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A GEOMETRIC PROBLEM IN FUNCTION THEORY

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Abstract. The Riemann Mapping Theorem asserts that there is an analytic function in the unit disk taking its boundary values on any given simple smooth curve in the plane. A theorem of Szegö says that an analytic function in the disk can have boundary values lying on prescribed circles about the origin. We shall prove that the circles of Szegö's theorem can be replaced by other curves, giving a common generalization of these theorems.

1. Introduction

For each t, $0 \leq t < 2\pi$, let Γ_t be a curve in the complex plane. Is there a function f, analytic in the unit disk and belonging to some appropriate function class, whose boundary function has the property that $f(e^{it})$ lies on Γ_t for a.e. t?

If Γ_t is the same simple closed and sufficiently smooth curve Γ for each t, then the Riemann Mapping Theorem gives an answer, as well as other information. If Γ_t is a circle about the origin of radius $w(e^{it})$, a well-known theorem of G. Szegö says that such a function exists in the Hardy space H^1, provided that w and $\log w$ are summable functions.

If we replace the circles even by ellipses, there is no obvious result; our problem is to find whether something more general is true. This paper presents two methods to approach the question and establishes positive results for various kinds of curves. Section 2 treats lines, Section 3 hyperbolas and ellipses, Section 4 regions that are convex but fairly general, and Section 5 mentions some further results that are not proved.

Suppose the family (Γ_t) is defined by a functional relation

$$F(x, y, e^{it}) = 0. \tag{1.1}$$

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Our problem is to find a function f in some Hardy class whose real and imaginary parts g and $\tilde{g}$ satisfy

$$F(g, \tilde{g}, e^{it}) = 0. \tag{1.2}$$

Here $\tilde{g}$ is the trigonometric conjugate function of g (which is defined to have mean value 0, or to make $f(0)$ real). Since the conjugate function is given in terms of g by a singular integral, in principle (1.2) is an integral equation. The results below provide solutions for some classes of such integral equations.

We assume that the reader is familiar with the Riemann mapping theorem and Szegö's theorem, and with the elementary facts about conjugate functions and analytic functions in the unit disk.

2. Lines

There is an easy case of the problem. Let each Γ_t be a vertical line with abscissa $w(e^{it})$. If w is summable, then its conjugate function $\tilde{w}$ exists, belongs to L^p for each $p < 1$, and $w + i\tilde{w}$ is the required function f. Then f belongs to H^p for each $p < 1$, and if w belongs to a better Lebesgue class, so does f. Obviously the statement can be made as well for horizontal lines.

Now let Γ_t be a line in the plane for each t. We parametrize Γ_t: its points are $q(u) = z_t + ue^{i\alpha(e^{it})}$, where for each t, z_t is the point on the line closest to the origin, $\alpha(e^{it})$ a real number of modulus at most $\pi/2$ determined by the slope of the line, and u a real parameter. We assume that z_t depends measurably on t (so $\alpha(e^{it})$ does too). As q traverses Γ_t, $\exp(-i\alpha(e^{it})+\tilde{\alpha}(e^{it}))q$ describes a horizontal line V_t. If g is a function on the circle taking its values on V_t, then $f = g\exp(i\alpha(e^{it}) - \tilde{\alpha}(e^{it}))$ takes its values on Γ_t. This line of reasoning will lead to the conclusion that f belongs to H^p for each $p < \frac{1}{2}$ if z_t is summable. Better conclusions follow if we assume that $|\alpha|$ is bounded away from $\pi/2$ and $|z_t|$ is smaller. But this method will not reach the right result, which should be rotationally invariant, and which we conjecture to be:

Let a family of lines Γ_t be given by equations $ax + by = c$ where a, b, c are real measurable functions of e^{it}, $a^2 + b^2 = 1$, $c \geq 0$. Suppose that c is summable. Then there is a function f in H^1 taking its values in the family of lines a.e.

Sometimes we can treat half-lines. If Γ_t is the ray from the origin with argument $\theta(\mathrm{e}^{it})$ ($|\theta(t)| < \pi$), then $f = \exp(i\theta - \tilde{\theta})$ is the required function. But if Γ_t is a vertical half-line based on the real axis with abscissa $w(e^{it})$, the function conjugate to w is not usually bounded unless w is sufficiently smooth. If it is, we can add a constant to the conjugate function to make it positive; but if $\tilde{w}$ is unbounded negatively, there is no solution.

3. Hyperbolas and ellipses

For some families of curves, fixed point methods provide a solution to our problem. Suppose that the family (Γ_t) is defined by (1.1) and that we can solve (1.2) for g: $g = G(\tilde{g}, e^{it})$. Define the transformation

$$S(g) = G(\tilde{g}, e^{it}). \tag{3.1}$$

Then a fixed point for S is a solution. If S is a strict contraction in a complete metric space, then Banach's fixed-point theorem shows that there is such a function. We show now how this is the case for a class of hyperbolas.

Fix a positive constant a. For each c, $(\Re z)^2 - a(\Im z)^2 = c^2$ describes a hyperbola in the complex plane. Write $c^2 = w(e^{it})$ and let Γ_t be the right branch of the corresponding hyperbola. We suppose that w belongs to L^1.

Define a map S of L^2 (here the space of real square-summable functions on the circle) into itself: $S(g) = (a\tilde{g}^2 + w)^{\frac{1}{2}}$. We assert that if $a < 1$, then S is a strict contraction: for all g, h in the space

$$\|S(g) - S(h)\|^2 \le a\|g - h\|^2. \tag{3.2}$$

Taking this for granted, S has a unique fixed point g, and $f = g + i\tilde{g}$ belongs to the Hardy space H^2. Therefore

$$(\Re f)^2 = g^2 = S(g)^2 = a\tilde{g}^2 + w = a(\Im f)^2 + w \tag{3.3}$$

and f is a solution under these hypotheses. Note that $f(0) = g(0)$ is real, a small bonus.

We show that S is a contraction. For any positive numbers r, s, u with $r > s$ we have

$$(r+u)^{\frac{1}{2}} - (s+u)^{\frac{1}{2}} < r^{\frac{1}{2}} - s^{\frac{1}{2}}. \tag{3.4}$$

Using this fact, for g, h in L^2 we have

$$\begin{aligned} \|Sg - Sh\|^2 &= \int \Big((a\tilde{g}^2 + w)^{\frac{1}{2}} - (a\tilde{h}^2 + w)^{\frac{1}{2}} \Big)^2 d\sigma \\ &< a \int (|\tilde{g}| - |\tilde{h}|)^2 d\sigma \\ &= a \int \tilde{g}^2 + \tilde{h}^2 - 2|\tilde{g}\tilde{h}| d\sigma \\ &\le a \int |\tilde{g} - \tilde{h}|^2 d\sigma. \end{aligned} \tag{3.5}$$

(σ is normalized Lebesgue measure on the circle.)

If g and h have mean value 0, the right side equals $a\|g - h\|^2$ (because the Fourier coefficients of a function and its conjugate have the same modulus, except for the mean value), and (3.2) is proved. But adding a constant to $g - h$ can only increase its norm, so the proof is finished.

The restriction on a is due to a defect in the method, and doubtless the result holds for all a.

If a is a positive constant and w a positive function,

$$a(\Re z)^2 + (\Im z)^2 = w \tag{3.6}$$

describes a family of ellipses like the hyperbolas above. For $a = 1$ this is Szegö's problem and there is a solution f provided that w and $\log w$ are both summable. Here, if f solves the problem for a particular a and w, then if solves the problem for $1/a$ and w/a, so there is a duality between values of a that are smaller than 1 and those that are larger.

For values of a close to 1, and w bounded and bounded from 0, the fixed point method can be applied not in L^2, but in the space of bounded functions g that satisfy $ag^2 \leq w$. The details are laborious and the result not very strong; a better one will be proved in the next section. So we pass on.

4. Convex regions

We come to more serious things.

Say that a family of curves (Γ_t) is *measurable* if the set of all pairs (z, e^{it}) such that z belongs to Γ_t is a Borel set in $C \times T$ (C is the complex plane, T the unit circle).

Let (Γ_t) be a measurable family of simple closed curves in the complex plane, and let G_t be the simply connected region bounded by Γ_t, each containing the origin. For each t, denote by $W(e^{it})$ the greatest distance from the origin to a point of Γ_t, and by $w(e^{it})$ the least such distance. These functions are measurable.

THEOREM. *Suppose that each region G_t is convex and that W and $\log w$ are summable functions. Then there is a function f in H^1 such that $f(e^{it})$ belongs to Γ_t for a.e. t.*

The proof of the theorem follows the idea of a proof of the Riemann mapping theorem, but with complications.

Let $\mathcal{A}$ be the set of functions g in H^1 such that

$$g(e^{it}) \text{ is in } G_t \text{ or } \Gamma_t \text{ for a.e. } t$$

and

$$g(0) = 0, \quad g'(0) \geq 0.$$

We verify that $\mathcal{A}$ is not empty. Since w is a summable positive function with summable logarithm, there is an outer function h in H^1 such that $|h(e^{it})| = w(e^{it})$ a.e. and $h(0) > 0$. Now G_t contains every complex number of modulus less than $w(e^{it})$; hence its closure contains $e^{it}h(e^{it})$. The derivative of $(zh(z))$ is $h(0) > 0$, so this function belongs to $\mathcal{A}$.

The functions of $\mathcal{A}$ are bounded in the norm of H^1, because $|g| \leq w$ a.e. Therefore the numbers $g'(0)$ are bounded above for g in $\mathcal{A}$. Choose a sequence

(g_n) for which these derivatives tend to their upper bound. A subsequence (which we call (g_n) again) converges uniformly on compact subsets of the open disk to a limit function h, which vanishes at 0 and whose derivative at 0 is the upper bound of derivatives of functions in $\mathcal{A}$ at 0. We shall show that h belongs to $\mathcal{A}$.

H^1 is a dual Banach space, and it is easy to see that (g_n) is convergent in the weak star-topology to an element of H^1, whose coefficients match those of h. Thus h belongs to H^1 and is the limit of (g_n) in two senses. We shall show that $\mathcal{A}$ is closed in the star-topology, so that h is in $\mathcal{A}$.

First, $\mathcal{A}$ is norm-closed in H^1. For if g is a norm limit of a sequence (g_n) from $\mathcal{A}$, a subsequence converges pointwise a.e. to g, and the properties required for membership in $\mathcal{A}$ are preserved.

Now $\mathcal{A}$ is convex because the regions G_t are convex. A closed convex bounded set in the dual of a Banach space is star-closed, according to the Banach-Alaoglu theorem. This proves that $\mathcal{A}$ is star-closed.

We remark that this has been the most important use of the hypothesis of convexity of the regions G_t, and a weaker hypothesis would suffice if it implies that $\mathcal{A}$ is star-closed.

The difficult point of the proof is to show that the values of h lie on the curves Γ_t, because it is not obvious whether they lie in G_t or on its boundary.

Let F be the subset of T where $h(e^{it})$ belongs to G_t rather than to Γ_t. We want to show that F has measure 0. First we show that it is measurable. Since h belongs to H^1, we can take it to be a Borel function. Let k be the mapping from T into $C \times T$ that carries e^{it} to $(h(e^{it}), e^{it})$. It is easy to see that the inverse image under k of a product set is a Borel set in T, and from this fact, that k is a Borel function.

The set of (z, e^{it}) such that z belongs to Γ_t is a Borel set in the product space, by hypothesis. Its inverse image under k is thus a Borel set in T. This is the set where $h(e^{it})$ belongs to Γ_t, the complement of F, which is therefore measurable.

Lemma. *Suppose that F has positive measure. Then there is a subset E of F with positive measure and a positive ϵ such that $\lambda h(e^{it})$ is in G_t for all e^{it} in E and λ such that $|\lambda - 1| < \epsilon$.*

Proof. For each e^{it} in F there is a positive ϵ such that $\lambda h(e^{it})$ is in G_t for all λ such that $|\lambda - 1| < \epsilon$. Therefore there is a positive ϵ and a smaller set E of positive measure on which this is the case for all e^{it} at once, as was to be shown. □

We now prove that the extremal function h solves the problem. Assuming that F has positive measure, we shall construct a function in $\mathcal{A}$ whose derivative at 0 exceeds that of h, and this will prove that F has measure 0.

The mapping functions familiar from proofs of the Riemann Mapping Theorem will not serve now. We need a function q with the following properties:

q is an analytic mapping from the vertical strip $(0 < \Re z < 1)$ to the disk about the point 1 with radius ϵ (the number found above) such that the left boundary of the strip is mapped to the real interval $(1-\epsilon, 1)$, and $(0,1)$ into $(1, 1+\epsilon)$.

We find such a function in steps. First, $\tan z$ is a Riemann map from the strip $(-\pi/4 < \Re z < \pi/4)$ onto the unit disk. It maps the left half of the strip to the left half of the disk, the right half to the right half, and the imaginary axis to the vertical segment $(-i, i)$. The interval $(0, \pi/4)$ is mapped to $(0,1)$.

Restrict the tangent to the right half of the strip, mapping it to the right hemisphere of the unit disk. Then $\tan^2 z$ maps this smaller strip onto the unit disk, but the boundary on the left (the vertical coordinate axis) is carried to $(-1, 0)$ inside the disk, and the boundary on the right to the whole circle. Now $q(z) = 1 + \epsilon \tan^2 \pi z/4$ is the function we want.

Let e be the indicator function of the set E (of the lemma above). Form the analytic function $k = e + i\tilde{e}$, where $\tilde{e}$ is the function conjugate to e. Then $k(z)$ lies in the strip $(0 < \Re z < 1)$ for all z in the open unit disk; $k(e^{it})$ lies on the right boundary of the strip when e^{it} is in E, and on the left boundary when e^{it} belongs to the complementary subset of the circle.

$q \circ k(z)$ maps the unit disk into the disk of radius ϵ about the point 1. For $z = e^{it}$ in the complement of E, $q \circ k(z)$ takes values in $(1-\epsilon, 1)$; for points in E, its values lie on the boundary of the little disk. And, importantly, $q \circ k(0) = q(c)$ where c, the mean value of k, satisfies $0 < c < 1$, so that $q(c) = 1 + \tan^2 \pi c/4 > 1$.

The product $f = h(q \circ k)$ is analytic and bounded in the unit disk. Its boundary values on E are those of h (which lie in G_t and not close to the boundary) multiplied by a number in a small neighborhood of 1, so the product remains in G_t. In the complement of E, where $h(e^{it})$ lies in Γ_t or G_t, $q \circ k(e^{it})$ is in the interval $(1-\epsilon, 1)$, and the convexity of G_t ensures that the product is in G_t or on its boundary. $f(0) = 0$ because $h(0) = 0$. For its derivative we have

$$f'(0) = h'(0)(q \circ k(0)) + h(0)((q \circ k)'(0)). \tag{4.1}$$

The second term is 0; and the first term is positive (now we know that f belongs to $\mathcal{A}$), and indeed greater than $h'(0)$, which was supposedly the maximum over $\mathcal{A}$, because $q \circ k(0) > 1$. This contradiction proves the theorem.

The theorem gives a better result about ellipses than the fixed point method could give. Let the family have equations

$$ax^2 + by^2 = c^2 \tag{4.2}$$

where a, b, c are positive Borel functions with $a^2 + b^2 = 1$. The function w of the theorem is the smaller of $c/a^{\frac{1}{2}}$, $c/b^{\frac{1}{2}}$; and W is the larger of these quantities. (Which one is larger may not be the same at different points, of course.) We assume that W and $\log w$ are summable; it comes to the same to say that $c/a^{\frac{1}{2}}, c/b^{\frac{1}{2}}$ are summable, together with their logarithms. We check that the family is measurable in the sense above; then the theorem applies.

Identify the complex plane with R^2, and let $F(x, y, e^{it}) = ax^2 + by^2 - c^2$. The family of ellipses is the null set of F in $R^2 \times T$. That is, the family is the inverse image of (0) under the Borel function F, and consequently is a Borel set in the product space as we wished to show.

We can also treat some families of hyperbolas; in this case each G_t is unbounded, W is infinite, and the theorem does not apply directly. Let Γ_t be the right-hand branch of the hyperbola

$$x^2 - ay^2 = c^2, \tag{4.3}$$

where a and c are positive measurable functions on T, and G_t the region to the right of Γ_t, where the left side of (4.3) is greater than the right. The point $(1, 0)$ belongs to G_t if and only if $c < 1$; we assume this is the case. But we also want $w(e^{it})$, the distance from $(1, 0)$ to Γ_t, to have summable logarithm. This distance is $1 - c^2$ if the asymptotes to the hyperbola are steep; otherwise there is a closer point whose squared distance is $(a + 1)^{-1} - c^2/a$. This quantity must be positive (a new condition) and its logarithm must be summable.

Now apply the map $(z - 1)/(z + 1)$; each curve Γ_t is carried to a closed curve Γ'_t lying in the closed unit disk. The new curve bounds a region G'_t that contains the origin and is convex. To establish convexity, observe that G_t is the intersection of the halfplanes bounded by lines tangent to Γ_t. These lines are carried by the mapping to circles, and inspection of the possibilities shows that the image of G_t is inside these circles (rather than outside them). Their interiors have intersection G'_t, and the intersection of circular disks is convex.

The other hypotheses of the theorem are satisfied, so there is a function g belonging to H^∞ taking its values on the family (Γ'_t). The inverse conformal map carries g to f taking its values on the hyperbolas. We cannot specify a function class to which f belongs without more information. The asymptotes of G_t have slope $\pm 1/a^{\frac{1}{2}}$. If this function is bounded, all the hyperbolas lie in a sector of the plane with opening less than π, and $f(z)$ lies within this sector for $|z| < 1$. Such a function belongs to H^p for a value of p greater than 1, depending on the opening of the sector.

Note that this theorem allows a more general coefficient a than the result about hyperbolas obtained by the fixed point method, but needs a condition of boundedness on c that was not required before. Probably there is a result that includes both.

It is clear that the origin, in the enunciation of the theorem, could be replaced by any other point. It would be interesting to find a result in which this exceptional point can be allowed to depend on t.

5. Extensions of the theorem

There are two directions in which the main theorem can be strengthened: the convexity condition can be relaxed, and some result about uniqueness proved (as for both the Riemann mapping theorem and Szegö's theorem).

Convexity was needed to show that $\mathcal{A}$ was closed in a weak-star topology. With a good deal of complication, and assuming that all the Γ_t lie inside one circle, convexity can be replaced by this condition: each Γ_t is starlike with respect to each point of a disk centered at the origin. If moreover the disks all have the same diameter, then the function f in H^∞ of the conclusion is unique when normalized to have the form $e^{it}g(e^{it})$, where g is outer and $g(0) > 0$.

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