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WEIGHTED POINCARÉ INEQUALITIES FOR SOLUTIONS TO A-HARMONIC EQUATIONS

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ABSTRACT. We first prove a local A_r-weighted Poincaré inequality for solutions to A-harmonic equations of the form $d^\star A(x, d\omega) = B(x, d\omega)$. Then, as an application of this local result, we prove a global A_r-weighted Poincaré inequality for functions that are solutions to such equations in John domains.

1. Introduction

Poincaré inequalities are now ubiquitous in analysis. We mention only [9], [2], and especially [3] for geometric applications of these inequalities.

In contrast, we show here that, for certain A-harmonic tensors, a weak local Poincaré inequality holds in $\mathbb{R}^n$ for all positive exponents. This borrows results from [4], [5], [7] and [8].

Using this result we obtain a global weighted Poincaré inequality for A-harmonic functions in John domains for all positive exponents.

Throughout this paper we assume Ω is a connected open subset of $\mathbb{R}^n$. Let $e_1, e_2, \dots, e_n$ denote the standard unit basis of $\mathbb{R}^n$. For $l = 0, 1, \dots, n$, the linear space of l-vectors, spanned by the exterior products $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_l}$, corresponding to all ordered l-tuples $I = (i_1, i_2, \dots, i_l)$, $1 \le i_1 < i_2 < \cdots < i_l \le n$, is denoted by $\wedge^l = \wedge^l(\mathbb{R}^n)$. The Grassman algebra $\wedge = \oplus \wedge^l$ is a graded algebra with respect to the exterior products. For $\alpha = \sum \alpha^I e_I \in \wedge$ and $\beta = \sum \beta^I e_I \in \wedge$, the inner product in $\wedge$ is given by $\langle \alpha, \beta \rangle = \sum \alpha^I \beta^I$, where the summation is over all l-tuples $I = (i_1, i_2, \dots, i_l)$ and all integers $l = 0, 1, \dots, n$. We define the Hodge star operator $\star \colon \wedge \to \wedge$ by the rule $\star 1 = e_1 \wedge e_2 \wedge \cdots \wedge e_n$ and $\alpha \wedge \star\beta = \beta \wedge \star\alpha = \langle \alpha, \beta \rangle (\star 1)$ for all $\alpha, \beta \in \wedge$. Hence the norm of $\alpha \in \wedge$ is given by the formula $|\alpha|^2 = \langle \alpha, \alpha \rangle = \star(\alpha \wedge \star\alpha) \in \wedge^0 = \mathbb{R}$. The Hodge star is an isometric isomorphism on $\wedge$ with $\star \colon \wedge^l \to \wedge^{n-l}$ and $\star\star(-1)^{l(n-l)} \colon \wedge^l \to \wedge^l$.

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Let $0 < p < \infty$. We denote the weighted L^p-norm of a measurable function f over E by

$$\|f\|_{p,E,w} = \left(\int_E |f(x)|^p w(x) dx \right)^{1/p}.$$

A differential l-form ω on Ω is a Schwartz distribution on Ω with values in $\wedge^l(\mathbb{R}^n)$. We denote the space of differential l-forms by $D'(\Omega, \wedge^l)$. We write $L^p(\Omega, \wedge^l)$ for the l-forms $\omega(x) = \sum_I \omega_I(x) dx_I = \sum \omega_{i_1 i_2 \ldots i_l}(x) dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_l}$ with $\omega_I \in L^p(\Omega, \mathbb{R})$ for all ordered l-tuples I. Thus $L^p(\Omega, \wedge^l)$ is a Banach space with norm

$$\|\omega\|_{p,E} = \left(\int_E |\omega(x)|^p dx \right)^{1/p} = \left(\int_E \left(\sum_I |\omega_I(x)|^2 \right)^{p/2} dx \right)^{1/p}.$$

Similarly, $W_p^1(\Omega, \wedge^l)$ is the space of those differential l-forms on Ω whose coefficients are in $W_p^1(\Omega, \mathbb{R})$. The notations $W^1_{p,\mathrm{loc}}(\Omega, \mathbb{R})$ and $W^1_{p,\mathrm{loc}}(\Omega, \wedge^l)$ are self-explanatory. We denote by $d\colon D'(\Omega, \wedge^l) \to D'(\Omega, \wedge^{l+1})$ the exterior derivative for $l = 0, 1, \ldots, n$. Its formal adjoint operator $d^\star\colon D'(\Omega, \wedge^{l+1}) \to D'(\Omega, \wedge^l)$ is given by $d^\star = (-1)^{nl+1} \star d \star$ on $D'(\Omega, \wedge^{l+1})$, $l = 0, 1, \ldots, n$.

We consider here solutions to the equation

$$d^\star A(x, d\omega) = B(x, d\omega), \tag{1.1}$$

where $A\colon \Omega \times \wedge^l(\mathbb{R}^n) \to \wedge^l(\mathbb{R}^n)$ satisfies the conditions

$$|A(x,\xi)| \le a|\xi|^{p-1}, \ \langle A(x,\xi), \xi \rangle \ge |\xi|^p \text{ and } |B(x,\xi)| \le b|\xi|^{p-1} \tag{1.2}$$

for almost every $x \in \Omega$ and all $\xi \in \wedge^l(\mathbb{R}^n)$. Here $a > 0$ is a constant and $1 < p < \infty$ is a fixed exponent associated with (1.1). Henceforth, p will denote this exponent. A solution to (1.1) is an element of the Sobolev space $W^1_{p,\mathrm{loc}}(\Omega, \wedge^{l-1})$ such that

$$\int_\Omega \langle A(x, d\omega), d\varphi \rangle + \langle B(x, d\omega), \varphi \rangle = 0$$

for all $\varphi \in W^1_{p,\mathrm{sloc}}(\Omega, \wedge^{l-1})$ with compact support.

Definition 1.3. We call u an A-harmonic tensor in Ω if u satisfies the A-harmonic equation (1.1) in Ω.

Example 1.4. We call u a p-harmonic function if u satisfies the p-harmonic equation

$$\mathrm{div}(\nabla u |\nabla u|^{p-2}) = 0$$

with $p > 1$.

2. The local weighted Poincaré inequality

For a measurable set $E \subset \mathbb{R}^n$ we write $|E|$ for the n dimensional Lebesgue measure of E. Throughout $Q \subset \mathbb{R}^n$ is a cube and σQ, $\sigma > 0$, denotes the cube with the same center as Q and volume $|\sigma Q| = \sigma^n |Q|$.

Definition 2.1. Let $r > 1$. We say that the weight $w(x) \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ satisfies the A_r condition, and write $w \in A_r$, if $w(x) > 0$ a.e. and

$$\sup_Q \left(\frac{1}{|Q|}\int_Q w dx\right)\left(\frac{1}{|Q|}\int_Q \left(\frac{1}{w}\right)^{1/(r-1)} dx\right)^{(r-1)} < \infty$$

for all $Q \subset \mathbb{R}^n$.

See [1] and [2] for the basic properties of A_r-weights.

We also need the following lemma, which is a reverse Hölder inequality [1].

Lemma 2.2. *If $w \in A_r$, then there exist constants $\beta > 1$ and C, independent of w, such that*

$$\|w\|_{\beta,Q} \le C|Q|^{(1-\beta)/\beta}\|w\|_{1,Q}$$

for all $Q \subset \mathbb{R}^n$.

The following Lemma 2.3 appears in [8].

Lemma 2.3. *Let u be an A-harmonic tensor in Ω, $\sigma > 1$, and $0 < s, t < \infty$. Then there exists a constant C, independent of u, such that*

$$\|u\|_{s,Q} \le C|Q|^{(t-s)/st}\|u\|_{t,\sigma Q}$$

for all Q with $\sigma Q \subset \Omega$.

Lemma 2.4 contains the classical Poincaré inequality as well as a generalization to differential forms given in [4]. When ω is a function, we denote its average value over Q by

$$\omega_Q = |Q|^{-1}\int_Q \omega(y)\, dy.$$

Otherwise ω_Q is the exterior derivative of a suitable transform of ω and plays the role of average value in the Poincaré inequality; see [4].

Lemma 2.4. *Let $u \in D'(Q, \wedge^l)$ and $du \in L^q(Q, \wedge^{l+1})$. Then $u - u_Q$ is in $W^1_q(Q, \wedge^l)$ with $1 < q < \infty$ and*

$$\|u - u_Q\|_{q,Q} \le C(n,q)|Q|^{1/n}\|du\|_{q,Q}$$

for Q in $\mathbb{R}^n$, $l = 0, 1, \ldots, n$.

We next state a Caccioppoli-type inequality. For this result see [8] and [7].

Lemma 2.5. *Let u be a solution to (1.1) in Ω and let $\sigma > 1$. There exists a constant C, depending only on a, b, p and n, such that*

$$\|du\|_{p,Q} \le C|Q|^{-1/n}\|u\|_{p,\sigma Q} \tag{2.6}$$

for all Q with $\sigma Q \subset \Omega$.

We also need the following result from [5].

Lemma 2.7. *Suppose that $|v| \in L^s_{\mathrm{loc}}(\Omega)$, $\sigma > 1$, and $0 < t < s$. If there exists a constant A such that*

$$\|v\|_{s,Q} \le A|Q|^{(t-s)/st}\|v\|_{t,2Q} \tag{2.8}$$

for all cubes Q with $2Q \subset \Omega$, then for all $r > 0$ there exists a constant B, depending only on σ, n, s, t, r and A, such that

$$\|v\|_{s,Q} \le B|Q|^{(r-s)/sr}\|v\|_{r,\sigma Q}$$

for all Q with $\sigma Q \subset \Omega$.

Lemma 2.9. *Suppose that u is a solution to (1.1), $\sigma > 1$, and $q > 0$. There exists a constant C, depending only on σ, n, p, a, b and q, such that*

$$\|du\|_{p,Q} \le C|Q|^{(q-p)/pq}\|du\|_{q,\sigma Q} \tag{2.10}$$

for all Q with $\sigma Q \subset \Omega$.

Proof. By Lemmas 2.5, 2.3 and 2.4 with $p' = (p+1)/2$,

$$\begin{aligned}\|du\|_{p,Q} &\le C_1|Q|^{1/n}\|u - u_{\sigma Q}\|_{p,\sqrt{\sigma}Q} \\ &\le C_2|Q|^{(p'-p)/pp'}\|u - u_{\sigma Q}\|_{p',\sigma Q} \\ &\le C_3|Q|^{(p'-p)/pp'}\|du\|_{p',\sigma Q}.\end{aligned}$$

Thus du satisfies the reverse Hölder inequality (2.8), and (2.10) follows from Lemma 2.7. □

We also require a result from [7].

Lemma 2.11. *There exists a constant C, depending only on n and q, such that*

$$\|v - v_Q\|_{q,Q} \le C\|v - c\|_{q,Q}$$

for all $v \in L^q(Q,\Lambda)$ and all $c \in \mathcal{D}'(Q,\Lambda)$ with $dc = 0$. Here $1 < q < \infty$ and v_Q is the average value of v over Q or the exterior derivative of a suitable transform of v.

We now have the following local weighted Poincaré inequality for A-harmonic tensors.

THEOREM 2.12. *Let $u \in D'(\Omega, \wedge^l)$ be an A-harmonic tensor in a domain $\Omega \subset \mathbb{R}^n$, and $du \in L^s(\Omega, \wedge^{l+1})$, $l = 0, 1, \ldots, n$. Assume that $\sigma > 1$, $0 < s < \infty$, and $w \in A_r$ for some $r > 1$. Then*

$$\|u - u_Q\|_{s,Q,w} \le C|Q|^{1/n}\|du\|_{s,\sigma Q,w} \tag{2.13}$$

for all cubes Q with $\sigma Q \subset \Omega$. Here C is a constant independent of u.

Proof. Choose $t = s\beta/(\beta - 1)$, where β is the exponent in Lemma 2.2. Then $0 < s < t$ and $\beta = t/(t - s)$. By Lemma 2.2 and Hölder's inequality,

$$\begin{aligned} \|u - u_Q\|_{s,Q,w} &= \left(\int_Q \left(|u - u_Q|w^{1/s}\right)^s\right)^{1/s} \\ &\le \|u - u_Q\|_{t,Q}\|w\|_{\beta,Q}^{1/s} \\ &\le C|Q|^{(1-\beta)/\beta s}\|w\|_{1,Q}^{1/s}\|u - u_Q\|_{t,Q}. \end{aligned} \tag{2.14}$$

Next choose $\alpha = s/r$ so that $\alpha < s < t$. If $\alpha > 1$ and $t > 1$, then using Lemmas 2.11, 2.3 and 2.4, we have

$$\begin{aligned} \|u - u_Q\|_{t,Q} &\le C\|u - u_{\sigma Q}\|_{t,Q} \\ &\le C|Q|^{(\alpha-t)/\alpha t}\|u - u_{\sigma Q}\|_{\alpha,\sigma Q} \\ &\le C|Q|^{(\alpha t + n\alpha - nt)/n\alpha t}\|du\|_{\alpha,\sigma Q}. \end{aligned} \tag{2.15}$$

If $t \le 1$, then first

$$\begin{aligned} \|u - u_Q\|_{t,Q} &\le C|Q|^{(2-t)/2t}\|u - u_Q\|_{2,Q} \\ &\le C|Q|^{(2-t)/2t}\|u - u_{\sigma Q}\|_{2,\sqrt{\sigma}Q} \\ &\le C|Q|^{(\alpha-t)/\alpha t}\|u - u_Q\|_{\alpha,\sigma Q}, \end{aligned}$$

and again (2.15) follows.

If $\alpha \le 1$, then using Lemmas 2.3, 2.11 and 2.4, we have

$$\begin{aligned} \|u - u_Q\|_{t,Q} &\le C|Q|^{(p-t)/pt}\|u - u_Q\|_{p,\sqrt{\sigma}Q} \\ &\le C|Q|^{(p-t)/pt}\|u - u_{\sqrt{\sigma}Q}\|_{p,\sqrt{\sigma}Q} \\ &\le C|Q|^{(p-t)/pt}|Q|^{1/n}\|du\|_{p,\sqrt{\sigma}Q}. \end{aligned} \tag{2.16}$$

Applying (2.10), (2.16) becomes

$$\|u - u_Q\|_{t,Q} \le C|Q|^{(\alpha t + n\alpha - nt)/n\alpha t}\|du\|_{\alpha,\sigma Q}. \tag{2.17}$$

Next, we have

$$\|du\|_{\alpha,\sigma Q} \le \|du\|_{s,\sigma Q,w}\|1/w\|_{\alpha/(s-\alpha),\sigma Q}^{1/s}. \tag{2.18}$$

Combining (2.14), (2.15), (2.17) and (2.18), we obtain

$$\tag{2.19} \begin{aligned}&\|u-u_Q\|_{s,Q,w}\\ &\quad \le C|Q|^{(\alpha-n)/n\alpha}\left(\|w\|_{1,Q}\|1/w\|_{\alpha/(s-\alpha),\sigma Q}\right)^{1/s}\|du\|_{s,\sigma Q,w}.\end{aligned}$$

Finally, Definition 2.1 gives the desired result

$$\|u-u_Q\|_{s,Q,w} \le C|Q|^{1/n}\|du\|_{s,\sigma Q,w}. \qquad \square$$

3. A global result in John domains

We now consider solutions u to div $A(x,\nabla u) = B(x,\nabla u)$ in $\Omega \subset \mathbb{R}^n$, which we call A-harmonic functions. We write $d\mu = w\,dx$ and denote the μ-average of the function u over the cube Q by

$$u_{Q,\mu} = \frac{1}{\mu(Q)}\int_Q u\,d\mu.$$

We assume that $0 < \mu(Q) < \infty$ for all Q.

Definition 3.1. A δ-John domain is a bounded domain $\Omega \subset \mathbb{R}^n$ with John center x_0 if every point $x \in \Omega$ can be joined to x_0 by a continuous curve $\gamma \subset \Omega$ for which $d(\xi,\partial\Omega) \ge \delta|\xi - x|$ for all $\xi \in \gamma$.

We define the sharp norm of a real-valued function f over E by

$$\|f\|^\sharp_{p,E,w} = \mathrm{Inf}_{a\in\mathbf{R}}\left(\int_E |f-a|^p d\mu\right)^{1/p}.$$

To obtain a global result we need the following result from [6]:

Theorem 3.2. *Suppose that f and g are measurable in a δ-John domain Ω with distinguished cube $Q_0 \subset \Omega$ and $0 < q < \infty$. If, for some constant A,*

$$\|f\|^\sharp_{q,Q,w} \le A\|g\|_{q,\sigma Q,w}$$

for all cubes Q with $\sigma Q \subset \Omega$, then there exists a constant B, depending only on n, q, σ and δ, such that

$$\|f\|^\sharp_{q,\Omega,w} \le AB\|g\|_{q,\Omega,w}.$$

(See also [5].)

Together with the local result, this gives a Poincaré inequality over John domains.

Theorem 3.3. *Suppose that u is an A-harmonic function in a δ-John domain Ω, $0 < q < \infty$, and $w \in A_r(\Omega)$. There exists a constant C, depending only q, δ, n, p and r, such that*

$$\|u\|^\sharp_{q,\Omega,w} \le C|\Omega|^{1/n}\|\nabla u\|_{q,\Omega,w}.$$

We remark that in the case $q \geq 1$ and $0 < \mu(E) < \infty$, we have

$$\|f\|^{\sharp}_{q,E,\mu} \leq \|f - f_{E,\mu}\|^{\sharp}_{q,E,\mu} \leq 2\|f\|^{\sharp}_{q,E,\mu}.$$

(See [6].) Thus we have the following corollary.

Corollary 3.4. *In addition to the hypotheses of Theorem 3.3, assume that $q \geq 1$ and $u_{\Omega,\mu} < \infty$. Then*

$$\|u - u_{\Omega,\mu}\|_{q,\Omega,w} \leq 2C|\Omega|^{1/n}\|\nabla u\|_{q,\Omega,w}.$$

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