is defined as a space of distributions on  $\partial\Omega$ , in terms of atoms, in the following sense. Each distribution  $f \in H^p(\partial\Omega)$  can be written as  $f = \sum_{j=0}^{\infty} \nu_j a_j$ , where  $\{\nu_j\} \in \ell^p$ , the  $a_j$ 's are  $p$ -atoms and the series is assumed to converge in the sense of distributions (see Section 1 for the precise definition).

In this paper we prove that, for  $0 < p \leq 1$ , the Hardy space  $\mathcal{H}^p(\Omega)$  continuously embeds into  $H^p(\partial\Omega)$ . In other words, every function  $f \in \mathcal{H}^p(\Omega)$  has boundary values that belong to  $H^p(\partial\Omega)$ , so that it admits an atomic decomposition.

To be more precise, for  $g$  a distribution on  $\partial\Omega$ ,  $P_S(g)$  is the holomorphic function in  $\Omega$  defined for  $z \in \Omega$  by

$$P_S(g)(z) = \left\langle g, \overline{S_\Omega(z, \cdot)} \right\rangle,$$

where  $S_\Omega(z, \zeta)$  denotes the Szegő kernel, which is  $C^\infty(\partial\Omega)$  in the second variable whenever  $z \in \Omega$ .

We prove that any  $f \in \mathcal{H}^p(\Omega)$  can be written as  $P_S(\sum \nu_j a_j)$ , where  $\sum \nu_j a_j$  is a distribution that belongs to  $H^p(\partial\Omega)$ . Moreover, we have  $P_S(\sum \nu_j a_j) = \sum \nu_j P_S(a_j)$  in the  $\mathcal{H}^p(\Omega)$ -sense, that is, the series converges to  $f$  in the  $\mathcal{H}^p(\Omega)$ -norm.

On the other hand, we prove that the Szegő projection  $P_S$  maps continuously  $H^p(\partial\Omega)$  into  $\mathcal{H}^p(\Omega)$ , for  $0 < p \leq 1$ .

Moreover, we prove that on  $\mathcal{H}^p(\Omega)$  a (weak) factorization theorem holds. More precisely, we show that, given any function  $f \in \mathcal{H}^p(\Omega)$ , there exist  $f_j, g_j \in \mathcal{H}^{2p}(\Omega)$  such that  $f = \sum_{j=0}^{\infty} f_j g_j$  and  $\sum_{j=0}^{\infty} \|f_j\|_{\mathcal{H}^{2p}} \|g_j\|_{\mathcal{H}^{2p}} \leq c \|f\|_{\mathcal{H}^p(\Omega)}$ , with  $c$  independent of  $f$ .

Finally, in Section 8 we extend the above results to so-called  $H$ -domains, a class of smooth, bounded domains of finite type that includes the strongly pseudoconvex domains and the convex domains of finite type. Such domains are a natural extension of the mentioned domains and were studied in [BPS3].

These results extend classical results to the case of convex domains of finite type, and more generally to the case of  $H$ -domains. The atomic decomposition of the holomorphic Hardy spaces was first proved in the case of the unit ball by Garnett and Latter [GL], and later extended to strongly pseudoconvex domains and domains of finite type in  $\mathbb{C}^2$  by Krantz and Li [KL2], and independently by Dafni [D] in the case of strongly pseudoconvex domains. Related

results about duality between  $\mathcal{H}^1$  and  $BMO$  appear in [KL1] for strongly pseudoconvex domains and domains of finite type in  $\mathbb{C}^2$ , and in [KL3] for convex domains of finite type. The factorization theorem is classical in dimension 1, while in several variables it was first proved by Coifman, Rochberg and Weiss [CRW] in the case of the unit ball for the space  $\mathcal{H}^1(\Omega)$ . In [KL2] this theorem was extended to the case of strongly pseudoconvex domains for all spaces  $\mathcal{H}^p(\Omega)$ ,  $0 < p \leq 1$ , and in [BPS2] to convex domains of finite type for  $\mathcal{H}^1(\Omega)$ .

Applications of these results to the regularity properties of small Hankel operators have been given in [CRW], [KL2] and [BPS2], just to name a few. Another classical characterization of Hardy spaces, in terms of the area integral, was given in [KL4] for the case of convex domains of finite type in  $\mathbb{C}^n$ . Finally, we mention that the optimal approach regions for a Fatou-type theorem for  $\mathcal{H}^p$  functions on a convex domain of finite type have been described in [DFi].

The geometry of convex domains of finite type was first described by McNeal [Mc]. This description was later applied by McNeal and Stein to the analysis of the mapping properties of the Bergman projection [McS1] and the Szegő projection [McS2]. In the case of strongly pseudoconvex domains and finite type domains in  $\mathbb{C}^2$  the geometry was determined by canonical vector fields. The natural quasi-distance on  $\partial\Omega$  was the control distance determined by these vector fields, i.e., the Carnot metric. The situation of convex domains of finite type is essentially more general. The “weight” of each vector field may vary from point to point, and one needs to take into consideration the different order of contact of complex lines with  $\partial\Omega$ . For these reasons it is natural to consider a *diameter function*  $\tau(\zeta, \lambda, r)$ , which gives the diameter of the largest one-dimensional disc in the direction of  $\lambda$  with center at  $\zeta$  that fits inside the region  $\{z' : \varrho(z') < r\}$ . Here  $\varrho$  denotes a fixed smooth defining function for  $\Omega$ .

As a consequence, in order to define the cancellation property for  $p$ -atoms for small values of  $p$ , we need to consider the pairing between functions  $f \in \mathcal{H}^p(\Omega)$  and smooth bump functions whose derivatives in *all tangential directions* can be controlled in terms of the diameter function.

We remark that, although the main lines of the proof of the atomic decomposition are standard, some of the classical arguments have been greatly simplified. On the other hand, in order to prove the factorization theorem we adapt an idea from [BPS2], where the result was proved in the case  $p = 1$ . The proof relies neither on the explicit expression of the Szegő kernel, as in [CRW], nor on its asymptotic expansion, as in [KL4]. Instead, it is based on a recent result by Diederich and Fornæss on the existence of support functions on convex domains of finite type.

We use the notation  $A \lesssim B$  to indicate that  $A \leq c \cdot B$ , where the constant  $c$  does not depend on the important parameters on which the functions  $A$  and

$B$  depend. (Typically, the constant  $c$  will only depend on the geometry of the domain  $\Omega$ .) We use the symbols  $\gtrsim$  and  $\approx$  with similar, obvious meanings.

### 1. Basic facts and notation

Let  $\Omega$  be a smoothly bounded convex domain in  $\mathbb{C}^n$ . A point  $\zeta \in \partial\Omega$  is said to be of finite type if the order of contact of complex lines with  $\partial\Omega$  at this point is finite (see [BS] and the references therein). The type of the point is the least upper bound of the various orders of contact. We say that  $\Omega$  is of finite type  $M_\Omega$  if every point on  $\partial\Omega$  is of finite type  $\leq M_\Omega$ .

Let  $\Omega = \{z \in \mathbb{C}^n : \varrho(z) < 0\}$ . There exists  $\varepsilon_0 > 0$  such that for  $|\varepsilon| \leq \varepsilon_0$  the sets  $\Omega_\varepsilon = \{z \in \mathbb{C}^n : \varrho(z) < \varepsilon\}$  are all convex and the normal projection  $\pi: \overline{U} \rightarrow \partial\Omega$  is well defined and smooth, where  $U = \{z \in \mathbb{C}^n : \delta(z) < \varepsilon_0\}$ . The basic geometric facts about convex domains of finite type were first proved by McNeal [Mc]; see also [McS1], [McS2], and [DFo]. We recall the results needed for the present work and take this opportunity to review the main elements of the construction and to set some notation.

For  $z \in U$  and  $\lambda \in \mathbb{C}^n$  a unit vector, we denote by  $\tau(z, \lambda, r)$  the distance from  $z$  to the surface  $\{z' : \varrho(z') = \varrho(z) + r\}$  along the complex line determined by  $\lambda$ .

For each  $z \in U$  and  $r < \varepsilon_0$  there exists a special set of coordinates  $\{w_1^{z,r}, \dots, w_n^{z,r}\}$ , which we call *r-extremal*. The first vector  $v^{(1)}$  is given by the direction transversal to the boundary, in the sense that the shortest distance from  $z$  to the set  $\{z' : \varrho(z') = \varrho(z) + r\}$  is attained in the complex line determined by  $v^{(1)}$ .

The vector  $v^{(2)}$  is chosen among the vectors orthogonal to  $v^{(1)}$  in such a way that  $\tau(z, v^{(2)}, r)$  is maximal. We repeat this process until we obtain an orthonormal basis  $\{v^{(1)}, \dots, v^{(n)}\}$ . We denote by  $(w_1, \dots, w_n)$  the coordinates with respect to this basis. Notice that these coordinates  $(w_1, \dots, w_n) = (w_1^{z,r}, \dots, w_n^{z,r})$  depend on  $z$  and  $r$ . However, the transversal direction  $w_1$  does not depend on  $r$ .

For  $k = 1, \dots, n$ , we set

$$(1.1) \quad \tau_k(z, r) = \tau(z, v^{(k)}, r)$$

and define the polydisc

$$(1.2) \quad Q(z, r) = \{w : |w_k| < \tau_k(z, r), k = 1, \dots, n\}.$$

The basic relations among these quantities are given in the following proposition; see [McS2, Prop. 1.1] and also [BPS2, Lemma 2.1].

**PROPOSITION 1.1.** *There exists a constant  $C > 0$  depending only on  $\Omega$  such that for any unit vector  $\lambda \in \mathbb{C}^n$ ,  $0 < r \leq \varepsilon_0$ ,  $z \in U$ , and  $0 < \eta < 1$  we have:*

$$(i) \quad \eta^{1/2} \tau(z, \lambda, r) \lesssim \tau(z, \lambda, \eta r) \lesssim \eta^{1/M_\Omega} \tau(z, \lambda, r);$$

- (ii)  $\eta^{1/2}Q(z, C^{-1}r) \subset Q(z, \eta r) \subset \eta^{1/M_\Omega}Q(z, Cr)$ ;
- (iii) if  $w \in Q(z, \delta)$  then  $\tau(z, \lambda, r) \approx \tau(w, \lambda, r)$ .

We define the quasi-distance  $d_b : U \times U$  by

$$(1.3) \quad d_b(z, w) = \inf\{\delta : w \in Q(z, \delta)\}$$

and set

$$(1.4) \quad d(z, w) = d_b(z, w) + \delta(z) + \delta(w).$$

Notice that  $d$  is initially defined on  $U \times U$ . We extend this function to  $\mathbb{C}^n \times \mathbb{C}^n$  by setting

$$d(z, w) = \psi(\varrho(z))\psi(\varrho(w))d(z, w) + (1 - \psi(\varrho(z)))(1 - \psi(\varrho(w)))|z - w|,$$

where  $\psi$  is a smooth cut-off function on  $\mathbb{R}$  such that  $\psi(t) = 1$  for  $|t| \leq \varepsilon_0/2$  and  $\psi(t) = 0$  for  $|t| \geq \varepsilon_0$ .

On the boundary we will use a family of “balls” centered at  $\zeta \in \partial\Omega$  and of radius  $\delta$ , defined as

$$B(\zeta, \delta) = Q(\zeta, \delta) \cap \partial\Omega.$$

For any unit vector  $\lambda$  we introduce the differential operator

$$(1.5) \quad L_\lambda = (\partial_\lambda \varrho)\partial_{x_1} - (\partial_{x_1} \varrho)\partial_\lambda,$$

where  $w_1 = x_1 + iy_1$  is the transversal direction fixed earlier. Here  $\partial_\lambda$  is the standard vector field defined by  $\lambda$  as  $\partial_\lambda f = \langle \lambda, df \rangle$ , for the real differential  $df$  of a smooth function  $f$ , where  $\langle \cdot, \cdot \rangle$  denotes the usual pairing between a one-form and a vector.

Notice that  $L_\lambda$  is always a tangential vector field. If  $\lambda \in S^{2n-1}$  is itself tangent to  $\partial\Omega$ , then  $L_\lambda$  is the directional derivative in the direction  $\lambda$ .

For  $\Lambda = (\lambda_1, \dots, \lambda_q)$  a  $q$ -list of vectors in  $S^{2n-1}$  and  $\mu = (\mu_1, \dots, \mu_q)$  a  $q$ -index we set  $|\mu| = \mu_1 + \dots + \mu_q$ ,

$$(1.6) \quad L_\Lambda^\mu = L_{\lambda_1}^{\mu_1} \dots L_{\lambda_q}^{\mu_q},$$

and

$$(1.7) \quad \tau^\mu(z, \Lambda, \delta) = \tau(z, \lambda_1, \delta)^{\mu_1} \dots \tau(z, \lambda_q, \delta)^{\mu_q}.$$

We recall the fundamental estimates for the Szegő kernel and its derivatives called interior estimates of  $S$ -type (see Definition 4 and Theorem 3.6 in [McS2]):

$$(1.8) \quad \left| L_{\Lambda, z}^\mu L_{\Lambda', z'}^{\mu'} S_\Omega(z, z') \right| \lesssim \frac{\tau^{-\mu}(z, \Lambda, \delta) \tau^{-\mu'}(z', \Lambda', \delta)}{\sigma(B(\pi(z), \delta))},$$

where  $\delta = d(z, z')$ ,  $z, z' \in \bar{\Omega} \times \bar{\Omega} \setminus \Delta_{\partial\Omega}$ , with  $\Delta_{\partial\Omega}$  denoting the diagonal on  $\partial\Omega$ .

We conclude this section by giving the definition of the real-variable Hardy spaces. We first introduce the notion of  $p$ -atoms. Let  $\zeta_0 \in \partial\Omega$ ,  $r_0 < \varepsilon_0$ , and let  $N$  be a positive integer. On  $\mathcal{C}^\infty(B(\zeta_0, r_0))$  we introduce the norm

$$(1.9) \quad \|\phi\|_{\mathcal{S}_N(B(\zeta_0, r_0))} = \sup_{\Lambda} \sum_{|\mu|=N} \|L_{\Lambda}^{\mu} \phi\|_{L^{\infty}(B(\zeta_0, r_0))} \tau^{\mu}(\zeta_0, \Lambda, r_0).$$

DEFINITION 1.2. We set

$$\ell_0 = [(1 - 1/p)(M_{\Omega} + 2n - 2)], \quad N_p = \ell_0 + 1,$$

where  $[x]$  denotes the integral part of  $x$ . A measurable function  $a$  on  $\partial\Omega$  is called a  $p$ -atom if it is either the constant function on  $\partial\Omega$  equal to  $\sigma(\partial\Omega)^{-1/p}$ , or satisfies the following conditions:

- (i)  $\text{supp } a \subseteq B(\zeta_0, r_0)$ ;
- (ii)  $|a(\zeta)| \leq \sigma(B(\zeta_0, r_0))^{-1/p}$ ;
- (iii)  $\int_{\partial\Omega} a(\zeta) d\sigma(\zeta) = 0$ ;
- (iv) for all  $\phi \in \mathcal{C}^\infty(B(\zeta_0, r_0))$  we have

$$\left| \int_{\partial\Omega} a(\zeta) \phi(\zeta) d\sigma(\zeta) \right| \leq \|\phi\|_{\mathcal{S}_{N_p}(B(\zeta_0, r_0))} \sigma(B(\zeta_0, r_0))^{1-1/p}.$$

Notice that condition (iv) replaces the classical *higher moment* condition and is in the same spirit as the analogous condition in [KL2]. The difference here lies in the choice of the norm  $\|\cdot\|_{\mathcal{S}_{N_p}}$ .

**Real-variable Hardy spaces.** Let  $0 < p \leq 1$ . The real Hardy space  $H^p(\partial\Omega)$  is the space of distributions  $f$  on  $\partial\Omega$  which can be written as

$$(1.10) \quad f = \sum_{j=0}^{\infty} \nu_j a_j,$$

where  $\sum |\nu_j|^p < \infty$ , the  $a_j$ 's are  $p$ -atoms, and the series is assumed to converge in the sense of distributions.

With a standard abuse of notation, the “norm” on  $H^p(\partial\Omega)$  is defined as

$$\|f\|_{H^p}^p = \inf \left\{ \sum_j |\nu_j|^p : f = \sum_j \nu_j a_j \right\}.$$

Setting  $d(f, g) = \|f - g\|_{H^p}^p$ , we see that  $H^p(\partial\Omega)$  is a complete metric space. This implies that the series in (1.10) converges in *norm*. This is in fact obvious since  $\|\sum_{j=m_1}^{m_2} \nu_j a_j\|_{H^p}^p \leq \sum_{j=m_1}^{m_2} |\nu_j|^p$ , which tends to 0 as  $m_1 \rightarrow \infty$ . This

implies that  $\sum_{j=0}^{\infty} \nu_j a_j$  necessarily converges in norm, and hence in the sense of distributions, to  $f$ .<sup>1</sup>

We point out that this definition of  $H^p(\partial\Omega)$  is consistent with the definition given in [CW] in the case of a space of homogeneous type for values of  $p$  close to 1; see also [KL5].

### 2. Statement of the main results

The main results of the present work are the following.

**THEOREM 2.1.** *Let  $\Omega$  be a smoothly bounded convex domain of finite type. Let  $0 < p \leq 1$ . Then there exists a constant  $c$  depending only on  $p$  and  $\Omega$  such that the following holds. Given any  $f \in \mathcal{H}^p(\Omega)$  there exist constants  $\nu_j$  and  $p$ -atoms  $a_j$  such that  $\sum_{j=0}^{\infty} \nu_j a_j \in H^p(\partial\Omega)$  and*

$$f = P_S \left( \sum_{j=0}^{\infty} \nu_j a_j \right) = \sum_{j=0}^{\infty} \nu_j P_S(a_j),$$

and moreover

$$\sum_{j=0}^{\infty} |\nu_j|^p \leq c \|f\|_{\mathcal{H}^p(\Omega)}^p.$$

**THEOREM 2.2.** *Let  $\Omega$  be a smoothly bounded convex domain of finite type. Let  $0 < p \leq 1$  and let  $H^p(\partial\Omega)$  be the real-variable Hardy space on  $\partial\Omega$ . Then*

$$P_S : H^p(\partial\Omega) \rightarrow \mathcal{H}^p(\Omega)$$

is bounded.

Finally, we have the following factorization theorem.

**THEOREM 2.3.** *Let  $\Omega$  be a smoothly bounded convex domain of finite type. Let  $0 < p \leq 1$ ,  $1 < q < \infty$ , and let  $q'$  be the conjugate exponent. There exists a constant  $c$  depending only on  $p, q$  and  $\Omega$  such that the following holds. Given any  $f \in \mathcal{H}^p(\Omega)$  there exist  $f_j \in \mathcal{H}^{pq}$ ,  $g_j \in \mathcal{H}^{pq'}$ ,  $j = 1, 2, \dots$ , such that*

$$f = \sum_{j=0}^{\infty} f_j g_j$$

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<sup>1</sup>In order to clarify one point on which there is a little bit of confusion in the literature, we remark that in [MTW, p. 513] it was shown that a generic  $H^p$ -function  $f$ , which is an infinite sum of atoms, cannot be written as a finite sum of atoms  $\sum_{j=1}^N \nu_j b_j$ , with  $\sum_{j=1}^N |\nu_j|^p \approx \|f\|_{H^p}^p$ . Of course, this fact does not contradict the norm convergence of the series in (1.10).

and

$$\sum_{j=0}^{\infty} \|f_j\|_{\mathcal{H}^{pq}} \|g_j\|_{\mathcal{H}^{pq'}} \leq c \|f\|_{\mathcal{H}^p(\Omega)}.$$

We remark that this theorem was proved in [BPS2] for  $p = 1$  by a different, more indirect, method. Our proof relies on the atomic decomposition obtained in Theorem 2.1.

The above theorems are valid on a class of smoothly bounded domains of finite type that includes the convex domains as well as the strongly pseudoconvex domains. These domains were introduced in [BPS3] and are called  $H$ -domains. In the final section of this paper, we discuss the extension of these theorems to  $H$ -domains.

### 3. Proof of Theorem 2.2

Let  $f = \sum_{j=0}^{\infty} \nu_j a_j \in H^p(\partial\Omega)$ . By definition,

$$\begin{aligned} P_S \left( \sum_{j=0}^{\infty} \nu_j a_j \right) (z) &= \left\langle \sum_{j=0}^{\infty} \nu_j a_j, \overline{S_{\Omega}(z, \cdot)} \right\rangle \\ &= \sum_{j=0}^{\infty} \left\langle \nu_j a_j, \overline{S_{\Omega}(z, \cdot)} \right\rangle \\ &= \sum_{j=0}^{\infty} \nu_j P_S(a_j)(z), \end{aligned}$$

since the series converges in the sense of distributions. It remains to prove that this last term belongs to  $\mathcal{H}^p(\Omega)$ , with norm controlled by  $\|f\|_{H^p(\partial\Omega)}$ .

We claim that there exists  $C > 0$  such that  $\|P_S(a)\|_{\mathcal{H}^p(\Omega)} \leq C$  for any  $p$ -atom  $a$ . From this it follows that, for any  $m_1, m_2 \in \mathbb{N}$  with  $m_1 \leq m_2$ ,

$$\left\| \sum_{m_1}^{m_2} \nu_j P_S(a_j) \right\|_{\mathcal{H}^p}^p \leq C^p \sum_{m_1}^{m_2} |\nu_j|^p.$$

Hence, by the assumption on  $\{\nu_j\}$  and the completeness of  $\mathcal{H}^p(\Omega)$ , one gets that  $P_S(f) = \sum_{j=0}^{\infty} \nu_j P_S(a_j)$  belongs to  $\mathcal{H}^p(\Omega)$ . Moreover,

$$\|P_S(f)\|_{\mathcal{H}^p}^p \lesssim \sum_j |\nu_j|^p$$

whenever  $f = \sum_j \nu_j a_j$ , which gives the desired estimate.



Thus, we only need to estimate  $\|P_S(a)\|_{\mathcal{H}^p}$  for any  $p$ -atom  $a$ . Let  $\text{supp } a \subseteq B(\zeta_0, \delta)$ , where  $\zeta_0$  and  $\delta = r_0$  are as in Definition 1.2. Let  $\varepsilon > 0$ . Then

$$\begin{aligned} \int_{\partial\Omega_\varepsilon} |P_S(a)(\zeta)|^p d\sigma_\varepsilon(\zeta) &= \int_{\partial\Omega} |P_S(a)(\zeta - \varepsilon\nu(\zeta))|^p d\sigma(\zeta) \\ &= \left( \int_{B(\zeta_0, 2\delta)} + \int_{eB(\zeta_0, 2\delta)} \right) |P_S(a)(\zeta_\varepsilon)|^p d\sigma(\zeta) \\ &= I + II, \end{aligned}$$

where  $\nu(\zeta)$  denotes the outward unit normal at  $\zeta \in \partial\Omega$  and  $\zeta_\varepsilon = \zeta - \varepsilon\nu(\zeta)$ . Since  $p < 2$ ,

$$\begin{aligned} I &\leq \left( \int_{B(\zeta_0, 2\delta)} |P_S(a)(\zeta_\varepsilon)|^2 d\sigma(\zeta) \right)^{p/2} \cdot \sigma(B(\zeta_0, 2\delta))^{1-p/2} \\ &\leq c \|a\|_{L^2(\partial\Omega)}^p \sigma(B(\zeta_0, 2\delta))^{1-p/2} \\ &\leq c, \end{aligned}$$

since  $\|a\|_{L^2(\partial\Omega)} \leq \sigma(B(\zeta_0, 2\delta))^{1/2-1/p}$  and  $P_S$  maps  $L^2(\partial\Omega_\varepsilon)$  into  $L^2(\partial\Omega)$  with norm independent of  $\varepsilon$ , as a consequence of the  $T(1)$ -theorem of David and Journé and of the results in [McS2].

Next, set  $E_k = \{\zeta \in \partial\Omega : 2^k\delta \leq d(\zeta, \zeta_0) \leq 2^{k+1}\delta\}$ . Then

$$II = \sum_{k=0}^{\infty} \int_{E_k} |P_S(a)(\zeta - \varepsilon\nu(\zeta))|^p d\sigma_\varepsilon(\zeta),$$

and

$$\begin{aligned} |P_S(a)(\zeta_\varepsilon)| &= \left| \int_{\partial\Omega} S_\Omega(\zeta_\varepsilon, w) a(w) d\sigma(w) \right| \\ &\leq \|S_\Omega(\zeta_\varepsilon, \cdot)\|_{\mathcal{S}_{N_p}(B(\zeta_0, \delta))} \cdot \sigma(B(\zeta_0, 2\delta))^{1-1/p}. \end{aligned}$$

Recall that the Szegő kernel satisfies the estimate (1.8), so that

$$|L_\Lambda^\mu S_\Omega(\zeta_\varepsilon, w)| \lesssim \frac{\tau^{-\mu}(w, \Lambda, d(\zeta_\varepsilon, w))}{\sigma(B(w, d(\zeta_\varepsilon, w)))},$$

and notice that, for  $\zeta \in E_k$  and  $w \in B(\zeta_0, \delta)$ ,

$$\begin{aligned} d(\zeta_\varepsilon, w) &= \varepsilon + d_b(\zeta, w) \geq \varepsilon + d_b(\zeta, \zeta_0) - d_b(\zeta_0, w) \\ &\geq \varepsilon + 2^k\delta - \delta \geq \delta 2^{k-1}. \end{aligned}$$

Therefore, for  $\zeta \in E_k$ ,

$$\begin{aligned} &\|S_\Omega(\zeta_\varepsilon, \cdot)\|_{\mathcal{S}_{N_p}(B(\zeta_0, \delta))} \\ &= \sup_\Lambda \sum_{|\mu|=N_p} \|L_\Lambda^\mu S_\Omega(\zeta_\varepsilon, \cdot)\|_{L^\infty(B(\zeta_0, \delta))} \cdot \tau^\mu(\zeta_0, \Lambda, \delta) \end{aligned}$$

$$\begin{aligned}
&= \sup_{\Lambda} \sum_{|\mu|=N_p} \sup_{w \in B(\zeta_0, \delta)} \frac{\tau^{-\mu}(w, \Lambda, d(\zeta_\varepsilon, w))}{\sigma(B(w, d(\zeta_\varepsilon, w)))} \cdot \tau^\mu(\zeta_0, \Lambda, \delta) \\
&\lesssim \sup_{\Lambda} \sum_{|\mu|=N_p} \sup_{w \in B(\zeta_0, \delta)} \frac{\tau^\mu(\zeta_0, \Lambda, \delta)}{\tau^\mu(w, \Lambda, \delta 2^{k-1})} \cdot \frac{1}{\sigma(B(w, \delta 2^{k-1}))} \\
&\lesssim \sup_{\Lambda} \sum_{|\mu|=N_p} \frac{\tau^\mu(\zeta_0, \Lambda, \delta)}{\tau^\mu(\zeta_0, \Lambda, \delta 2^{k-1})} \cdot \frac{1}{\sigma(B(w, \delta 2^{k-1}))} \\
&\lesssim \sum_{|\mu|=N_p} 2^{-k|\mu|/M_\Omega} \frac{1}{\sigma(B(w, \delta 2^{k-1}))}.
\end{aligned}$$

Returning to the estimation of  $II$ , we obtain

$$\begin{aligned}
II &= \sum_{k=1}^{\infty} \int_{E_k} |P_S(a)(\zeta_\varepsilon)|^p d\sigma_\varepsilon(\zeta) \\
&\lesssim \sum_k \|S_\Omega(\zeta_\varepsilon, \cdot)\|_{\mathcal{S}_{N_p}(B(\zeta_0, \delta))}^p \sigma(B(\zeta_0, \delta))^{p-1} \sigma(E_k) \\
&\lesssim \sum_k \sum_{|\mu|=N_p} 2^{-kp|\mu|/M_\Omega} \frac{\sigma(B(\zeta_0, \delta))^{p-1}}{\sigma(B(w, \delta 2^k))^{p-1}} \\
&\lesssim \sum_k \sum_{|\mu|=N_p} 2^{-k[(|\mu|/M_\Omega)p + (1 + (2n-2)/M_\Omega)(p-1)]} \\
&\leq c,
\end{aligned}$$

since the term in brackets in the exponent is positive, due to our choice of  $N_p$ .  $\square$

#### 4. Maximal functions and a partition of unity

In the proof of the atomic decomposition of  $\mathcal{H}^p(\Omega)$  we are going to use some maximal operators that now we introduce. These operators are standard variants of classical ones; see [FS], [St2], and also [KL2].

Given  $\zeta \in \partial\Omega$  we define the *approach region*  $\mathcal{A}_\gamma(\zeta)$  as the subset of  $\Omega$  given by

$$\mathcal{A}_\gamma(\zeta) = \{z \in \Omega : d(\zeta, \pi(z)) < \gamma\delta(z)\}.$$

We define the *non-tangential maximal function*

$$(4.1) \quad f_\gamma^*(\zeta) = \sup_{z \in \mathcal{A}(\zeta)} |f(z)|,$$

and the tangential variant

$$(4.2) \quad f_N^{**}(\zeta) = \sup_{w \in \Omega} \left( \frac{\delta(w)}{\delta(w) + d(\zeta, \pi(w))} \right)^N |f(w)|.$$

We consider a space of *smooth bump functions at  $\zeta$* , defined by

$$\mathcal{K}_\gamma^M(\zeta) = \{g \in C^\infty(\partial\Omega) : \text{supp } g \subseteq B(\zeta_0, t_0), \zeta_0 \in \mathcal{A}_\gamma(\zeta) \text{ and } \|g\|_{M, \zeta_0, t_0} \leq 1\},$$

where

$$(4.3) \quad \|g\|_{M, \zeta_0, t_0} = \sup_{\Lambda, |\mu| \leq M} \tau^\mu(\zeta_0, \Lambda, t_0) \sigma(B(\zeta_0, t_0)) \|L_\Lambda^\mu g\|_{L^\infty(B(\zeta_0, t_0))}.$$

Following [McS2, Def. 2], we say that a function  $\psi$  is a smooth bump function of order  $N$  on  $B(\zeta_0, t_0) \subseteq \partial\Omega$  if  $\psi \in C^\infty(B(\zeta_0, t_0))$  and

$$\sup_{\Lambda} |L_\Lambda^\mu \psi(z)| \tau^\mu(\zeta_0, \Lambda, t_0) \leq C_\psi,$$

for all  $z \in B(\zeta_0, t_0)$  and  $|\mu| \leq N$ . If  $C_\psi = 1$ ,  $\psi$  is called a *normalized* smooth bump function of order  $N$ .

The *grand maximal function* is defined as

$$(4.4) \quad K_{\gamma, M}(f)(\zeta) = \sup_{g \in \mathcal{K}_\gamma^M(\zeta)} \left| \int_{\partial\Omega} f(w)g(w) d\sigma(w) \right|.$$

LEMMA 4.1. *With the above definitions, there exist  $c = c(\Omega)$  and  $N = N(\gamma, M)$  such that*

$$K_{\gamma, M}f(\zeta) \lesssim f_{c\gamma}^*(\zeta) + f_N^{**}(\zeta).$$

*Proof.* We wish to estimate  $|\int_{\partial\Omega} f(w)g(w) d\sigma(w)|$  for  $g \in \mathcal{K}_\gamma^M(\zeta)$ . Given such a function  $g$ , there exist  $\zeta_0$  and  $t_0$  such that  $\text{supp } g \subseteq B(\zeta_0, t_0)$  and  $d(\zeta, \zeta_0) < \gamma t_0$ ; i.e.,  $\zeta_0 - t_0\nu(\zeta_0) \in \mathcal{A}_\gamma(\zeta)$ . By Lemma 6.5 in [BPS1] (which requires only the holomorphy of  $f$ ) and by integration by parts, for any positive integer  $k$  there exists a differential operator  $Y_{k+1}$  with smooth coefficients, of order  $k+1$ , such that

$$\left| \int_{\partial\Omega} f(w)g(w) d\sigma(w) \right| = \left| \int_{\Omega} f(w)Y_{k+1}\tilde{g}(w)\delta^k(w) dV(w) \right|.$$

Here  $\tilde{g}$  is a smooth extension of  $g$ , say

$$\tilde{g}(w) = \begin{cases} g(\pi(w))g_1(\varrho(w)) & \text{if } |\varrho(w)| < 2t_0, \\ 0 & \text{otherwise,} \end{cases}$$

and  $g_1 \in C_0^\infty([-2t_0, 2t_0])$ ,  $g_1(t) = 1$  if  $|t| \leq t_0/2$ . The positive integer  $k$  will be selected later.

Notice that  $|g_1^{(j)}(t)| \leq c_j t_0^{-j}$  and hence

$$\begin{aligned} |Y_{k+1}\tilde{g}(w)| &\lesssim \sum_{j=0}^{k+1} \frac{1}{t_0^j} \sup_{\Lambda, |\mu|=k+1-j} \|L_{\Lambda}^{\mu} g\|_{L^{\infty}(B(\zeta_0, t_0))} \\ &\lesssim \sum_{j=0}^{k+1} \frac{1}{t_0^j} \sup_{\Lambda, |\mu|=k+1-j} \frac{1}{\tau^{\mu}(\zeta_0, \Lambda, t_0)} \cdot \frac{1}{\sigma(B(\zeta_0, t_0))} \\ &\lesssim \frac{1}{t_0^{k+1}} \cdot \frac{1}{\sigma(B(\zeta_0, t_0))}. \end{aligned}$$

Now we write

$$\begin{aligned} \left| \int_{\partial\Omega} f(w)g(w) d\sigma(w) \right| &\leq \int_{\Omega_{t_0}} |f(w)Y_{k+1}\tilde{g}(w)|\delta^k(w) dV(w) \\ &\quad + \int_{\Omega \setminus \Omega_{t_0}} |f(w)Y_{k+1}\tilde{g}(w)|\delta^k(w) dV(w) \\ &= I + II. \end{aligned}$$

Notice that  $w \in \text{supp } \tilde{g}$  implies that  $d(\pi(w), \zeta) \leq c\gamma t_0$ . Thus, if  $w \in \Omega_{t_0} \cap \text{supp } \tilde{g}$  then  $w \in \mathcal{A}_{c\gamma}(\zeta)$ . Hence,

$$\begin{aligned} I &\lesssim f_{c\gamma}^*(\zeta) \int_{B(\zeta_0, t_0)} \int_{t_0}^{2t_0} \frac{t^k}{t_0^{k+1}\sigma(B(\zeta_0, t))} dt d\sigma(w) \\ &\lesssim f_{c\gamma}^*(\zeta). \end{aligned}$$

Moreover, since

$$\begin{aligned} |f(w)| &\leq f_N^{**}(\zeta) \left( 1 + \frac{d(\pi(w), \zeta)}{\delta(w)} \right)^N \\ &\lesssim f_N^{**}(\zeta) \left( 1 + \frac{t_0}{\delta(w)} \right)^N, \end{aligned}$$

we have

$$\begin{aligned} II &\lesssim \int_{B(\zeta_0, t_0)} \int_{t_0}^{\infty} |f(w)| \frac{t^k}{t_0^{k+1}\sigma(B(\zeta_0, t))} dt d\sigma(w) \\ &\lesssim \int_{t_0}^{\infty} f_N^{**}(\zeta) \left( 1 + \frac{t_0}{t} \right)^N \frac{t^k}{t_0^{k+1}} dt \\ &\lesssim f_N^{**}(\zeta), \end{aligned}$$

if  $k - N < -1$ . □

The next two lemmas are classical, and they hold on any smoothly bounded domain. The first is due to Stein; see [St1, Sec. 9]. The second is a version of a result of Fefferman and Stein; see [FS, Lemma VI.1].

LEMMA 4.2. *Let  $\Omega$  be any smoothly bounded domain,  $0 < p \leq \infty$ . Then*

$$\|f_\gamma^*\|_{L^p(\partial\Omega)} \lesssim \|f\|_{\mathcal{H}^p(\Omega)}.$$

LEMMA 4.3. *Let  $\Omega$  be any smoothly bounded domain,  $0 < p \leq \infty$ . Then, for  $N$  large enough,*

$$\|f_N^{**}\|_{L^p(\partial\Omega)} \lesssim \|f\|_{\mathcal{H}^p(\Omega)}.$$

We now introduce a smooth partition of unity on any open set  $\mathcal{O} \subseteq \partial\Omega$ .

LEMMA 4.4. *Let  $\mathcal{O} \subseteq \partial\Omega$  be an open set. Then there exist a collection of balls  $B_i = B(\zeta_i, r_i)$ , functions  $\phi_i \in C^\infty(\partial\Omega)$ ,  $i = 0, 1, 2, \dots$ , and constants  $\alpha > 1 > \beta > 0$ , depending only on  $\Omega$ , such that the following conditions hold:*

- (i)  $0 \leq \phi_i \leq 1$ ;
- (ii)  $\text{supp } \phi_i \subseteq B_i$ ;
- (iii)  $\phi_i = 1$  on  $\frac{1}{\alpha}B_i$ ;
- (iv)  $\sum_{i=0}^\infty \phi_i = \chi_{\mathcal{O}}$ ;
- (v) *for each  $i$  there exists  $\zeta_i \in {}^c\mathcal{O}$  such that for any integer  $N$  we have  $c_N \phi_i / \|\phi_i\|_{L^1} \in \mathcal{K}_\alpha^N(\zeta_i)$  for some  $c_N = c(N, \Omega)$ .*

*Proof.* By [McS2, Prop. 1.9], given any  $\zeta_0, \delta > 0$  and  $C > 1$ , there exist normalized smooth bump functions of order  $N$  on  $B(\zeta_0, C\delta)$  that are identically equal to 1 on  $B(\zeta_0, \delta)$ .

Given this result, the proof now proceeds as the proof of Lemma 4.3 in [KL2]. We give the details for the sake of completeness.

Let  $\{B(\zeta_i, r_i)\}$  be a sequence of balls satisfying, for some  $0 < \beta < 1 < \alpha$ , the following conditions:

- (a)  $\cup_i \beta B(\zeta_i, r_i) = \mathcal{O}$ ;
- (b)  $B(\zeta_i, r_i) \subseteq \mathcal{O}$ ,  $\alpha B(\zeta_i, r_i) \cap {}^c\mathcal{O} \neq \emptyset$ ;
- (c)  $\frac{1}{\alpha}B(\zeta_i, r_i)$  are pairwise disjoint;
- (d) no point in  $\mathcal{O}$  lies in more than  $N_\Omega$  of the balls  $B(\zeta_i, r_i)$ .

Such a sequence exists; an example is a Whitney covering for  $\mathcal{O}$ .

Given  $B_i = B(\zeta_i, r_i)$  let  $\psi_i$  be a normalized smooth bump function supported in  $B_i$  that equals 1 on  $\frac{1}{\alpha}B_i$ . By condition (d) above,

$$1 \leq \sum_i \psi_i(\zeta) \leq N_\Omega \quad \text{for all } \zeta \in \mathcal{O}.$$

Now set

$$\phi_i(\zeta) = \psi_i(\zeta) / \sum_j \psi_j(\zeta).$$

Then (i)–(iv) are clearly satisfied, so we only need to check (v).

By condition (b) above, there exists  $\xi_i \in \alpha B(\zeta_i, r_i) \cap {}^c\mathcal{O}$ . Then  $\zeta_i - r_i \nu(\zeta_i) \in \mathcal{A}_\gamma(\xi_i)$ ,  $\text{supp } \phi_i \subseteq B_i$ , with  $\gamma > 1$ , and

$$\begin{aligned} \sup_{\Lambda, |\mu| \leq N} \tau^\mu(\xi_i, \Lambda, r_i) \sigma(B(\xi_i, r_i)) \|L_\Lambda^\mu \phi_i\|_{L^\infty(\partial\Omega)} &\lesssim c_N \sigma(B(\xi_i, r_i)) \\ &\lesssim c_N \|\phi_i\|_{L^1(\partial\Omega)}. \end{aligned} \quad \square$$

### 5. Beginning of the proof of Theorem 2.1

We let  $k_0$  be the least integer such that

$$(5.1) \quad \|K_{\gamma, M}(f) + f_\gamma^*\|_{L^p(\partial\Omega)} \leq 2^{k_0}.$$

For a positive integer  $k$  define

$$(5.2) \quad \mathcal{O}_k = \{z \in \partial\Omega : K_{\gamma, M}f(z) + f_\gamma^*(z) > 2^{k_0+k}\}.$$

For each  $k$  we fix a Whitney covering and a partition of  $\chi_{\mathcal{O}_k}$ ,  $\{\phi_i^k\}$ , as in Lemma 4.4.

Let  $\zeta_0 \in \partial\Omega$  and  $B(\zeta_0, r_0)$  be fixed. The ball  $B(\zeta_0, r_0)$  is contained in the polydisc  $Q(\zeta_0, r_0)$ . By the results in [Mc] there exists an  $r_0$ -extremal (orthonormal) basis  $\{v^{(1)}, \dots, v^{(n)}\}$  on  $Q(\zeta_0, r_0)$ , where  $v^{(1)}$  is transversal to the boundary. We denote by  $(w_1, \dots, w_n)$  the coordinates with respect to this basis and we define  $V_\ell(\zeta_0, r_0)$  to be the space of all polynomials of degree  $\leq \ell$  in  $\text{Im } w_1, w_2, \overline{w_2}, \dots, w_n, \overline{w_n}$ .

We remark that  $V_\ell(\zeta_0, r_0)$  does not depend on  $r_0$  since  $v^{(1)}$  does not depend on  $r_0$ . Hence we simply write  $V_\ell(\zeta_0)$ .

Let  $\phi_0$  be a smooth bump function supported on  $B(\zeta_0, r_0)$ . We denote by  $L_{\phi_0}^2(d\sigma)$  the  $L^2$ -space with respect to the probability measure  $(\phi_0/\|\phi_0\|_{L^1})d\sigma$ . We let  $\mathcal{P}_{\phi_0}$  denote the orthogonal projection of  $L_{\phi_0}^2(d\sigma)$  onto  $V_\ell(\zeta_0)$  (which is obviously contained in  $L_{\phi_0}^2(d\sigma)$ ).

In what follows, we will write  $w = s + it$ ,  $s = (s_1, s')$ ,  $t = (t_1, t')$ ,  $s_1, t_1 \in \mathbb{R}$ ,  $s', t' \in \mathbb{R}^{n-1}$ , so that  $V_\ell(\zeta_0)$  is the set of all polynomials in  $s', t_1, t'$ , of degree  $\leq \ell$ .

Let  $\{\pi_J\}$  be an orthonormal basis for  $V_{\ell_0}$ , where  $\ell_0 = [(1 - 1/p)(M_\Omega - 2n - 2)]$  and  $J = (j', j) = (0, j'_2, \dots, j'_n, j_1, \dots, j_n)$ ,  $|J| = j'_2 + \dots + j'_n + j_1 + \dots + j_n$ .

LEMMA 5.1. *Let  $\pi_J$  be as above. Then:*

- (i)  $|\pi_J(z)| \leq c_J$  on the support of  $\phi_0$ ;
- (ii)  $|L_\Lambda^\mu \pi_J(z)| \leq c_{J, \mu} \tau^{-\mu}(\zeta_0, \Lambda, r_0)$  on the support of  $\phi_0$ .

*Proof.* We begin with (i). Recall that  $\zeta_0$  and  $r_0$  are fixed. Then  $B := B(\zeta_0, r_0) = Q(\zeta_0, r_0) \cap \partial\Omega$ , where  $Q(\zeta_0, r_0)$  is the polydisc centered at  $\zeta_0$  and of polyradius  $\tau_1(\zeta_0, r_0) = r_0, \tau_2(\zeta_0, r_0), \dots, \tau_n(\zeta_0, r_0)$ .

Let

$$\pi_J(w) = \sum_{|\gamma'| + |\gamma| \leq \ell_0} a_{\gamma', \gamma} s^{\gamma'} t^\gamma,$$

where  $\gamma' = (0, \gamma'_2, \dots, \gamma'_n)$ ,  $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)$ . Then

$$\begin{aligned} \|\pi_J\|_{L^\infty(B)} &\leq \sup_{(s', t) \in B} \left| \sum_{|\gamma'| + |\gamma| \leq \ell_0} a_{\gamma', \gamma} s^{\gamma'} t^\gamma \right| \\ &= \sup_{(\tilde{s}', \tilde{t}) \in \tilde{B}} \left| \sum_{|\gamma'| + |\gamma| \leq \ell_0} a_{\gamma', \gamma} \tilde{s}^{\gamma'} \tilde{t}^\gamma \tau^{\gamma' + \gamma}(\zeta_0, r_0) \right|, \end{aligned}$$

where  $\tilde{s} = (0, s'_2/\tau_2, \dots, s'_n/\tau_n)$ ,  $\tilde{t} = (t_1/\tau_1, t_2/\tau_2, \dots, t_n/\tau_n)$ ,  $\tau_j = \tau_j(\zeta_0, r_0)$ , and  $\tilde{B} = (-1, 1)^{2n-1}$ .

Hence, using the equivalence of all norms on a finite dimensional vector space, we have

$$\begin{aligned} \|\pi_J\|_{L^\infty(B)} &\leq \sup_{(s', t) \in B} \left| \sum_{|\gamma'| + |\gamma| \leq \ell_0} a_{\gamma', \gamma} s^{\gamma'} t^\gamma \right| \\ &\lesssim \left\| \sum_{|\gamma'| + |\gamma| \leq \ell_0} a_{\gamma', \gamma} \tau^{\gamma' + \gamma}(\zeta_0, r_0) \tilde{s}^{\gamma'} \tilde{t}^\gamma \right\|_{L^2(\tilde{B})} \\ &= \sigma(B)^{-1/2} \left\| \sum_{|\gamma'| + |\gamma| \leq \ell_0} a_{\gamma', \gamma} s^{\gamma'} t^\gamma \right\|_{L^2(B)} \\ &\approx \left\| \sum_{|\gamma'| + |\gamma| \leq \ell_0} a_{\gamma', \gamma} s^{\gamma'} t^\gamma \right\|_{L^2_{\phi_0}(B)} \\ &= 1. \end{aligned}$$

Here we use the fact that, since  $\phi_0 = 1$  on  $B(\zeta_0, r_0)$  and  $\text{supp } \phi_0 \subseteq \alpha B$ ,  $\|\phi_0\|_{L^1} \approx \sigma(B(\zeta_0, r_0))$ . This proves (i).

It remains to prove the estimates on the derivatives of  $\pi_J$ . We are going to show that, for any  $(\gamma', \gamma)$  such that  $|\gamma'| + |\gamma| \leq \ell_0$ , there exists  $c_{\gamma', \gamma}^J$  so that

$$(5.3) \quad |a_{\gamma', \gamma}| \leq \frac{c_{\gamma', \gamma}^J}{\tau^{\gamma' + \gamma}(\zeta_0, r_0)}.$$

Assuming this estimate for the moment, it follows that, for any  $\beta', \beta$  with  $|\beta'| + |\beta| \leq \ell_0$ ,

$$\begin{aligned} \left| \partial_s^{\beta'} \partial_t^\beta \pi_J(w) \right| &= \left| \sum_{\gamma' \geq \beta', \gamma \geq \beta, |\gamma'| + |\gamma| \leq \ell_0} c_{\beta', \beta} a_{\gamma', \gamma} s^{\gamma' - \beta'} t^{\gamma - \beta} \right| \\ &\leq \frac{c_{J, \gamma', \gamma}}{\tau^{\gamma' + \gamma}(\zeta_0, r_0)} \cdot \tau^{\gamma' - \beta' + \gamma - \beta}(\zeta_0, r_0) \\ &\leq \frac{c_{J, \gamma', \gamma}}{\tau^{\beta' + \beta}(\zeta_0, r_0)}. \end{aligned}$$

The estimate for any derivative  $L_\lambda^\mu$  follows from the fact that any  $L_\lambda$  is a linear combination of the  $\partial_{s_j}$ 's and the  $\partial_{t_k}$ 's, say  $L_\lambda = \sum_{j=2}^n a_j \partial_{s_j} + \sum_{k=1}^n b_k \partial_{t_k}$ , and that

$$\frac{1}{\tau(\zeta_0, \lambda, r_0)} \approx \sum_{j=1}^n \frac{|a_j| + |b_j|}{\tau_j(\zeta_0, r_0)},$$

where  $a_1 = 0$ ; see [McS2, Prop. 1.1].

It remains to prove (5.3). Since the  $\pi_J$ 's are real-analytic, they satisfy the mean-value property. In particular,

$$\tau^{\gamma'+\gamma}(\zeta_0, r_0) \left| \partial_s^{\gamma'} \partial_t^\gamma \pi_J(\zeta_0) \right| \leq \frac{c}{\sigma(B(\zeta_0, r_0))} \int_{B(\zeta_0, r_0)} |\pi_J(w)| d\sigma(w) \leq \tilde{c}_J,$$

so that

$$|a_{\gamma', \gamma}| = \left| \partial_s^{\gamma'} \partial_t^\gamma \pi_J(\zeta_0) \right| \leq \frac{\tilde{c}_J}{\tau^{\gamma'+\gamma}(\zeta_0, r_0)},$$

which proves (5.3) and finishes the proof.  $\square$

LEMMA 5.2. *With the notation fixed above, there exists  $c > 0$  such that for  $f \in \mathcal{H}^p(\Omega)$*

$$|\mathcal{P}_{\phi_i^k}(f)(z) \phi_i^k(z)| \leq c2^k.$$

Furthermore,

$$\left| \mathcal{P}_{\phi_j^{k+1}} \left( \left( f - \mathcal{P}_{\phi_j^{k+1}}(f) \right) \phi_i^k \right) (z) \phi_j^{k+1}(z) \right| \leq c2^{k+1}.$$

*Proof.* We use Lemma 5.1 with  $\zeta_i^k = \zeta_0$ ,  $\phi_i^k = \phi_0$ , and  $B_i^k(\zeta_i^k, r_i^k) = B_0$ . Then

$$\mathcal{P}_{\phi_i^k}(f)(z) = \sum_{|J| \leq \ell_0} c_J(f) \pi_J(z),$$

where

$$c_J(f) = \int_{\partial\Omega} f(w) \overline{\pi_J(w)} \phi_i^k(w) d\sigma(w).$$

The estimates on the  $\pi_J$ 's and  $\phi_i^k$  imply that  $\pi_J \cdot \phi_i^k \in \mathcal{K}_\gamma^M(\zeta_i^k)$ , so that

$$|c_J(f)| \leq K_{\gamma, M}(f)(\zeta_i^k) \leq c2^k.$$

This estimate, together with another application Lemma 5.1, imply the required bound on  $\mathcal{P}_{\phi_i^k}(f) \phi_i^k$ .



Next we prove the second estimate in the statement. Set  $h = (f - \mathcal{P}_{\phi_j^{k+1}}(f))\phi_i^k$ . Then

$$\begin{aligned} c_J^{(k+1)}(h) &= \int_{\partial\Omega} (f - \mathcal{P}_{\phi_j^{k+1}}(f))(w)\phi_i^k(w)\overline{\pi_J^{(k+1)}(w)}d\sigma_{\phi_j^{k+1}}(w) \\ &= \int_{\partial\Omega} f(w)\phi_i^k(w)\overline{\pi_J^{(k+1)}(w)}\phi_j^{k+1}(w)\frac{d\sigma(w)}{\|\phi_j^{k+1}\|_{L^1}} \\ &\quad - \int_{\partial\Omega} \mathcal{P}_{\phi_j^{k+1}}(f)(w)\phi_i^k(w)\phi_j^{k+1}(w)\overline{\pi_J^{(k+1)}(w)}\frac{d\sigma(w)}{\|\phi_j^{k+1}\|_{L^1}} \\ &= I + II. \end{aligned}$$

By condition (v) in Lemma 4.4, we have, for some  $\xi_j^{k+1} \in {}^c\mathcal{O}_{k+1}$ ,

$$|I| \leq K_{\gamma,M}(f)(\xi_j^{k+1}) \leq 2^{k+1},$$

by construction. On the other hand, by the previous bound,

$$\left| \mathcal{P}_{\phi_j^{k+1}}(f)(w)\phi_j^{k+1}(w) \right| \leq c2^{k+1},$$

while  $|\pi_J^{(k+1)}(w)| \leq c$ , so that the desired conclusion follows.  $\square$

### 6. Atomic decomposition

In this section we define the atomic decomposition of a given function  $f \in \mathcal{H}^p(\Omega)$ , and finish the proof of Theorem 2.1. As mentioned above,  $f$  admits boundary values defined a.e. on  $\partial\Omega$ , which we also denote by  $f$ .

We write

$$\begin{aligned} f &= \left( f - \sum_{i=0}^{\infty} f\phi_i^k \right) + \sum_{i=0}^{\infty} f\phi_i^k \\ &= f_k + \sum_{i=0}^{\infty} f\phi_i^k \\ &= f_k + \sum_{i=0}^{\infty} \mathcal{P}_{\phi_i^k}(f)\phi_i^k + \sum_{i=0}^{\infty} (f - \mathcal{P}_{\phi_i^k}(f))\phi_i^k \\ &= h_k + \sum_{i=0}^{\infty} (f - \mathcal{P}_{\phi_i^k}(f))\phi_i^k, \end{aligned}$$

where  $h_k = f_k + \sum_{i=0}^{\infty} \mathcal{P}_{\phi_i^k}(f)\phi_i^k$ .

Notice that

$$(6.1) \quad \left| \sum_{i=0}^{\infty} \mathcal{P}_{\phi_i^k}(f)(z)\phi_i^k(z) \right| \leq c_02^k,$$

since no point in  ${}^c\mathcal{O}_k$  lies in more than  $N_\Omega$  of the balls  $B(\zeta_i^k, r_i^k)$ . Moreover, observe that

$$\text{supp} \left( \sum_{i=0}^{\infty} (f - \mathcal{P}_{\phi_i^k}(f)) \phi_i^k \right) \subseteq \mathcal{O}_k,$$

so that the function on the left-hand side above tends to 0 pointwise as  $k \rightarrow \infty$ . This implies that  $h_k \rightarrow f$  pointwise a.e., so that the following equality holds a.e.:

$$(6.2) \quad f = h_0 + \sum_{k=0}^{\infty} (h_{k+1} - h_k).$$

Now notice that

$$(6.3) \quad f - h_k = \sum_{i=0}^{\infty} (f - \mathcal{P}_{\phi_i^k}(f)) \phi_i^k$$

and

$$(6.4) \quad \sum_{i=0}^{\infty} \mathcal{P}_{\phi_j^{k+1}} \left( (f - \mathcal{P}_{\phi_j^{k+1}}(f)) \phi_i^k \right) = \mathcal{P}_{\phi_j^{k+1}} (f - \mathcal{P}_{\phi_j^{k+1}}(f)) = 0.$$

Then, using (6.3) and (6.4) we write

$$\begin{aligned} h_{k+1} - h_k &= \sum_{i=0}^{\infty} (f - \mathcal{P}_{\phi_i^k}(f)) \phi_i^k - \sum_{j=0}^{\infty} (f - \mathcal{P}_{\phi_j^{k+1}}(f)) \phi_j^{k+1} \\ &= \sum_{i=0}^{\infty} \left\{ (f - \mathcal{P}_{\phi_i^k}(f)) \phi_i^k \right. \\ &\quad \left. - \sum_{j=0}^{\infty} \left( (f - \mathcal{P}_{\phi_j^{k+1}}(f)) \phi_j^k - \mathcal{P}_{\phi_j^{k+1}} \left( (f - \mathcal{P}_{\phi_j^{k+1}}(f)) \phi_i^k \right) \right) \phi_j^{k+1} \right\} \\ &= \sum_{i=0}^{\infty} b_i^k, \end{aligned}$$

where we have set

$$(6.5) \quad b_i^k = (f - \mathcal{P}_{\phi_i^k}(f)) \phi_i^k - \sum_{j=0}^{\infty} \left( (f - \mathcal{P}_{\phi_j^{k+1}}(f)) \phi_j^k - \mathcal{P}_{\phi_j^{k+1}} \left( (f - \mathcal{P}_{\phi_j^{k+1}}(f)) \phi_i^k \right) \right) \phi_j^{k+1}.$$

Hence

$$(6.6) \quad f = h_0 + \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} b_i^k.$$

Now let

$$(6.7) \quad a_0 = \frac{1}{\nu_0} h_0, \quad a_i^k = \frac{1}{\nu_i^k} b_i^k,$$

where

$$\nu_0 = \|h_0\|_{L^\infty(\partial\Omega)} \sigma(\partial\Omega)^{1/p}, \quad \nu_i^k = 2^{k_0+k+1} \sigma(B_i^k)^{1/p}.$$

Then equation (6.6) becomes

$$f = \nu_0 a_0 + \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \nu_i^k a_i^k.$$

The remainder of this section is devoted to proving that this representation is the desired atomic decomposition of  $f$ .

**Estimate for  $h_0$ .** By definition,

$$h_0 = f_0 + \sum_{i=0}^{\infty} \mathcal{P}_{\phi_i^0}(f) \phi_i^0.$$

By (6.1) we have

$$\left| \sum_{i=0}^{\infty} \mathcal{P}_{\phi_i^0}(f) \phi_i^0 \right| \leq c,$$

and, by definition,

$$|f_0| \leq |f^*|_{c\mathcal{O}_0} \leq 2^{k_0}.$$

Thus  $\|h_0\|_{L^\infty} \leq c2^{k_0}$ , so that  $a_0$  is an atom supported in  $\partial\Omega$ .

**Size estimates for the  $b_i^k$ 's.** We have

$$\begin{aligned} |b_i^k| &= \left| \left( f - \mathcal{P}_{\phi_i^k}(f) \right) \phi_i^k \right. \\ &\quad \left. - \sum_{j=0}^{\infty} \left( \left( f - \mathcal{P}_{\phi_j^{k+1}}(f) \right) \phi_i^k - \mathcal{P}_{\phi_j^{k+1}} \left( f - \mathcal{P}_{\phi_j^{k+1}}(f) \right) \phi_i^k \right) \phi_j^{k+1} \right| \\ &\leq \left| \left( f - \mathcal{P}_{\phi_i^k}(f) \right) \phi_i^k - \sum_{j=0}^{\infty} \left( f - \mathcal{P}_{\phi_j^{k+1}}(f) \right) \phi_i^k \phi_j^{k+1} \right| \\ &\quad + \left| \sum_{j=0}^{\infty} \left( \mathcal{P}_{\phi_j^{k+1}} \left( f - \mathcal{P}_{\phi_j^{k+1}}(f) \right) \phi_i^k \right) \phi_j^{k+1} \right|. \end{aligned}$$

The second term on the right-hand side is bounded by  $c_0 c 2^{k+1}$  by Lemma 5.2, while the first term is bounded by

$$\begin{aligned}
& \left| \sum_{j=0}^{\infty} \left( (f - \mathcal{P}_{\phi_i^k}(f)) - (f - \mathcal{P}_{\phi_j^{k+1}}(f)) \phi_i^k \right) \phi_j^{k+1} \right| + \left| (f - \mathcal{P}_{\phi_i^k}(f)) \phi_i^k \chi_{\mathcal{O}_{k+1}} \right| \\
& \leq \left| \sum_{j=0}^{\infty} (\mathcal{P}_{\phi_j^{k+1}}(f) - \mathcal{P}_{\phi_i^k}(f)) \phi_i^k \phi_j^{k+1} \right| + \left| (f - \mathcal{P}_{\phi_i^k}(f)) \phi_i^k \chi_{\mathcal{O}_k \setminus \mathcal{O}_{k+1}} \right| \\
& \leq \left| \sum_{j=0}^{\infty} \mathcal{P}_{\phi_j^{k+1}}(f) \phi_j^{k+1} \right| + \left| \mathcal{P}_{\phi_i^k}(f) \phi_i^k \right| + |f \chi_{\mathcal{O}_k \setminus \mathcal{O}_{k+1}}| + \left| \mathcal{P}_{\phi_i^k}(f) \phi_i^k \right| \\
& \leq c_0 c 2^k + f^* \chi_{\mathcal{O}_k \setminus \mathcal{O}_{k+1}} \\
& \leq c_0 c 2^k,
\end{aligned}$$

where we have again used Lemma 5.2.

**Support of the  $b_i^k$ 's.** The first term in (6.5) is supported in  $B_i^k$ . To ensure that the terms in the series are not identically 0, the condition  $B_i^k \cap B_j^{k+1} \neq \emptyset$  must be satisfied for some  $j$ .

We claim that if  $B_i^k \cap B_j^{k+1} \neq \emptyset$ , then  $r_j^{k+1} \leq c \alpha r_i^k$ . Indeed, let  $w \in B_i^k \cap B_j^{k+1}$ . Since  $\mathcal{O}_{k+1} \subseteq \mathcal{O}_k$ , we have

$$\begin{aligned}
Cr_j^{k+1} & \leq d(B_j^{k+1}, \partial \mathcal{O}_{k+1}) \leq d(B_j^{k+1}, \partial \mathcal{O}_k) \\
& \leq d(w, \partial \mathcal{O}_k) \leq d(B_i^k, \partial \mathcal{O}_k) + 2d(w, \zeta_i^k) \\
& \leq c \alpha r_i^k,
\end{aligned}$$

since, by the Whitney property, one has  $Cr_i^k \leq d(B_i^k, \partial \mathcal{O}_k) \leq \alpha r_i^k$ .

**Moment condition.** We wish to estimate

$$\left| \int_{\partial \Omega} b_i^k(w) \phi(w) d\sigma(w) \right|$$

for  $\phi \in \mathcal{S}_{N_p}(B_i^k)$ .

On  $B_i^k$  we work in local coordinates and use the Taylor expansion of order  $\ell_0$  of  $\phi$  around  $\zeta_i^k$ . We denote by  $S_\phi^{\ell_0}(\zeta_i^k)$  the corresponding Taylor polynomial. Notice that  $S_\phi^{\ell_0}(\zeta_i^k) \in V_{\ell_0}(\zeta_i^k)$ .

By the definition of  $\mathcal{S}_{N_p}(B_i^k)$  we have

$$\|\phi - S_\phi^{\ell_0}(\zeta_i^k)\|_{L^\infty(\partial \Omega)} \leq \|\phi\|_{\mathcal{S}_{N_p}(B(\zeta_0, r_0))}.$$

By construction, the first term in  $b_i^k$  is orthogonal to  $V_{\ell_0}(\zeta_i^k)$ , and each non-vanishing term in the series is orthogonal to some  $V_{\ell_0}(\zeta_j^{k+1})$ . Hence, if  $B_i^k \cap$

$B_j^{k+1} \neq \emptyset$  then  $r_j^{k+1} \leq C\alpha r_i^k$  and  $B_j^{k+1} \subseteq C'\alpha B_i^k$ . In this case it follows that  $V_{\ell_0}(\zeta_j^{k+1}) \subseteq V_{\ell_0}(\zeta_i^k)$ , since the same coordinate system works. Therefore,

$$\begin{aligned} \left| \int_{\partial\Omega} b_i^k(w)\phi(w) d\sigma(w) \right| &= \left| \int_{\partial\Omega} b_i^k(w) \left( \phi(w) - S_\phi^{\ell_0}(\zeta_i^k) \right) d\sigma(w) \right| \\ &\lesssim \|\phi\|_{S_{N^p}(B_i^k)} \cdot \sigma(B_i^k) \cdot 2^{k+k_0}, \end{aligned}$$

by the size estimate for the  $b_i^k$ 's.

**Coefficients in  $\ell^p$ .** It remains to prove that  $\{\nu_i^k\} \in \ell^p$ . We have

$$\begin{aligned} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} |\nu_i^k|^p &\leq c^p \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} 2^{k+k_0} \sigma(B_i^k) \\ &\leq c^p \sum_{k=0}^{\infty} 2^{k+k_0} \sigma(\mathcal{O}_k) \\ &\leq c^p \int_1^{\infty} t^{p-1} \sigma(\{\zeta \in \partial\Omega : K_\alpha^M f(\zeta) + f_\alpha^*(\zeta) \geq t\}) dt \\ &\leq c^p \|f\|_{\mathcal{H}^p}^p. \end{aligned}$$

**Convergence in  $\mathcal{H}^p(\Omega)$ -norm.** We have shown that the boundary value of  $f \in \mathcal{H}^p(\Omega)$ , which we also denote by  $f$ , admits a decomposition  $f = \sum_j \nu_j a_j$  with  $p$ -atoms  $a_j$  and constants  $\nu_j$  such that  $\sum_j |\nu_j|^p < \infty$ .

This equality, which stems from (6.6), holds a.e.; see (6.2). We will show that it is also valid in the distribution sense. This together with Theorem 2.2 implies that the above series converges to  $f$  in the  $\mathcal{H}^p(\Omega)$ -norm.

We now show that the equality

$$f = h_0 + \sum_{k=0}^{\infty} (h_{k+1} - h_k),$$

which holds a.e., also holds in the distribution sense.

Recall that  $h_{k+1} - h_k = \sum_{i=0}^{\infty} b_i^k$ , so it suffices to show that  $\sum_{k=0}^m (h_{k+1} - h_k)$  converges in the sense of distributions. We have

$$\begin{aligned} (6.8) \quad \left| \int_{\partial\Omega} \left( \sum_{k=\ell}^m (h_{k+1} - h_k) \right) (z)\psi(z) d\sigma(z) \right| &= \left| \sum_{k=\ell}^m \int_{\partial\Omega} \sum_{i=0}^{\infty} b_i^k \psi(z) d\sigma(z) \right| \\ &= \left| \sum_{k=\ell}^m \sum_{i=0}^{\infty} \int_{\partial\Omega} b_i^k \psi(z) d\sigma(z) \right| \\ &\lesssim \sum_{k=\ell}^m \sum_{i=0}^{\infty} 2^{k+k_0} \sigma(B_i^k) \|\psi\|_{S_{N^p}(B_i^k)} \end{aligned}$$

$$\begin{aligned}
&\lesssim \sum_{k=\ell}^m \sum_{i=0}^{\infty} 2^{k+k_0} \sigma(B_i^k) \|\psi\|_{\mathcal{C}^{\ell_0+1}(\partial\Omega)} (r_i^k)^{[(1/p-1)(M_\Omega+2n-2)]+1} \\
&\lesssim \sum_{k=\ell}^m \sum_{i=0}^{\infty} 2^{k+k_0} \sigma(B_i^k) \|\psi\|_{\mathcal{C}^{\ell_0+1}(\partial\Omega)} (r_i^k)^{(1/p-1)(2n-1)},
\end{aligned}$$

since  $[(1/p-1)(M_\Omega+2n-2)]+1 \geq (1/p-1)(M_\Omega+2n-1) \geq (1/p-1)(2n-1)$ .

Hence, using the inequality  $1/p \geq 1$  twice, the left-hand side in (6.8) is bounded by a constant times

$$\begin{aligned}
&\sum_{k=\ell}^m \sum_{i=0}^{\infty} 2^{k+k_0} \sigma(B_i^k)^{1/p} \|\psi\|_{\mathcal{C}^{\ell_0+1}(\partial\Omega)} \\
&\lesssim \sum_{k=\ell}^m 2^k \left( \sum_{i=0}^{\infty} \sigma(B_i^k)^{1/p} \right)^{1/p} \\
&\lesssim \left( \sum_{k=\ell}^m 2^{kp} \sigma(\mathcal{O}_k) \right)^{1/p} \\
&\lesssim \left( \sum_{k=\ell}^{\infty} \int_{2^{k-1}}^{2^k} t^{p-1} \sigma(\{f^* + K_\alpha^M f \geq t\}) dt \right)^{1/p} \\
&\lesssim \left( \int_{2^{\ell-1}}^{\infty} t^{p-1} \int_{\{f^* + K_\alpha^M f \geq t\}} d\sigma dt \right)^{1/p} \\
&\lesssim \left( \int_{\mathcal{O}_{\ell-1}} |f^* + K_\alpha^M f|^p d\sigma \right)^{1/p},
\end{aligned}$$

which tends to 0 as  $\ell \rightarrow \infty$ .

This finishes the proof of the atomic decomposition.  $\square$

## 7. Proof of Theorem 2.3

The method used in the proof of the atomic decomposition yields the same result using atoms with arbitrarily large order of cancellation. Hence we call an atom  $a$  a  $(k, p)$ -atom if  $k \geq N_p$ ,  $a$  is a  $p$ -atom and, moreover, satisfies

$$\left| \int_{\partial\Omega} a(\zeta) \phi(\zeta) d\sigma(\zeta) \right| \leq \|\phi\|_{\mathcal{S}_k(B(\zeta_0, r_0))} \sigma(B(\zeta_0, r_0))^{1-1/p},$$

for all  $\phi \in \mathcal{C}^\infty(B(\zeta_0, r_0))$ , where  $B(\zeta_0, r_0)$  is the support of  $a$ .

**LEMMA 7.1.** *Let  $a$  be a  $(k, p)$ -atom, having support in  $B = B(\zeta_0, r_0)$ , and let  $A = P_S(a)$ . Then  $A$  satisfies the following estimates:*

$$(i) \|A\|_{\mathcal{H}^{2p}} \leq c\sigma(B)^{-1/2p};$$

(ii) for  $C > 1$  and  $d(\zeta, \zeta_0) \geq Cr_0$ ,

$$|A(\zeta)| \leq c \left( \frac{r_0}{d(\zeta, \zeta_0)} \right)^{1+(k-2+2n)/M_\Omega} \sigma(B(\zeta_0, r_0))^{-1/p};$$

(iii) for  $C > 1$ ,  $\ell$  a positive integer and  $d(\zeta, \zeta_0) \geq Cr_0$ ,

$$|\nabla^\ell A(\zeta)| \leq c \frac{r_0^{1+(k-2+2n)/M_\Omega}}{d(\zeta, \zeta_0)^{1-\ell+(k-2+2n)/M_\Omega}} \sigma(B(\zeta_0, r_0))^{-1/p}.$$

In the following we will set  $\beta = 1 + (k - 2 + 2n)/M_\Omega$ .

*Proof.* The proof follows the same lines as that of [BPS2, Lemma 4.7].

The estimate (i) is the same as the estimate for  $\|A\|_{\mathcal{H}^p}$ .

Denote by  $S_k$  the Taylor polynomial of the function  $w \mapsto S_\Omega(\zeta, w)$  around  $\zeta_0$  of order  $k$ . Then  $\|S_\Omega(\zeta, w) - S_k(w)\|_{L^\infty} \leq \|S_\Omega(\zeta, \cdot)\|_{\mathcal{S}_k(B(\zeta_0, r_0))}$ . Hence,

$$\begin{aligned} |A(\zeta)| &= \left| \int_{\partial\Omega} a(w) S_\Omega(\zeta, w) d\sigma(w) \right| \\ &= \left| \int_{\partial\Omega} a(w) (S_\Omega(\zeta, w) - S_k(w)) d\sigma(w) \right| \\ &\leq \|S_\Omega(\zeta, \cdot)\|_{\mathcal{S}_k(B(\zeta_0, r_0))} \sigma(B(\zeta_0, r_0))^{1-1/p}. \end{aligned}$$

Using the estimate (1.8) we have

$$\begin{aligned} \|S_\Omega(\zeta, \cdot)\|_{\mathcal{S}_k(B(\zeta_0, r_0))} &= \sum_{|\mu|=k} \sup_{\Lambda} \|L_{\Lambda, w}^\mu S_\Omega(\zeta, \cdot)\|_{L^\infty(B)} \tau^\mu(\zeta_0, \Lambda, r_0) \\ &\lesssim \sum_{|\mu|=k} \sup_{w \in B} \sup_{\Lambda} \frac{\tau^\mu(\zeta_0, \Lambda, r_0)}{\tau^\mu(w, \Lambda, d(\zeta, w))} \cdot \frac{1}{\sigma(B(w, d(\zeta, w)))}. \end{aligned}$$

Recall that, if  $d(\zeta, \zeta_0) \geq Cr_0$  and  $w \in B(\zeta_0, r_0) \subseteq B(\zeta_0, d(\zeta_0, \zeta)/C)$ , then  $d(\zeta_0, \zeta) \approx d(\zeta, w)$ . Thus,

$$\|S_\Omega(\zeta, \cdot)\|_{\mathcal{S}_k(B(\zeta_0, r_0))} \lesssim \left( \frac{r_0}{d(\zeta, \zeta_0)} \right)^{k/M_\Omega} \cdot \sigma(B(\zeta_0, d(\zeta, \zeta_0)))^{-1},$$

so that

$$|A(\zeta)| \lesssim \left( \frac{r_0}{d(\zeta, \zeta_0)} \right)^\beta \cdot \sigma(B(\zeta_0, r_0))^{-1/p}.$$

The estimates for the derivatives of  $A$  follow in the same fashion. □

**Completion of the proof of Theorem 2.3.** By the atomic decomposition, it suffices to factorize each holomorphic atom  $A = P_S(a)$ , where  $a$  is a  $(k, p)$ -atom with  $k$  large enough and with support in some ball  $B(\zeta_0, r_0)$ .

To obtain this factorization we use a recent result of Diederich and Fornæss [DFo] on the existence of support functions on convex domains of finite type.

To be precise, there exists a neighborhood  $U$  of  $\partial\Omega$  and a function  $H: \Omega \times U \rightarrow \mathbb{C}$  such that  $H \in \mathcal{C}^\infty(\Omega \times U)$ ,  $H(\cdot, w)$  is holomorphic for each  $w \in U$ , and

$$d(z, w) \lesssim |H(z, w)| \lesssim d(z, w),$$

on  $\Omega \times U$ . This function  $H$  was used in [BPS2] to prove the factorization theorem for  $\mathcal{H}^1(\Omega)$ .

We set  $H_0 = H(\cdot, \tilde{\zeta}_0)$ , where  $\tilde{\zeta}_0 = \zeta_0 - r_0\nu(\zeta_0)$ . We write  $A = AH_0^\ell H_0^{-\ell}$ , with  $\ell$  to be selected later.

We prove the result for  $q = q' = 2$ , the general case being completely analogous. Writing  $\zeta_\varepsilon = \zeta - \varepsilon\nu_\zeta$ , we have

$$\begin{aligned} \|H_0^{-\ell}\|_{\mathcal{H}^{2p}}^{2p} &\lesssim \sup_{\varepsilon>0} \int_{\partial\Omega} d(\zeta_\varepsilon, \tilde{\zeta}_0)^{-2p\ell} d\sigma(\zeta) \\ &\lesssim r_0^{-2p\ell} \sigma(B), \end{aligned}$$

by a standard integration result; see [BPS2, Lemma 2.2].

On the other hand, for a fixed constant  $C > 1$ , we have

$$\begin{aligned} \|AH_0^\ell\|_{\mathcal{H}^{2p}}^{2p} &\lesssim \sup_{\varepsilon_0>\varepsilon>0} \int_{B(\zeta_0, Cr_0)} |A(\zeta_\varepsilon)|^{2p} d(\zeta_\varepsilon, \tilde{\zeta}_0)^{2p\ell} d\sigma(\zeta) \\ &\quad + \frac{1}{\sigma(B)} \int_{cB(\zeta_0, Cr_0)} \frac{r_0^{2p\beta}}{d(\zeta, \zeta_0)^{2p\beta}} d(\zeta, \tilde{\zeta}_0)^{2p\ell} d\sigma(\zeta) \\ &\lesssim (Cr_0)^{2p\ell} \|A\|_{\mathcal{H}^p}^{2p} \\ &\quad + \frac{Cr_0^{2p\ell}}{\sigma(B)} \int_{cB(\zeta_0, Cr_0)} \frac{r_0^{2p\beta}}{d(\zeta, \zeta_0)^{2p\beta}} \left(1 + \frac{d(\zeta, \zeta_0)}{r_0}\right)^{2p\ell} d\sigma(\zeta) \\ &\lesssim \frac{r_0^{2p\ell}}{\sigma(B)} \end{aligned}$$

for  $k$  large enough, where we have again used Lemma 2.2 in [BPS2].

This finishes the proof of the factorization theorem.  $\square$

## 8. Final remarks

We now introduce the class of  $H$ -domains.

**DEFINITION 8.1.** We say that  $\Omega$  is an  $H$ -domain if it is a smooth bounded domain of finite type and the following condition holds: For each  $\zeta \in \partial\Omega$  there exist a neighborhood  $V_\zeta$  and a biholomorphic map  $\Phi_\zeta$  defined on  $V_\zeta$  such that  $\Phi_\zeta(\Omega \cap V_\zeta)$  is geometrically convex.

This class of domains, which obviously includes the strongly pseudoconvex domains and the convex domains of finite type, has been analyzed in [BPS3]. In this paper, it was shown that for these domains the Szegő and Bergman



kernels satisfy the same local estimates (1.8) as in the case of the convex domains of finite type or strongly convex domains. Therefore, Theorems 2.1–2.3 immediately extend to the case of  $H$ -domains, since all the arguments in their proofs are based on local estimates for the kernels; see [BPS3].

## REFERENCES

- [BS] H. P. Boas and E. J. Straube, *On equality of line type and variety type of real hypersurfaces in  $\mathbb{C}^n$* , J. Geom. Anal. **2** (1992), 95–98.
- [BPS1] A. Bonami, M. M. Peloso, and F. Symesak, *Powers of the Szegő kernel and Hankel operators on Hardy spaces*, Michigan Math. J. **46** (1999), 225–250.
- [BPS2] ———, *Factorization of Hardy spaces and Hankel operators on convex domains in  $\mathbb{C}^n$* , J. Geom. Anal. **11** (2001), 363–367.
- [BPS3] ———, *On Hankel operators on Hardy and Bergman spaces and related questions*, Actes des Rencontre d’Analyse Complexe dédiées à Jean Poly (to appear).
- [CRW] R. Coifman, R. Rochberg, and G. Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. Math. **103** (1976), 611–635.
- [CW] R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. **83** (1977), 569–645.
- [D] G. Dafni, *Hardy spaces on some pseudoconvex domains*, J. Geom. Anal. **4** (1994), 273–316.
- [DFi] F. Di Biase and B. Fisher, *Boundary behaviour of  $\mathcal{H}^p$  functions on convex domains of finite type in  $\mathbb{C}^n$* , Pacific J. Math. **183** (1998), 25–38.
- [DFo] K. D. Diederich and J. E. Fornæss, *Support functions for convex domains of finite type*, Math. Z. **230** (1999), 145–164.
- [FS] C. Fefferman and E. M. Stein, *Hardy spaces of several variables*, Acta Math. **232** (1972), 137–193.
- [GL] J. B. Garnett and R. H. Latter, *The atomic decomposition for Hardy spaces in several complex variables*, Duke J. Math. **45** (1978), 815–845.
- [G] D. Goldberg, *A local version of real Hardy spaces*, Duke J. Math. **46** (1979), 27–42.
- [KL1] S. G. Krantz and S-Y. Li, *A note on Hardy spaces and functions of bounded mean oscillation on domains in  $\mathbb{C}^n$* , Michigan Math. J. **41** (1994), 71–97.
- [KL2] ———, *On decomposition theorems for Hardy spaces on domains in  $\mathbb{C}^n$  and applications*, J. Fourier Anal. and Appl. **2** (1995), 65–107.
- [KL3] ———, *Duality theorems for Hardy and Bergman spaces on convex domains of finite type in  $\mathbb{C}^n$* , Ann. Inst. Fourier (Grenoble) **45** (1995), 1305–1327.
- [KL4] ———, *Area integral characterization for functions in Hardy spaces on domains in  $\mathbb{C}^n$* , Complex Variables Theory Appl. **32** (1997), 373–399.
- [KL5] ———, *Hardy classes, integral operators, and duality on spaces of homogeneous type*, preprint.
- [Mc] J. D. McNeal, *Estimates on the Bergman kernels of convex domains*, Adv. Math. **109** (1994), 108–139.
- [McS1] J. D. McNeal and E. M. Stein, *Mapping properties of the Bergman projection on convex domains of finite type*, Duke J. Math. **73** (1994), 177–199.
- [McS2] ———, *The Szegő projection on convex domains*, Math. Z. **224** (1997), 519–553.
- [MTW] Y. Meyer, M. Taibleson, and G. Weiss, *Some functional analytic properties of the spaces  $B_q$  generated by blocks*, Indiana Univ. Math. J. **34** (1985), 493–515.
- [St1] E. M. Stein, *Boundary behaviour of holomorphic functions of several complex variables*, Math. Notes, no. 11, Princeton University Press, Princeton, NJ, 1972.

- [St2] ———, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Math. Series, vol. 43, Princeton University Press, Princeton, NJ, 1993.

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