Illinois Journal of Mathematics
Volume 46, Number 1, Spring 2002, Pages 129–143
S 0019-2082

GORENSTEIN LIAISON AND SPECIAL LINEAR CONFIGURATIONS

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Abstract. Liaison theory has been extensively studied during the past decades. In codimension 2, the theory has reached a very satisfactory state, but in higher codimensions there are still many open problems. In this paper we prove that two unions $V = \bigcup_{i=1}^{k} L_i$ and $V' = \bigcup_{i=1}^{k'} L'_i$ of independent linear varieties of dimension $d \geq 1$ in $\mathbb{P}^n$ are in the same G-liaison class if and only if $k = k'$ or, equivalently, if V and V' have isomorphic deficiency modules $M^i(V) \cong M^i(V')$, $i = 1, \ldots, d$. We also describe the G-liaison classes of arithmetically Buchsbaum divisors on rational normal scrolls.

Introduction

Liaison theory has been extensively studied during the past decades; in codimension 2, the theory has reached a very satisfactory state, but in higher codimensions there are still many open questions and problems. Much of the theory has been built around linking with complete intersections schemes, which in codimension 2 coincide with Gorenstein schemes, and recent attention has been focused on Gorenstein liaison. The results given in [7], [3], and [11], among other papers, suggest that Gorenstein liaison is a more natural approach in codimension ≥ 3 and that much of the codimension 2 case can be carried over naturally to higher codimensions.

This paper addresses a basic question about liaison of linear configurations, namely, whether a set C of t independent linear spaces of dimension $d \geq 1$ in $\mathbb{P}^n$ can be G-linked to any other set of t independent linear spaces of dimension d. (Here "independent" means that the spaces span a linear space of dimension $dt + t - 1$.) It is well known that any two skew lines in $\mathbb{P}^3$ are in the same CI-liaison class; Migliore [9] conjectured that a pair of skew lines in $\mathbb{P}^4$ can be CI-linked to another pair of skew lines if and only if they are

Received February 14, 2001; received in final form July 2, 2001.
2000 *Mathematics Subject Classification.* 14H45, 14C10, 14C20.
The research of the first author was partially supported by Comissionat per a Universitats i Recerca del Dpt. de Presidència de la Generalitat de Catalunya. The research of the second author was partially supported by DGICYT PB97-0893.

contained in the same hyperplane $H \subset \mathbb{P}^4$. On the other hand, Lesperance [8, Theorem 1.5] and Hartshorne [6, Proposition 3.1] recently proved that we can G-link a pair of skew lines in $\mathbb{P}^4$ to any other pair of skew lines in $\mathbb{P}^4$. In this paper we generalize this result and prove that two unions $V = \bigcup_{i=1}^{k} L_i$ and $V' = \bigcup_{i=1}^{k'} L'_i$ of independent linear varieties of dimension $d \geq 1$ in $\mathbb{P}^n$ are in the same G-liaison class if and only if $k = k'$ or, equivalently, if V and V' have isomorphic deficiency modules $M^i(V) \cong M^i(V')$, $i = 1, \dots, d$, up to shifts and dual (see Theorem 3.5).

As a consequence, we obtain that all reduced curves $C \subset \mathbb{P}^{2n}$ with Rao module K^{n-1} concentrated in degree 0 belong to the same G-liaison class. We point out that while much work has recently been devoted to G-liaison of codimension c arithmetically Cohen-Macaulay schemes in $\mathbb{P}^n$, the results presented here are among the first for non-arithmetically Cohen-Macaulay schemes of codimension $c > 2$ in $\mathbb{P}^n$, $n > 4$.

To prove these results we first describe the G-liaison classes of arithmetically Buchsbaum divisors on rational normal scrolls (see Theorem 2.4 and Remark 2.5). As main ingredient in the proof we use the fact that, roughly speaking, Gorenstein liaison is being developped as a theory about generalized divisors on arithmetically Cohen-Macaulay schemes which collapses to the setting of CI-liaison theory as a theory of generalized divisors on complete intersections (for more details see [7] and [5]).

The structure of the paper is as follows. In Section 1, we review the basic facts on Gorenstein liaison that we will need in the sequel. In Section 2, we describe the G-liaison classes of arithmetically Buchsbaum divisors on rational normal scrolls. Section 3 is the heart of the paper. In this section we study the G-liaison classes of independent linear configurations; in particular, we prove the result announced above.

1. Preliminaries

Throughout this paper, $\mathbb{P}^n$ will be the n-dimensional projective space over an algebraically closed field K of characteristic zero, $R = K[X_0, \dots, X_n]$, and $\mathfrak{m} = (X_0, \dots, X_n)$ its homogeneous maximal ideal. By a *subscheme* $V \subset \mathbb{P}^n$ we mean an equidimensional closed subscheme. For a subscheme V of $\mathbb{P}^n$ we denote by I_V its ideal sheaf and by $I(V)$ its saturated homogeneous ideal; note that $I(V) = H^0_*(I_V) := \bigoplus_{t \in \mathbb{Z}} H^0(\mathbb{P}^n, I_V(t))$.

If Z is any closed subscheme of $\mathbb{P}^n$ we denote by $\langle Z \rangle$ the span of Z, i.e., the smallest linear subspace of $\mathbb{P}^n$ containing Z as a subscheme.

Given a closed subscheme $V \subset \mathbb{P}^n$ of dimension $d \geq 1$, we define its *deficiency modules* as

$$M^i(V) := H^i_*(I_V) = \oplus_{t \in \mathbb{Z}} H^i(\mathbb{P}^n, I_V(t))$$

for $1 \leq i \leq d$. When V is a curve, its deficiency module $M^1(V)$ is also called *Rao module* (or *Hartshorne-Rao module*) and is denoted simply by $M(V)$.

A closed subscheme $V \subset \mathbb{P}^n$ is said to be arithmetically Cohen-Macaulay (briefly ACM) if its homogeneous coordinate ring is a Cohen-Macaulay ring, i.e., if $\dim R/I(V) = \operatorname{depth} R/I(V)$. We recall that a subscheme $V \subset \mathbb{P}^n$ of dimension $d \geq 1$ is arithmetically Cohen-Macaulay (briefly ACM) if and only if all its deficiency modules vanish.

A closed subscheme $V \subset \mathbb{P}^n$ of codimension c is *arithmetically Gorenstein* (briefly AG) if its saturated homogeneous ideal $I(V)$ has a minimal free graded R-resolution of the following type:

$$0 \to R(-t) \to F_{c-1} \to \ldots \to F_1 \to F_0 \to I(V) \to 0.$$

In other words, $V \subset \mathbb{P}^n$ is AG if and only if V is ACM and the last module in the minimal free resolution of its saturated ideal has rank one. For instance, any complete intersection scheme is arithmetically Gorenstein; the converse is true only in codimension 2.

In the following definition we collect the main concepts about Gorenstein liaison that we will need in this paper.

Definition 1.1 (see also [7, Definitions 2.3, 2.4 and 2.10]). We say that two subschemes V_1 and V_2 of $\mathbb{P}^n$ are *directly Gorenstein linked*, or simply *directly G-linked*, by an arithmetically Gorenstein scheme $X \subset \mathbb{P}^n$ if $I(X) \subset I(V_1) \cap I(V_2)$ and we have $I(X) : I(V_1) = I(V_2)$ and $I(X) : I(V_2) = I(V_1)$. We denote this by $V_1 \overset{X}{\sim} V_2$, and we say that V_2 is the *residual* to V_1 in X. We say that two subschemes V_1 and V_2 are *G-linked* if there exists a sequence of schemes $A_1, \ldots, A_r$ such that A_i is directly G-linked to A_{i+1}, $A_1 = V_1$, and $A_r = V_2$. We call *G-liaison* the equivalence relation generated by G-links. If X is a complete intersection, we say that V_1 and V_2 are *complete intersection linked*, or simply *CI-linked*. We call *CI-liaison* the equivalence relation generated by CI-links.

We say that V_1 and V_2 are *CI-bilinked* (resp. *G-bilinked*) if V_1 is linked to V_2 in two steps by complete intersection schemes (resp. arithmetically Gorenstein schemes). We say that two schemes are *evenly linked* (in terms of CI-liaison or G-liaison) if they can be linked by an even number of direct links.

The authors of [7] have developed the main tools we will need in this paper. These tools concern G-links produced using divisors on ACM schemes satisfying the property G_1 (Gorenstein in codimension 1). A *divisor* D on a scheme S of $\mathbb{P}^n$ will mean a pure codimension 1 subscheme D of S with no embedded components. For more details on the theory of generalized divisors on schemes satisfying the property G_1 see [5].

Proposition 1.2. *Let $X \subset \mathbb{P}^n$ be an ACM subscheme satisfying property G_1 and let K_X be a canonical divisor on X and H the hyperplane section. Then any element of the linear system $|dH - K_X|$ is arithmetically Gorenstein.*

Proof. See [7, Corollary 5.5]. □

Proposition 1.3. *Let $X \subset \mathbb{P}^n$ be an ACM subscheme satisfying property G_1 and let $C \subset X$ be an effective divisor. Take any divisor C_1 in the linear system $|C + tH|$, where H is a hyperplane section of X and $t \in \mathbb{Z}$. Then C and C_1 are G-bilinked. (Notice that if $t = 0$ then C and C_1 are linearly equivalent.)*

Proof. See [7, Corollary 5.13]. □

We end this section by recalling a close relationship between G-linked schemes and their deficiency modules.

Theorem 1.4 (Hartshorne-Schenzel). *Let $V_1, V_2 \subset \mathbb{P}^n$ be schemes of the same dimension d such that $V_1 \overset{X}{\sim} V_2$, where X is an AG scheme. Then there is an integer p such that for all $i = 1, \ldots, d$*

$$M^{d-i+1}(V_2) \cong M^i(V_1)^\vee(p),$$

where $M^i(V)^\vee := \operatorname{Hom}_K(M^i(V), K)$.

We are trying to gain some insight into a possible converse of this result. In particular, we ask:

Question 1.5. Let V_1, V_2 be two curves in $\mathbb{P}^n$ having isomorphic Rao modules (up to shifts and dual). Are V_1 and V_2 in the same G-liaison class?

In order to shed light on this problem, we will prove that two disjoint unions $C = \bigcup_{i=1}^{k} L_i$ (resp. $C' = \bigcup_{i=1}^{k'} L'_i$) of k (resp. k') independent lines in $\mathbb{P}^N$ belong to the same G-liaison class if and only if $M(C) \cong M(C')$ (see Theorem 3.5).

The above theorem of Hartshorne-Schenzel gives rise to the following definition:

Definition 1.6. Let $\mathcal{L}$ be an even G-liaison class. We denote by $\mathcal{L}^0$ the set of schemes $V \in \mathcal{L}$ whose configuration of deficiency modules coincides with the leftmost one (see [10, Section 5.4]), and we call the elements in $\mathcal{L}^0$ *minimal* elements.

For more information about Gorenstein liaison, see [10] and [7].

2. G-liaison classes of arithmetically Buchsbaum divisors on rational normal scrolls

In [3, Section 3] the authors studied the G-liaison classes of ACM divisors on rational normal scrolls. In this section we are going to study G-liaison classes of arithmetically Buchsbaum divisors on rational normal scrolls. While much work has recently been devoted to G-liaison of codimension c schemes in $\mathbb{P}^n$, our results are among the first for non ACM schemes of codimension $c > 2$ in $\mathbb{P}^n$, $n \geq 3$ (see [6] and [8] for some examples for non ACM curves in $\mathbb{P}^4$).

Let us recall the basic facts on rational normal scrolls; we refer to [4] for more details. Let $\mathcal{E} = \oplus_{i=0}^{k} \mathcal{O}_{\mathbb{P}^1}(a_i)$ be a rank $k+1$ vector bundle on $\mathbb{P}^1$, where $0 \leq a_0 \leq \cdots \leq a_k$, and $a_k > 0$. Let $\mathbb{P}(\mathcal{E}) = \operatorname{Proj}(\operatorname{Sym}(\mathcal{E})) \to \mathbb{P}^1$ be the projectivized vector bundle and let $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ be its tautological line bundle. Then $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is generated by global sections and defines a rational map $\mathbb{P}(\mathcal{E}) \to \mathbb{P}^{\sum a_i + k}$. We write $S(\mathcal{E})$ or $S(a_0, \ldots, a_k)$ for the image of this map, which is a variety of dimension $k+1$ and degree $c := \sum a_i$. A *rational normal scroll* is one of the varieties $S(\mathcal{E})$.

Rational normal scrolls $S = S(a_0, \ldots, a_k) \subset \mathbb{P}^{c+k}$ are reduced and irreducible varieties of *minimal degree*, i.e., $\deg S = \operatorname{codim} S + 1$. They are ACM and, in fact, are standard determinantal varieties defined by the 2×2 minors of a $2 \times c$ matrix with suitable linear entries. Moreover, the singularities of rational normal scrolls occur in codimension greater than or equal to 2, so they satisfy property G_1.

The divisor class group of a rational normal scroll $S = S(a_0, \ldots, a_k) \subset \mathbb{P}^{c+k}$, $k \geq 1$, is generated by the hyperplane section H and a linear subspace $F \subset S$ of dimension k. The canonical class of S is $K_S \sim -(k+1)H + (c-2)F$.

We will study the G-liaison classes of arithmetically Buchsbaum schemes on rational normal scrolls. To this end we first review the definition of arithmetically Buchsbaum schemes.

Definition 2.1. A curve C in $\mathbb{P}^n$ is *arithmetically Buchsbaum* (or briefly AB) if $M(C)$ is annihilated by all linear forms of R (and hence by all homogeneous polynomials). A subscheme $V \subset \mathbb{P}^n$ of dimension $d > 1$ is *arithmetically Buchsbaum* (or AB) if $M^i(V)$ is annihilated by all linear forms for all $i = 1, \ldots, d$ and if the general proper hyperplane section of V is again arithmetically Buchsbaum.

We remark that if C is an arithmetically Buchsbaum curve, then $M(C)_t = 0$ for $t < 0$, i.e., C does not have deficiency in negative degrees (see [10, Remark 1.3.11(b)] or [1, Remark 1.4]).

To simplify the notation, we will assume that all AB schemes considered here are not ACM.

The following result of Nagel gives a classification of arithmetically Buchsbaum divisors on rational normal scrolls in terms of the linear equivalence classes.

PROPOSITION 2.2. *Let $X \subset \mathbb{P}^{c+k}$ be a reduced, effective divisor on a rational normal scroll $S = S(a_0, \dots, a_k)$, $c = \sum_{i=0}^{k} a_i$. Assume that $d := \deg X > c$ and write $d = t \cdot c + b$, where $1 - c \le b \le 0$. Then X is arithmetically Buchsbaum if and only if either*

(i) $X \sim (t-1)H + bF$, $1 < b \le a_0 + 1$, *or*

(ii) $X \sim (t+1)H + bF$, $1 - c - a_0 \le b \le -c$.

Furthermore, in case (i) *we have*

$$H^i_*(I_X) \cong \begin{cases} K^{b-1}(-t+1), & i = 1, \\ 0, & 1 < i \le k. \end{cases}$$

Proof. See [12, Theorem 5.10]. □

REMARK 2.3. From Proposition 2.2 it follows that a singular rational normal scroll does not contain arithmetically Buchsbaum divisors which are not ACM.

We are now ready to state the main result of this section.

THEOREM 2.4. *Let X, $X' \subset \mathbb{P}^{c+k}$ be two arithmetically Buchsbaum schemes of dimension k lying on a rational normal scroll $S = S(a_0, \dots, a_k)$, $c = \sum_{i=0}^{k} a_i$. Then X and X' have isomorphic deficiency modules, up to shifts and dual, if and only if X and X' belong to the same G-liaison class.*

Proof. If X and X' are G-linked, then their deficiency modules are isomorphic (up to shifts and dual) by the Hartshorne-Schenzel Theorem (Theorem 1.4). It remains therefore to prove the converse.

Since rational normal scrolls are ACM schemes and satisfy condition G_1, we can apply Propositions 1.2 and 1.3.

We first note that we may assume that X and X' are reduced and have degree greater than c. Indeed, if necessary, we can add hyperplane sections and take a general element Y, resp. Y', in the linear system $|X + nH|$, resp. $|X' + n'H|$. By Proposition 1.3, these new divisors Y, Y', will be in the same G-liaison class as X, X'.

If X (resp. X') is in case (ii), i.e., if $X \sim (t+1)H + bF$ with $1 - c - a_0 \le b \le -c$ (resp. $X' \sim (t'+1)H + b'F$ with $1 - c - a_0 \le b' \le -c$), then we can perform a direct G-link using an AG divisor of type $mH - K_S \sim (m+k+1)H - (c-2)F$ containing X (resp. X') for $m \gg 0$, and the residual will be linearly equivalent to $(l - t - 1)H + (-b - c + 2)F$ (resp. $(l - t' - 1)H + (-b' - c + 2)F$). Thus, the residual is in case (i), and it therefore suffices to consider this case.

Now, assuming that X and X' are in case (i), i.e., that $X \sim (t-1)H + bF$ and $X' \sim (t'-1)H + b'F$ with $1 < b, b' \le a_0 + 1$, the last assertion of Proposition 2.2 implies that they have the same deficiency modules (up to shifts) if and only if $b = b'$. But in this case Proposition 1.3 implies that X and X' are in the same G-liaison class. □

Remark 2.5. (i) In the proof of Theorem 2.4 we have seen that on a rational normal scroll $S(a_0, \dots, a_k) \subset \mathbb{P}^{c+k}$ there are exactly a_0 different G-liaison classes containing arithmetically Buchsbaum divisors. We denote these classes by $\mathcal{L}_t$, $1 \le t \le a_0$. They correspond to the modules K^t, $1 \le t \le a_0$, concentrated in only one degree of the first deficiency module. Moreover, $t+1$ fibers, $(t+1)F$, of a rational normal scroll $S(a_0, \dots, a_k)$ are minimal schemes (in the sense of Definition 1.6) in the corresponding even G-liaison class.

(ii) At present, we cannot say that two arithmetically Buchsbaum schemes with isomorphic deficiency modules (up to shifts and dual) lying on two different rational normal scrolls are in the same G-liaison class. However, in the next section we will show that this is indeed the case (see Proposition 3.9).

Remark 2.6. In the case of ACM divisors on rational normal scrolls, the authors proved a similar result (see [3, Theorems 4.7 and 4.10] and [2, Theorem 3.2.3]).

3. G-liaison of independent linear varieties

It is well known that any two skew lines in $\mathbb{P}^3$ are in the same CI-liaison class. Recently it has been proved that any two skew lines in $\mathbb{P}^4$ are in the same G-liaison class (see [8]). Thus, it is natural to ask:

Question 3.1. Let $C = L_1 \cup \dots \cup L_n$ and $C' = L'_1 \cup \dots \cup L'_n$ be the union of n skew independent lines in $\mathbb{P}^N$, with $N \ge 2n-1$. Are C and C' in the same G-liaison class?

In this section, we answer this question in the affirmative. In fact, in Theorem 3.5 we show that any two configurations of k independent linear varieties of dimension $d \ge 1$ lying in $\mathbb{P}^N$, where $N \ge dk+k-1$, belong to the same G-liaison class.

Definition 3.2. We say that k linear varieties $L_1, \dots, L_k \subset \mathbb{P}^N$ of dimension $\dim L_i = d_i$ are *independent* if their linear span has maximal dimension, i.e., if $\langle L_1 \cup \dots \cup L_k \rangle \cong \mathbb{P}^{\sum d_i + k - 1}$.

Notice that independent linear varieties are always disjoint, but not vice versa. For example, three disjoint lines in $\mathbb{P}^4$ are not independent.

By a result of the first author (see [1, Theorem 2.3]), the union of k independent lines $X = L_1 \cup \dots \cup L_k \subset \mathbb{P}^N$, $N \ge 2k-1$, is an arithmetically Buchsbaum curve. In fact, proceeding as in [1] we can prove the following:

Lemma 3.3. *Let $X = L_1 \cup \dots \cup L_k \subset \mathbb{P}^N$ be the union of k independent linear varieties of dimension $d \ge 1$, with $N \ge dk+k-1$. Then X is an arithmetically Buchsbaum scheme and its deficiency modules are given by*

$$M^1(X)_t = \begin{cases} K^{k-1}, & t = 0, \\ 0, & t \ne 0, \end{cases}$$

and $M^i(X) = 0$ for $1 < i \leq d$. In particular, X is a minimal scheme in its even G-liaison class.

The following proposition will be the key ingredient in the proof of Theorem 3.5.

PROPOSITION 3.4. *Let $L_1, \dots, L_{k+1} \subset \mathbb{P}^N$ be a set of $k+1$ linear varieties of dimension $d \geq 1$ such that for any k-tuple $1 \leq i_1 < i_2 \cdots < i_k \leq k+1$, $L_{i_1}, \dots, L_{i_k}$ are independent and $\langle L_1 \cup \cdots \cup L_{k+1} \rangle \cong \mathbb{P}^{dk+k-1+\varepsilon}$ with $0 \leq \varepsilon \leq d$. Then there is a rational normal scroll*

$$S_\varepsilon := S(\underbrace{k-1, \dots, k-1}_{d+1-\varepsilon}, \underbrace{k, \dots, k}_{\varepsilon}) \subset \mathbb{P}^{dk+k-1+\varepsilon}$$

of dimension $d+1$ containing $L_1 \cup \cdots \cup L_{k+1}$.

Proof. Without loss of generality we may assume that we have fixed coordinates $X_0, X_1, \dots, X_N$ in $\mathbb{P}^N$ so that the linear span $\langle L_1 \cup \cdots \cup L_{k+1} \rangle$ is defined by $X_{dk+k+\varepsilon} = \cdots = X_N = 0$, the linear variety $H := \langle L_1 \cup \cdots \cup L_k \rangle$ is defined by $X_{dk+k} = \cdots = X_{dk+k-1+\varepsilon} = \cdots = X_N = 0$ and for any $i = 0, \dots, k-1$ the linear variety L_{i+1} is defined in H by

$$L_{i+1} : \ X_j = 0 \text{ for } 0 \leq j \leq id + i - 1 \ , (i+1)d + i + 1 \leq j \leq dk + k - 1.$$

We take $d+1$ points in L_{k+1}, $M_i = (m_i^0 : m_i^1 : \cdots : m_i^{dk+k-1} : m_i^{dk+k} : \cdots : m_i^{dk+k+\varepsilon-1} : 0 \cdots : 0)$ such that $\langle M_1 \cup \cdots \cup M_{d+1} \rangle = L_{k+1}$ and $\langle M_1 \cup \cdots \cup M_{d-\varepsilon+1} \rangle = L_{k+1} \cap H$. For each $i = 1, \dots, d-\varepsilon+1$ we consider the points $P_i^1 = (m_i^0 : \cdots : m_i^d : 0 : \cdots : 0) \in L_1, \dots, P_i^k = (0 : \cdots : 0 : m_i^{d(k-1)+k-1} : \cdots : m_i^{dk+k-1} : 0 : \cdots : 0) \in L_k$, and for $i = d-\varepsilon+2, \dots, d+1$ we take general points $P_i^j \in L_j$ such that $\langle P_1^j \cup \cdots \cup P_{d+1}^j \rangle = L_j$ for $j = 1, \dots, k$. Set

$$\Pi_i := \langle P_i^1 \cup P_i^2 \cup \cdots \cup P_i^k \cup M_i \rangle \subset \mathbb{P}^{dk+k-1+\varepsilon} \text{ for } i = 1, \dots, d+1.$$

Thus, the linear space Π_i has dimension $k-1$ for $i = 1, \dots, d-\varepsilon+1$, and dimension k for $i = d-\varepsilon+2, \dots, d+1$. Since $\langle \bigcup_{\substack{j=1,\dots,k \\ i=1,\dots,d+1}} P_i^j \rangle = \mathbb{P}^{dk+k-1}$, the linear spaces $\Pi_1, \dots, \Pi_{d+1}$ are independent. Moreover, for all $i = 1, \dots, d+1$, we can consider rational normal curves D_i in Π_i through the points $P_i^1, \dots, P_i^k, M_i$. (Note that D_i has degree $k-1$ if $1 \leq i \leq d-\varepsilon+1$ and degree k if $d-\varepsilon+2 \leq i \leq d+1$.) In fact, since for $i = 1, \dots, d-\varepsilon+1$ the points $P_i^1, \dots, P_i^k, M_i$ form a projective coordinate system of the projective space Π_i, we can fix $k+1$ points $Q_0, Q_1, \dots, Q_k$ in $\mathbb{P}^1$ so that we get morphisms $\phi_i : \mathbb{P}^1 \to \Pi_i$ defining the curves D_i with $\phi_i(Q_0) = M_i$ and $\phi_i(Q_j) = P_i^j$ for all $j = 1, \dots, k$, and $i = 1, \dots, d-\varepsilon+1$. In addition, since for $i = d-\varepsilon+2, \dots, d+1$, the points $P_i^1, \dots, P_i^k, M_i$ are part of a projective coordinate system of the projective space Π_i, we can also consider morphisms $\phi_i : \mathbb{P}^1 \to \Pi_i$ defining the curves D_i with $\phi_i(Q_0) = M_i$ and $\phi_i(Q_j) = P_i^j$ for all $j = 1, \ldots, k$ and $i = d-\varepsilon+2, \ldots, d+1$.

According to [4, p. 46],

$$S_\varepsilon := \bigcup_{p\in\mathbb{P}^1} \langle \phi_1(p) \cup \cdots \cup \phi_{d+1}(p)\rangle$$

is a rational normal scroll of type

$$S(\underbrace{k-1, \ldots, k-1}_{d-\varepsilon+1}, \underbrace{k, \ldots, k}_{\varepsilon}) \subset \mathbb{P}^{dk+k-1+\varepsilon},$$

and by construction it contains the linear subspaces

$$\begin{aligned}
\langle \phi_1(Q_1) \cup \cdots \cup \phi_{d+1}(Q_1)\rangle &= \langle P_{1_1} \cup \cdots \cup P_{1_{d+1}}\rangle = L_1,\\
&\vdots\\
\langle \phi_1(Q_k) \cup \cdots \cup \phi_{d+1}(Q_k)\rangle &= \langle P_{k_1} \cup \cdots \cup P_{k_{d+1}}\rangle = L_k,\\
\langle \phi_1(Q_0) \cup \cdots \cup \phi_{d+1}(Q_0)\rangle &= \langle M_1 \cup \cdots \cup M_{d+1}\rangle = L_{k+1}.
\end{aligned}$$

This completes the proof. □

THEOREM 3.5. *Let $V = \bigcup_{i=1}^k L_i$ and $V' = \bigcup_{i=1}^{k'} L'_i$ with $k, k' \geq 2$ be two unions of k (resp. k') independent linear varieties of dimension $d \geq 1$ in $\mathbb{P}^N$. Then V and V' belong to the same G-liaison class if and only if $k = k'$.*

REMARK 3.6. Notice that if $k = 1$ then V is a complete intersection. For $d = 0$, we have a slightly different result. Indeed, for any n, a set of n independent points in $\mathbb{P}^N$, $N \geq n-1$, is glicci, i.e., belongs to the G-liaison class of a complete intersection.

Proof of Theorem 3.5. Because of the independence of the linear varieties L_i (resp. L'_i), the linear varieties L_i (resp. L'_i) are disjoint and $N \geq \max(dk+k-1, dk'+k'-1)$.

Assume that V and V' are in the same G-liaison class. The G-liaison invariance of the deficiency modules $M^i(V)$ together with Lemma 3.3 yields $k = k'$. To prove the converse we distinguish two cases, depending on whether $L_1 \cup \cdots \cup L_k \cup L'_1 \cup \cdots \cup L'_k$ span a linear space of dimension $\leq dk+k-1+d$ (Case 1) or a linear space of dimension $> dk+k-1+d$ (Case 2).

Case 1: Let $L_1 \cup \cdots \cup L_k \cup L'_1 \cup \cdots \cup L'_k$ span a linear space of dimension $dk+k-1+\varepsilon$ with $0 \leq \varepsilon \leq d$.

We first note that we may assume that V and V' are disjoint and that for any $i = 1, \ldots, k$, $L'_{j_1} \cup \cdots \cup L'_{j_{i-1}} \cup L_{s_i} \cup \cdots \cup L_{s_k}$ with $1 \leq j_1 \cdots \leq j_{i-1} \leq k$ and $1 \leq s_i \cdots \leq s_k \leq k$ is a disjoint union of k independent linear varieties of dimension d . Indeed, otherwise we consider a third union of k independent linear varieties of dimension d, $V'' = L''_1 \cup \cdots \cup L''_k \subset \mathbb{P}^{dk+k-1+\varepsilon}$, which satisfies this condition with respect to V and V'. Thus, if we can show that V and V'' are G-linked, and that V'' is G-linked to V', then it follows that V and V' belong to the same G-liaison class.

According to Proposition 3.4, there exists a smooth rational normal scroll

$$S_\varepsilon := S(\underbrace{k-1,\ldots,k-1}_{d+1-\varepsilon},\underbrace{k,\ldots,k}_{\varepsilon}) \subset \mathbb{P}^{dk+k-1+\varepsilon}$$

containing $L_1 \cup \cdots \cup L_k \cup L_1'$. By the assumptions we have made at the beginning of the proof and by Lemma 3.3, $V_1 := L_1' \cup L_2 \cup \cdots \cup L_k$ is an AB scheme. Moreover, $V_1 = L_1' \cup L_2 \cup \cdots \cup L_k$ and $V = L_1 \cup \cdots \cup L_k$ belong to the same linear system $|kF|$ of the rational normal scroll S_ε. Therefore, applying Theorem 2.4, we conclude that V and V_1 are in the same G-liaison class.

Iterating this process, we obtain configurations of k independent linear varieties of dimension d, $V_i = L_1' \cup \cdots \cup L_i' \cup L_{i+1} \cup \cdots \cup L_k$, $i = 1, \ldots, k-1$, which are all arithmetically Buchsbaum and in the G-liaison class of V. Repeating the process above one more time, we see that V_{k-1} belongs to the same G-liaison class as V'. This yields the result in the first case.

Case 2: Let $L_1 \cup \ldots L_k \cup L_1' \cup \cdots \cup L_k'$ span a linear space of dimension $n > dk + k - 1 + d$.

We can assume that V and V' are disjoint, because we can always consider a third union of k independent linear varieties of dimension d, say V'', that are disjoint to V and V'.

Let $L_1'' \subset \mathbb{P}^N$ be a linear variety of dimension d which intersects the linear spaces $\langle L_1 \cup \cdots \cup L_k \rangle$ and $\langle L_1' \cup \cdots \cup L_k' \rangle$ in points and does not intersect either $\langle L_2 \cup \cdots \cup L_k \rangle$ or $\langle L_2' \cup \cdots \cup L_k' \rangle$. Thus, the linear space $\langle L_1'' \cup L_1 \cup \cdots \cup L_k \rangle$ has dimension $dk+k-1+d$, and by the first case it follows that $V = L_1 \cup \cdots \cup L_k$ is in the same G-liaison class as $L_1'' \cup L_2 \cup \cdots \cup L_k$. Similarly, $V' = L_1' \cup \cdots \cup L_k'$ is in the same G-liaison class as $L_1'' \cup L_2' \cup \cdots \cup L_k'$. Repeating this process k times with suitable linear varieties $L_2'', \ldots, L_k''$ of dimension d, we obtain that V and V' belong to the same G-liaison class as the AB scheme $C'' = L_1'' \cup \cdots \cup L_k''$. This completes the proof. $\square$

Remark 3.7. Without the hypothesis that the varieties be independent, the above result is not true. For instance, if in $\mathbb{P}^5$, $R = K[X_0, X_1, X_2, X_3, X_4, X_5]$, and we consider the union $V = L_1 \cup L_2 \cup L_3 \subset \mathbb{P}^5$ with $L_1 : X_0 = X_1 = X_2 = X_3 = 0$, $L_2 : X_0 = X_1 = X_4 = X_5 = 0$, $L_3 : X_2 = X_3 = X_4 = X_5 = 0$, then V is the union of three independent lines and V has Rao module K^2 concentrated in degree 0. On the other hand, if we consider $V' = L_1' \cup L_2' \cup L_3' \subset \mathbb{P}^5$ with $L_1' = L_1$, $L_2' = L_2$ and L_3' defined by $X_0 = X_2 = X_4 = X_3 + X_5 = 0$, then V' is the disjoint union of three lines which are not independent (since $L_1' \cup L_2' \cup L_3'$ generate the hyperplane defined by $X_0 = 0$) and the Rao module of V' is

$$M(V')_t = \begin{cases} K^2, & t = 0, \\ K, & t = 1, \\ 0, & t \neq 0, 1. \end{cases}$$

Thus, by the Hartshorne-Schenzel Theorem, V and V' cannot be in the same G-liaison class.

Corollary 3.8. *Let $C = C_1 \cup C_2 \subset \mathbb{P}^N$ be the disjoint union of two rational normal curves C_1 and C_2 of degrees d_1 and d_2, respectively, with $N \geq d_1 + d_2 + 1$. Assume that $\langle C_1 \rangle \cap \langle C_2 \rangle = \emptyset$. Then C is an arithmetically Buchsbaum curve and in the G-liaison class of any two skew lines in $\mathbb{P}^N$.*

Proof. The fact that C is AB is given in [1, Theorem 2.3]. Moreover, C has Hartshorne-Rao module K concentrated in degree 0.

Let $H_i := \langle C_i \rangle \cong \mathbb{P}^{d_i}$ for $i = 1, 2$; then $H_1 \cap H_2 = \emptyset$ by hypothesis. We consider a scroll $S := S(d_1, d_2) \subset \mathbb{P}^N$ of dimension 2 containing C_1 and C_2 (see the geometric construction of rational normal scrolls in [4, p. 46]). Then, since by Theorem 2.4 any two curves with the same Rao module in this scroll lie in the same G-liaison class, C is in the G-liaison class of two fibers of this scroll. As a consequence, by Theorem 3.5, C is in the G-liaison class of any two skew lines. □

In Remark 2.5 we have pointed out that there are exactly a_0 different G-liaison classes $\mathcal{L}_t$, $1 \leq t \leq a_0$, containing arithmetically Buchsbaum schemes of codimension c lying on a rational normal scroll $S = S(a_0, \ldots, a_k) \subset \mathbb{P}^{c+k}$. It is natural to ask if these classes coincide with the a_0 different G-liaison classes containing AB schemes of codimension c lying on another rational normal scroll $S' = S(a_0, \ldots, a_k) \subset \mathbb{P}^{c+k}$. The answer is yes; more precisely, we have the following result:

Proposition 3.9. *Let X, $X' \subset \mathbb{P}^N$ be two arithmetically Buchsbaum schemes of dimension k. Assume that X (resp. X') lies on a rational normal scroll $S = S(a_0, \ldots, a_k) \subset \mathbb{P}^N$ (resp. $S' = S(a'_0, \ldots, a'_k) \subset \mathbb{P}^N$). Then X and X' have isomorphic deficiency modules, up to shifts and dual, if and only if X and X' belong to the same G-liaison class.*

Proof. If X and X' belong to the same G-liaison class, then their deficiency modules are isomorphic (up to shifts and dual) by Theorem 1.4.

Now assume that X and X' have isomorphic deficiency modules (up to shifts and dual). Since $X \subset S$ is AB, by Proposition 2.2 it follows that X is linearly equivalent to $aH_S + bF_S$ on S for some $a \geq 0$ and either

(i) $1 < b \leq a_0 + 1$, or
(ii) $1 - c - a_0 \leq b \leq -c$.

Moreover, in case (i) we have

$$H^i_*(I_X) \cong \begin{cases} K^{b-1}(l), & i=1, \\ 0, & 1 < i \leq k. \end{cases}$$

for some $l \in \mathbb{Z}$. It can also be checked that in case (ii)

$$H^i_*(I_X) \cong \begin{cases} K^{-b-c+2}(m), & i=k, \\ 0, & 1 \leq i < k, \end{cases}$$

for some $m \in \mathbb{Z}$. In case (ii), we can perform a G-link as in the proof of Theorem 2.4, and X is in the same G-liaison class as a divisor $\widetilde{X} \sim \tilde{a}H_S + (-b-c+2)F_S$ which is in case (i).

We do the same with X': we write $X' \sim a'H_{S'} + b'F_{S'}$ with $a' \geq 0$ and either

(a) $1 < b' \leq a'_0 + 1$, or
(b) $1 - c' - a'_0 \leq b' \leq -c'$, where $c' = \sum a'_i$.

If X is in case (i) and X' is in case (a), then X (resp. X') is in the G-liaison class of any b different fibers of S (resp. b' different fibers of S') by Theorem 2.4. Let Y (resp. Y') denote b (resp. b') different fibers of S (resp. S'). Then Y (resp. Y') is also an AB subscheme of $\mathbb{P}^N$. Hence, by [1] (or rather [2]), Y (resp. Y') is a configuration of b (resp. b') independent linear varieties of dimension k of $\mathbb{P}^N$. Moreover, since we are assuming that X and X' have isomorphic deficiency modules (up to shifts and dual), Y and Y' have also isomorphic deficiency modules (up to shifts and dual), so $b = b'$. By Theorem 3.5, Y is in the same G-liaison class as Y', and we are done.

If X is in case (i) and X' is in case (b), let Y be b different fibers of S and let Y' be $-c' - b' + 2$ different fibers of S'. Then X is in the G-liaison class of Y and X' is in the G-liaison class of Y'. Notice that Y and Y' are AB and are independent configurations of linear varieties (see [1] or [2]). Moreover, since X and X' have isomorphic deficiency modules (up to shifts and dual), we have $b = -c' - b' + 2$ and Theorem 3.5 implies that Y and Y' belong to the same G-liaison class. Hence X and X' are also in the same G-liaison class.

If X is in case (ii) and X' is in case (a), we interchange the roles of X and X' in the argument above, and we are done.

If X is in case (ii) and X' is in case (b), then X (resp. X') is in the G-liaison class of $-c - b + 2$ (resp. $-c' - b' + 2$) different fibers of S (resp. S'). Arguing as above, we conclude the proof of the Proposition. $\square$

As another consequence of Theorem 3.5, and following the proof of [8, Theorem 3.6], we obtain a partial generalization of a result of Hartshorne (see [6, Proposition 3.1(c)]) and Lesperance ([8, Theorem 3.6]). To this end, we need the following lemma:

LEMMA 3.10. *Let $C = L_1 \cup \cdots \cup L_{n+1} \subset \mathbb{P}^n$ be a non-degenerate (i.e., $H^0(I_C(1)) = 0$) closed polygonal curve. Then C is an arithmetically Gorenstein curve.*

Proof. Without loss of generality we can fix coordinates $X_0, \dots, X_n$ in $\mathbb{P}^n$ so that the lines L_i, $i = 1, \dots, n+1$, are defined by

$$L_i : \ X_{i-1} = X_i = \cdots = X_{n+i-2} = 0,$$

where $X_{n+j} = X_{j-1}$ for all $j \geq 1$. The homogeneous ideal $I(C)$ is generated by the $(n-2)(n+1)/2$ quadrics X_iX_j with $|j-i| \geq 2$. Moreover, we have $\deg(C) = n+1$, $p_a(C) = 1$, and $M(C) = H^1_*(I_C) = 0$. Hence C is an ACM curve.

In addition, the general hyperplane section of C is the union of $n+1$ general points in $\mathbb{P}^{n-1}$, and is known to be an AG scheme (see [10, Example 4.1.11(b)]). Therefore, by [10, Theorem 1.3.5] C is an arithmetically Gorenstein curve. $\square$

COROLLARY 3.11. *Let $C \subset \mathbb{P}^{2n}$, $n \geq 2$, be a reduced curve with Rao module K^{n-1} concentrated in degree 0. Then C is in the G-liaison class of any n independent lines.*

Proof. Since C is reduced with Rao module $M_0 \cong K^{n-1}$, C must have n connected components. By [1, Corollary 2.6] and Remark 2.7 these components must be of the form $C = L_1 \cup \cdots \cup L_{n-1} \cup C_0$, where $L_1, \dots, L_{n-1}$ are independent lines, C_0 is a curve living on a plane H_0 and $H_0 \cap \langle L_1 \cup \cdots \cup L_{n-1} \rangle = \emptyset$. Let $e := \deg C_0$; we can assume $e \geq 2$ because the case $e = 1$ follows from Theorem 3.5.

Taking coordinates in $\mathbb{P}^{2n}$, we can also assume that $\langle L_1 \cup \cdots \cup L_{n-1} \rangle \cong \mathbb{P}^{2n-3}$ is defined by $X_0 = X_1 = X_2 = 0$ and H_0 is defined by $X_3 = \cdots = X_{2n} = 0$. Since the lines $L_1, \dots, L_{n-1}$ are independent, $H^1(I_{L_1\cup\cdots\cup L_{n-1}}(t)) = 0$ for all $t \geq 1$ (see Lemma 3.3). Moreover, $H^2(I_{L_1\cup\cdots\cup L_{n-1}}(t)) = 0$ for all $t \geq -1$. Thus $I_{L_1\cup\cdots\cup L_{n-1}}$ is 2-regular and the homogeneous ideal of $L_1 \cup \cdots \cup L_{n-1} \subset \mathbb{P}^{2n-3}$ is generated by quadrics. Let $Q \in (X_3, \dots, X_{2n})_2$ be a quadric in $I(L_1 \cup \cdots \cup L_{n-1})$. We can also suppose that $I(C_0) = (X_0^e + G, X_3, \dots, X_{2n})$, where G is a form of degree e in the variables X_1, X_2.

We consider the point $P = (1, 0, \dots, 0) \in H_0$, $P \notin C_0$, and the planes $\Pi_i := \langle L_i \cup P \rangle$ for $i = 1, \dots, n-1$, and $\Pi_n := H_0$. We also consider n planes, $\Gamma_1, \dots, \Gamma_n$, through the point P such that for $i = 1, \dots, n-1$, the intersections $\Gamma_i \cap \Pi_i$, $\Gamma_i \cap \Pi_{i+1}$, $\Gamma_n \cap \Pi_n$, and $\Gamma_n \cap \Pi_1$ are lines, and any other intersection among these planes is exactly the point P. (We can construct these planes $\Gamma_1, \dots, \Gamma_n$ by taking lines through P in the planes Π_i.)

We define S to be the surface $S = \Pi_1 \cup \cdots \cup \Pi_n \cup \Gamma_1 \cup \cdots \cup \Gamma_n$. Thus S is a cone from P over a union of $2n$ lines in $\mathbb{P}^{2n-1}$ satisfying the hypothesis of Lemma 3.10, and hence is an arithmetically Gorenstein surface.

Let $F := X_0^e + G + G'Q \in I(C)_e$ be a form of degree e cutting S properly (where $G' \in R_{e-2}$ is a general form). Then $S \cap F$ is an AG scheme that links C to a scheme $D \subset \Pi_1 \cup \cdots \cup \Pi_{n-1} \cup \Gamma_1 \cup \cdots \cup \Gamma_n$.

Let $X := \Pi_1 \cup \cdots \cup \Pi_{n-1} \cup \Gamma_1 \cup \cdots \cup \Gamma_n$. Then X is an ACM surface (because it is the residual to the plane Π_n in S) and satisfies property G_1. Let H be the general hyperplane section on X and let K_X be a canonical divisor on X. Take an integer $m \gg 0$ such that $mH - K_X$ is effective. Since the divisor $L_1 + \cdots + L_{n-1} + D$ is linearly equivalent to eH, by Proposition 1.2 the divisor $E := L_1 + \cdots + L_{n-1} + D + mH - K_X$ is AG and links D to $L_1 + \cdots + L_{n-1} + mH - K_X$.

Now let $G \in I(L_1 \cup \cdots \cup L_{n-1})_1$ be a linear form cutting X properly. Then $H_G + mH - K_X$ is AG and contains $L_1 + \cdots + L_{n-1} + mH - K_X$, so it links this divisor to a union Y of n independent lines in the hyperplane defined by G (i.e., $Y = G \cap (\Gamma_1 \cup \cdots \cup \Gamma_n)$). Finally, applying Theorem 3.5, we obtain the result. □

REMARK 3.12. In $\mathbb{P}^N$, $N \geq 2n$, any curve C of the form $C = C_0 \cup L_1 \cup \cdots \cup L_n$ as in the proof of Corollary 3.11, spans a space $\mathbb{P}^{2n}$, and hence is also in the G-liaison class of any n independent lines.

REMARK 3.13. We remark that all curves C in the Corollary above are arithmetically Buchsbaum and have Rao module K^{n-1} concentrated in degree 0. Since an arithmetically Buchsbaum curve cannot have a Rao module different from zero in negative degrees, any such curve is minimal in its even liaison class. Moreover, C belongs to the G-liaison class of the union of n independent lines, C_0, and by [1, Remark 2.7], C_0 has minimal degree among the reduced minimal curves.

Finally, it is well known that if $V \subset \mathbb{P}^n$ is an integral arithmetically Buchsbaum scheme of degree d and codimension c, then its Castelnuovo-Mumford regularity, $\mathrm{reg}(V)$, is bounded above by $\mathrm{reg}(V) \leq \lceil (d-1)/c \rceil + 1$ (see [12, Theorem 1.2]), where $\lceil m \rceil$ is the smallest integer $\geq m$ for $m \in \mathbb{Q}$. By a result of Nagel (see [12, Theorem 1.3]) we obtain:

COROLLARY 3.14. *Let $V \subset \mathbb{P}^n$ be an integral AB scheme of codimension $c < n$ and degree $d > \max(2c(c+1), 14)$. Assume that V has maximum Castelnuovo-Mumford regularity, i.e., that $\mathrm{reg}(V) = \lceil (d-1)/c \rceil + 1$. Then V is in the G-liaison class of any configuration of independent linear varieties of codimension c having deficiency modules isomorphic to those of V (up to shifts and dual).*

Proof. A result of Nagel (see [12, Theorem 1.3]) says that, under the hypothesis of the Corollary, V is a divisor on a variety of minimal degree. A variety of minimal degree is either a rational normal scroll, a cone over a quadric hypersurface, or a cone over the Veronese surface in $\mathbb{P}^5$ (see [4], for instance). But any divisor on a cone over a quadric hypersurface or over the Veronese surface is ACM (see [12, Proposition 2.9]), and we have assumed that the AB schemes in this paper are not ACM. Thus V is a divisor on a rational normal scroll, and the result follows from Remark 2.5 and Theorem 3.5. □

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