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EXIT TIMES FROM CONES IN $\mathbf{R}^n$ OF SYMMETRIC STABLE PROCESSES

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ABSTRACT. Let T_θ be the first exit time of an n-dimensional symmetric stable process with parameter $\alpha \in (0,2)$ from a cone of angle θ, $0 < \theta < \pi$. Then there exists a constant $p(\theta,\alpha,n)$ such that for x in the cone, $E^x T_\theta^p < \infty$ if $p < p(\theta,\alpha,n)$, and $E^x T_\theta^p = \infty$ if $p > p(\theta,\alpha,n)$. We characterize $p(\theta,\alpha,n)$ in terms of the principal eigenvalue of an operator and give upper and lower bounds for it. We also present a generalization of this result to more general cones in $\mathbf{R}^n$. These results extend the two-dimensional results of R. D. DeBlassie to n dimensions and more general cones.

1. Introduction

Let X_t be a symmetric stable process with parameter α, that is, a process X_t in $\mathbf{R}^n$ with stationary independent increments and whose transition density $p_\alpha(t, x-y)$ is determined by its Fourier transform

$$\exp\Big[-\frac{t|y|^\alpha}{2^{\alpha/2}\Gamma(\alpha/2)}\Big] = \int_{\mathbf{R}^n} e^{iu\cdot y} p_\alpha(t,u)du,$$

where $\alpha \in (0,2]$. These processes have right continuous sample paths and their transition densities satisfy the scaling property

$$p_\alpha(t,x,y) = t^{-n/\alpha} p_\alpha(1, t^{-1/\alpha}x, t^{-1/\alpha}y)\ .$$

When $\alpha = 2$ the process X_t is just an n-dimensional Brownian motion B_t.

For $x \in \mathbf{R}^n \setminus \{0\}$ we let $\varphi(x)$ be the angle between x and the point $(0,\ldots,0,1)$. A cone of angle θ is the region in $\mathbf{R}^n$ given by

$$W_\theta = \{x \in \mathbf{R}^n : x \neq 0, \pi - \theta < \varphi(x) \leq \pi\},$$

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where $0 < \theta < \pi$.

Let $\tau_{\theta,\alpha} = \inf\{t > 0 : X_t \notin W_\theta\}$ be the first time X_t exits W_θ. In the Brownian motion case, D. L. Burkholder [3, p. 192] proved that for all $x \in W_\theta$

$$E^x \tau_{\theta,2}^p < \infty \Longleftrightarrow p < p(\theta, n), \tag{1}$$

where $p(\theta, n)$ is defined in terms of the smallest zero of a certain hypergeometric function. In particular, $p(\theta, 2) = \pi/(4\theta)$. In his proof, Burkholder used his two-sided L^p inequalities for stopping times of Brownian motion to reduce the problem to the study of a Dirichlet problem for the Laplacian in cones.

Burkholder's result was extend to more general cones in [1] and [4], where the authors use separation of variables to explicitly solve the heat equation and then obtain the exact moments of integrability of the exit times. More precisely, let D be an open connected subset of $\mathbf{S}^{n-1}$. The cone generated by D is

$$C = \Big\{x \in \mathbf{R}^n \setminus \{0\} : \frac{x}{\|x\|_{\mathbf{R}^n}} \in D\Big\}.$$

If D is also regular for the Dirichlet problem with respect to $\Delta_{\mathbf{S}^{n-1}}$, the Laplace-Beltrami operator in $\mathbf{S}^{n-1}$, then it is well known that there exists a complete set of orthonormal eigenfunctions m_j with corresponding eigenvalues $0 < \eta_1 \le \eta_2 \le \eta_3 \cdots$ satisfying

$$\begin{cases} \Delta_{\mathbf{S}^{n-1}} m_j = \eta_j m_j \text{ in } D, \\ m_j = 0 \text{ on } \partial D. \end{cases}$$

Let $\tau_{C,\alpha} = \inf\{t > 0 : X_t \notin C\}$. R. Bañuelos and R. Smits [1] and R. D. DeBlassie [4] proved that for all $x \in C$

$$E^x \tau_{C,2}^p < \infty \Longleftrightarrow p < \frac{a_1}{2},$$

where $a_1 = \sqrt{\eta_1 + (\frac{n}{2} - 1)^2} - \frac{n}{2} + 1$. For a complete account of the literature on moments of exit times of Brownian motion and conditioned Brownian motion in cones we refer the reader to [1].

It is not possible to adapt the aforementioned techniques to general symmetric α-stable processes because they all involve, at some point, separation of variables, and for $\alpha \in (0, 2)$ the generator of X_t is an integral operator. However, using a result of S. A. Molchanov and E. Ostrovskii and the Donsker-Varadhan theory of large deviations, R. D. DeBlassie [5] was able to obtain an analogue of Burkholder's result for two-dimensional symmetric α-stable processes. More precisely, he proved that for any $\theta \in (0, \pi)$ and any $\alpha \in (0, 2)$ there exists $p(\theta, \alpha, 2)$ such that for all $x \in W_\theta$

$$E^x \tau_{\theta,\alpha}^p \begin{cases} < \infty & \text{if } p < p(\theta, \alpha, 2), \\ = \infty & \text{if } p > p(\theta, \alpha, 2). \end{cases}$$

Moreover he showed that $E^x\tau_{\theta,\alpha}^{p(\theta,1,2)} = \infty$. However his proof did not settle the case $p = p(\theta,\alpha,2)$ for $\alpha \neq 1$. DeBlassie characterized $p(\theta,\alpha,2)$ in terms of the principal eigenvalue of a degenerate non-self-adjoint differential operator, and he obtained explicit upper and lower bounds for $p(\theta,\alpha,2)$. In this paper we extend DeBlassie's result to the n-dimensional case and to more general cones, and we obtain a new upper bound for the corresponding critical value of integrability. Our main results are as follows:

THEOREM 1. *Let $\theta \in (0,\pi)$ and $\alpha \in (0,2)$. Then there exists $p(\theta,\alpha,n)$ such that for all $x \in W_\theta$*

$$E^x\tau_{\theta,\alpha}^p \begin{cases} < \infty & \text{if } p < p(\theta,\alpha,n), \\ = \infty & \text{if } p > p(\theta,\alpha,n). \end{cases}$$

Furthermore, for all $x \in W_\theta$

$$E^x\tau_{\theta,\alpha}^{p(\theta,1,n)} = \infty.$$

In addition, let $\delta = \delta(\theta) = (\sin\theta)/(1+\cos\theta)$ and $H_\delta = \mathbf{R}_+^n \setminus [\,\{0\}\times B(0,\delta)^c\,]$, where $B(0,\delta)$ is the open ball in $\mathbf{R}^{n-1}$ of radius δ centered at the origin and $\mathbf{R}_+^n$ is the closed half space $\{(x_1,\dots,x_n) \in \mathbf{R}^n : x_1 \geq 0\}$. Then

$$p(\theta,\alpha,n) = \frac{[4\lambda_\delta + (n-\alpha)^2]^{1/2} - (n-\alpha)}{2\alpha},$$

where λ_δ is the principal eigenvalue, in H_δ, of the differential operator L given by

$$(2)\quad \frac{a(v)^2}{4}\left[4v_1\frac{\partial^2}{\partial v_1^2} + \sum_{k=2}^{n}\frac{\partial^2}{\partial v_i^2} + \frac{2(1-n+\alpha)}{a(v)}\sum_{k=1}^{n} v_i\frac{\partial}{\partial v_i} + 2(2-\alpha)\frac{\partial}{\partial v_1}\right],$$

where $a(v) = 1 + v_1 + v_2^2 + \cdots + v_n^2$.

The Donsker-Varadhan theory of large deviations [6] gives the following characterization for λ_δ:

$$\lambda_\delta = -\sup_{U\in\mathcal{K}(\delta)}\ \sup_{\mu(\overline{U})=1}\ [-I(\mu)].$$

Here the μ's are probabilities measures, $\mathcal{K}(\delta)$ is the class of bounded open sets U with C^∞ boundary and $U \subset \overline{U} \subset H_\delta$, and $I(\mu)$ is the Donsker-Varadhan I-function associated to L:

$$I(\mu) = -\inf\left\{\int \frac{Lf}{f}\,d\mu : \ f - c \in C_0^2(\mathbf{R}_+^n) \text{ for some } c \in \mathbf{R}\right\}.$$

The following result gives further properties of the constants $p(\theta,\alpha,n)$.

THEOREM 2. *For $\theta \in (0,\pi)$ we have:*

(i) *$\theta \to p(\theta,\alpha,n)$ is continuous and decreasing;*

(ii) *$p(\theta,\alpha,n) < 1$;*
(iii) *if $\theta \in (0,\frac{\pi}{2})$, then $p(\theta,\alpha,n) > \frac{1}{2} = p(\frac{\pi}{2},\alpha,n)$;*
(iv) *if $\theta \in (\frac{\pi}{2},\pi)$, then*

$$p(\theta,\alpha,n) \geq \frac{[(n+2-\alpha)\alpha\delta(\theta)^2 + (n-\alpha)^2]^{\frac{1}{2}} - (n-\alpha)}{2\alpha}.$$

The proofs of Theorem 1 and Theorem 2 follow closely the two-dimensional proofs of DeBlassie [5]. Since the arguments in [5] strongly depend on the form of the operator L and the simple geometry of unit circle, it is not immediately clear that they also apply to the n-dimensional case. However, once we are able to identify the corresponding operators in higher dimensions, the arguments of DeBlassie, although more technical in several places, can be easily adapted to prove Theorems 1 and 2. At the end of this section we will give a brief outline of the proof of these theorems; we refer the reader to [8] for a complete proof. DeBlassie's argument also implies the following extension of the results in [1] and [4] to general n-dimensional symmetric α-stable processes.

Theorem 3. *Let D be a proper open connected subset of $\mathbf{S}^{n-1}$ such that $(0,\ldots,0,1)$ is not in the closure of D, and consider the cone C in $\mathbf{R}^n$ generated by D. If $\alpha \in (0,2)$ and $\tau_{C,\alpha} = \inf\{t>0 : X_t \notin C\}$ is the first time X_t exits C, then there exists $0 < p(C,\alpha,n) < 1$ such that for all $x \in C$*

$$E^x \tau_{C,\alpha}^p \begin{cases} < \infty & \text{if } p < p(\theta, C, n), \\ = \infty & \text{if } p > p(C,\alpha,n). \end{cases}$$

If H_C is the complement in $\mathbf{R}^n_+$ of the stereographic projection of $\{0\} \times C$, then

$$p(C,\alpha,n) = \frac{[4\lambda_C + (n-\alpha)^2]^{1/2} - (n-\alpha)}{2\alpha},$$

where λ_C is the principal eigenvalue, in H_C, of the differential operator L. Furthermore, if D is also regular with respect to the Laplace-Beltrami operator in $\mathbf{S}^{n-1}$, then for all $x \in C$

$$E^x \tau_{C,\alpha}^{p(C,1,n)} = \infty.$$

Notice that since D is a proper open connected subset of $\mathbf{S}^{n-1}$ such that $(0,\ldots,0,1)$ is not in the closure of D, there exist $\theta, \theta' \in (0,\pi)$ such that $C \subset W_\theta$ and a rotation of $W_{\theta'}$ is contained in C. Thus

$$0 < p(\theta,\alpha,n) < p(C,\alpha,n) < p(\theta',\alpha,n) < 1.$$

As a matter of fact we could use parts (iii) and (iv) of Theorem 2 to obtain better lower bounds for $p(C,\alpha,n)$ in terms of θ. However the only upper bound provided by Theorem 2 is $p(C,\alpha,n) < 1$. Our next result gives an upper bound for $p(C,\alpha,n)$ in terms of the corresponding value for the Brownian motion $p(C,2,n)$. This result is new even in the two-dimensional case.

THEOREM 4. *Let C be a cone generated by a proper open connected subset D of $\mathbf{S}^{n-1}$ such that $(0,\dots,0,1)$ is not in the closure of D. If D is regular for the Laplace-Beltrami operator in $\mathbf{S}^{n-1}$, then*

$$p(C,\alpha,n) \le \frac{2}{\alpha} p(C,2,n).$$

In particular,

$$p(\theta,\alpha,n) \le \frac{2}{\alpha} p(\theta,2,n).$$

Theorem 4 immediately follows from the next result, which we believe is of independent interest.

THEOREM 5. *Let D be a domain in $\mathbf{R}^n$, and let $\tau_{D,\alpha}$ be the first time the α-stable process X_t exits D. If ϕ_u is the local time at 0 of a Bessel process of parameter $2-\alpha$ starting at 0, then*

$$E^x[\,\tau_{D,\alpha}^p\,] \;\ge\; p\,E[\,\phi_1^p\,]\; E^x[\,\tau_{D,2}^{p\alpha/2}\,].$$

We will prove Theorem 5 in §2 using the representation of the symmetric α-stable processes given by S. A. Molchanov and E. Ostrovskii [9]. As we have mentioned above, the values of $p(C,2,n)$ are given in terms of η_1, the first Dirichlet eigenvalue of $\Delta_{\mathbf{S}^{n-1}}$ in D. In general it is not easy to find good approximations of η_1. Nevertheless, in the case when C is a straight cone the values of $p(\theta,2,n)$ can be explicitly given in terms of zeros of certain hypergeometric functions.

For the convenience of the reader we now give a brief outline of the proof of Theorems 1 and 2. Let Y_t^1 be a Bessel process with parameter $2-\alpha$, $0<\alpha<2$, and let $B_t=(Y_t^2,\dots,Y_t^{n+1})$ be an n-dimensional Brownian motion independent of Y_t^1. The generator of the process $Y_t=(Y_t^1,\dots,Y_t^{n+1})$ is given by

$$\overline{G} = \frac{1}{2}\left[\frac{1-\alpha}{y_1}\frac{\partial}{\partial y_1} + \sum_{i=1}^{n+1}\frac{\partial^2}{\partial y_i^2}\right]. \tag{3}$$

Also Y_t has state space $\mathbf{R}_+^{n+1}$ if $Y_0^1 \ge 0$.

Molchanov and Ostrovskii [9] proved that if σ_t is the inverse local time of the Bessel process Y_t^1 at 0 and $Y_0^1=0$, then

$$X_t = (Y_{\sigma_t}^2,\dots,Y_{\sigma_t}^{n+1})$$

is an n-dimensional symmetric stable processes with parameter α.

Define

$$Z_t = (\,[Y_t^1]^2, Y_t^2,\dots,Y_t^{n+1}\,),$$
$$\widetilde{\tau}_{\theta,\alpha} = \inf\{\,t>0 : Z_t \in \{0\}\times W_\theta^c\,\}.$$

Note that $\widetilde{\tau}_{\theta,\alpha}$ is the first time Z_t (or Y_t) hits $\{0\} \times W_\theta^c$. Since $Z_t \in \{0\} \times W_\theta^c$ if only if

$$Y_t^1 = 0 \text{ and } (Y_t^2, \dots, Y_t^{n+1}) \notin W_\theta,$$

it is clear that $(0, X_{\tau_{\theta,\alpha}})$ and $Z_{\widetilde{\tau}_{\theta,\alpha}}$ have the same distribution. Let $\mathcal{F}_t = \sigma(Z_t)$. Then $(Z_t^2, \cdots, Z_t^n)$ is an n-dimensional $\mathcal{F}_t$-Brownian motion and $\widetilde{\tau}_{\theta,\alpha}$ is an $\mathcal{F}_t$-stopping time. Thus by Theorem 3.1 and Remark 3.1 in [3] we have that for every $p > 0$,

$$E^{(0,x)} |Z_{\widetilde{\tau}_{\theta,\alpha}}|^{2p} < \infty \iff E^{(0,x)} \widetilde{\tau}_{\theta,\alpha}^p < \infty.$$

Therefore

$$E^x |X_{\tau_{\theta,\alpha}}|^{2p} < \infty \iff E^{(0,x)} \widetilde{\tau}_{\theta,\alpha}^p < \infty. \tag{4}$$

We will prove that there exists a constant $\beta(\theta, \alpha, n) < \alpha$ such that

$$E^{(0,x)} \widetilde{\tau}_{\theta,\alpha}^{p/2} \begin{cases} < \infty & \text{if } p < \beta(\theta, \alpha, n), \\ = \infty & \text{if } p > \beta(\theta, \alpha, n). \end{cases}$$

Thus Theorem 1 will follow from Theorems 3.1 and 3.2 in [2]. The problem has therefore been reduced to a study of $\widetilde{\tau}_{\theta,\alpha}$, a stopping time of a diffusion. Unfortunately, our arguments do not settle the case $p = p(\theta, \alpha, n)$ except when $\alpha = 1$. The main difficulty is that the operators involved in our proof are non-self-adjoint, and thus the classical eigenfunction expansion of the heat kernel is not available.

Recall that the Laplacian in polar coordinates is given by

$$\Delta_{\mathbf{R}^{n+1}} = \frac{\partial^2}{\partial r^2} + \frac{n}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{\mathbf{S}^n},$$

where $\Delta_{\mathbf{S}^n}$ is the Laplace-Beltrami operator in $\mathbf{S}^n$ and $r = (y_1^2 + \cdots + y_{n+1}^2)^{1/2}$. Let $e_{n+1} = (0, \dots, 0, 1) \in \mathbf{R}^{n+1}$ and consider the change of variables Ψ from $\mathbf{R}^{n+1} \setminus \{te_{n+1} : t > 0\}$ to $\mathbf{R}^{n+1}$, given by

$$\Psi(y_1, \dots, y_{n+1}) = \left(r, \frac{y_1}{r - y_{n+1}}, \dots, \frac{y_n}{r - y_{n+1}} \right) = (r, u_1, \dots, u_n).$$

This function is usually called the stereographic projection. Under this transformation $\mathbf{R}_+^{n+1} \setminus [\,\{0\} \times W_\theta^c\,]$ is taken to $(0, \infty) \times H_\delta$ and $[\,\{0\} \times W_\theta^c\,] \setminus \{te_{n+1}, t > 0\}$ is taken to $(0, \infty) \times \{0\} \times B(0, \delta)^c$. In addition, in these coordinates

$$\Delta_{\mathbf{S}^n} = \frac{(\,|u|^2 + 1\,)^n}{4} \sum_{k=1}^n \frac{\partial}{\partial u_k} \Big[(\,|u|^2 + 1\,)^{2-n} \Big] \frac{\partial}{\partial u_k};$$

see [7]. A straightforward computation shows that the generator of Y_t reduces to

$$\widehat{G} = \frac{\partial^2}{\partial r^2} + \frac{n+1-\alpha}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\,\widehat{L}, \tag{5}$$

where $\widehat{L}$ is given by

$$\frac{(|u|^2+1)^2}{4}\left[\frac{1-\alpha}{u_1}\frac{\partial}{\partial u_1}+\sum_{k=1}^{n}\frac{\partial^2}{\partial u_i^2}\;+\;\frac{2(1-n+\alpha)}{|u|^2+1}u_i\frac{\partial}{\partial u_i}\right]. \tag{6}$$

Take $v=(u_1^2,u_2,\ldots,u_n)$. Then

$$(v_1,\ldots,v_n)=\left(\frac{y_1^2}{(r-y_{n+1})^2},\ldots,\frac{y_n}{r-y_{n+1}}\right). \tag{7}$$

In these coordinates $\widehat{G}$ becomes

$$G=\frac{\partial^2}{\partial\rho^2}+\frac{n+1-\alpha}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\,L, \tag{8}$$

where L is the operator given in (2).

Now, $u(t,z)=P^z(\widetilde{\tau}_{\theta,\alpha}>t)$ solves the initial boundary value problem

$$\begin{cases}\left(\frac{\partial}{\partial t}-G\right)u(t,z)=0, & (t,z)\in(0,\infty)\times\mathbf{R}_+^{n+1}\setminus[\{0\}\times W_\theta^c],\\ u(0,z)=1, & z\notin\{0\}\times W_\theta^c,\\ u(t,z)=0, & (t,z)\in(0,\infty)\times\{0\}\times W_\theta^c.\end{cases}$$

Since this problem involves a degenerate non-self-adjoint differential operator, the classical methods for resolving the eigenvalue problem are not available. However, Donsker and Varadhan have characterized the principal eigenvalue for such operators, and we can employ their machinery. It can be proved that Z_t has a skew product representation $(\,R_t,V(A_t)\,)$, where R and V are independent, and A is a continuous strictly increasing process which is independent of V. Ignoring for the moment the fact that V can explode and defining $\eta_\delta=\inf\{t>0:V_t\in\{0\}\times B(0,\delta)^c\}$ we have

$$\begin{aligned}P^z(\widetilde{\tau}_{\theta,\alpha}>t)&=P^z(\,V(A_s)\notin\{0\}\times B(0,\delta)^c,\;0\le s\le t\,)\\&=P^z(\,\eta_\delta>A_t\,)\\&=\int_0^\infty P^z(\,\eta_\delta>a\,)d_a(A_t\le a).\end{aligned}$$

The Donsker–Varadhan theory of large deviations gives the asymptotic behavior of $P_z(\eta_\delta>s)$ as $s\to\infty$, and the process A_t is easy to analyze. Hence we are able to decide whether $E_z\tau_\theta^p<\infty$ or not.

2. Proof of Theorem 5

Recall that

$$X_t=(Y_{\sigma_t}^2,\ldots,Y_{\sigma_t}^{n+1}),$$

where $(Y_t^2,\ldots,Y_t^{n+1})$ is an n-dimensional Brownian motion and σ_t is the inverse of ϕ_u, the local time at 0 of a Bessel process of parameter $2-\alpha$ starting at 0.

Let $x \in D$ and note that $\sigma_t = \sup\{u : \phi_u \leq t\}$. Since X_t has right-continuous paths and σ_t is nondecreasing, we have

$$\begin{aligned} P^x\{\,\tau_{D,\alpha} > t\,\} &= P^x\{\,X_s \in D, 0 \leq s \leq t\,\} \\ &= P^x\{\,B_{\sigma_s} \in D, 0 \leq s \leq t\,\} \\ &\geq P^x\{\,\tau_{D,2} > \sigma_s, 0 \leq s \leq t\,\} \\ &= P^x\{\,\tau_{D,2} > \sigma_t\,\}. \end{aligned}$$

Let $g_{\alpha/2}(t,u)$ be the density of σ_t. Since σ_t and $(Y_t^2, \ldots, Y_t^{n+1})$ are independent, we obtain

$$P^x\{\,\tau_{D,\alpha} > t\,\} \;\geq\; P^x\{\,\tau_{D,2} > \sigma_t\,\} = \int_0^\infty P^x\{\,\tau_{D,2} > u\,\} g_{\alpha/2}(t,u)du.$$

Thus Fubini's Theorem implies that

$$\begin{aligned} E^x[\,\tau_{D,\alpha}^p\,] &= \int_0^\infty pt^{p-1}P^x\{\,\tau_{D,\alpha} > t\,\}dt \\ &\geq \int_0^\infty pt^{p-1}\int_0^\infty P^x\{\,\tau_{D,2} > u\,\}g_{\alpha/2}(t,u)dudt \\ &= \int_0^\infty P^x\{\,\tau_{D,2} > u\,\}\int_0^\infty pt^{p-1}g_{\alpha/2}(t,u)dtdu. \end{aligned} \tag{9}$$

It is well known that if D is the unit ball in $\mathbf{R}^n$, then all moments of $\tau_{D,\alpha}$ are finite. Therefore for every $p > 0$

$$\int_0^\infty pt^{p-1}g_{\alpha/2}(t,u)dt < \infty,$$

a.s. in u, and hence

$$\lim_{t\to\infty} t^p g_{\alpha/2}(t,u) = 0.$$

Let $f_\alpha(u,t)$ be the density of ϕ_u. By the proof of Theorem 1 in [9] we have

$$\frac{d}{dt}g_{\alpha/2}(t,u) = -\frac{d}{du}f_\alpha(u,t)$$

and

$$E[\phi_u^p] = E[\phi_1^p]u^{p\alpha/2} < \infty.$$

Thus integration by parts gives

$$\begin{aligned} \int_0^\infty pt^{p-1}g_{\alpha/2}(t,u)dt &= \lim_{t\to\infty} t^p g_{\alpha/2}(t,u) - \int_0^\infty t^p \frac{d}{dt}g_{\alpha/2}(t,u)dt \\ &= \int_0^\infty t^p \frac{d}{du}f_\alpha(u,t)dt \\ &= \frac{d}{du}E[\,\phi_u^p\,] = \frac{p\alpha}{2}u^{\frac{p\alpha}{2}-1}E[\phi_1^p]. \end{aligned}$$

By (9) we conclude that

$$E^x[\,\tau_{D,\alpha}^p\,] \geq \int_0^\infty pP^x\{\,\tau_{D,2} > u\,\}\frac{p\alpha}{2}u^{\frac{p\alpha}{2}-1}E[\phi_1^p]du = pE^x[\,\tau_{D,2}^{p\alpha/2}\,]E[\phi_1^p],$$

which proves the result. □

In the case when D is a two-dimensional cone Theorem 5 implies that

$$p_{\theta,\alpha} \leq \frac{\pi}{2\theta\alpha}.$$

Notice that if $\theta\alpha$ is small enough, then $\frac{\pi}{2\alpha\theta} > 1$, and in this case we have strict inequality in Theorem 4 for p close enough to $\frac{\pi}{2\alpha\theta}$.

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