Illinois Journal of Mathematics
Volume 46, Number 1, Spring 2002, Pages 93–109
S 0019-2082

ESTIMATES OF FUNCTIONS WITH VANISHING PERIODIZATIONS

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ABSTRACT. We prove that if a function $f \in L^p(\mathbb{R}^d)$ has vanishing periodizations then $\|f\|_{p'} \lesssim \|f\|_p$, provided $1 \le p < 2d/(d+2)$ and $d \ge 3$.

1. Introduction

Let $f \in L^1(\mathbb{R}^d)$. Define a family of its periodizations with respect to a rotated integer lattice by

$$g_\rho(x) = \sum_{\nu \in \mathbb{Z}^d} f(\rho(x-\nu)) \tag{1}$$

for all rotations $\rho \in \mathrm{SO}(d)$. We have the trivial estimate $\|g_\rho\|_1 \le \|f\|_1$, and $\widehat{g_\rho}(m) = \hat{f}(\rho m)$, where $m = (m_1, \dots, m_d) \in \mathbb{Z}^d$. The author has shown recently that g_ρ is in $L^2([0,1]^d \times \mathrm{SO}(d))$ if and only if $f \in L^2(\mathbb{R}^d)$, provided $d \ge 5$. The requirement $f \in L^1(\mathbb{R}^d)$ can be replaced by $f \in L^p(\mathbb{R}^d)$ for a certain range of p; for details see [6] and [7].

The main object of our research are functions f whose periodizations g_ρ vanish identically for a.e. rotations $\rho \in \mathrm{SO}(d)$. This property is equivalent to the statement that $\hat{f}$ vanishes on all spheres of radius $|m| = (m_1^2 + \cdots + m_d^2)^{1/2}$, where $m \in \mathbb{Z}^d$. Such functions are closely related to the Steinhaus tiling set problem (see [4] and [5]): Does there exists a (measurable) set $E \subset \mathbb{R}^d$ such that every rotation and translation of E contains exactly one integer lattice point? M. Kolountzakis [4] showed that if $f \in L^1$ and $|x|^\alpha f(x) \in L^1$ for a certain $\alpha > 0$ and f has constant periodizations, then $\hat{f} \in L^1$ in the case of dimension $d = 2$. Kolountzakis and Wolff [5, Theorem 1] proved that if the periodizations of a function from $L^1(\mathbb{R}^d)$ are constant, then the function is continuous and, in fact, bounded, provided that the dimension d is at least three. Here we generalize the latter result for functions f in $L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$:

Received January 30, 2001; received in final form April 21, 2002.
2000 *Mathematics Subject Classification.* 42B35.
This research was partially conducted by the author for the Clay Mathematics Institute and partially supported by NSF grant DMS 97-29992.

Theorem 1. *Let $d \geq 3$ and let $f \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, $1 \leq p < 2d/(d+2)$, have identically vanishing periodizations. Then $f \in L^{p'}(\mathbb{R}^d)$, and*

$$\|f\|_{p'} \leq C\|f\|_p,$$

where C depends only on d and p.

The main reason for the condition $d \geq 3$ is due to the famous result of Lagrange stating that every positive integer can be represented as a sum of four squares, and that every integer of the form $8k+1$ can be written as a sum of three squares. Since relatively few integers can be represented as sums of two squares, we will show in Section 3 that the result of Kolountzakis and Wolff does not hold if $d = 2$. This is why there is no analogous theorem for $d = 2$. Another reason why the dimension d has to be at least 3 is because we consider the family of periodizations with respect to the group of rotations $\mathrm{SO}(d)$. This leads to estimates involving the decay of spherical harmonics. For $d = 2$ the rate of decay is not fast enough, although it is almost fast enough. In the case $d = 2$ the range for p in the theorem becomes $1 \leq p < 1$, and hence is empty.

Remark 1. There is no essential difference between the case of identically vanishing periodizations and the case where the functions g_ρ are trigonometric polynomials of uniformly bounded degrees for all $\rho \in \mathrm{SO}(d)$.

Corollary 1. *If $p \leq r \leq p'$, then under the conditions of Theorem 1 we have*

$$\|f\|_r \leq C\|f\|_p,$$

where C depends only on d and p.

We will show in Section 3 that the range of r in this result is sharp.

We will use the notation $x \lesssim y$ if $x \leq Cy$ for some constant $C > 0$ independent from x and y, and we write $x \sim y$ if $x \lesssim y$ and $y \lesssim x$ both hold.

2. Proof of the theorem

We define functions $h, h_1, h_2 : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{C}$ by

$$\begin{aligned} h(y,t) &= \int \hat{f}(\xi) e^{i2\pi y\cdot\xi} d\sigma_t(\xi) \\ &= \int_{\mathbb{R}^d} f(x)\widehat{d\sigma_t}(y-x)dx \\ &= \int_{\mathbb{R}^d} f(y-x)\widehat{d\sigma_t}(x)dx, \end{aligned} \tag{2}$$

$$h_1(y,t) = \int_{|x|\leq 1} f(y-x)\widehat{d\sigma_t}(x)dx, \tag{3}$$

$$h_2(y,t) = \int_{|x|>1} f(y-x)\widehat{d\sigma_t}(x)dx, \tag{4}$$

where $d\sigma_t$ is the Lebesgue surface measure on a sphere of radius t. Clearly, $h = h_1 + h_2$. To proceed further we will need certain technical estimates involving the functions h_1 and h_2; these are given in two lemmas below. The proof of the theorem itself begins after Remark 2 following Lemma 2. The Fourier transforms in the two lemmas below are taken with respect to variable t, except in the second part of the proof of Lemma 2. The $L^{p'}$ norms are taken with respect to the variable y. We will use some techniques of Kolountzakis and Wolff [5] and Kovrijkine [6], [7].

Lemma 1. *Let $q : \mathbb{R} \to \mathbb{R}$ be a Schwartz function supported in $[1/2, 2]$, let $f \in L^p(\mathbb{R}^d)$, where $1 \le p \le 2$, and let $b \in [0, 1)$. Define $H_{1,N} : \mathbb{R}^d \times \mathbb{R} \to \mathbb{C}$ by*

$$H_{1,N}(y,t) = \frac{1}{\sqrt{t+b}} h_1(y, \sqrt{t+b}) q\left(\frac{\sqrt{t+b}}{N}\right).$$

Then

$$\sum_{l \ge 0} \sum_{\nu \ne 0} \|\hat{H}_{1,2^l}(y,\nu)\|_{p'} \le C\|f\|_p, \tag{5}$$

where C depends only on q and d.

Proof. It will be enough to show that

$$\sum_{\nu \ne 0} \|\hat{H}_{1,N}(y,\nu)\|_{p'} \le \frac{C\|f\|_p}{N}. \tag{6}$$

We have

$$|\hat{H}_{1,N}(y,\nu)| \le \frac{C}{|\nu|^k} \int \left| \frac{\partial^k}{\partial t^k} H_{1,N}(y,t) \right| dt \tag{7}$$

for $\nu \ne 0$. Applying Minkowski's inequality to (7) we obtain

$$\|\hat{H}_{1,N}(y,\nu)\|_{p'} \le \frac{C}{|\nu|^k} \int \left\| \frac{\partial^k}{\partial t^k} H_{1,N}(y,t) \right\|_{L^{p'}(dy)} dt. \tag{8}$$

We need to estimate the integrand on the right side of (8). To do so we will first estimate the $L^{p'}$ norm of derivatives of $h_1(y,t)$ when $t \ge 1$. We have

$$\left\| \frac{\partial^k}{\partial t^k} h_1(y,t) \right\|_{p'} \lesssim t^{d-1} \|f\|_p \tag{9}$$

with an implicit constant depending only on k and d. In order to obtain (9), we rewrite the definition (3) of h_1 as

$$h_1(y,t) = \int_{|x|\le 1} f(y-x)\widehat{d\sigma_t}(x)dx$$
$$= t^{d-1}\int_{\mathbb{R}^d} f(y-x)\cdot \chi_{\{|x|\le 1\}}\int_{|\xi|=1} e^{-i2\pi t x\cdot\xi} d\sigma(\xi)dx,$$

differentiate the last expression k times, and apply Young's inequality.

We can easily prove by induction that

$$\frac{d^k}{dt^k}\left(\frac{h_1(\sqrt{t+b})}{\sqrt{t+b}}\right) = \sum_{i=0}^{k} C_{i,k}\frac{h_1^{(i)}(\sqrt{t+b})}{(\sqrt{t+b})^{2k+1-i}}. \tag{10}$$

Combining (10) and (9) we obtain for $t\sim N^2$

$$\left\|\frac{\partial^k}{\partial t^k}\left(\frac{h_1(y,\sqrt{t+b})}{\sqrt{t+b}}\right)\right\|_{p'} \le CN^{d-k-2}\|f\|_p \tag{11}$$

with C depending only on k and d.

Since $q((\sqrt{t+b})/N) = q(\sqrt{t'+b'}) = \tilde{q}(t')$ with $t' = t/N^2$ and $b' = b/N^2$ and $\tilde{q}(t')$ is a Schwartz function supported in $t'\sim 1$, we have

$$\left|\frac{d^k}{dt^k}q\left(\frac{(\sqrt{t+b})}{N}\right)\right| = N^{-2k}\left|\frac{d^k}{dt'^k}\tilde{q}(t')\right| \le CN^{-2k} \tag{12}$$

with C depending only on k and q.

Now $q((\sqrt{t+b})/N)$ is supported in $t\sim N^2$. Hence we obtain from (11) and (12)

$$\left\|\frac{\partial^k}{\partial t^k}H_{1,N}(y,t)\right\|_{p'} = \left\|\frac{d^k}{dt^k}\left(\frac{h_1(y,\sqrt{t+b})}{\sqrt{t+b}}q\left(\frac{\sqrt{t+b}}{N}\right)\right)\right\|_{p'} \le CN^{d-2-k}\|f\|_p \tag{13}$$

with C depending only on k, d and q. Since $H_{1,N}(y,t)$ is also supported in $t\sim N^2$, we have

$$\int \left\|\frac{\partial^k}{\partial t^k}H_{1,N}(y,t)\right\|_{L^{p'}(dy)} dt \le CN^{d-k}\|f\|_p.$$

Substituting this estimate into (8) we obtain

$$\|\hat{H}_{1,N}(y,\nu)\|_{p'} \le \frac{CN^{d-k}\|f\|_p}{|\nu|^k} \tag{14}$$

for every $\nu\ne 0$.

Summing (14) over all $\nu\ne 0$ and putting $k=d+1$ we obtain our desired result

$$\sum_{\nu\ne 0}\|\hat{H}_{1,N}(y,\nu)\|_{p'} \le \frac{C\|f\|_p}{N},$$

where C depends only on q and d. The assertion of the lemma follows by summing over dyadic values N. □

The next lemma will be proven using the methods of the Stein-Tomas restriction theorem (see [1, p. 104]).

LEMMA 2. *Let* $q : \mathbb{R} \to \mathbb{R}$ *be a Schwartz function supported in* $[1/2, 2]$, *let* $f \in L^p(\mathbb{R}^d)$, *where* $1 \le p < 2d/(d+2)$ *and let* $b \in [0, 1)$. *Define* $H_{2,N} : \mathbb{R}^d \times \mathbb{R} \to \mathbb{C}$ *by*

$$H_{2,N}(y,t) = \frac{1}{\sqrt{t+b}} h_2(y, \sqrt{t+b}) q\left(\frac{\sqrt{t+b}}{N}\right).$$

Then we have

$$\sum_{\nu \neq 0} \left\| \sum_{l \geq 0} \hat{H}_{2,2^l}(y,\nu) \right\|_{p'} \leq C \|f\|_p \tag{15}$$

with C *depending only on* p, q *and* d.

Proof. We have

$$\begin{aligned} \hat{H}_{2,N}(y,\nu) &= \int H_{2,N}(y,t) e^{-i2\pi\nu t} dt \\ &= 2e^{i2\pi\nu b} \int Nq(t) h_2(y,tN) e^{-i2\pi\nu(Nt)^2} dt \\ &= 2e^{i2\pi\nu b} \int Nq(t) e^{-i2\pi\nu(Nt)^2} \int_{|x|>1} f(y-x) \widehat{d\sigma_{Nt}}(x) dx dt \\ &= 2e^{i2\pi\nu b} \int_{|x|>1} f(y-x) \int Nq(t) e^{-i2\pi\nu(Nt)^2} (Nt)^{d-1} \widehat{d\sigma}(Ntx) dt dx \\ &= (D_{N,\nu} * f)(y), \end{aligned} \tag{16}$$

where

$$D_{N,\nu}(x) = 2e^{i2\pi\nu b} \chi_{\{|x|>1\}} \int Nq(t) e^{-i2\pi\nu(Nt)^2} (Nt)^{d-1} \widehat{d\sigma}(Ntx) dt. \tag{17}$$

Set

$$K_\nu(x) = \sum_{l \geq 0} D_{2^l,\nu}(x). \tag{18}$$

We need to estimate

$$\left\| \sum_{l \geq 0} \hat{H}_{2,2^l}(y,\nu) \right\|_{p'} = \|K_\nu * f\|_{p'}.$$

If $p' = \infty$ or $p' = 2$, then

$$\|K_\nu * f\|_\infty \le \|K_\nu\|_\infty \|f\|_1$$
$$\|K_\nu * f\|_2 \le \|\hat{K}_\nu\|_\infty \|f\|_2.$$

We first show that

$$\|K_\nu\|_\infty \le \left\| \sum_{l \ge 0} |D_{2^l,\nu}|(x) \right\|_\infty \le C|\nu|^{-d/2} \tag{19}$$

To this end we need to estimate $D_{N,\nu}$.

We will use the well-known fact that $\widehat{d\sigma}(x) = \mathrm{Re}(B(|x|))$ with $B(r) = a(r)e^{i2\pi r}$ and $a(r)$ satisfying

$$|a^k(r)| \le \frac{C}{r^{(d-1)/2+k}}, \tag{20}$$

with C depending only on k and d. We now estimate the integral in (17) with $B(|x|)$ instead of $\widehat{d\sigma}(x)$:

$$\begin{aligned} &\int Nq(t)e^{-i2\pi\nu(Nt)^2}(Nt)^{d-1}a(N|x|t)e^{i2\pi N|x|t}dt \\ &= \frac{N^{\frac{d+1}{2}}}{|x|^{\frac{d-1}{2}}} \int q(t)e^{-i2\pi\nu(Nt)^2} t^{d-1} a(N|x|t)(N|x|)^{\frac{d-1}{2}} e^{i2\pi N|x|t} dt \\ &= \frac{N^{\frac{d+1}{2}}}{|x|^{\frac{d-1}{2}}} e^{i2\pi\frac{|x|^2}{4\nu}} \int q(t)a(N|x|t)(N|x|)^{\frac{d-1}{2}} t^{d-1} e^{-i2\pi\nu N^2(t-\frac{|x|}{2\nu N})^2} dt \\ &= \frac{N^{\frac{d+1}{2}}}{|x|^{\frac{d-1}{2}}} e^{i2\pi\frac{|x|^2}{4\nu}} \int \phi(t,|x|) e^{-i2\pi\nu N^2(t-\frac{|x|}{2\nu N})^2} dt, \end{aligned} \tag{21}$$

where $\phi(t,|x|) = q(t)a(N|x|t)(N|x|)^{(d-1)/2}t^{d-1}$ is a Schwartz function with respect to the variable t supported in $[1/2, 2]$, which, by (20), is bounded, together with each derivative, uniformly in t, $|x| \ge 1$, and N. Note that we used here the fact that $N|x| \ge 1$. We can say even more. Let $|x| = c \cdot r$, where $c \ge 2$ and $r \ge 1/2$. Then all partial derivatives of $\phi(t, c\cdot r)$ with respect to t and r are also bounded uniformly in t, r, c and N. Hence $\phi(t, c\cdot t)$ is a Schwartz function supported in $[1/2, 2]$ which is bounded, together with each derivative, uniformly in t, c and N. We will use this fact later to estimate $\hat{K}_\nu$.

Fix some x with $|x| \ge 1$. In the calculations below we will write $\phi(t)$ instead of $\phi(t, |x|)$ for simplicity. From the method of stationary phase (see

[3, Theorem 7.7.3]) it follows that if $k \geq 1$ then

$$\left| \int \phi(t) e^{-i2\pi\nu N^2 (t-\frac{|x|}{2\nu N})^2} dt - \sum_{j=0}^{k-1} c_j (\nu N^2)^{-j-1/2} \phi^{(2j)} \left(\frac{|x|}{2\nu N} \right) \right| \leq c_k (|\nu| N^2)^{-k-1/2} \tag{22}$$

with some constants c_j.

Since ϕ is supported in $[1/2, 2]$, we conclude from (22) that

$$\left| \int \phi(t) e^{-i2\pi\nu N^2 (t-\frac{|x|}{2\nu N})^2} dt \right| \leq \begin{cases} C(|\nu| N^2)^{-1/2} & \text{if } N \in [\frac{|x|}{4\nu}, \frac{|x|}{\nu}], \\ C_k (|\nu| N^2)^{-k-1/2} & \text{if } N \notin [\frac{|x|}{4\nu}, \frac{|x|}{\nu}]. \end{cases} \tag{23}$$

Replacing in (17) $\widehat{d\sigma}(x)$ by $(B(|x|) + \bar{B}(|x|))/2$, it follows from (23) that

$$|D_{N,\nu}(x)| \leq \frac{N^{\frac{d+1}{2}}}{|x|^{\frac{d-1}{2}}} \begin{cases} C(|\nu| N^2)^{-1/2} & \text{if } N \in [\frac{|x|}{4|\nu|}, \frac{|x|}{|\nu|}], \\ C_k (|\nu| N^2)^{-k-1/2} & \text{if } N \notin [\frac{|x|}{4|\nu|}, \frac{|x|}{|\nu|}]. \end{cases} \tag{24}$$

The number of dyadic $N \in [\frac{|x|}{4\nu}, \frac{|x|}{\nu}]$ is at most 3. Therefore choosing $k \geq (d-1)/2$ and summing (24) over all dyadic N we have

$$|K_\nu(x)| \leq \sum_{l \geq 0} |D_{2^l,\nu}(x)| \leq C|\nu|^{-d/2}$$

with C depending only on d and q. Thus we have proved (19).

We now show that

$$\|\hat{K}_\nu\|_\infty \leq \left\| \sum_{l \geq 0} |\hat{D}_{2^l,\nu}|(y) \right\|_\infty \leq C. \tag{25}$$

Since supp $\phi \in [1/2, 2]$, we can rewrite (22) using a stronger version of the method of stationary phase (see [3, Theorems 7.6.4, 7.6.5, 7.7.3]).

$$\left| \int \phi(t) e^{-i2\pi\nu N^2 \left(t-\frac{|x|}{2\nu N}\right)^2} dt - \sum_{j=0}^{k-1} c_j (\nu N^2)^{-j-1/2} \phi^{(2j)} \left(\frac{|x|}{2\nu N} \right) \right| \leq \frac{c_k (|\nu| N^2)^{-k-1/2}}{\max(1, \frac{|x|}{8N|\nu|})^k},$$

where the numbers c_j are suitable constants. Therefore, for $\nu > 0$,

$$D_{N,\nu}(x) = \chi_{\{|x|>1\}} \frac{N^{\frac{d+1}{2}}}{|x|^{\frac{d-1}{2}}} e^{i2\pi \frac{|x|^2}{4\nu}} \sum_{j=0}^{k-1} c_j (\nu N^2)^{-j-1/2} \phi^{(2j)} \left(\frac{|x|}{2\nu N} \right) + \phi_k(x), \tag{26}$$

where

$$|\phi_k(x)| \leq \chi_{\{|x|>1\}} \frac{N^{\frac{d+1}{2}}}{|x|^{\frac{d-1}{2}}} \frac{c_k (|\nu| N^2)^{-k-1/2}}{\max(1, \frac{|x|}{8N|\nu|})^k}.$$

If $\nu < 0$ then we simply replace $\phi^{(2j)}(|x|/(2\nu N))$ by $\bar{\phi}^{(2j)}(-|x|/(2\nu N))$. We further assume that $\nu > 0$. Choosing $k \geq (d+2)/2$ we have

$$\|\hat{\phi}_k\|_\infty \leq \|\phi_k\|_1 = \int\limits_{|x|\leq 8\nu N} |\phi_k| dx + \int\limits_{|x|>8\nu N} |\phi_k| dx \leq \frac{C}{N}, \tag{27}$$

where C depends only on d and q. We can ignore the factor $\chi_{\{|x|>1\}}$ in front of the sum in (26) because if $|x|/(2\nu N) \in [1/2, 2]$, then $|x| \geq \nu N \geq 1$. We will only consider the term $j = 0$ in the sum; the other terms can be treated similarly. The Fourier transform of

$$\frac{N^{\frac{d+1}{2}}}{|x|^{\frac{d-1}{2}}} e^{i2\pi \frac{|x|^2}{4\nu}} (\nu N^2)^{-1/2} \phi\left(\frac{|x|}{2\nu N}\right)$$

at a point y is equal to

$$\begin{aligned} &N^{\frac{d+1}{2}} (2\nu N)^{\frac{d+1}{2}} (\nu N^2)^{-1/2} \int_{\mathbb{R}^d} \psi(|x|) e^{i2\pi\nu N^2 |x|^2} e^{-i2\pi 2\nu N x\cdot y} dx \\ &\qquad = C(\nu N^2)^{d/2} e^{-i2\pi\nu|y|^2} \int_{\mathbb{R}^d} \psi(|x|) e^{i2\pi\nu N^2 |x-\frac{y}{N}|^2} dx, \end{aligned} \tag{28}$$

where $\psi(t) = \phi(t, 2\nu N t)t^{-(d-1)/2}$ is a Schwartz function supported in $[1/2, 2]$ whose derivatives and the function itself are bounded uniformly in t, ν and N (see the remark after (21)). The same holds for the partial derivatives of $\psi(|x|)$. Applying the stationary phase method for $\mathbb{R}^d$ (see [3, Theorem 7.7.3]), we get

$$\left| \int_{\mathbb{R}^d} \psi(|x|) e^{i2\pi\nu N^2 |x-\frac{y}{N}|^2} dx \right| \leq \begin{cases} C(\nu N^2)^{-d/2} & \text{if } N \in [\frac{|y|}{2}, 2|y|], \\ C_k(\nu N^2)^{-k-d/2} & \text{if } N \notin [\frac{|y|}{2}, 2|y|], \end{cases} \tag{29}$$

Therefore the absolute value of (28) can be bounded from above by

$$\leq \begin{cases} C & \text{if } N \in [\frac{|y|}{2}, 2|y|], \\ C_k(\nu N^2)^{-k} & \text{if } N \notin [\frac{|y|}{2}, 2|y|], \end{cases} \tag{30}$$

Similar inequalities hold for the Fourier transforms of the other terms in the sum in (26). The number of dyadic values $N \in [|y|/2, 2|y|]$ is bounded by 3. Using (27), choosing $k \geq 1$ in (30), and summing over all dyadic N, we obtain

$$\sum_{l\geq 0} |\hat{D}_{2^l,\nu}(y)| \leq C \tag{31}$$

with C depending only on d and q, provided $\nu \neq 0$. Thus we have proved (25).

Using (19) and (25) and interpolating between $p = 1$ and $p = 2$, we obtain

$$\|K_\nu * f\|_{p'} \leq C|\nu|^{-\alpha_p} \|f\|_p, \tag{32}$$

where $\alpha_p = (d/2)(2-p)/p$. Note that $\alpha_p > 1$ if $p < 2d/(d+2)$. Summing (32) over all $\nu \neq 0$ yields the desired inequality

$$\sum_{\nu \neq 0} \left\| \sum_{l \geq 0} \hat{H}_{2,2^l}(y,\nu) \right\|_{p'} \leq C\|f\|_p. \qquad \square$$

REMARK 2. It is clear from the proof that we have the same inequality if the summation over $l \geq 0$ is replaced by a summation over any subset of the nonnegative integers.

We are now in a position to proceed with the proof of the theorem. Let $q : \mathbb{R} \to \mathbb{R}$ be a fixed nonnegative Schwartz function supported in $[1/2, 2]$ such that

$$q(t) + q(t/2) = 1$$

when $t \in [1, 2]$. It follows that

$$\sum_{l \geq 0} q\left(\frac{t}{2^l}\right) = 1 \tag{33}$$

when $t \geq 1$. Define

$$q_0(t) = 1 - \sum_{l \geq 0} q\left(\frac{t}{2^l}\right)$$

for $t \geq 0$. It is clear that $q_0(|x|)$ is a Schwartz function supported in $|x| \leq 1$. Let $\psi(t) = q_0(t) + q(t)$. Then

$$\psi_k(t) = \psi\left(\frac{t}{2^k}\right) = q_0(t) + \sum_{l \geq 0}^{k} q\left(\frac{t}{2^l}\right)$$

and $\psi(|x|)$ is a Schwartz function supported in $|x| \leq 2$ such that $\psi(|x|) = 1$ if $|x| \leq 1$. Therefore

$$\int \hat{f}(x) e^{2\pi x \cdot y} \psi\left(\frac{|x|}{2^k}\right) dx = (f * \widehat{\psi_k})(y)$$

converges to f in L^p as $k \to \infty$. To prove that $f \in L^{p'}$ and $\|f\|_{p'} \lesssim \|f\|_p$ it will be enough to show that

$$\|f * \widehat{\psi_k}\|_{p'} \leq C\|f\|_p;$$

an application of Fatou's lemma to a subsequence of $f * \widehat{\psi_k}$ converging a.e. to f will then yield the assertion.

We have

$$(f*\widehat{\psi_k})(y) = (f*\widehat{q_0})(y) + \sum_{l\geq 0}^{k} \int \hat{f}(x)e^{2\pi x\cdot y} q\left(\frac{|x|}{2^l}\right)dx \tag{34}$$

$$= (f*\widehat{q_0})(y) + \sum_{l\geq 0}^{k} \int_0^\infty q\left(\frac{t}{2^l}\right) \int \hat{f}(\xi)e^{i2\pi y\cdot\xi} d\sigma_t(\xi)dt$$

$$= (f*\widehat{q_0})(y) + \sum_{l\geq 0}^{k} \int_0^\infty q\left(\frac{t}{2^l}\right) h(y,t)dt.$$

By Young's inequality we have

$$\|f*\widehat{q_0}\|_{p'} \lesssim \|f\|_p \tag{35}$$

for $1 \leq p \leq 2$. It thus remains to estimate the sum over l.

A well-known result in number theory due to Lagrange states that every positive integer can be represented as a sum of four squares (see [2, p. 25]). Moreover, there exists an infinite arithmetic progression of positive integers (e.g., integers of the form $8n+1$) which can be represented as sums of three squares (see [2, p. 38]). We will only use the latter result. By rescaling we can assume that $\hat{f}$ vanishes on all spheres of radius $\sqrt{n+b}$, where n is a nonnegative integer and $0 < b < 1$ is a fixed number. Therefore $h(y,\sqrt{n+b}) = 0$ for all $y \in \mathbb{R}^d$. Making a change of variables and keeping in mind that q is supported in $[1/2,2]$, we rewrite the terms in the sum as follows:

$$\int_0^\infty q\left(\frac{t}{N}\right) h(y,t)dt = \int \frac{1}{2\sqrt{t+b}} q\left(\frac{\sqrt{t+b}}{N}\right) h(y,\sqrt{t+b})dt.$$

An application of Poisson's summation formula gives

$$\begin{aligned} 0 &= \sum_n \frac{1}{\sqrt{n+b}} q\left(\frac{\sqrt{n+b}}{N}\right) h(y,\sqrt{n+b}) \\ &= \sum_\nu \left(\frac{1}{\sqrt{t+b}} q\left(\frac{\sqrt{t+b}}{N}\right) h(y,\sqrt{t+b})\right)^\wedge (\nu) \\ &= \int \frac{1}{\sqrt{t+b}} q\left(\frac{\sqrt{t+b}}{N}\right) h(y,\sqrt{t+b})dt + \sum_{\nu\neq 0} \hat{H}_{1,N}(y,\nu) + \sum_{\nu\neq 0} \hat{H}_{2,N}(y,\nu), \end{aligned}$$

where

$$H_{i,N}(y,t) = \frac{1}{\sqrt{t+b}} q\left(\frac{\sqrt{t+b}}{N}\right) h_i(y,\sqrt{t+b}), \quad i = 1,\ 2.$$

Applying Lemmas 1 and 2, along with Remark 2, we can bound the sum by

$$\begin{aligned}\left\|\sum_{l\geq 0}^{k}\int_0^\infty q\left(\frac{t}{2^l}\right)h(y,t)dt\right\|_{p'} &\leq \sum_{l\geq 0}\sum_{\nu\neq 0}\|\hat{H}_{1,2^l}(y,\nu)\|_{p'}\\ &\quad+\sum_{\nu\neq 0}\left\|\sum_{l\geq 0}^{k}\hat{H}_{2,2^l}(y,\nu)\right\|_{p'}\\ &\leq C\|f\|_p.\end{aligned}$$

Combining (34), (35), and the last inequality, we obtain the desired inequality

$$\|f*\widehat{\psi_k}\|_{p'}\leq C\|f\|_p,$$

from which the statement of the theorem follows. □

REMARK 3. We say that a function $f\in L^p$ has vanishing periodizations if there exists a sequence of Schwartz functions f_k with vanishing periodizations converging to f in L^p. It follows from Theorem 1 that $f\in L^{p'}$ and the functions f_k converge to f in $L^{p'}$ if $d\geq 3$ and $1\leq p<2d/(d+2)$.

3. Counterexamples and open questions

When $d=1$ or $d=2$, Theorem 1 does not apply. The case $d=1$ is not interesting. We can easily construct examples of functions f with vanishing periodizations such that their L^p norms are not bounded by their L^q norms, for any given pair $p\neq q$.

We now show that, when $d=2$, the assertion of Theorem 1 does not hold. More precisely, Lemma 3 below shows that if $1\leq p<2$, then the inequality

$$\|f\|_{p'}\lesssim\|f\|_p$$

does not hold for functions with vanishing periodizations. This lemma deals with a sequence of functions f_n such that $\hat{f}_n$ vanishes on all circles of radius $\sqrt{l^2+k^2}$. Denote by X_2 the Banach space of functions from $L^1(\mathbb{R}^2)$ whose Fourier transforms vanish on all circles of radius $\sqrt{l^2+k^2}$, i.e.,

$$X_2=\{f\in L^1(\mathbb{R}^2):\hat{f}(\mathbf{r})=0 \text{ if } |\mathbf{r}|=\sqrt{l^2+k^2},(k,l)\in\mathbb{Z}^2\}.$$

The lemma depends in crucial way on the following fact from number theory (see [2, p. 22]):

The number of integers in $[n,2n]$ which can be represented as sums of two squares is $n\epsilon_n$, where $\epsilon_n\lesssim 1/\ln^{1/2}n\to 0$ as $n\to\infty$.

We only need that $\lim_{n\to\infty}\epsilon_n=0$.

Lemma 3. *Let $1 \le p < 2$ and $d = 2$. Then there exists a sequence of Schwartz functions $f_n \in X_2$ such that*

$$\lim_{n\to\infty} \frac{\|f_n\|_{p'}}{\|f_n\|_p} = \infty.$$

Proof. Let $a_1 < a_2 < a_3 < \cdots$ be the enumeration of the numbers $a_m = \sqrt{l^2+k^2}$ in ascending order, and set $\delta_m = a_{m+1} - a_m$. As mentioned above, the number of a_m in the interval $[\sqrt{n}, 2\sqrt{n}]$ is $n\epsilon_n$. Let a_{m_0} and a_{m_1} denote, respectively, the smallest and largest elements a_m in this interval. Then

$$\sum_{m=m_0}^{m_1-1} \delta_m = a_{m_1} - a_{m_0} \sim \sqrt{n}.$$

Let

$$\delta = \frac{C}{\sqrt{n}\epsilon_n} \tag{36}$$

with a small enough constant $C > 0$ so that if

$$M = \{m_0 \le m < m_1 : \delta_m \ge \delta\}$$

then

$$\sqrt{n} \lesssim \sum_{m\in M} \delta_m.$$

This is possible since $m_1 - m_0 \sim n\epsilon_n$. Choose coordinate axes x and y. We will construct functions $\hat{f}_n$ supported in $\bigcup_{m\in M} R_m$, where R_m is a largest possible rectangle inscribed between circles of radius a_m and a_{m+1} with sides parallel to the coordinate axes. Then R_m is of size $\sim \delta_m \times \sqrt{\delta_m a_m} \gtrsim \delta_m \times \sqrt{\delta\sqrt{n}} \gtrsim \delta_m \times 1$. We split each rectangle R_m further into $[\delta_m/\delta]$ smaller rectangles r of the same size $\sim \delta \times 1$. The number of these rectangles r is

$$N = \sum_{m\in M} \left[\frac{\delta_m}{\delta}\right] \sim \sum_{m\in M} \frac{\delta_m}{\delta} \sim \frac{\sqrt{n}}{1/\sqrt{n}\epsilon_n} = n\epsilon_n, \tag{37}$$

since $\delta_m \ge \delta$ for $m \in M$. Enumerate these rectangles by r_k, $k = 1, \ldots, N$. Let r_k be centered at $(\lambda_k, 0)$. It is clear that $|\lambda_k - \lambda_l| \ge \delta$ for $k \ne l$. Let ϕ be a nonnegative Schwartz function on $\mathbb{R}$ supported in $[-1/2, 1/2]$. Then $\check{\phi}(x) \ge C > 0$ if x is small enough. Define $\hat{f}_n$ by

$$\hat{f}_n(x,y) = \sum_{k=1}^{N} \phi\left(\frac{x-\lambda_k}{\delta}\right)\phi(y). \tag{38}$$

The kth term in (38) is supported in r_k. Therefore, $\hat{f}_n$ is a Schwartz function supported in $\bigcup_{m\in M} R_m$. Hence $\hat{f}_n$ vanishes on all circles of radius a_l. Taking the inverse Fourier transform of (38), we get

$$f_n(\xi,\eta) = \delta\check{\phi}(\xi\delta)\check{\phi}(\eta)\sum_{k=1}^{N} e^{i\lambda_k\xi}. \tag{39}$$

Assume first that $p' < \infty$. Then

$$\begin{aligned}\int |f_n(\xi,\eta)|^{p'}\,d\xi d\eta &\geq \|\check{\phi}\|_{p'}^{p'}\delta^{p'} \int\limits_{|\xi|\leq(100^{-1})/\sqrt{n}} |\check{\phi}(\xi\delta)|^{p'}\left|\sum_{k=1}^{N} e^{i\lambda_k\xi}\right|^{p'} d\xi \\ &\gtrsim \delta^{p'}N^{p'}\frac{1}{\sqrt{n}} \sim (\sqrt{n})^{p'-1},\end{aligned}$$

where the second step follows from the bound

$$\left|\sum_{k=1}^{N} e^{i\lambda_k\xi}\right| \geq \left|\sum_{k=1}^{N}\cos(\lambda_k\xi)\right| \gtrsim N$$

since $|\lambda_k\xi| \leq 1/50$, and the third step follows from (36) and (37). Therefore

$$\|f_n\|_{p'} \gtrsim (\sqrt{n})^{1/p}. \tag{40}$$

If $p' = \infty$ we obtain in a similar way that

$$\|f_n\|_\infty \geq |f_n(0)| \gtrsim \sqrt{n}. \tag{41}$$

We now estimate the L^p norm from above. Set

$$g(x) = \sum_{k=1}^{N} e^{i(\lambda_k/\delta)\xi}.$$

Since $|(\lambda_k - \lambda_l)/\delta| \geq \delta/\delta = 1$ for $k \neq l$, we have

$$\int_I |g|^2 \sim N$$

for any interval I of length 4π (see [8, Theorem 9.1]). Therefore,

$$\int_I |g|^p \leq |I|^{1-2/p}\left(\int_I |g|^2\right)^{p/2} \lesssim N^{p/2} \tag{42}$$

for any interval I of length 4π. Since $\check{\phi}$ is a Schwartz function, we have

$$|\check{\phi}(x)| \lesssim \frac{1}{1+x^2}.$$

Therefore

$$\int |f_n(\xi,\eta)|^p d\xi d\eta = \|\check{\phi}\|_p^p \delta^{p-1} \int |\check{\phi}(\xi)|^p \cdot \left|\sum_{k=1}^{N} e^{i(\lambda_k/\delta)\xi}\right|^p d\xi$$

$$= C\delta^{p-1} \sum_{l=-\infty}^{\infty} \int\limits_{l4\pi}^{(l+1)4\pi} |\check{\phi}(\xi)|^p \cdot |g(\xi)|^p d\xi$$

$$\lesssim \delta^{p-1} \sum_{l=-\infty}^{\infty} \frac{1}{(1+l^2)^p} N^{p/2}$$

$$\lesssim \sqrt{n}\epsilon_n^{1-p/2},$$

where the last step follows from (36) and (37). Hence

$$\|f_n\|_p \lesssim (\sqrt{n})^{1/p} \epsilon_n^{(2-p)/2p}. \tag{43}$$

Dividing (40) by (43) we obtain the desired result

$$\frac{\|f_n\|_{p'}}{\|f_n\|_p} \geq \frac{(\sqrt{n})^{1/p}}{(\sqrt{n})^{1/p}\epsilon_n^{(2-p)/(2p)}} = \frac{1}{\epsilon_n^{(2-p)/(2p)}} \to \infty,$$

as $n \to \infty$ since $p < 2$. □

Corollary 2. *There exists a function $f \in X_2$ such that*

$$\|f\|_{L^\infty(D(0,1))} = \infty.$$

Proof. It follows immediately from the lemma and (41) that if $p = 1$ then

$$\sup_{f \in X_2} \frac{\|f\|_{L^\infty(D(0,1))}}{\|f\|_1} = \infty.$$

We claim that there exists a function $f \in X_2$ such that $\|f\|_{L^\infty(D(0,1))} = \infty$. Suppose, to get a contradiction, that this is not true. Then the restriction operator

$$T : f \to f|_{D(0,1)}$$

maps X_2 to $L^\infty(D(0,1))$. Note that if $f_n \to f$ in L^1 and $f_n \to g$ in $L^\infty(D(0,1))$, then $f = g$ a.e. on $D(0,1)$. An application of the Closed Graph Theorem shows that T is a bounded operator acting from X_2 to $L^\infty(D(0,1))$. This contradicts Corollary 2, and thus proves our claim. □

Obviously, this function f is not continuous. Therefore the theorem of Kolountzakis and Wolff mentioned in the Introduction does not hold for dimension 2.

Remark 4. It is an open problem whether, for $f \in X_2$, the inequality

$$\|f\|_r \lesssim \|f\|_p$$

holds when $1 \le p < 2$ and $p < r < p'$.

We now show that the range of r in Corollary 1 is sharp. We need to consider two cases, $r > p'$ and $r < p$. In the first case the argument is similar to the one given in the previous lemma, and we therefore give only a sketch. We will deal with a sequence of functions f_n such that the functions $\hat{f}_n$ vanish on all circles of radius $\sqrt{m_1^2 + \cdots + m_d^2}$. Denote by X_d the Banach space of functions from $L^1(\mathbb{R}^d)$ whose Fourier transforms vanish on all circles of radius $\sqrt{m_1^2 + \cdots + m_d^2}$, i.e.,

$$X_d = \{f \in L^1(\mathbb{R}^d) : \hat{f}(\mathbf{r}) = 0 \text{ if } |\mathbf{r}| = \sqrt{m_1^2 + \cdots + m_d^2},\ (m_1, \ldots, m_d) \in \mathbb{Z}^d\}.$$

We will construct a sequence of Schwartz functions f_n with Fourier transforms supported outside of spheres of radius $\sqrt{m}$. Therefore these functions automatically belong to X_d.

LEMMA 4. *Let $1 < p \le 2$ and $r > p'$. Then there exists a sequence of Schwartz functions $f_n \in X$ such that*

$$\lim_{n\to\infty} \frac{\|f_n\|_r}{\|f_n\|_p} = \infty.$$

Proof. A maximal rectangle inscribed between spheres of radius $\sqrt{n}$ and $\sqrt{n+1}$ has dimensions $\sim (1/\sqrt{n}) \times 1 \times 1 \times \cdots \times 1$. Let r_k denote parallel identical rectangles inscribed between spheres of radius $\sqrt{n+k}$ and $\sqrt{n+k+1}$, for $k = 0, 1, \ldots, n-1$, with dimensions $\sim (1/\sqrt{n}) \times 1 \times 1 \times \cdots \times 1$, and centered at $(\lambda_k, 0, 0, \ldots, 0)$. It is clear that $\lambda_{k+1} - \lambda_k \sim 1/\sqrt{n}$. Let ϕ be a nonnegative Schwartz function on $\mathbb{R}$ supported in $[-1/100, 1/100]$. We have $\check{\phi}(x) \ge C > 0$ when x is small enough. Define $\hat{f}_n$ by

$$\hat{f}_n(x_1, x_2, \ldots, x_d) = \sum_{k=0}^{n-1} \phi((x_1 - \lambda_k)\sqrt{n}) \prod_{l=2}^{d} \phi(x_l). \tag{44}$$

The kth term in (44) is supported in r_k. Therefore, $\hat{f}_n$ is a Schwartz function vanishing on all spheres of radius $\sqrt{m}$. Taking the inverse Fourier transform of (44), we get

$$f_n(y_1, y_2, \ldots, y_d) = \prod_{l=2}^{d} \check{\phi}(y_l) \frac{1}{\sqrt{n}} \check{\phi}\left(\frac{y_1}{\sqrt{n}}\right) \sum_{k=0}^{n-1} e^{i\lambda_k y_1}. \tag{45}$$

Arguing as in the proof of Lemma 3, we obtain

$$\|f_n\|_r \gtrsim (\sqrt{n})^{1/r'}$$

and

$$\|f_n\|_p \lesssim (\sqrt{n})^{1/p}.$$

Therefore,

$$\frac{\|f_n\|_r}{\|f_n\|_p} \gtrsim (\sqrt{n})^{(1/p')-(1/r)} \to \infty$$

as $n \to \infty$, since $r > p'$. □

The case when $r < p$ is very simple. Let

$$\hat{f}(x) = \phi\left(\frac{x - x_0}{\epsilon}\right),$$

where ϕ is a Schwartz function supported in $B^d(0,1)$ so that $\hat{f}$ is supported in a small ball $B^d(x_0, \epsilon)$ placed between two fixed spheres of radius $\sqrt{n}$ and $\sqrt{n+1}$. Then $f(y) = \epsilon^d \check{\phi}(\epsilon y)$ and

$$\frac{\|f\|_r}{\|f\|_p} \sim \frac{\epsilon^{d/r'}}{\epsilon^{d/p'}} \to \infty$$

as $\epsilon \to 0$, since $r < p$. Note that we did not impose any restriction on p here.

We now show that Theorem 1 does not hold if $p > 2$. More precisely, let $p > 2$ and $r \neq p$. Then the following inequality is not true for functions with vanishing periodizations:

$$\|f\|_r \lesssim \|f\|_p.$$

Since we have already dealt with the case when $r < p$, we only need to consider the case $r > p$. The argument is almost the same as in the proof of Lemma 4. We construct a sequence of Schwartz functions f_n with Fourier transforms vanishing on all spheres of radius $\sqrt{m}$ and such that $\|f_n\|_r \gtrsim (\sqrt{n})^{1/r'}$ and $\|f_n\|_p \leq \|\hat{f}_n\|_{p'} \lesssim (\sqrt{n})^{1/p'}$. Therefore

$$\frac{\|f_n\|_r}{\|f_n\|_p} \gtrsim (\sqrt{n})^{(1/p)-(1/r)} \to \infty.$$

REMARK 5. Since Theorem 1 trivially holds for $p = 2$, it is natural to expect that it also holds for $1 \leq p \leq 2$. However, the question whether the theorem holds for $2d/(d+2) \leq p < 2$ is still open.

Another interesting question is whether the inequality

$$\|\hat{f}\|_p \lesssim \|f\|_p \tag{46}$$

holds for some range of $p < 2$ if f has vanishing periodizations. It would then follow that

$$\|\hat{f}\|_r \lesssim \|f\|_p \tag{47}$$

for $p \leq r \leq p'$. From Theorem 1 we see that (47) holds when $2 \leq r \leq p'$, $1 \leq p < 2d/(d+2)$ and $d \geq 3$, since $\|f\|_2 \lesssim \|f\|_p$.

Our final open question is whether the following inequalities are true for functions with not necessarily vanishing periodizations g_ρ:

$$\|f\|_{p'} \lesssim \|f\|_p + \|g\|_{p'}$$

and

$$\|g\|_{p'} \lesssim \|f\|_p + \|f\|_{p'}$$

for some range of $p \leq 2d/(d+1)$, where

$$\|g\|_{p'} = \left(\int_{\rho \in \mathrm{SO}(d)} \|g_\rho\|_{p'}^p d\rho \right)^{1/p} .$$

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