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A CONVEXITY THEOREM FOR TORUS ACTIONS ON CONTACT MANIFOLDS

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Abstract. We show that the image cone of a moment map for an action of a torus on a contact compact connected manifold is a convex polyhedral cone and that the moment map has connected fibers provided the dimension of the torus is bigger than 2 and that no orbit is tangent to the contact distribution. This may be considered as a version of the Atiyah–Guillemin–Sternberg convexity theorem for torus actions on symplectic cones and as a direct generalization of the convexity theorem of Banyaga and Molino for completely integrable torus actions on contact manifolds.

1. Introduction

The goal of the paper is to prove a convexity theorem for torus actions on contact manifolds. Recall that a *contact form* on a manifold M of dimension $2n+1$ is a 1-form α such that $\alpha \wedge d\alpha^n \neq 0$. A (co-oriented) *contact structure* on a manifold M is a subbundle ξ of the tangent bundle TM which is given as the kernel of a contact form. Note that if f is any nowhere vanishing function and α is a contact form, then $\ker \alpha = \ker f\alpha$. Thus a co-oriented contact structure is a conformal class of contact forms. One can show that a hyperplane subbundle ξ of TM is a co-oriented contact structure if and only if its annihilator ξ° in T^*M is a trivial line bundle and $\xi^\circ \setminus 0$ is a symplectic submanifold of the punctured cotangent bundle $T^*M \setminus 0$ (we use 0 as a shorthand for the image of the zero section). In fact, the map $\psi_\alpha : M \times \mathbb{R} \to \xi^\circ$, $(m,t) \mapsto t\alpha_m$, defines a trivialization, and the pull-back by ψ_α of the tautological 1-form on T^*M is $t\alpha$. The symplectic manifold $(M \times (0,\infty), d(t\alpha))$ is called the *symplectization* of (M,α).

Recall that a *symplectic cone* is a symplectic manifold (N,ω) with a proper action of the real line which expands the symplectic form exponentially. For example, the action of $\mathbb{R}$ on $M \times (0,\infty)$ given by $s \cdot (m,t) = (m, e^s t)$ makes

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the symplectization $(M \times (0, \infty), d(t\alpha))$ of (M, α) into a symplectic cone. Conversely a symplectic cone is the symplectization of a contact manifold.

Throughout the paper α will always denote a contact form and ξ will always denote a co-oriented contact structure. We will refer either to a pair (M, α) or to a pair (M, ξ) as a contact manifold.

An action of a Lie group G on a contact manifold (M, ξ) is *contact* if the action preserves the contact structure. It is not hard to show that if additionally the action of G is proper (for example if G is compact) and preserves the co-orientation of ξ (for example if G is connected), then it preserves a contact form α with $\xi = \ker \alpha$ (see [L]).

Contact moment maps. We now recall the notion of a moment map for an action of a group on a contact manifold. An action of a Lie group G on a manifold M naturally lifts to a Hamiltonian action on the cotangent bundle T^*M. The corresponding moment map $\Phi : T^*M \to \mathfrak{g}^*$ is given by

$$\langle \Phi(q,p), A \rangle = \langle p, A_M(q) \rangle, \tag{1.1}$$

for all vectors $A \in \mathfrak{g}$, all points $q \in M$ and all covectors $p \in T^*_q M$. Here and elsewhere in the paper A_M denotes the vector field induced on M by $A \in \mathfrak{g}$.

If the action of the Lie group G on the manifold M preserves a contact distribution ξ, then the lifted action preserves the annihilator $\xi^\circ \subset T^*M$. Moreover, if the action of G preserves a co-orientation of ξ then it preserves the two components of $\xi^\circ \setminus 0$. Denote one of the components by ξ°_+. In this case we define the *moment map* Ψ for the action of G on (M, ξ) to be the restriction of Φ to ξ°_+:

$$\Psi = \Phi|_{\xi^\circ_+}.$$

An invariant contact form α on M defining the contact distribution ξ is a nowhere zero section of $\xi^\circ \to M$. We may assume that $\alpha(M) \subset \xi^\circ_+$. In this case we get a map $\Psi_\alpha : M \to \mathfrak{g}^*$ by composing Ψ with α: $\Psi_\alpha = \Psi \circ \alpha$. It follows from (1.1) that

$$\langle \Psi_\alpha(x), A \rangle = \alpha_x(A_M(x)) \tag{1.2}$$

for all $x \in M$ and all $A \in \mathfrak{g}$. Recall that the choice of a contact form on M establishes a bijection between the space of smooth functions on M and the space of contact vector fields. It is easy to check that for any $A \in \mathfrak{g}$ the contact vector field corresponding to the function $\langle \Psi_\alpha, A \rangle$ is A_M. Thus it makes sense to think of Ψ_α as the moment map defined by the contact form α and of Ψ as the moment map defined by the contact distribution ξ. The image $\Psi_\alpha(M)$ depends on the action and the contact form, while the image $\Psi(\xi^\circ_+)$ depends only on the action and the contact distribution. Clearly the two sets are related:

$$\Psi(\xi^\circ_+) = \mathbb{R}^+ \Psi_\alpha(M).$$

Definition 1.1. Let (M, ξ) be a co-oriented contact manifold with an action of a Lie group G preserving the contact structure ξ and its co-orientation. Let ξ°_+ denote a component of $\xi^\circ \setminus 0$, the annihilator of ξ minus the zero section. Let $\Psi : \xi^\circ_+ \to \mathfrak{g}^*$ denote the corresponding moment map. The *moment cone* $C(\Psi)$ is the set

$$C(\Psi) := \Psi(\xi^\circ_+) \cup \{0\}.$$

Note that if α is an invariant contact form with $\xi = \ker \alpha$ and $\alpha(M) \subset \xi^\circ_+$, and if $\Psi_\alpha : M \to \mathfrak{g}^*$ is the moment map defined by α, then $C(\Psi) = \{tf \mid f \in \Psi_\alpha(M),\, t \in [0, \infty)\}$.

We can now state the main result of the paper.

Theorem 1.2. *Let (M, ξ) be a co-oriented contact manifold with an effective action of a torus G preserving the contact structure and its co-orientation. Let ξ°_+ be a component of the annihilator of ξ in T^*M minus the zero section: $\xi^\circ \setminus 0 = \xi^\circ_+ \sqcup (-\xi^\circ_+)$. Assume that M is compact and connected and that the dimension of G is bigger than 2. If 0 is not in the image of the contact moment map $\Psi : \xi^\circ_+ \to \mathfrak{g}^*$ then the fibers of Ψ are connected and the moment cone $C(\Psi) = \Psi(\xi^\circ_+) \cup \{0\}$ is a convex rational polyhedral cone.*

Remark 1.3. A polyhedral set in $\mathfrak{g}^*$ is the intersection of finitely many closed half-spaces. A polyhedral set is rational if the annihilators of codimension one faces are spanned by vectors in the *integral lattice* $\mathbb{Z}_G$ of $\mathfrak{g}$, that is, by vectors in the kernel of $\exp : \mathfrak{g} \to G$. The whole space $\mathfrak{g}^*$ is trivially a rational polyhedral cone. Note that a rational polyhedral cone C in $\mathfrak{g}^*$ is of the form

$$C = \bigcap_i \{v_i \geq 0\}$$

for some finite collection of vectors $v_1, \ldots, v_r$ in the integral lattice $\mathbb{Z}_G$.

Remark 1.4. For actions of tori of dimension less than or equal than 2, the fibers of the corresponding moment maps need not be connected. For actions of two-dimensional tori the moment cone need not be convex. In fact, it is easy to construct an example of an effective 2-torus action on an overtwisted 3-sphere so that the image cone is not convex. It is also easy to construct examples of moment maps for actions of 2-tori and circles with non-connected fibers (the convexity result for circles is trivial). See [L].

Theorem 1.2 extends known convexity results for Hamiltonian torus actions on symplectic manifolds. Such results have a long history. Atiyah [A] and, independently, Guillemin and Sternberg [GS] proved that for Hamiltonian torus actions on compact symplectic manifolds the image of the moment map is a rational polytope and that the fibers of the moment map are connected. The assumption of compactness of the manifold has been subsequently weakened by de Moraes and Tomei [MT], by Prato [P], by Hilgert, Neeb, and Plank [HNP] using the methods of [CDM], and by Lerman, Meinrenken, Tolman and Woodward [LMTW] to the point where it is enough to assume that the moment map is *proper* as a map from a symplectic manifold M to a *convex* open subset U of the dual of the Lie algebra $\mathfrak{g}^*$. The conclusion is that the fibers of the moment map are connected and that the intersection of the image of the moment map with U is a convex locally polyhedral set. Note that the hypotheses of Theorem 1.2 only guarantee that the moment map $\Psi : \xi^\circ_+ \to \mathfrak{g}^*$ is proper as a map into $\mathfrak{g}^* \setminus \{0\}$, which is certainly not convex.

Theorem 1.2 is a direct generalization of a convexity theorem of Banyaga and Molino [BM2]:

Theorem 1.5 (Banyaga–Molino). *Let (M, ξ) be a co-oriented contact manifold with an effective contact action of a torus G preserving the co-orientation. Assume that M is compact and connected, that the dimension of G is bigger than 2 and that $\dim M + 1 = 2 \dim G$. Then the moment cone $C(\Psi)$ is a convex rational polyhedral cone.*

Remark 1.6. It is easy to show the hypotheses of the Banyaga–Molino theorem guarantee that the image of the moment map does not contain the origin:

Lemma 1.7. *Let (M, ξ) be a co-oriented contact manifold with an effective action of a torus G preserving the contact structure and its co-orientation. Let α be an invariant contact form with $\ker \alpha = \xi$ and let $\Psi_\alpha : M \to \mathfrak{g}^*$ be the corresponding moment map. If $\dim M + 1 = 2 \dim G$ then $\Psi_\alpha(x) \neq 0$ for any $x \in M$.*

Proof. Suppose not. Then for some point $x \in M$ the orbit $G \cdot x$ is tangent to the contact distribution. Therefore the tangent space $\zeta_x := T_x(G \cdot x)$ is isotropic in the symplectic vector space (ξ_x, ω_x) where $\omega_x = d\alpha_x|_\xi$.

We now argue that this forces the action of G not to be effective. More precisely we argue that the slice representation of the connected component of identity H of the isotropy group of the point x is not effective. The group H acts on ξ_x preserving the symplectic form ω_x and preserving $\zeta_x = T_x(G \cdot x)$. Since ζ_x is isotropic, $\xi_x = (\zeta_x^\omega / \zeta_x) \oplus (\zeta_x \times \zeta_x^*)$ as a symplectic representation of H. Here ζ_x^ω denotes the symplectic perpendicular to ζ_x in (ξ_x, ω_x). Note that since G is a torus, the action of H on ζ_x is trivial. Hence it is trivial on ζ_x^*.

Observe next that the dimension of the symplectic vector space $V =: \zeta_x^\omega / \zeta_x$ is $\dim \xi_x - 2 \dim \zeta_x = \dim M - 1 - 2(\dim G - \dim H) = (\dim M - 1) - (\dim M + 1) + 2 \dim H = 2 \dim H - 2$. On the other hand, since H is a compact connected Abelian group acting symplecticly on V, its image in the group of symplectic linear transformations $\mathrm{Sp}(V)$ lies in a maximal torus T of a maximal compact subgroup of $\mathrm{Sp}(V)$. The dimension of T is $\dim V/2 = \dim H - 1$. Therefore the representation of H on V is not faithful. Since the fiber at x of the normal bundle of $G \cdot x$ in M is $(T_x M/\xi_x) \oplus (\xi_x/\zeta_x) \simeq \mathbb{R} \oplus (V \oplus \zeta_x^*)$, the slice representation of H is not faithful. Consequently the action of G in not effective in a neighborhood of an orbit $G \cdot x$. This is a contradiction. $\square$

Remark 1.8. The paper [BM2] is not published. It is a revision of [BM1], which is not widely available, but has an extensive review in Math. Reviews (MR 94c53029). Theorem 1.5 is cited without proof in [B]. Providing an independent and easily accessible proof of Theorem 1.5 is one of the motivations for this paper.

Remark 1.9. I do not know if the condition that no orbit is tangent to the contact distribution is necessary for Theorem 1.2 to hold.

A note on notation. Throughout the paper the Lie algebra of a Lie group denoted by a capital Roman letter will be denoted by the same small letter in the fraktur font: thus $\mathfrak{g}$ denotes the Lie algebra of a Lie group G, etc. The vector space dual to $\mathfrak{g}$ is denoted by $\mathfrak{g}^*$. The identity element of a Lie group is denoted by 1. The natural pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$ will be denoted by $\langle \cdot, \cdot \rangle$.

When a Lie group G acts on a manifold M we denote the action by an element $g \in G$ on a point $x \in G$ by $g \cdot x$; $G \cdot x$ denotes the G-orbit of x, and so on. The vector field induced on M by an element X of the Lie algebra $\mathfrak{g}$ of G is denoted by X_M. The isotropy group of a point $x \in M$ is denoted by G_x; the Lie algebra of G_x is denoted by $\mathfrak{g}_x$ and is referred to as the isotropy Lie algebra of x. We recall that $\mathfrak{g}_x = \{X \in \mathfrak{g} \mid X_M(x) = 0\}$.

If P is a principal G-bundle then $[p, m]$ denotes the point in the associated bundle $P \times_G M = (P \times M)/G$ which is the orbit of $(p, m) \in P \times M$.

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2. Torus actions on contact manifolds

We now proceed with a proof of Theorem 1.2. The methods we use is a mixture of the ideas from [CDM] and [LMTW].

Recall that M denotes a compact connected manifold with an effective action of a torus G ($\dim G > 2$) preserving a co-oriented contact distribution ξ. Choose a G-invariant contact form α with $\ker \alpha = \xi$. Let $\Psi_\alpha : M \to \mathfrak{g}^*$ be the corresponding moment map; it is defined by equation (1.2). Recall also that we assume that $0 \notin \Psi_\alpha(M)$. Note that this condition amounts to saying that no orbit of G is tangent to the contact distribution ξ; thus it is a condition on a contact distribution and not on a particular choice of a contact form representing the distribution.

Next fix an inner product on the dual of the Lie algebra $\mathfrak{g}^*$. Since $\Psi_\alpha(x) \neq 0$ for all x we can define a new contact form α' by

$$\alpha'_x := \frac{1}{\|\Psi_\alpha(x)\|}\alpha_x.$$

Then the corresponding moment map $\Psi_{\alpha'}$ satisfies $\|\Psi_{\alpha'}(x)\| = 1$ for all $x \in M$. We assume from now on that we have chosen an invariant contact form α in such a way that the corresponding moment map Ψ_α sends M to the unit sphere $S := \{f \in \mathfrak{g}^* \mid \|f\| = 1\}$.

Lemma 2.1. *Let (M, ξ) be a co-oriented contact manifold with an effective contact action of a torus G. Assume that no orbit of G is tangent to the contact distribution. Let α be a G-invariant contact form defining ξ normalized so that the image of M under the corresponding moment map Ψ_α lies in the unit sphere S in $\mathfrak{g}^*$. Let $H \subset \mathfrak{g}^*$ be an open half-space, i.e., suppose that for some $0 \neq v \in \mathfrak{g}$ we have $H = \{f \in \mathfrak{g}^* \mid \langle f, v\rangle > 0\}$.*

For any connected component N of $\Psi_\alpha^{-1}(H)$, the fibers of $\Psi_\alpha|_N$ are connected.

Lemma 2.2. *Let M, ξ, G, α and Ψ_α be as in Lemma 2.1 above. Let H be an open half-space and N a component of $\Psi_\alpha^{-1}(H)$. Then $\Psi_\alpha(N)$ is a convex rational polyhedral subset of $H \cap S \subset \mathfrak{g}^*$ with open interior.*

Remark 2.3. A subset W of the unit sphere $S = \{f \in \mathfrak{g}^* \mid \|f\| = 1\}$ is convex iff there is a convex cone $C \subset \mathfrak{g}^*$ (with the vertex at the origin) so that $W = S \cap C$. Equivalently, W is convex if for any two points $x, y \in W$ there is a geodesic of length $\leq \pi$ connecting x to y and lying entirely in W.

A subset W of S (respectively of $H \cap S$) is rational polyhedral if there exist vectors $v_1, \dots v_k$ in the integral lattice $\mathbb{Z}_G = \ker\{\exp : \mathfrak{g} \to G\}$ such that

$$W = \{f \in S \mid \langle f, v_i\rangle \geq 0, \quad 1 \leq i \leq k\}$$

(respectively $W = \{f \in S \cap H \mid \langle f, v_i\rangle \geq 0, \quad 1 \leq i \leq k\}$).

Proof of Lemmas 2.1 and 2.2. Consider the symplectization $(M \times \mathbb{R}, d(e^t\alpha))$ of (M, α). As usual t denotes the coordinate on $\mathbb{R}$. The contact action of G on M extends trivially to a Hamiltonian action on the symplectization. The corresponding moment map $\Phi : M \times \mathbb{R} \to \mathfrak{g}^*$ is given by

$$\Phi(x, t) = e^t \Psi_\alpha(x).$$

The symplectic manifold $(N \times \mathbb{R}, d(e^t\alpha)|_{N\times\mathbb{R}})$ is a symplectization of $(N, \alpha|_N)$. The manifold $N \times \mathbb{R}$ is a connected symplectic manifold with a Hamiltonian action of G, the map $\Phi_N := \Phi|_{N\times\mathbb{R}}$ is a corresponding moment map for the action of G. Moreover, it has the following two properties:

1. $\Phi_N(N \times \mathbb{R})$ is contained in the convex open subset H of $\mathfrak{g}^*$;
2. $\Phi_N : N \times \mathbb{R} \to H$ is proper.

Therefore Theorem 4.3 of [LMTW] applies. We conclude that the fibers of Φ_N are connected and that the image $\Phi_N(N \times \mathbb{R})$ is convex.

Next, since the action of the torus G on M is effective, it is free on a dense open subset of M. This is a consequence of the principal orbit type theorem and the fact that G is abelian. Consequently the action of G on $N \times \mathbb{R}$ is free on a dense open subset. Hence the image $\Phi_N(N \times \mathbb{R})$ has non-empty interior. Also, since M is compact and G is abelian, the number of subgroups of G that occur as isotropy groups of points of M is finite. Therefore not only does [LMTW, Theorem 4.3] imply that $\Phi_N(N \times \mathbb{R})$ is the intersection a locally polyhedral subset of $\mathfrak{g}^*$ with the open half-space H, but that in fact $\Phi_N(N \times \mathbb{R}) = \Phi(N \times \mathbb{R})$ is a polyhedral cone. □

LEMMA 2.4. *Let M, G, α and Ψ_α be as in Lemma 2.1 above. Define an equivalence relation $\sim$ on M by declaring the equivalence classes to be the connected components of the fibers of the moment map Ψ_α. Let $\overline{M} = M/\sim$.*

Then $\overline{M}$ is a compact path connected space and the moment map $\Psi_\alpha : M \to \mathfrak{g}^*$ descends to a continuous map $\overline{\Psi} : \overline{M} \to S$, where as before S is the unit sphere in $\mathfrak{g}^*$ centered at 0.

Moreover, $\overline{M}$ is a length space and $\overline{\Psi} : \overline{M} \to S$ is locally an isometric embedding. More precisely, for any open half-space H and any connected component N of $\overline{\Psi}^{-1}(H)$ the map $\overline{\Psi}|_N : N \to S$ is an isometric embedding.

Our proof of Lemma 2.4 uses length spaces, the notion that is due to Gromov [G1, G2]. We therefore briefly summarize the relevant facts. The treatment follows D. Burago, Yu. Burago and S. Ivanov [BBI].

2.1. Digression: length structures and length spaces. Let X be a topological space. Consider a class $\mathcal{A}$ of continuous paths in X which is closed under restrictions, concatenations and reparameterizations. Suppose that there is a map $L : \mathcal{A} \to [0, \infty]$ (the "length") satisfying the following conditions for any curve $\gamma : [a, b] \to X$ in $\mathcal{A}$:

(a) $L(\gamma) = L(\gamma|_{[a,c]}) + L(\gamma|_{[c,b]})$ for any $c \in (a, b)$.
(b) The function $L_t := L(\gamma|_{[a,t]})$ is a continuous function of $t \in [a, b]$.
(c) If $\varphi : [c, d] \to [a, b]$ is monotone and continuous, then $L(\gamma) = L(\gamma \circ \varphi)$.
(d) If a sequence of curves $\gamma_i \in \mathcal{A}$ converges to γ uniformly, then $L(\gamma) \leq \liminf L(\gamma_i)$.
(e) If $U \subset X$ is a proper open subset, and $p \in U$ is a point then the number

$$\inf\{L(\gamma) \mid \gamma : [a, b] \to X, \gamma \in \mathcal{A}, \gamma(a) = p, \gamma(b) \notin U\}$$

is positive.

Definition 2.5. The triple $(X, \mathcal{A}, L)$, where X is a topological space, $\mathcal{A}$ is a class of continuous curves in X and $L : \mathcal{A} \to [0, \infty]$ is a map satisfying the conditions above, is called a *length structure*.

Let $(X, \mathcal{A}, L)$ be a length structure. Suppose that for any two points $x, y \in X$ there is a path $\gamma \in \mathcal{A}$ starting at x and ending at y. We then define the *distance* $d_L : X \times X \to [0, \infty]$ by

$$d_L(x, y) = \inf\{L(\gamma) \mid \gamma : [a, b] \to X, \gamma(a) = x, \gamma(y) = b, \gamma \in \mathcal{A}\}.$$

One can check that if $d_L(x, y) < \infty$ for all $x, y \in X$ then d_L is a metric.

Suppose (X, d) is a metric space. Then we can take $\mathcal{A}$ to be the set of rectifiable paths and $L = L_d : \mathcal{A} \to [0, \infty]$ to be the length functional. Then $(X, \mathcal{A}, L)$ is a length structure. Note that in general $d_L(x, y) \geq d(x, y)$ for $x, y \in X$. If $d_L = d$ then (X, d) is called a *length space*. A unit sphere S in a normed finite dimensional vector space with the standard metric induced by the embedding is an example of a length space.

Definition 2.6. Let $(X, \mathcal{A}, L)$ be a length structure. Let $\gamma : [a, b] \to X$ be a curve in $\mathcal{A}$. It is a *geodesic* if for any $c, d \in [a, b]$ with $|c - d|$ sufficiently small $L(\gamma|_{[c,d]}) = d_L(\gamma(c), \gamma(d))$.

Remark 2.7. We think of geodesics as maps, not as subsets. Also, from now on all geodesics are parameterized by arc length.

If (X, d) is a compact connected metric space then a version of the Hopf-Rinow theorem holds, and so any two points of X can be connected by a geodesic. See, for example, Proposition 3.7 in [BH]. This ends our digression on length spaces.

Proof of Lemma 2.4. It is clear that $\overline{M}$ is a compact path-connected topological space and that the moment map $\Psi_\alpha : M \to \mathfrak{g}^*$ descends to a continuous map $\overline{\Psi} : \overline{M} \to S = \{\|f\| = 1\}$. Moreover, by Lemmas 2.1 and 2.2, for any open half-space $H \subset \mathfrak{g}^*$ and any component Z of $\overline{\Psi}^{-1}(H)$, the map $\overline{\Psi} : Z \to S \cap H$ is a topological embedding which is a homeomorphism on an open dense set.

This gives us a way to define a length structure on $\overline{M}$: We define the class $\mathcal{A}$ to be the set of all curves $\overline{\gamma} : [a, b] \to \overline{M}$ such that $\overline{\Psi} \circ \overline{\gamma}$ is a rectifiable curve in the unit sphere S. For $\overline{\gamma} \in \mathcal{A}$ we set $L(\overline{\gamma}) = L_S(\overline{\Psi} \circ \overline{\gamma})$, where L_S is the length functional on the rectifiable curves in the sphere defined by the standard metric. Let d_L be the corresponding metric on $\overline{M}$. Then, since for any half-space H and any component Z of $\overline{\Psi}^{-1}(H)$ the set $\overline{\Psi}(Z)$ is convex in the sphere S, the map $\overline{\Psi} : Z \to S$ is an *isometric* embedding. Thus $\overline{\Psi} : \overline{M} \to S$ is locally an isometric embedding. □

Corollary 2.8. *Let $\overline{M}$, $\overline{\Psi}$ and S be as in Lemma 2.4. If $\overline{\gamma}$ is a geodesic in $\overline{M}$ then $\overline{\Psi}\circ\overline{\gamma}$ is a geodesic in S.*

Remark 2.9. Since $\overline{\Psi}$ is a local isometry it maps geodesics in $\overline{M}$ to geodesics in the unit sphere S of the *same length*. In particular, if the end points of a (nonconstant) geodesic $\overline{\gamma}$ in $\overline{M}$ are sent by $\overline{\Psi}$ to the same point in the sphere, then $\overline{\Psi}\circ\overline{\gamma}$ multiply covers a great circle and consequently the length of $\overline{\gamma}$ is an integer multiple of 2π.

We emphasize that Lemmas 2.1 and 2.2 can be restated for the induced map $\overline{\Psi}:\overline{M}\to S$ of Lemma 2.4 as follows:

Lemma 2.10. *For any open half-space H and any connected component N of $\overline{\Psi}^{-1}(H)$ the map $\overline{\Psi}|_N\to S$ is an isometric embedding.*

Lemma 2.11. *For any open half-space H and any connected component N of $\overline{\Psi}^{-1}(H)$ the set $\overline{\Psi}(N)$ is a convex polyhedral subset of the sphere S with non-empty interior.*

As a consequence of Lemmas 2.10 and 2.11 we get:

Corollary 2.12. *Let $\overline{\Psi}:\overline{M}\to S$ be as in Lemma 2.4. Suppose the points $x_1, x_2\in\overline{M}$ lie in the same connected component of $\overline{\Psi}^{-1}(H)$ for some open half-space H.*

If $\overline{\Psi}(x_1)=\overline{\Psi}(x_2)$ then $x_1=x_2$. If $\overline{\Psi}(x_1)\neq\overline{\Psi}(x_2)$ then there is a geodesic $\overline{\gamma}$ in $\overline{M}$ connecting x_1 to x_2. Moreover we may choose $\overline{\gamma}$ such that $\overline{\Psi}\circ\overline{\gamma}$ is a geodesic in S lying entirely in the half-space H and connecting $\overline{\Psi}(x_1)$ and $\overline{\Psi}(x_2)$.

As a consequence of Lemma 2.4 we get:

Corollary 2.13. *Any two points in $\overline{M}$ can be connected by a short geodesic, i.e., for any two points $x, y\in\overline{M}$ there is a geodesic $\overline{\gamma}$ with $\overline{\gamma}(0)=x$ and $\overline{\gamma}(d)=y$, where d is the distance between x and y (recall that all geodesics are parameterized by arc length).*

Remark 2.14. Such a geodesic in $\overline{M}$ need not be unique. For example, consider the unit co-sphere bundle M in the cotangent bundle of a flat torus G. Then $M=G\times S$, $\Psi: G\times S\to S\subset\mathfrak{g}^*$ is the projection and $\overline{M}$ is the unit sphere S. In this case for any point $x\in\overline{M}=S$ there are infinitely many geodesics of length π connecting x and $-x$.

The following lemma uses the notation above.

Lemma 2.15. *Suppose x_1, x_2 are two points in $\overline{M}$ connected by a path $\overline{\gamma}$ with the property that $\overline{\Psi} \circ \overline{\gamma}$ lies entirely in some open half-space H. Then the points x_1, x_2 lie in the same connected component of $\overline{\Psi}^{-1}(H)$.*

Proof. The image of $\overline{\gamma}$ lies in a connected component of $\overline{\Psi}^{-1}(H)$. □

Lemma 2.16 below is the main technical tool for proving the connectedness of fibers of moment maps.

Lemma 2.16. *Let $\overline{\Psi} : \overline{M} \to S$ be as in Lemma 2.4. Suppose $\overline{\gamma}_1$, $\overline{\gamma}_2$ are two distinct geodesics in $\overline{M}$ with $\overline{\gamma}_1(0) = \overline{\gamma}_2(0)$, and suppose that $\overline{\Psi} \circ \gamma_1$ and $\overline{\Psi} \circ \gamma_2$ trace out two* distinct *great circles in the unit sphere S. Then $\overline{\gamma}_2(0) = \overline{\gamma}_2(2\pi)$ (and so $\overline{\gamma}_1(0) = \overline{\gamma}_1(2\pi)$).*

Remark 2.17. Note that the assumption $\dim G > 2$ is crucial for the lemma to make sense.

Proof of Lemma 2.16. The idea of the proof is to show that there is an open half-space H containing $\overline{\Psi}(\overline{\gamma}_2(0))$ such that $\overline{\gamma}_2(0)$ and $\overline{\gamma}_2(2\pi)$ lie in the same connected component of $\overline{\Psi}^{-1}(H)$. For then, by Corollary 2.12, $\overline{\gamma}_2(0) = \overline{\gamma}_2(2\pi)$.

Given a path $\overline{\gamma}_i$ in $\overline{M}$ we write γ_i for the path $\overline{\Psi} \circ \overline{\gamma}_i$ in S.

Since by assumption the geodesics γ_1 and γ_2 trace out two distance great circles in S, $\gamma_1(\frac{\pi}{2}) \neq \pm\gamma_2(\frac{\pi}{2})$. On the other hand, we clearly have $\gamma_1(0) = -\gamma_1(\pi) = -\gamma_2(\pi)$, $\gamma_1(2\pi) = \gamma_2(2\pi) = \gamma_1(0)$, $\gamma_1(\frac{3\pi}{2}) = -\gamma_1(\frac{\pi}{2})$, and $\gamma_2(\frac{3\pi}{2}) = -\gamma_2(\frac{\pi}{2})$.

Since $\gamma_1(\frac{\pi}{2}) \neq \pm\gamma_2(\frac{\pi}{2})$, there is an open half-space H_1 containing the points $\gamma_1(0)$, $\gamma_1(\frac{\pi}{2})$ and $\gamma_2(\frac{\pi}{2})$. By Lemma 2.15, $\overline{\gamma}_1(\frac{\pi}{2})$ and $\overline{\gamma}_2(\frac{\pi}{2})$ lie in the same connected component of $\overline{\Psi}^{-1}(H_1)$ as $\overline{\gamma}_1(0)$. By Corollary 2.12 there a geodesic $\overline{\sigma}_1$ in $\overline{M}$ connecting $\overline{\gamma}_1(\frac{\pi}{2})$ to $\overline{\gamma}_2(\frac{\pi}{2})$ such that $\sigma_1 := \overline{\Psi} \circ \overline{\sigma}_1$ traces out a short geodesic connecting $\gamma_1(\frac{\pi}{2})$ to $\gamma_2(\frac{\pi}{2})$.

Choose an open half-space H_2 containing the points $\gamma_1(\frac{\pi}{2})$, $\gamma_2(\frac{\pi}{2})$ and $\gamma_1(\pi) = \gamma_2(\pi)$. Note that by construction $\overline{\gamma}_1(\frac{\pi}{2})$ is connected to $\overline{\gamma}_2(\frac{\pi}{2})$ by $\overline{\sigma}_1$, $\overline{\gamma}_1(\frac{\pi}{2})$ is connected to $\overline{\gamma}_1(\pi)$ by a piece of $\overline{\gamma}_1$ and $\overline{\gamma}_2(\frac{\pi}{2})$ is connected to $\overline{\gamma}_2(\pi)$ by a piece of $\overline{\gamma}_2$. By Lemma 2.15 $\overline{\gamma}_1(\pi)$ and $\overline{\gamma}_2(\pi)$ lie in the same connected component of $\overline{\Psi}^{-1}(H_2)$. By Corollary 2.12 we have $\overline{\gamma}_1(\pi) = \overline{\gamma}_2(\pi)$.

Choose a half-space H_3 containing $\gamma_1(\pi)$, $\gamma_1(\frac{\pi}{2})$ and $\gamma_2(\frac{3\pi}{2})$. Since $\overline{\gamma}_1(\pi) = \overline{\gamma}_2(\pi)$, since $\overline{\gamma}_1(\pi)$ is connected to $\overline{\gamma}_1(\frac{\pi}{2})$ by a piece of $\overline{\gamma}_1$ and since $\overline{\gamma}_2(\pi)$ is connected to $\overline{\gamma}_2(\frac{3\pi}{2})$ by a piece of $\overline{\gamma}_2$, $\overline{\gamma}_1(\frac{\pi}{2})$ and $\overline{\gamma}_2(\frac{3\pi}{2})$ lie in the same connected component of $\overline{\Psi}^{-1}(H_3)$. By Corollary 2.12 there a geodesic $\overline{\sigma}_2$ in $\overline{M}$ connecting $\overline{\gamma}_1(\frac{\pi}{2})$ to $\overline{\gamma}_2(\frac{3\pi}{2})$ such that $\sigma_2 := \overline{\Psi} \circ \overline{\sigma}_2$ traces out a short geodesic connecting $\gamma_1(\frac{\pi}{2})$ to $\gamma_2(\frac{3\pi}{2})$.

Finally choose a half-space H_4 containing $\gamma_1(0) = \gamma_2(0) = \gamma_2(2\pi)$, $\gamma_1(\frac{\pi}{2})$ and $\gamma_2(\frac{3\pi}{2})$. Arguing as above we see that $\overline{\gamma}_2(0)$ and $\overline{\gamma}_2(2\pi)$ lie in the same connected component of $\overline{\Psi}^{-1}(H_4)$. Hence, by Corollary 2.12, $\overline{\gamma}_2(0) = \overline{\gamma}_2(2\pi)$. □

LEMMA 2.18. *The fibers of the map $\overline{\Psi} : \overline{M} \to S$ are connected, i.e., $\overline{\Psi}$ is an embedding.*

Proof. Suppose $x_1, x_2 \in \overline{M}$ are two points with $\overline{\Psi}(x_1) = \overline{\Psi}(x_2)$. We want to show that $x_1 = x_2$. Suppose not. Then the distance d between x_1 and x_2 is positive. Let $\overline{\gamma}_1$ be a short geodesic connecting x_1 and x_2, so that $\overline{\gamma}_1(0) = x_1$ and $\overline{\gamma}_1(d) = x_2$. Then $\gamma_1 := \overline{\Psi} \circ \overline{\gamma}_1$ is a geodesic in the unit sphere S starting and ending at $\gamma_1(0)$. Therefore γ_1 multiply covers a great circle in S (and so d is an integer multiple of 2π).

Suppose that we can construct a geodesic $\overline{\gamma}_2$ connecting x_1 to x_2 so that $\gamma_2 := \overline{\Psi} \circ \overline{\gamma}_2$ covers a great circle distinct from the one covered by γ_1. Then by Lemma 2.16 $\overline{\gamma}_1(0) = \overline{\gamma}_1(2\pi)$, contradicting the choice of $\overline{\gamma}_1$ as a short geodesic.

Now we construct $\overline{\gamma}_2$ with the required properties. Pick an open half-space H containing $\gamma_1(0)$. Let N denote the connected component of $\overline{\Psi}^{-1}(H)$ containing x_1. By Lemma 2.11 the set $\overline{\Psi}(N)$ is convex with nonempty interior. Pick a point y in N so that $\overline{\Psi}(y)$ is not in the image of the geodesic γ_1. By Corollary 2.12 there is a geodesic $\overline{\sigma}$ connecting x_1 to y with the image of $\sigma := \overline{\Psi} \circ \overline{\sigma}$ lying entirely in H. Let $\overline{\tau}$ be a short geodesic connecting y to x_2. If the image of $\tau := \overline{\Psi} \circ \overline{\tau}$ lies entirely in a half-space containing $\overline{\Psi}(x_2)$ and $\overline{\Psi}(y)$ then by Lemma 2.15 we have $x_1 = x_2$.

Otherwise τ traces out a long geodesic connecting $\overline{\Psi}(y)$ to $\overline{\Psi}(x_2) = \gamma_1(0)$. If $\overline{\tau}$ passes through x_1 then the piece of $\overline{\tau}$ starting at x_1 and ending at x_2 is the desired geodesic $\overline{\gamma}_2$. If $\overline{\tau}$ does not pass through x_1, concatenate $\overline{\sigma}$ with $\overline{\tau}$. The concatenation $\overline{\gamma}_2$ is the desired geodesic. □

LEMMA 2.19. *The image of the map $\overline{\Psi} : \overline{M} \to S$ is convex.*

Proof. Suppose f_1, f_2 are two points in the image of $\overline{\Psi}$. Then either f_1 and f_2 lie in some open half-space H or $f_1 = -f_2$. In the former case, by Lemma 2.18, $N = \overline{\Psi}^{-1}(H)$ is connected. Hence, by Lemma 2.11, $\overline{\Psi}(N) = H \cap \overline{\Psi}(\overline{M})$ is convex and consequently $\overline{\Psi}(\overline{M})$ is convex.

In the latter case we argue as follows. The sets $\overline{\Psi}^{-1}(f_i)$, $i = 1, 2$ consists of single points; denote these points by x_i. Connect x_1 and x_2 by a short geodesic $\overline{\gamma}$. Then the image of $\gamma = \overline{\Psi} \circ \overline{\gamma}$ contains an arc of a great circle in S passing through f_1 and $f_2 = -f_1$ (in fact it follows from the proof of Lemma 2.16 that the image of γ is exactly such an arc). □

LEMMA 2.20. *Let $\Psi_\alpha : M \to \mathfrak{g}^*$ be a moment map as in Lemma 2.1. The corresponding moment cone $C(\Psi)$ is a rational convex polyhedral cone. That is either $C(\Psi) = \mathfrak{g}^*$ or there exist vectors $v_1, \dots, v_k$ in the integral lattice $\mathbb{Z}_G$ of the torus G such that*

$$C(\Psi) = \bigcap_i \{v_i \geq 0\}.$$

Proof. By Lemmas 2.11 and 2.18 for any open half-space H of $\mathfrak{g}^*$ there exist vectors $v_1, \dots, v_r$ in the integral lattice $\mathbb{Z}_G$ (r depends on H) such that

$$C(\Psi) \cap H = \left(\bigcap_i \{v_i \geq 0\} \right) \cap H.$$

Moreover, we may and will assume that the set of v_i's is minimal. Thus no v_i is strictly positive on $C(\Psi) \cap H$. Since the moment cone is a cone on a compact set, there exist finitely many open half-spaces $H^1, \dots, H^s$ such that $\bigcup_\beta H^\beta$ contains $C(\Psi) \setminus \{0\}$. For each such half-space H^β, let $v_1^\beta, \dots, v_{r(\beta)}^\beta$ be the minimal set of integral vectors so that

$$C(\Psi) \cap H^\beta = \left(\bigcap_i \{v_i^\beta \geq 0\} \right) \cap H^\beta.$$

We claim that

$$C(\Psi) = \bigcap_{i,\beta} \{v_i^\beta \geq 0\}.$$

As a first step we argue that for any i, β we have

$$C(\Psi) \subset \{v_i^\beta \geq 0\}.$$

By choice of v_i^β there exists a point $x \in C(\Psi) \cap H^\beta$ such that $v_i^\beta(x) = 0$ (since $x \in H^\beta$, $x \neq 0$). Suppose there exists a point $y \in C(\Psi)$ with $v_i^\beta(y) < 0$. Since $C(\Psi)$ is convex, $tx + (1-t)y \in C(\Psi)$ for all $t \in [0,1]$. On the other hand, $v_i^\beta(tx + (1-t)y) = (1-t)v_i^\beta(y) < 0$ for all $t \in [0,1)$. Since H^β is open there is $\epsilon > 0$ so that $tx + (1-t)y \in H^\beta$ for all $t \in (\epsilon, 1]$. Therefore for all $t \in (\epsilon, 1)$ we have

$$tx + (1-t)y \in H^\beta \cap C(\Psi) \subset \{v_i^\beta \geq 0\},$$

which is a contradiction. We conclude that

$$C(\Psi) \subset \bigcap_{i,\beta} \{v_i^\beta \geq 0\}.$$

Next we argue that the reverse inclusion, i.e., $\bigcap_{i,\beta} \{v_i^\beta \geq 0\} \subset C(\Psi)$, holds as well. By construction, for each β

$$C(\Psi) \cap H^\beta = \left(\bigcap_i \{v_i^\beta \geq 0\} \right) \cap H^\beta.$$

Since $\bigcup_\beta H^\beta \cup \{0\}$ covers the image cone $C(\Psi)$, we have

$$\begin{aligned} C(\Psi) = C(\Psi) \cap \left(\bigcup_\beta H^\beta \cup \{0\} \right) &= \{0\} \cup \bigcup_\beta (C(\Psi) \cap H^\beta) \\ &= \bigcup_\beta \left(\bigcap_i \{v_i^\beta \geq 0\} \cap (H^\beta \cup \{0\} \right) \\ &\supseteq \left(\bigcap_{i,\beta} \{v_i^\beta \geq 0\} \right) \cap \left(\bigcup_\beta H^\beta \cup \{0\} \right). \end{aligned}$$

Therefore

$$C(\Psi) = \left(\bigcap_{i,\beta} \{v_i^\beta \geq 0\} \right) \cap \left(\bigcup_\beta H^\beta \cup \{0\} \right). \tag{2.1}$$

Finally, since $\bigcap_{i,\beta}\{v_i^\beta \geq 0\}$ is closed and convex, its intersection with the unit sphere $S \cap \bigcap_{i,\beta}\{v_i^\beta \geq 0\}$ is closed and connected. On the other hand,

$$\begin{aligned} S \cap \bigcap_{i,\beta} \{v_i^\beta \geq 0\} = &\left(S \cap \bigcap_{i,\beta} \{v_i^\beta \geq 0\} \cap \left(\bigcup_\beta H^\beta \right) \right) \\ &\sqcup S \cap \left(\bigcap_{i,\beta} \{v_i^\beta \geq 0\} \setminus \left(\bigcup_\beta H^\beta \right) \right). \end{aligned} \tag{2.2}$$

It follows from (2.1) and (2.2) that the set $S \cap \bigcap_{i,\beta}\{v_i^\beta \geq 0\}$ is a disjoint union of two closed sets. Therefore the set $S \cap \left(\bigcap_{i,\beta}\{v_i^\beta \geq 0\} \setminus \cup_\beta H^\beta \right)$ is empty. We conclude that

$$C(\Psi) = \bigcap_{i,\beta} \{v_i^\beta \geq 0\} \cap \left(\bigcup_\beta H^\beta \cup \{0\} \right) = \bigcap_{i,\beta} \{v_i^\beta \geq 0\}. \qquad \square$$

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