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THE EULER CLASS AS A COHOMOLOGY GENERATOR

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ABSTRACT. We show that a generator of the cohomology group $H^n(S^n)$ cannot be realized as the Euler class of a vector bundle over the n-sphere unless n equals 2, 4, or 8.

A basic question in algebraic topology is which cohomology classes can be realized as characteristic classes. It is known that if X is a CW-complex, then for any cohomology class $\alpha \in H^k(X)$, there exists a vector bundle over X whose Euler class is a multiple of α [4]. What does not seem to be known, however, is whether α itself can occur as an Euler class, even in the simplest possible non-trivial case when X is a sphere.

In this note, we give a geometric proof of the fact that a generator of $H^n(S^n)$ cannot in general occur as the Euler class of a bundle over S^n. More precisely, when $n \neq 2, 4, 8$, then the Euler class of any vector bundle over S^n must be an even multiple of a generator of $H^n(S^n)$. This in turn implies that the Stiefel-Whitney class of any bundle over a sphere is trivial, provided n is not one of these exceptional values. Another consequence is that any rank $4n$ bundle over S^{4n} with trivial Pontrjagin class is equivalent to a pullback $f^*\tau$ of the tangent bundle τ, for some map $f : S^{4n} \to S^{4n}$.

We also examine some extensions of these results to the non-spherical case.

1. The Euler class of a vector bundle over a sphere

We begin by recalling a geometric way of computing the Euler number of a rank n bundle ξ over S^n: Denote by p and q a pair of antipodal points, and by U_+ and U_- their complements in S^n. Since U_+, U_- are contractible, there are trivializations

$$\phi_+ : U_+ \times \mathbb{R}^n \to \pi^{-1}(U_+), \quad \phi_- : U_- \times \mathbb{R}^n \to \pi^{-1}(U_-).$$

Restricting to the equator, we have a map $\phi_-^{-1} \circ \phi_+ : S^{n-1} \times \mathbb{R}^n \to S^{n-1} \times \mathbb{R}^n$ sending (p, u) to $(p, g(p)u)$ with $g : S^{n-1} \to SO(n)$. g is called the *clutching map* of E, and its significance lies in that free homotopy classes of such maps classify vector bundles over S^n up to isomorphism type.

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Now fix a vector $u \in S^{n-1}$, and define a map $f : S^{n-1} \to S^{n-1}$ by $f(x) = g(x)u$, $x \in S^{n-1}$. The following lemma follows from the arguments in [5], but as it is not explicitly argued, we provide a proof here for convenience of the reader:

Lemma 1.1. *The degree of f equals, up to sign, the Euler number of ξ.*

Proof. Endow ξ with a Riemannian connection ∇, and let Σ_p denote the unit sphere in the tangent space of S^n at p. Fix a unit vector u in the fiber E_p of ξ over p. If $\gamma_x : [0, \pi] \to S^n$ denotes the half-great circle from p to q in direction $x \in \Sigma_p$, and P_{γ_x} is parallel translation along γ_x, we obtain a map from Σ_p to the unit sphere in the fiber of ξ over q by assigning to $x \in \Sigma_p$ the vector $P_{\gamma_x} u$. Both domain and range are $(n-1)$-spheres, and by [2, Theorem 11.16] the degree of this map equals (up to sign) the Euler number of ξ.

On the other hand, we may use the connection to obtain trivializations of the oriented orthonormal frame bundle $\mathrm{Fr}(\xi)$ of ξ: Identify an oriented orthonormal frame b_p at p with a linear isometry $b_p : \mathbb{R}^n \to E_p$, so that any frame at p can be written as $b_p \circ h$ for a unique $h \in SO(n)$. If γ^{pr} denotes the minimal geodesic from p to $r \in U_+$, then the map

$$\begin{aligned} \phi_+ : U_+ \times SO(n) &\to \mathrm{Fr}(\xi) \\ (r, h) &\mapsto P_{\gamma^{pr}}(b_p \circ h) \end{aligned}$$

is a trivialization of $\mathrm{Fr}(\xi)|_{U_+}$. Choosing another frame b_q at q yields a similar trivialization ϕ_- of $\mathrm{Fr}(\xi)|_{U_-}$.

Observe that if $\phi_+(r, h) = \phi_-(r, \bar{h})$, then

$$\bar{h} = (b_q^{-1} \circ P_{\gamma_x} \circ b_p) \circ h,$$

where $x \in \Sigma_p$ is the unique vector for which the geodesic γ_x passes through r. Identifying Σ_p with the equator S^{n-1} via $x \mapsto \gamma_x(\pi/2)$, we see that the clutching map of ξ is given by

$$g(x) = b_q^{-1} \circ P_{\gamma_x} \circ b_p,$$

and the lemma follows. □

Theorem 1.2. *If $n \neq 2, 4, 8$, then the Euler class of any rank k vector bundle over S^n is an even multiple of a generator of $H^k(S^n)$.*

Proof. We need of course only consider the case when $k = n$, and show that the degree of the map f from Lemma 1.1 is even. Now, $f = \pi \circ g$, where g is the clutching map of the bundle and $\pi : SO(n) \to S^{n-1}$ denotes the principal fibration

$$SO(n-1) \to SO(n) \to S^{n-1}.$$

It therefore remains to show that for non-exceptional values of n, $\operatorname{im} \pi_\# = 2\mathbb{Z} \subset \pi_{n-1}(S^{n-1}) = \mathbb{Z}$.

To see this, notice that in the portion

$$\cdots \to \pi_{n-1}(SO(n-1) \xrightarrow{\pi_\#} \pi_{n-1}(S^{n-1}) \xrightarrow{\partial} \pi_{n-2}(SO(n-1)) \to \cdots$$

of the long exact homotopy sequence of this fibration, ∂ is not trivial: In fact, if $h : S^{n-1} \to BSO(n-1)$ is a classifying map for the tangent bundle of the $(n-1)$-sphere, then $h_\# = \partial$ under the isomorphism $\pi_{n-1}(BSO(n-1)) \simeq \pi_{n-2}(SO(n-1))$. Thus, ∂ vanishes only when TS^{n-1} is trivial; i.e., when $n = 2, 4,$ or 8. Furthermore, for even n, $\pi_{n-2}(SO(n-1))$ is either $\mathbb{Z}_2$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$; see [7]. Together with exactness, this implies that $\operatorname{im} \pi_\# = \ker \partial = 2\mathbb{Z}$, as claimed. □

Remark. Conversely, given an even multiple $2ku$ of the standard generator u of $H^n(S^n)$, there exists a vector bundle over S^n whose Euler class equals $2ku$: Just consider $f^*\tau$, where τ is the tangent bundle of S^n and $f : S^n \to S^n$ is a map of degree k.

2. Some applications

Theorem 1.2 has several direct consequences. Among them is the following:

Corollary 2.1. *If $n \neq 2, 4,$ or 8, then the Stiefel-Whitney class $w(\xi)$ of any vector bundle ξ over S^n is trivial.*

Proof. The top Stiefel-Whitney class is the reduction mod 2 of the Euler class, and thus vanishes by Theorem 1.2. This concludes the argument when the rank of the bundle is $\leq n$. If the bundle has rank $n + k$, then ξ^{n+k} splits as a Whitney sum $\xi^n \oplus \epsilon^k$ of a rank k bundle ξ^n and a trivial bundle ϵ^k. But then $w(\xi^{n+k}) = w(\xi^n)$, and the claim follows. □

Let γ^k denote the canonical bundle over $BSO(k)$. Recall that every oriented rank k bundle over S^n is equivalent to the pullback $f^*\gamma^k$ for some $f : S^n \to BSO(k)$, and that two such bundles $f^*\gamma^k$, $g^*\gamma^k$ are equivalent iff $[f] = [g] \in \pi_n(BSO(k)) \simeq \pi_{n-1}(SO(k))$. Furthermore, these bundles are determined up to finite ambiguity by their characteristic classes; i.e., the map c which assigns to $[f] \in \pi_{n-1}(SO(k))$ the pair $(l, m) \in \mathbb{Z} \oplus \mathbb{Z}$, where l denotes the $[n/4]$-th Pontrjagin number and m the Euler number of $f^*\gamma^k$, is a homomorphism with finite kernel; see, for example, [1].

Corollary 2.2. *Let ξ denote an oriented rank $4m$ bundle over S^{4m}. If the Pontrjagin class of ξ is zero, then ξ is the pullback $f^*\tau$ of the tangent bundle τ for some $f : S^{4m} \to S^{4m}$.*

Proof. Since $\pi_{4m-1}(SO(4m)) = \mathbb{Z} \oplus \mathbb{Z}$ (see [7]), the homomorphism c from above is one-to-one. Consider first the case when $m > 2$: According to the remark following Theorem 1.2, there exists a map $f : S^{4m} \to S^{4m}$ such that $f^*\tau$ has the same Euler class as ξ. But $f^*\tau$ has trivial Pontrjagin class, so the statement follows. Next, suppose $m = 1$. From the arguments on page 246 of [6], it is easy to see that $c : \pi_3(SO(4)) \to \mathbb{Z} \oplus \mathbb{Z}$ is given by $c(k,l) = (-4k - 2l, l)$. If the rank 4 bundle ξ has trivial first Pontrjagin class, then $l = -2k$. Since the tangent bundle τ has $(k,l) = (-1,2)$, ξ is equivalent to $f^*\tau$, where $f : S^4 \to S^4$ has degree $-k$. Finally, the case when $m = 2$ is argued in exactly the same way; see, for example, [3] for an explicit description of $c : \pi_7(SO(8)) \to \mathbb{Z} \oplus \mathbb{Z}$. □

The Hurewicz homomorphism allows us to also draw some conclusions in the non-spherical case:

Corollary 2.3. *Let M be simply connected and rationally $(k+1)/2$-connected, where $k \neq 2,\ 4,$ or 8. If $H^k(M)$ is not a torsion group, then there exists a cohomology class in $H^k(M)$ that cannot be realized as the Euler class of any bundle over M.*

Proof. The Hurewicz homomorphism $h : \pi_k(M) \to H_k(M)$ is a $\mathcal{C}$-epimorphism under the above hypotheses; i.e., $H_k(M)/h(\pi_k(M)$ is finite; see [4]. By the universal coefficient theorem, $H_k(M)$ is not a torsion group, so there must exist some $\sigma \in h(\pi_k(M))$ of infinite order. Invoking once again the universal coefficient theorem, there exists some $\alpha \in H^k(M)$ such that $\langle \alpha, \sigma \rangle = 1$. If $f : S^k \to M$ satisfies $f_*[S^n] = \sigma$ (i.e., $h[f] = \sigma$), then $1 = \langle \alpha, f_*[S^k] \rangle = \langle f^*\alpha, [S^k] \rangle$, and $f^*\alpha$ is a generator of $H^k(S^k)$; since $f^*\alpha$ cannot be realized as an Euler class, neither can α. □

Example. Consider the Stiefel manifold $V_{2k,k}$ of k-frames in $\mathbb{R}^{2k}$, with k even, $k \neq 2,\ 4,$ or 8. It is well-known that $V_{2k,k}$ is $(k-1)$-connected, and that $\pi_k(V_{2k,k}) = \mathbb{Z}$; see, for example, [8]. By Hurewicz and the universal coefficient theorem, $H^k(V_{2k,k}) = \mathbb{Z}$. A generator of the latter group cannot be realized as an Euler class.

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