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AN ELEMENTARY GIT CONSTRUCTION OF THE MODULI SPACE OF STABLE MAPS

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Abstract. This paper provides an elementary construction of the moduli space of stable maps $\overline{M}_{0,0}(\mathbb{P}^r, d)$ as a sequence of "weighted blow-ups along regular embeddings" of a projective variety. This is a corollary to a more general GIT construction of $\overline{M}_{0,n}(\mathbb{P}^r, d)$ that places stable maps, the Fulton-MacPherson space $\mathbb{P}^1[n]$, and curves $\overline{M}_{0,n}$ into a single context.

0. Introduction

Given a projective space $\mathbb{P}^r$ and a class $d \in A_1(\mathbb{P}^r) \cong \mathbb{Z}$, an *n-pointed, stable map of degree d* consists of the data $\{\mu : C \to \mathbb{P}^r, \{p_i\}_{i=1}^n\}$, where:

- C is a complex, projective, connected, reduced, n-pointed, genus 0 curve with at worst nodal singularities.
- $\{p_i\}$ are smooth points of C.
- $\mu : C \to \mathbb{P}^r$ is a morphism.
- $\mu_*[C] = dl$, where l is a line generator of $A_1(\mathbb{P}^r)$.
- If μ collapses a component E of C to a point, then E must contain at least three special points (nodes or marked points).

We say that two stable maps are *isomorphic* if there is an isomorphism of the pointed domain curves $f : C \to C'$ that commutes with the morphisms to $\mathbb{P}^r$. Then there is a projective coarse moduli space $\overline{M}_{0,n}(\mathbb{P}^r, d)$ that parametrizes stable maps up to isomorphism [3]. The open locus $M_{0,n}(\mathbb{P}^r, d)$ corresponds to maps with a smooth domain while the *boundary* is naturally broken into divisors $D(N_1, N_2, d_1, d_2)$, where $N_1 \cup N_2$ is a partition of $\{1, 2, \dots, n\}$ and $d_1 + d_2 = d$. This corresponds to maps where the domain curve has two components, one of degree d_1 with the points of N_1 on it.

Similarly, we can define stable maps to $\mathbb{P}^r \times \mathbb{P}^1$ of bi-degree $(d, 1)$, and look at the corresponding coarse moduli space $\overline{M}_{0,n}(\mathbb{P}^r \times \mathbb{P}^1, (d, 1))$. The

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boundary again is broken into divisors. When no confusion is possible, we write $D(N_1, N_2, d_1, d_2)$ in place of $D(N_1, N_2, (d_1, 1), (d_2, 0))$.

In [14], Pandharipande constructs the open $M_{0,0}(\mathbb{P}^r, d) \subset \overline{M}_{0,0}(\mathbb{P}^r, d)$ as the GIT quotient of the open basepoint free locus $U(1, r, d) \subset \oplus_0^r H^0(\mathbb{P}^1, \mathcal{O}(d))$. We have a similar construction for the open pointed locus $M_{0,n}(\mathbb{P}^r \times \mathbb{P}^1, (d, 1)) \subset \overline{M}_{0,n}(\mathbb{P}^r \times \mathbb{P}^1, (d, 1))$. Our main result is the following theorem, which gives a construction of the compact $\overline{M}_{0,n}(\mathbb{P}^r, d)$ as a geometric quotient of $\overline{M}_{0,n}(\mathbb{P}^r \times \mathbb{P}^1, (d, 1))$ by $G = SL_2(\mathbb{C})$.

THEOREM 0.1. *Let E be an effective divisor such that $-E$ is ϕ-ample. Take a linearized line bundle $\mathcal{L} \in \mathrm{Pic}^G((\mathbb{P}^1)^n \times \mathbb{P}^r_d)$ such that*

$$((\mathbb{P}^1)^n \times \mathbb{P}^r_d)^{ss}(\mathcal{L}) = ((\mathbb{P}^1)^n \times \mathbb{P}^r_d)^s(\mathcal{L}) \neq \emptyset.$$

Then for each sufficiently small $\epsilon > 0$, the line bundle $\mathcal{L}' = \phi^*(\mathcal{L})(-\epsilon E)$ is ample and

$$\begin{aligned}(\overline{M}_{0,n}(\mathbb{P}^r \times \mathbb{P}^1, (d, 1)))^{ss}(\mathcal{L}') &= (\overline{M}_{0,n}(\mathbb{P}^r \times \mathbb{P}^1, (d, 1)))^s(\mathcal{L}') \\ &= \phi^{-1}\{((\mathbb{P}^1)^n \times \mathbb{P}^r_d)^{ss}(\mathcal{L})\}.\end{aligned}$$

There is a canonical identification

$$(\overline{M}_{0,n}(\mathbb{P}^r \times \mathbb{P}^1, (d, 1)))^s(\mathcal{L}')/G = \overline{M}_{0,n}(\mathbb{P}^r, d)$$

and a commutative diagram

$$\begin{array}{ccc}(\overline{M}_{0,n}(\mathbb{P}^r \times \mathbb{P}^1, (d, 1)))^s(\mathcal{L}') & \xrightarrow{f} & \overline{M}_{0,n}(\mathbb{P}^r, d) \\ \phi\big\downarrow & & \bar{\phi}\big\downarrow \\ ((\mathbb{P}^1)^n \times \mathbb{P}^r_d)^s(\mathcal{L}) & \longrightarrow & ((\mathbb{P}^1)^n \times \mathbb{P}^r_d)^s(\mathcal{L})/G\end{array}$$

where $\mathbb{P}^r_d := \mathbb{P}((H^0(\mathbb{P}^1, \mathcal{O}(d))^{r+1})$ and $\phi : \overline{M}_{0,n}(\mathbb{P}^r \times \mathbb{P}^1, (d, 1)) \to (\mathbb{P}^1)^n \times \mathbb{P}^r_d$ is the Givental contraction map [4].

The eventual goal would be to construct $\overline{M}_{0,0}(\mathbb{P}^r, d)$ as sequence of blow-ups of some projective variety. One benefit of such a construction is the ability to compute the Chow ring of $\overline{M}_{0,0}(\mathbb{P}^r, d)$, as Keel's Theorem 1 from the appendix of [9] gives the Chow ring of a blow-up. This cannot happen. First, $\overline{M}_{0,0}(\mathbb{P}^r, d)$ is not smooth. It has singularities at points corresponding to maps with nontrivial automorphisms. However, $\overline{M}_{0,0}(\mathbb{P}^r, d)$ is actually smooth when considered as a stack, and so at best we may hope for a stack analogue of a sequence of blow-ups mentioned above. The second issue seems more serious. There are no known maps from $\overline{M}_{0,0}(\mathbb{P}^r, d)$ to anything nice, and a birational map from $\overline{M}_{0,0}(\mathbb{P}^r, d)$ is exactly what is needed to carry out the above project.

As corollaries to our GIT construction, we are able to construct a birational map $\bar{\phi}$ from $\overline{M}_{0,0}(\mathbb{P}^r, d)$. Recently, progress has been made on understanding

$\overline{M}_{0,0}(\mathbb{P}^r \times \mathbb{P}^1, (d,1))$. For example, in [13], the above ϕ is factored into a sequence of intermediate moduli spaces such that the map between two successive spaces is a "weighted blow-up of a regular local embedding". As a corollary of the above theorem, we take the quotient of these intermediate spaces and factor $\bar{\phi}$.

In Section 1, we collect some preliminary results and definitions that will be used throughout the paper. Section 2 identifies the stable locus in $(\mathbb{P}^1)^n \times \mathbb{P}^r_d$ and explains how to pull it back to $\overline{M}_{0,n}(\mathbb{P}^r \times \mathbb{P}^1, (d,1))$. In Section 3, we prove the above theorem. Finally, in Section 4 we explain the factorization of φ from [13] and construct the intermediate spaces the induced quotient map. All work is done over $\mathbb{C}$.

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1. Preliminaries

Suppose that we want to compactify the space of n-pointed degree-d morphisms from $\mathbb{P}^1 \to \mathbb{P}^r$. Perhaps after the above discussion, one would expect that $\overline{M}_{0,n}(\mathbb{P}^r, d)$ correctly compactifies these objects. However, since we quotient out by isomorphisms, we only get degree d *un*-parametrized pointed morphisms to $\mathbb{P}^r$. Here we discuss two spaces that do correctly answer this question.

1.1. Linear sigma model. On one hand, an n-pointed, degree-d morphism f is given by $(r+1)$ homogeneous degree d polynomials in two variables, along with a choice of n distinct points on the domain $\mathbb{P}^1$. In the notation of [14], these maps correspond to the basepoint free locus

$$\begin{aligned} ((\mathbb{P}^1)^n \setminus \Delta) \times \mathbb{P}(U(1,r,d)) \subset (\mathbb{P}^1)^n \times \mathbb{P}\Big(\bigoplus_0^r H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))\Big) \\ := (\mathbb{P}^1)^n \times \mathbb{P}^r_d. \end{aligned}$$

Once we pick coordinates on $\mathbb{P}^1$, we can consider a closed point on the basepoint free locus as

$$[x_1 : y_1] \times \cdots \times [x_n : y_n] \times [f_0(x,y) : f_1(x,y) : \cdots : f_r(x,y)],$$

where $[x_s : y_s] \neq [x_t : y_t]$, the f_j do not have any common roots, and scaling does not change the map. The coefficients of these f_j determine a point in the projective space $\mathbb{P}^r_d := \mathbb{P}^{(r+1)(d+1)-1}$. We will sometimes write a^j_i for the coefficient $x^{d-i}y^i$ on f_j (after choosing the obvious coordinates on $\mathbb{P}^r_d$.) We thus have a simple compactification by allowing the $r+1$ forms to have common roots, and allowing the n points to come together. This space is sometimes referred to as the linear sigma model.

Moreover, there is a $G = SL_2(\mathbb{C})$ action on this space, similar to the action examined in [12] on binary quantics. On closed points, the action is given by

$$G \times ((\mathbb{P}^1)^n \times \mathbb{P}^r_d) \to (\mathbb{P}^1)^n \times \mathbb{P}^r_d$$

$$\begin{aligned} g \cdot &[[x_1 : y_1] \times \cdots \times [x_n : y_n] \times f_0(x : y), \ldots, f_r(x : y)] \\ &= [g[x_1 : y_1] \times \cdots \times g[x_n : y_n] \times f_0 \circ g^{-1}(x : y), \cdots : f_r \circ g^{-1}(x : y)], \end{aligned}$$

where g and g^{-1} act on $[x : y]$ by matrix multiplication.

1.2. The graph space. There is another, less simple (non-linear) compactification of $((\mathbb{P}^1)^n \setminus \Delta) \times \mathbb{P}(U(1, r, d))$. It is clear that this set equals $M_{0,n}(\mathbb{P}^r \times \mathbb{P}^1, (d, 1))$, and thus $\overline{M}_{0,n}(\mathbb{P}^r \times \mathbb{P}^1, (d, 1))$ provides another compactification.

We will refer to the domain curve C for a map in $\overline{M}_{0,n}(\mathbb{P}^r \times \mathbb{P}^1, (d, 1))$ as a *comb*. There is an obvious distinguished component C_0 on which $\mu|_{C_0}$ will be of degree $(d', 1)$. We will call this component the *handle*. The other components fit into *teeth* T_i, which are (perhaps reducible) genus-0, n_i-pointed curves meeting C_0 at unique points q_i. There is always a representative of the map so that the degree 1 part of μ restricted to the handle is the identity.

The action on $\overline{M}_{0,n}(\mathbb{P}^r \times \mathbb{P}^1, (d, 1))$ is induced by the action on the image $\mathbb{P}^1$. Namely we have

$$G \times \overline{M}_{0,n}(\mathbb{P}^r \times \mathbb{P}^1, (d, 1)) \to \overline{M}_{0,n}(\mathbb{P}^r \times \mathbb{P}^1, (d, 1))$$

$$g \cdot [\mu_1 \times \mu_2 : C \to \mathbb{P}^r \times \mathbb{P}^1] \to [\mu_1 \times g \circ \mu_2 : C \to \mathbb{P}^r \times \mathbb{P}^1]$$

1.3. The Givental map. In [4] Givental constructed a projective morphism that relates the graph space and the linear sigma model:

THEOREM 1.1 ([4]). *There is a projective morphism*

$$\varphi : \overline{M}_{0,0}(\mathbb{P}^r \times \mathbb{P}^1, (d, 1)) \to \mathbb{P}^r_d.$$

Set-theoretically, consider a point in $\overline{M}_{0,0}(\mathbb{P}^r \times \mathbb{P}^1, (d, 1))$. As mentioned above, there is a representative

$$[\mu : C \to \mathbb{P}^r \times \mathbb{P}^1 \text{ of bi-degree } (d, 1)]$$

and a component $C_0 \subset C$ such that $\mu|_{C_0}$ is the graph of $r + 1$ degree d' polynomials $(f_0, \ldots, f_r)$ with no common zero. On the teeth $T_1, \ldots, T_s$, μ has degree $(d_i, 0)$, respectively, and $d_1 + \cdots + d_s = d - d'$. Thus μ sends T_i into $\mathbb{P}^r \times z_i \subset \mathbb{P}^r \times \mathbb{P}^1$. Let h be a degree $d - d'$ form that vanishes at each z_i with multiplicity d_i. Then

$$\varphi(\mu) = [f_0 \cdot h, f_1 \cdot h, \ldots, f_r \cdot h] \in \mathbb{P}^r_d,$$

where we read off the coefficients to obtain the point in projective space.

Definition 1.2. Define the projective morphism ϕ to be the product of φ with the n morphisms $\pi_2 \circ ev_i : \overline{M}_{0,n}(\mathbb{P}^r, d) \to \mathbb{P}^r \times \mathbb{P}^1 \to \mathbb{P}^1$, where ev_i is evaluation at the n^{th} marked point. This product gives us a morphism from the graph space with marked points:

$$\phi : \overline{M}_{0,n}(\mathbb{P}^r \times \mathbb{P}^1, (d, 1)) \to (\mathbb{P}^1)^n \times \mathbb{P}^r_d$$

On the open locus, $M_{0,n}(\mathbb{P}^r \times \mathbb{P}^1, (d, 1))$, ϕ gives the isomorphism with $((\mathbb{P}^1)^n \setminus \Delta) \times \mathbb{P}(U(1, r, d))$ mentioned above.

The following lemma is needed when we take the quotients.

Lemma 1.3. *The above map*

$$\phi : \overline{M}_{0,n}(\mathbb{P}^r \times \mathbb{P}^1, (d, 1)) \to (\mathbb{P}^1)^n \times \mathbb{P}^r_d$$

is equivariant with respect to the above G actions.

Proof. We show that both the evaluation morphisms ev_i and the Givental φ map are equivariant. Then their product is equivariant as well.

Take a point in $\overline{M}_{0,n}(\mathbb{P}^r \times \mathbb{P}^1, (d, 1))$. Choose a representative $(\mu : C \to \mathbb{P}^r \times \mathbb{P}^1, \{p_i\})$. Write $C = C_0 \cup T_i$ as a comb, such that $T_i \cap C_0 = q_i$. Also write $\mu_i = \pi_1 \circ \mu|_{T_i}$.

If we look at the image of the above map under φ, we see that $\varphi(\mu)$ will be the product of $r + 1$ forms $(f_0, \ldots, f_r)$ of degree d' representing the handle and a form h of degree $d - d'$ that vanishes at the q_i with the correct degrees. We see that $g \cdot \varphi(\mu)$ will be the product of $(f_0 \cdot g^{-1}, \ldots, f_r \cdot g^{-1})$, which are $r + 1$ forms of degree d' with no common zero, with a form h' of degree $d - d'$ that vanishes at $g(q_i)$ with the same degree that h vanished at q_i.

We now need to calculate $\varphi(g \cdot \mu)$. With the above notation, we see that $g \cdot (\mu : C \to \mathbb{P}^r \times \mathbb{P}^1)$ sends $p \in T_i$ to $(\mu_i(p), g(q_i))$, and $p \in C_0$ to $(\mu_0(p), g(p))$. We find a representative of this new map for which the degree 1 part is the identity.

Take the curve $C' = g(C_0) \cup T_i$, where now the teeth T_i are glued to C_0 at $g(q_i)$. Define the map from $C' \to \mathbb{P}^r \times \mathbb{P}^1$ as $(\mu_0 \circ g^{-1}, id)$ on $g(C_0)$, and as μ_i on the other teeth. The corresponding map is isomorphic to $g \cdot \mu$.

We look at the image under the Givental map. The image will be the $r + 1$ degree d' forms $\mu_0 \circ g^{-1}$, along with a form h that vanishes at $g(q_i)$ with the correct degree. This is the same as $g \cdot \varphi(\mu)$. This shows that φ is equivariant.

That the evaluation morphisms are equivariant is immediate. □

1.4. The forgetful morphism. The second map that we will be interested in is the "forgetful" morphism

$$f : \overline{M}_{0,n}(\mathbb{P}^r \times \mathbb{P}^1, (d, 1)) \to \overline{M}_{0,n}(\mathbb{P}^r, d)$$

defined by forgetting the map to $\mathbb{P}^1$ and collapsing any components that become unstable. Moreover, since the G action on $\overline{M}_{0,n}(\mathbb{P}^r, d)$ is trivial, we automatically have that f is G equivariant.

2. Calculations on $(\mathbb{P}^1)^n \times \mathbb{P}^r_d$

Immediately, one would expect that $\overline{M}_{0,n}(\mathbb{P}^r, d)$ is the quotient of $\overline{M}_{0,n}(\mathbb{P}^r \times \mathbb{P}^1, (d,1))$ by G as G "takes into account" the map to $\mathbb{P}^1$. The question is: "How do we take the quotient?" We will use Geometric Invariant Theory (GIT) in order to find an open set in $\overline{M}_{0,n}(\mathbb{P}^r \times \mathbb{P}^1, (d,1))$ such that the quotient is $\overline{M}_{0,n}(\mathbb{P}^r, d)$.

All background concerning GIT will be taken from [2] and [12], though we recall the main theorem in its generality here for reference.

Theorem 2.1 ([2], [12]). *Let X be an algebraic variety, $\mathcal{L}$ a G-ample line bundle on X (i.e., G-linearized and G-effective). Then there are open sets $X^s(\mathcal{L}) \subset X^{ss}(\mathcal{L}) \subset X$ (the* stable and semi-stable loci*), such that the quotient*

$$\pi : X^{ss}(\mathcal{L}) \to X^{ss}(\mathcal{L})//G$$

is quasi-projective and a "good categorical quotient". This says (among other things) that for any other G-invariant morphism $g : X^{ss}(\mathcal{L}) \to Z$, there is a unique morphism $h : X^{ss}(\mathcal{L})//G \to Z$ satisfying $h \circ \pi = g$. If we restrict to $X^s(\mathcal{L})$, then we have a "geometric quotient". This says (among other things) that the geometric fibers are orbits of the geometric points of X, and the regular functions on $X^s(\mathcal{L})/G$ are G-equivariant functions on X.

When X is proper over $\mathbb{C}$ (as in this case) and $\mathcal{L}$ is ample, then $X^{ss}(\mathcal{L})//G$ will be projective [12].

We now consider the case when $G = SL_2(\mathbb{C})$ and $X = (\mathbb{P}^1)^n \times \mathbb{P}^r_d$ described above in Section 1. We see that in this case any line bundle admits a unique $SL_2(\mathbb{C})$ linearization.

Proposition 2.2.

$$\mathrm{Pic}^G((\mathbb{P}^1)^n \times \mathbb{P}^r_d) \cong \mathbb{Z}^{n+1}.$$

Proof. For any vector $\vec{k} = (k_1, \ldots, k_n, k_{n+1}) \in \mathbb{Z}^{n+1}$, we define a line bundle on $(\mathbb{P}^1)^n \times \mathbb{P}^r_d$ by

$$\mathcal{L}_{\vec{k}} = \bigotimes_{i=1}^{n+1} \pi_i^*(\mathcal{O}(k_i)),$$

where π_i is projection onto the i-th component. Every line bundle on $(\mathbb{P}^1)^n \times \mathbb{P}^r_d$ is isomorphic to $\mathcal{L}_{\vec{k}}$ for some choice of $\vec{k}$ ([5]). We need only show that each of these line bundles has one (and only one) linearization. However, since each π_i is G-equivariant, and each of the restrictions of $\mathcal{L}_{\vec{k}}$ to a factor has a unique linearization ([2]), $\mathcal{L}_{\vec{k}}$ has a canonical G-linearization. $\square$

Corollary 2.3.

$$\mathcal{L}_{\vec{k}} \textit{ is ample} \iff k_i > 0.$$

Proof. If all $k_i > 0$, then $\mathcal{L}_{\vec{k}}$ defines the projective embedding

$$(\mathbb{P}^1)^n \times \mathbb{P}^r_d \xrightarrow{\text{Veronese}} \prod_{i=1}^n \mathbb{P}^{(1+k_i)-1} \times \mathbb{P}^{\binom{rd+r+d+k_{n+1}}{rd+r+d}-1} \xrightarrow{\text{Segre}} \mathbb{P}^{\left((\prod_{i=1}^n 1+k_i)\times\binom{rd+r+d+k_{n+1}}{rd+r+d}\right)-1}.$$

On the other hand, if some multiple of $\mathcal{L}_{\vec{k}}$ defines a closed embedding, restricting it to any factor will be ample. But this is $\mathcal{O}_{\mathbb{P}^1}(k_i)$ (or $\mathcal{O}_{\mathbb{P}^r_d}(k_{n+1})$) and these are ample iff $k_i, k_{n+1} > 0$. □

In order to find the stable and semi-stable loci in $(\mathbb{P}^1)^n \times \mathbb{P}^r_d$, we will look at the image under the above Veronese/Segre maps. The main point is the following.

Proposition 2.4. *Let Ω be the composition of the Veronese and Segre maps above. Then*

$$((\mathbb{P}^1)^n \times \mathbb{P}^r_d)^{ss}(\mathcal{L}_{\vec{k}}) = \Omega^{-1}\left\{\left(\mathbb{P}^{\left((\prod_{i=1}^n 1+k_i)\times\binom{rd+r+d+k_{n+1}}{rd+r+d}\right)-1}\right)^{ss}(\mathcal{O}(1))\right\}$$

and similarly for the stable locus.

Proof. Call the image projective space $\mathbb{P}^{BIG}$. First, we show that there is an action of G on $\mathbb{P}^k$ such that the Veronese map $\mathbb{P}^1 \to \mathbb{P}^k$ is G equivariant. We need a representation of G in $GL(k+1)$. We write it out explicitly, choosing $[x, y]$ as coordinates on $\mathbb{P}^1$ and the obvious coordinates $[x^k : x^{k-1}y : \cdots : y^k]$ on $\mathbb{P}^k$, as follows:

$$\rho : G \to GL(k+1)$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to [a_{i,j}]_{i,j=0}^k,$$

where

$$a_{i,j} = \sum_{n=0}^{j} \binom{k-i}{n}\binom{i}{j-n} b^n a^{k-i-n} d^{j-n} c^{i-j+n}.$$

This is the coefficient of $x^{k-j}y^j$ in $(ax+by)^{k-i}(cx+dy)^i$, and it is a homomorphism. We can define the representation of G into $GL(\binom{rd+r+d+k_{n+1}}{rd+r+d})$ similarly. We now have representations

$$\rho_i : G \to GL(k_i + 1) \quad \text{and} \quad \rho_{n+1} : G \to GL\left(\binom{rd+r+d+k_{n+1}}{rd+r+d}\right).$$

We define the action on $\mathbb{P}^{BIG}$ by taking the tensor representation. This extends to an action on all of $\mathbb{P}^{BIG}$. Thus Ω is G-invariant by construction.

Take the composition $(\mathbb{P}^1)^n \times \mathbb{P}^r_d \to \Omega((\mathbb{P}^1)^n \times \mathbb{P}^r_d) \hookrightarrow \mathbb{P}^{BIG}$. We apply the following theorem from [12] to each of these arrows.

THEOREM 2.5 ([12, p. 46]). *Assume that $f : X \to Y$ is finite, G-equivariant with respect to actions of G on X and Y. If X is proper over k ($\mathbb{C}$ for us) and M is ample on Y, then*

$$X^{ss}(f^*M) = f^{-1}\{Y^{ss}(M)\}$$

and the same result holds for the stable locus.

Finally, that $\Omega^*\mathcal{O}(1) = \mathcal{L}_{\vec{k}}$ is obvious. □

We are now able to determine the stable and semi-stable locus in the linear-sigma model.

THEOREM 2.6. *Let $\vec{k} = (k_1, k_2, \dots, k_{n+1}) \in \mathbb{Z}_+^{n+1}$. Then $[x_1 : y_1] \times \cdots \times [x_n : y_n] \times [a_i^j] \in ((\mathbb{P}^1)^n \times \mathbb{P}_d^r)^{ss}(\mathcal{L}_{\vec{k}})$ (respectively $\in ((\mathbb{P}^1)^n \times \mathbb{P}_d^r)^{s}(\mathcal{L}_{\vec{k}})$) if for every point $p \in \mathbb{P}^1$*

$$\sum_{i \mid [x_i:y_i]=p} k_i + d_p \cdot k_{n+1} \leq \frac{1}{2}\left(\sum_{i=1}^{n} k_i + d \cdot k_{n+1}\right)$$

(respectively, if the above holds with strict inequality), where d_p is the degree of common vanishing of the forms $f_0, \dots, f_r$ at $p \in \mathbb{P}^1$.

Proof. We prove the theorem by first looking at the action of a maximal torus acting on $\mathbb{P}^{BIG}$. Here, there is only one line bundle, so everything is canonical. Then we pull back to find the corresponding locus in $(\mathbb{P}^1)^n \times \mathbb{P}_d^r$. We then move on to the entire group G.

Let T be the maximal torus of $SL_2(\mathbb{C})$, equal to the image of the 1-parameter subgroup

$$\lambda(t) = \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}.$$

We choose coordinates a_i^j on $\mathbb{P}_d^r$, where a_i^j is the coefficient of $x^{d-i}y^i$ in $f_j(x, y)$. Similarly, we choose the following coordinates on $\mathbb{P}^{BIG}$. For $0 \leq s_i \leq k_i$ ($1 \leq i \leq n$), and v_{ij} such that $\sum_{i=0}^{d}\sum_{j=0}^{r} v_{ij} = k_{n+1}$, we take the coordinate $x_i^{k_i - s_i} y_i^{s_i} (a_i^j)^{v_{ij}}$. Then T acts on $\mathbb{P}^{BIG}$ by

$$\lambda(t) \cdot (x_i^{k_i - s_i} y_i^{s_i} (a_i^j)^{v_{ij}}) \to t^{(\sum_{i=1}^{n} 2s_i - k_i) + (\sum_{ij} (d-2i) v_{ij})} x_i^{k_i - s_i} y_i^{s_i} (a_i^j)^{v_{ij}}.$$

By the above lemma, we know that it is enough to compute the semi-stable locus of this action on $\mathbb{P}^{BIG}$ and pull it back via the various inclusions and embeddings. Luckily we know how to compute the semi-stable locus of a torus acting on a projective space. From [2] we know that a point of projective space is stable (resp semi-stable) with respect to T if and only if $0 \in \text{interior}(\overline{wt})$ (respectively $0 \in \overline{wt}$). In our case, the weight set (wt) is the subset of

$$\left\{ -\sum_{i=1}^{n} k_i - d \cdot k_{n+1}, \dots, \sum_{i=1}^{n} k_i + d \cdot k_{n+1} \right\}$$

consisting of powers of t such that the coordinate $x_i^{k_i-s_i}y_i^{s_i}(a_i^j)^{v_{ij}}$ is non zero.

If the point is unstable, then all the powers $(\sum_{i=1}^n 2s_i-k_i)+(\sum_{ij}(d-2i)v_{ij})$ are < 0 (or all are > 0.) So $x_i^{k_i-s_i}y_i^{s_i}(a_i^j)^{v_{ij}} = 0$ if

$$\begin{aligned}
&0 \leq \left(\sum_{i=1}^n (2s_i - k_i) + \sum_{ij}(d-2i)v_{ij}\right) \iff \\
&0 \leq 2\sum_{i=1}^n (s_i - k_i) - 2\sum_{ij} i \cdot v_{ij} + \sum_{i=1}^n k_i + dk_{n+1} \iff \\
&\sum_{ij} i \cdot v_{ij} + \sum_{i=1}^n (k_i - s_i) \leq \frac{1}{2}\left(\sum_{i=1}^n k_i + dk_{n+1}\right).
\end{aligned}$$

Define the following sets in $(\mathbb{P}^1)^n \times \mathbb{P}^r_d$:

$$US = \left\{ [x_1 : y_1] \times \cdots \times [x_n : y_n] \times [a_i^j] \;\middle|\; x_i^{k_i-s_i}y_i^{s_i}(a_i^j)^{v_{ij}} = 0 \text{ if } \sum_{ij} i \cdot v_{ij} + \sum_{i=1}^n (k_i - s_i) \leq \frac{1}{2}\left(\sum_{i=1}^n k_i + dk_{n+1}\right) \right\}$$

and

$$X = \left\{ [x_1 : y_1] \times \cdots \times [x_n : y_n] \times [a_i^j] \;\middle|\; \frac{1}{2}\left(\sum_{i=1}^n k_i + dk_{n+1}\right) < \sum_{[x_i:y_i]=[1:0]} k_i + k_{n+1} \cdot d_{[1:0]} \right\}.$$

We show that $US = X$.

First, assume that $X \subset US$. Let $x = \{[x_1 : y_1] \times \cdots \times [x_n : y_n] \times [a_i^j]\}$ be in $US \setminus X$. So, $x_i^{k_i-s_i}y_i^{s_i}(a_i^j)^{v_{ij}} = 0$ if

$$\sum_{ij} i \cdot v_{ij} + \sum_{i=1}^n (k_i - s_i) \leq \frac{1}{2}\left(\sum_{i=1}^n k_i + dk_{n+1}\right).$$

But we also have

$$\sum_{[x_i:y_i]=[1:0]} k_i + k_{n+1} \cdot d_{[1:0]} \leq \frac{1}{2}\left(\sum_{i=1}^n k_i + dk_{n+1}\right).$$

Then, take $s_i = 0$ if $[x_i : y_i] = [1 : 0]$. And at least one of the $a^j_{d_{[1:0]}} \neq 0$. For that value of j, let $v_{ij} = k_{n+1}$. Then we have

$$\sum_{i=0}^n (k_i - s_i) + \sum_{ij} i \cdot v_{ij} = \sum_{[x_i:y_i]=[1:0]} k_i + k_{n+1} \cdot d_{[1:0]} \leq \frac{1}{2}\left(\sum_{i=1}^n k_i + dk_{n+1}\right).$$

The coordinate $x_i^{k_i-s_i}y_i^{s_i}(a_i^j)^{v_{ij}}$ is $\neq 0$ by construction, which says that $x \notin US$, a contradiction. Thus $US \subseteq X$.

Now, assume that $US \subset X$ Take $y = \{[x_1 : y_1] \times \cdots \times [x_n : y_n] \times [a_i^j]\} \in X \setminus US$. So $x_i^{k_i-s_i}y_i^{s_i}(a_i^j)^{v_{ij}} \neq 0$, but

$$\sum_{ij} i \cdot v_{ij} + \sum_{i=1}^{n}(k_i - s_i)\} \leq \frac{1}{2}\left(\sum_{i=1}^{n} k_i + dk_{n+1}\right).$$

Combining this with the fact that $y \in X$, we see that

$$\sum_{i=1}^{n}(k_i - s_i) + \sum_{ij} i \cdot v_{ij} < \sum_{[x_i:y_i]=[1:0]} k_i + k_{n+1} \cdot d_{[1:0]}.$$

Then, for all i with $[x_i : y_i] = [1 : 0]$, we must have $s_i = 0$. Thus,

$$\sum_{[x_i:y_i]=[1:0]} k_i \leq \sum_{i=1}^{n}(k_i - s_i).$$

Similarly, since $(a_i^j)^{v_{ij}} \neq 0$, we know that if $(a_i^j) = 0$, then $v_{ij} = 0$. Thus,

$$\sum_{j=0}^{r}\sum_{i=0}^{d} i \cdot v_{ij} = \sum_{j=0}^{r}\sum_{i=d_{[1:0]}}^{d} i \cdot v_{ij} \geq d_{[1:0]} \sum_{ij} v_{ij} = d_{[1:0]} \cdot k_{n+1}.$$

Combining these gives our contradiction, showing that $X \subseteq US$ as desired.

If we repeat this calculation, replacing the condition $(\sum_{i=1}^{n} 2s_i - k_i) + (\sum_{ij}(d-2i)v_{ij}) < 0$ with $(\sum_{i=1}^{n} 2s_i - k_i) + (\sum_{ij}(d-2i)v_{ij}) > 0$, we get the following lemma.

Lemma 2.7. $[x_1 : y_1] \times \cdots \times [x_n, y_n] \times [a_i^j]$ *is unstable with respect to* T *if*

$$\sum_{i \mid [x_i:y_i]=[1:0]} k_i + k_{n+1} \cdot d_{[1:0]} > \frac{1}{2}\left(\sum_{i=1}^{n} k_i + dk_{n+1}\right)$$

or

$$\sum_{i \mid [x_i:y_i]=[0:1]} k_i + k_{n+1} \cdot d_{[0:1]} > \frac{1}{2}\left(\sum_{i=1}^{n} k_i + dk_{n+1}\right).$$

We are now ready to move on to stability with respect to G. Suppose that $[x_1 : y_1] \times \cdots \times [x_n, y_n] \times [a_i^j]$ is stable with respect to G and there is a point p in $\mathbb{P}^1$ such that

$$\sum_{[x_i:y_i]=p} k_i + k_{n+1} \cdot d_p > \frac{1}{2}\left(\sum_{i=1}^{n} k_i + dk_{n+1}\right).$$

Let $g \in G$ map $p \to [1:0]$. Then $g \cdot [x_1 : y_1] \times \cdots \times [x_n, y_n] \times [a_i^j]$ is unstable with respect to T, and $[x_1 : y_1] \times \cdots \times [x_n, y_n] \times [a_i^j]$ is unstable with respect to G, contradicting the assumption.

Now assume that $[x_1 : y_1] \times \cdots \times [x_n, y_n] \times [a_i^j]$ is unstable, but has no point p such that

$$\sum_{i|[x_i:y_i]=p} k_i + k_{n+1} \cdot d_p > \frac{1}{2}\left(\sum_{i=1}^{n} k_i + dk_{n+1}\right).$$

Then there is some maximal torus T' for which $[x_1 : y_1] \times \cdots \times [x_n, y_n] \times [a_s]$ is unstable. For any maximal torus in G, there is $g \in G$ such that $gT'g^{-1} = T$. Then we have that $g \cdot [x_1 : y_1] \times \cdots \times [x_n, y_n] \times [a_i^j]$ is unstable with respect to T, hence must have either $[1 : 0]$ or $[0 : 1]$ satisfying Lemma 2.7. Then $[x_1 : y_1] \times \cdots \times [x_n, y_n] \times [a_i^j]$ has $g^{-1}[1 : 0]$ satisfying Lemma 2.7. $\square$

We are now ready to describe the chamber decomposition of the ample cone of $\mathrm{Pic}^G((\mathbb{P}^1)^n \times \mathbb{P}^r_d)$. As a first step we normalize our line bundle so that we form the simplex

$$\Delta = \left\{(k_1, k_2, \ldots, k_{n+1}) \;\middle|\; \sum_{i=1}^{n} k_i + d \cdot k_{n+1} = 2\right\}.$$

Then for each subset $I \in (1, 2, \ldots, n)$ and each integer $0 \leq d_I \leq d$, we get a wall W_{I,d_I} given by

$$\sum_{i \in I} k_i + d_I \cdot k_{n+1} = 1$$

and the walls break Δ into chambers. Following [7], we mention the following obvious statements.

(1) We have
$$W_{S,d_S} = W_{S^c,d-d_S}.$$

(2) Each interior wall divides Δ into two parts
$$\left\{(k_1, k_2, \ldots, k_{n+1}) \;\middle|\; \sum_{i \in I} k_i + d_I \cdot k_{n+1} \leq 1\right\}$$
and
$$\left\{(k_1, k_2, \ldots, k_{n+1}) \;\middle|\; \sum_{i \in I} k_i + d_I \cdot k_{n+1} \geq 1\right\}.$$

(3) Two vectors $\vec{k} = (k_1, \ldots, k_{n+1})$ and $\vec{k'} = (k'_1, \ldots, k'_{n+1})$ lie in the same chamber if for all $I \subset \{1, 2, \ldots\}$ and $0 \leq d_I \leq d$, we have
$$\sum_{i \in I} k_i + d_I \cdot k_{n+1} \leq 1 \iff \sum_{i \in I} k'_i + d_I \cdot k'_{n+1} \leq 1.$$

This means that vectors in the same chamber will define the same stable and semi-stable loci, and hence the same quotient.

(4) There are semi-stable points that are not stable iff $\vec{k}$ lies on a wall.

Recall that our goal is not to take the GIT quotient of $(\mathbb{P}^1)^n \times \mathbb{P}^r_d$, but that of $\overline{M}_{0,n}(\mathbb{P}^r \times \mathbb{P}^1, (d,1))$. Up to this point we have not said anything about the stable or semi-stable loci in $\overline{M}_{0,n}(\mathbb{P}^r \times \mathbb{P}^1, (d,1))$. We are able to pull back the stable locus via ϕ, by the following theorem of Yi Hu [6].

Theorem 2.8 ([6]). *Let $\pi : Y \to X$ be a G-equivariant projective morphism between two (possibly singular) quasi-projective varieties. Given any linearized ample line bundle L on X, choose a relatively ample linearized line bundle M on Y. Assume moreover that $X^{ss}(L) = X^s(L)$. Then there exists an n_0 such that when $n \geq n_0$, we have*

$$Y^{ss}(\pi^* L^n \otimes M) = Y^s(\pi^* L^n \otimes M) = \pi^{-1}\{X^s(L)\}.$$

For example, the locus of maps in $\overline{M}_{0,0}(\mathbb{P}^r \times \mathbb{P}^1, (d,1))$ that are stable will be maps such that no tooth of the comb C has degree $\geq d/2$. Figure 1 below shows how the stable locus depends on the line bundle.

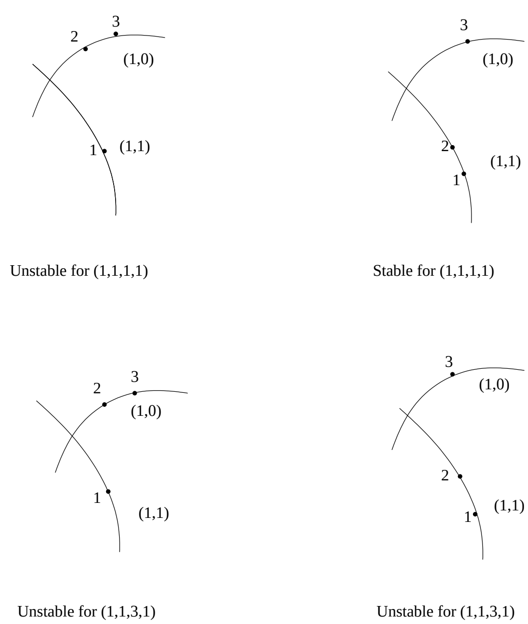

Figure 1. Stable locus of $\overline{M}_{0,3}(\mathbb{P}^r \times \mathbb{P}^1, (2,1))$

3. The geometric quotient

We are now ready to present our GIT description of $\overline{M}_{0,n}(\mathbb{P}^r, d)$. The construction is similar to that of $\overline{M}_{0,n}$ from [8]. We state it similarly. First let E be an effective divisor with support the full exceptional locus of ϕ, such that $-E$ is ϕ ample. Such an E exists by the following lemma from [11].

LEMMA 3.1 ([11, p. 70]). *Let $f : X \to Y$ be a birational morphism. Assume that Y is projective and X is $\mathbb{Q}$-factorial. Then there is an effective f-exceptional divisor E such that $-E$ is f-ample.*

$\overline{M}_{0,n}(\mathbb{P}^r \times \mathbb{P}^1, (d, 1))$ is $\mathbb{Q}$-factorial because it is locally the quotient of a smooth scheme by a finite group.

THEOREM 0.1. *For each linearized line bundle $\mathcal{L} \in \mathrm{Pic}^G((\mathbb{P}^1)^n \times \mathbb{P}^r_d)$ such that*

$$((\mathbb{P}^1)^n \times \mathbb{P}^r_d)^{ss}(\mathcal{L}) = ((\mathbb{P}^1)^n \times \mathbb{P}^r_d)^{s}(\mathcal{L}) \neq \emptyset$$

and for each sufficiently small $\epsilon > 0$, the line bundle $\mathcal{L}' = \phi^*(\mathcal{L})(-\epsilon E)$ is ample and

$$\begin{aligned}(\overline{M}_{0,n}(\mathbb{P}^r \times \mathbb{P}^1, (d, 1)))^{ss}(\mathcal{L}') &= (\overline{M}_{0,n}(\mathbb{P}^r \times \mathbb{P}^1, (d, 1)))^{s}(\mathcal{L}') \\ &= \phi^{-1}\{((\mathbb{P}^1)^n \times \mathbb{P}^r_d)^{ss}(\mathcal{L})\}.\end{aligned}$$

There is a canonical identification

$$(\overline{M}_{0,n}(\mathbb{P}^r \times \mathbb{P}^1, (d, 1)))^{s}(\mathcal{L}')/G = \overline{M}_{0,n}(\mathbb{P}^r, d)$$

and a commutative diagram

$$\begin{tikzcd} (\overline{M}_{0,n}(\mathbb{P}^r \times \mathbb{P}^1, (d, 1)))^{ss}(\mathcal{L}') \arrow[r, "f"] \arrow[d, "\phi"'] & \overline{M}_{0,n}(\mathbb{P}^r, d) \arrow[d] \\ ((\mathbb{P}^1)^n \times \mathbb{P}^r_d)^{ss}(\mathcal{L}) \arrow[r] & ((\mathbb{P}^1)^n \times \mathbb{P}^r_d)^{ss}(\mathcal{L})/G \end{tikzcd}$$

where ϕ is the generalized Givental map, f is the forgetful morphism.

Proof. To prove the first two statements, we apply the above quoted theorem of Hu [6] (Theorem 2.8). Following the notation from [8], let U be the semi-stable locus in $(\mathbb{P}^1)^n \times \mathbb{P}^r_d$ for the above action of G corresponding to $\mathcal{L}_{\vec{k}}$. Recall that this corresponds to $([x_i, y_i], f_0, \ldots f_r)$ such that for any $p \in \mathbb{P}^1$, we have

$$\sum_{i|[x_i:y_i]=p} k_i + k_{n+1} \cdot d_p \leq \frac{1}{2}\left(\sum_{i=1}^{n} k_i + k_{n+1} \cdot d\right).$$

Let $U' = \phi^{-1}(U)$. Let the corresponding quotients be Q and Q'. We have the obvious composition of G invariant maps:

$$U' \to U \to Q.$$

By the universal properties of GIT quotients, we get a proper birational map $Q' \to Q$. Similarly, since G acts trivially on $\overline{M}_{0,n}(\mathbb{P}^r, d)$, we have by the universal property again a proper birational map from $Q' \to \overline{M}_{0,n}(\mathbb{P}^r, d)$. We will show that this is an isomorphism by showing that both sides have the same Picard number. This is enough since both sides are $\mathbb{Q}$-factorial. We have

$$\rho(Q') = \rho(U') = \rho(U) + e(U) = \rho(Q) + e(U),$$

where $e(u)$ is the number of ϕ-exceptional divisors that meet U'. Since ϕ is an isomorphism on the open locus $M_{0,n}(\mathbb{P}^r \times \mathbb{P}^1, (d,1)) \subset \overline{M}_{0,n}(\mathbb{P}^r \times \mathbb{P}^1, (d,1))$, we need only look at the boundary divisors in $\overline{M}_{0,n}(\mathbb{P}^r \times \mathbb{P}^1, (d,1))$. We use Lemma 3.2 to see which divisors are exceptional.

LEMMA 3.2.

$$\phi(D(N_1, N_2, d_1, d_2)) \subset (\mathbb{P}^1)^n \times \mathbb{P}^r_d$$

has codimension $|N_2| + (r+1)d_2 - 1$.

Proof. The idea for this proof comes from Kirwan [10]. First notice that

$$\phi(D(N_1, N_2, d_1, d_2)) = (p_1, p_2, \ldots, p_n, f_0, \ldots, f_r),$$

where $p_i = p_j$ for $i, j \in N_2$, and each of the f_s vanishes at that point of multiplicity d_2.

First, we calculate the codimension of $(p_1, p_2, \ldots, p_n, f_0, \ldots, f_r)$, where each of the p_i is $[0:1]$, and where each of f_j has a zero of order d_2 at $[0:1]$ and one of order $d - d_2 = d_1$ at $[1:0]$ (i.e., each f_j consists of only the monomial $x^{d_2}y^{d_1}$). It is clear this has codimension $n + rd + r + d - r = n + (r+1)d$. If we remove the condition that each f_j have a root of order d_1 at $[1:0]$, then we allow each f_j to have higher powers of x. We also remove the condition that those p_i with $i \notin N_2$ are equal to $[0:1]$. Thus we see that the set of $(p_1, p_2, \ldots, p_n, f_0, \ldots, f_r)$ such that $[0:1] = p_i$ for $i \in N_2$, and each f_i vanishes at $[0:1]$ with multiplicity d_2 has codimension

$$n + (r+1)d - (r+1)d_1 - |N_1| = |N_2| + (r+1)d_2.$$

Finally, we act on this set by G. We subtract one from the above codimension because G has dimension two, but we do not count the two dimensional stabilizer of $[0:1]$. $\square$

Next we show that $\rho(Q')$ is independent of the chamber that $\mathcal{L}_{\vec{k}}$ comes from. We check that as we cross a wall W_{I,d_I}, $\rho(Q')$ does not change. Let our two open sets be U_1 and U_2. Recall that W_{I,d_I} breaks our chamber into two parts

$$\left\{ (k_1, \ldots, k_n, k_{n+1}) \;\middle|\; \sum_{i \in I} k_i + d_I \cdot k_{n+1} \leq 1 \right\}$$

and

$$\left\{(k_1, \ldots, k_n, k_{n+1}) \;\Big|\; \sum_{i \in I} k_i + d_I \cdot k_{n+1} \geq 1\right\},$$

so suppose that U_1 meets the first set. Notice that U_1' and U_2' meet the same divisors $D(N_1, N_2, d_1, d_2)$ except that U_1' meets $D(I^c, I, d - d_I, d_I)$ but not $D(I, I^c, d_I, d - d_I)$. Similarly, U_2' meets $D(I, I^c, d_I, d - d_I)$ but not $D(I^c, I, d - d_I, d_I)$.

If $2 < |I|, 1 < r$ and $1 \leq d_I \leq d$, or $r = 1$ and $1 < d_i \leq d$, then $Q_1 \dashrightarrow Q_2$ is a small modification (an isomorphism in codimension 1 in the notation of [11]). Hence $\rho(Q_1) = \rho(Q_2)$, and it is clear that $e(U_1) = e(U_2)$.

If $2 = |I|$ and $d_I = 0$, then we see that $Q_1 \dashrightarrow Q_2$ contracts the divisor $(p_1, \ldots, p_n, f_0, \ldots f_r)$, where $p_i = p_j, i, j \in I$. Therefore $\rho(Q_1) = \rho(Q_2) + 1$. However, by Lemma 3.2, we see that the divisor $D(I^c, I, d, 0)$ with $|I| = 2$ lying over U_1 is not exceptional, while its complement $D(I, I^c, 0, d)$ lying over U_2 is exceptional. Hence $e(U_2) = e(U_1) + 1$. Putting these together we see that $\rho(Q_1') = \rho(Q_2')$ as desired.

If $r = 1, |I| = 0, d_I = 1$, then we see that $Q_1 \dashrightarrow Q_2$ contracts the divisor $(p_1, \ldots, p_n, f_0, f_1)$, where f_0, f_1 have a common root. Therefore $\rho(Q_1) = \rho(Q_2) + 1$. However, by Lemma 3.2, we see that the divisor $D(N, 0, d-1, 1)$ lying over U_1 is not exceptional, while its complement $D(0, N, 1, d-1)$ is contracted. Hence $e(U_2) = e(U_1) + 1$. Putting these together we see that $\rho(Q_1') = \rho(Q_2')$ as desired.

Finally, we prove the theorem for one vector of one chamber. Here we look at all divisors $D(N_1, N_2, d_1, d_2) \subset \overline{M}_{0,n}(\mathbb{P}^r \times \mathbb{P}^1, (d, 1))$. We have 2^n ways to distribute the n points on the domain curve, and we can label the collapsed component with any degree $\leq d$. Hence there are $2^n(d+1)$ potential configurations. However the configurations $D(I, I^c, d, 0)$ are not stable maps if $|I| = n$ or $|I| = n - 1$. Hence there are $2^n(d+1) - n - 1$ total boundary divisors in $\overline{M}_{0,n}(\mathbb{P}^r \times \mathbb{P}^1, (d, 1))$. We need to determine how many are stable (with respect to the group). This requires several calculations, as a given linearization $\vec{k}$ may lie in a maximal chamber for certain values of d, n, but lie on a wall for others. All the calculations are very similar. Assume that $r > 1$.

Case 1: $d + n$ odd, $d > n$.. We choose the linearization corresponding to $(1, 1, 1, \ldots, 1, 1)$. We count the unstable divisors, i.e., the number of $D(N_1, N_2, d_1, d_2)$ such that

$$|N_2| + d_2 \geq \frac{d + n + 1}{2}.$$

Any divisor $D(N_1, N_2, d_1, d_2)$ with $\frac{d+n+1}{2} \leq d_2 \leq d$ is unstable. There are $2^n(\frac{d-n+1}{2})$ of these. Thus the total number of unstable divisors is

$$2^n\left(\frac{d-n+1}{2}\right) + \overbrace{(2^n - \binom{n}{0})}^{\#\text{ with } d_2=\frac{d+n-1}{2}} + \overbrace{(2^n - \binom{n}{0} - \binom{n}{1})}^{\#\text{ with } d_2=\frac{d+n-3}{2}} +$$

$$\cdots + \overbrace{2^n - \binom{n}{0} - \binom{n}{1} - \cdots - \binom{n}{n-1}}^{\#\text{ with } d_2=\frac{d-n+1}{2}}$$

$$= 2^n\left(\frac{d-n+1}{2}\right) + n2^n - n\binom{n}{0} - (n-1)\binom{n}{1} - \cdots - 1\binom{n}{n-1}$$

$$= 2^n\left(\frac{d-n+1}{2}\right) + n2^n - n\binom{n}{n} - (n-1)\binom{n}{n-1} - \cdots - 1\binom{n}{1}$$

$$= 2^n\left(\frac{d-n+1}{2}\right) + n2^n - \sum_{i=1}^{n}(i)\binom{n}{i}$$

$$= 2^n\left(\frac{d-n+1}{2}\right) + n2^n - n2^{n-1} = 2^{n-1}(d+1).$$

Hence the total number of stable divisors is $2^{n-1}(d+1)-1-n$ (stable with respect to the group). Of these all are ϕ-exceptional all except those for which $I^c = 2$ (by Corollary 3.2), so there are $2^{n-1}(d+1)-1-n-\binom{n}{2}$ ϕ-exceptional divisors. Thus, since $\rho(Q) = n+1$,

$$\rho(Q') = \rho(Q) + e(U) = n+1+2^{n-1}(d+1) - 1 - n - \binom{n}{2}$$
$$= 2^{n-1}(d+1) - \binom{n}{2}.$$

Case 2: $d+n$ odd, $d<n$.. We again choose the linearization corresponding to $(1,1,1,\ldots,1,1)$.

Case 3: $d+n$ even, n odd.. We choose the linearization corresponding to $(1,1,\ldots,1,2)$.

Case 4: $d+n$ even, n even.. We choose the linearization corresponding to $(1,2,2,\ldots,2,1)$.

Note that care must be taken when $n=2$ and $d=1,2$, since in this case $\rho((\mathbb{P}^1)^n \times \mathbb{P}^r_d)$ is equal to 2 for the given linearizations (instead of the expected number 3). This is because the unstable locus contains a divisor. However, we do not need to subtract out the divisor $D(0,2,d_1,d_2)$ for not being ϕ-exceptional, because it is unstable with respect to the group. Thus, the sums work out to be the same.

We have shown for every line bundle such that the stable locus equals the semi-stable locus that $\rho(Q') = 2^{n-1}(d+1) - \binom{n}{2}$. From [14] we have

$$\rho(\overline{M}_{0,n}(\mathbb{P}^r, d)) = 2^{n-1}(d+1) - \binom{n}{2}.$$

This completes the proof for $r > 1$.

When $r = 1$, we repeat the above construction. Here we see, by Corollary 3.2, that the divisor $D(N, 0, d-1, 1)$ is not ϕ-exceptional. So we subtract one from the above count of ϕ-exceptional divisors. Thus

$$\begin{aligned}\rho(Q') &= \rho(Q) + e(U) = n + 1 + 2^{n-1}(d+1) - 2 - n - \binom{n}{2} \\ &= 2^{n-1}(d+1) - \binom{n}{2} - 1.\end{aligned}$$

This agrees with the value of $\rho(\overline{M}_{0,n}(\mathbb{P}^1, d))$ obtained as an immediate consequence of Theorem 4.4 in [1].

In the case when $r = 0$, we have either $d = 0$ or the moduli space is empty. Thus $\mathbb{P}^0_0 = pt$. The calculation follows since now we are only dealing with stable curves, and not stable maps, and was proven originally in [8]. We obtain that

$$\rho(Q') = 2^{n-1} - \binom{n}{2} - 1 = \rho(\overline{M}_{0,n}). \qquad \square$$

There are three immediate corollaries that are interesting. The first is a new proof of a result of Keel and Hu [8]. By letting $d, r = 0$ in the above theorem we have:

Corollary 3.3 ([8]). *For each linearized line bundle $\mathcal{L} \in \mathrm{Pic}^G((\mathbb{P}^1)^n)$ such that*

$$((\mathbb{P}^1)^n)^{ss}(\mathcal{L}) = ((\mathbb{P}^1)^n)^s(\mathcal{L}) \neq \emptyset$$

and for each sufficiently small $\epsilon > 0$, the line bundle $\mathcal{L}' = \phi^*(\mathcal{L})(-\epsilon E)$ is ample and

$$(\mathbb{P}^1[n])^{ss}(\mathcal{L}') = (\mathbb{P}^1[n])^s(\mathcal{L}') = \phi^{-1}((\mathbb{P}^1)^n)^{ss}(\mathcal{L}).$$

There is a canonical identification

$$(\mathbb{P}^1[n])^s(\mathcal{L}')/G = \overline{M}_{0,n}$$

and a commutative diagram

$$\begin{array}{ccc}(\mathbb{P}^1[n])^{ss}(\mathcal{L}') & \xrightarrow{f} & \overline{M}_{0,n} \\ {\scriptstyle\phi}\big\downarrow & & \big\downarrow \\ ((\mathbb{P}^1)^n)^{ss}(\mathcal{L}) & \longrightarrow & ((\mathbb{P}^1)^n)^{ss}(\mathcal{L})/G)\end{array}$$

Proof. In the case when $d = 1$ and $r = 1$, we have that

$$\overline{M}_{0,n}(\mathbb{P}^1, 1) = \mathbb{P}^1[n],$$

where $\mathbb{P}^1[n]$ is the Fulton-MacPherson compactification of n points on $\mathbb{P}^1$. The Fulton-MacPherson map $\mathbb{P}^1[n] \to (\mathbb{P}^1)^n$ is exactly the product of evaluation morphisms ϕ. □

Secondly, we find that the Grassmannian of lines is a GIT quotient of a projective space.

Corollary 3.4. *The Grassmannian of lines in $\mathbb{P}^r$ is the GIT quotient of $\mathbb{P}^r_1 = \mathbb{P}^{2(r+1)-1}$ by the above action of G.*

Proof. We know that $\overline{M}_{0,0}(\mathbb{P}^r, 1) = M_{0,0}(\mathbb{P}^r, 1) = \mathbb{G}(1, r)$. By the theorem

$$(\overline{M}_{0,0}(\mathbb{P}^r \times \mathbb{P}^1, (1, 1)))^s/G = \mathbb{G}(1, r).$$

But $\overline{M}_{0,0}(\mathbb{P}^r \times \mathbb{P}^1, (1, 1))^s = M_{0,0}(\mathbb{P}^r \times \mathbb{P}^1, (1, 1)) \cong (\mathbb{P}^r_1)^s$. Note that when $n = 0$, there is only one ample line bundle (up to multiples) on the linear sigma model and it has a unique linearization. □

The third corollary constructs $\overline{M}_{0,0}(\mathbb{P}^r, d)$ as a sequence of intermediate moduli spaces. It requires more background, and is given in the next section.

4. Intermediate moduli spaces

In the case when $n = 0$, we obtain as a corollary a factorization of the induced map

$$\bar{\varphi} : \overline{M}_{0,0}(\mathbb{P}^r, d) \to (\mathbb{P}^r_d)^s / \operatorname{Aut}(\mathbb{P}^1)$$

into a sequence of intermediate moduli spaces, such that the map between successive spaces is "almost" a blow-up. We will use a factorization of the Givental map φ presented in [13]. We give the necessary definitions and results here.

Recall the construction of $\overline{M}_{0,0}(\mathbb{P}^r, d)$ presented by Fulton and Pandharipande in [3]. Given a basis of hyperplanes $\bar{t} \in H^0(\mathbb{P}^1, \mathcal{O}(1))$, there is an open subset $U_{\bar{t}} \subset \overline{M}_{0,0}(\mathbb{P}^r, d)$ such that if we pull back those hyperplanes, the corresponding domain curves along with the sections will be $(r + 1)d$ pointed stable *curves.* By choosing an ordering on the sections, we get an étale rigidification of this open set by a smooth moduli space, denoted by $M_{0,0}(\mathbb{P}^r, d, \bar{t})$, that is a $(\mathbb{C}^*)^r$ bundle over $\overline{M}_{0,d(r+1)}$:

$$\overline{M}_{0,d(r+1)} \supset B \xleftarrow{(\mathbb{C}^*)^r} M_{0,0}(\mathbb{P}^r, d, \bar{t}) \xrightarrow{(S_d)^{r+1}} U_{\bar{t}} \subset \overline{M}_{0,0}(\mathbb{P}^r, d).$$

$\overline{M}_{0,0}(\mathbb{P}^r, d)$ is then constructed by gluing together the $U_{\bar{t}}$ for different choices of $\bar{t}$.

In [13], Mustaţă constructs similar rigidifications of $\overline{M}_{0,0}(\mathbb{P}^r \times \mathbb{P}^1, (d,1))$ and $\mathbb{P}^r_d$ that are $(\mathbb{C}^*)^r$ bundles over $\mathbb{P}^1[d(r+1)]$ and $(\mathbb{P}^1)^{d(r+1)}$, where $\mathbb{P}^1[d(r+1)]$ is the Fulton-MacPherson compactification of $d(r+1)$ points on $\mathbb{P}^1$:

$$\begin{tikzcd}
\mathbb{P}^1[d(r+1)] \supset B \arrow[d, "F-M"'] & G(r,d,\bar{t}) \arrow[l, "(\mathbb{C}^*)^r"'] \arrow[r, "(S_d)^{r+1}"] \arrow[d, "\varphi(\bar{t})"'] & U_{\bar{t}} \subset \overline{M}_{0,0}(\mathbb{P}^r \times \mathbb{P}^1, (d,1)) \arrow[d, "\varphi"'] \\
(\mathbb{P}^1)^{d(r+1)} & \mathbb{P}^r_d(\bar{t}) \arrow[l, "(\mathbb{C}^*)^r"'] \arrow[r, "(S_d)^{r+1}"] & \mathbb{P}^r_d
\end{tikzcd}$$

The factorization of φ is obtained by gluing together the pull back of the following factorization of the Fulton-MacPherson map.

Definition/Theorem 4.1 ([13, p. 13]). *Consider a degree 1 morphism $\phi : C \to \mathbb{P}^1$ having as domain C a rational curve with N marked points. The morphism will be called n-stable if:*

(1) *Not more than $N - n$ of the marked points coincide.*
(2) *Any ending curve that is not the parametrized component contains more than $N - n$ points.*
(3) *All the marked points are smooth, and every component that is not the parametrized component has at least three distinct special points.*

There is a smooth projective moduli space $\mathbb{P}^1[N,n]$ for families of n-stable degree 1 morphisms. Moreover $\mathbb{P}^1[N,n]$ is the blow-up of $\mathbb{P}^1[N,n-1]$ along the strict transforms of the n-dimensional diagonals in $(\mathbb{P}^1)^N$.

There is an analogous factorization of $\varphi(\bar{t})$.

Definition/Theorem 4.2 ([13, p. 22]). *A $(\bar{t}, d, k)$-acceptable family of morphism over S is given by the following data:*

$$(\pi : \mathcal{C} \to S, \phi : \mathcal{C} \to \mathbb{P}^1, \{q_{i,j}\}_{0 \le i \le n, 1 \le j \le d}, \mathcal{L}, e),$$

where:

(1) *The family $(\pi : \mathcal{C} \to S, \phi : \mathcal{C} \to \mathbb{P}^1, \{q_{i,j}\}_{0 \le i \le n, 1 \le j \le d})$ is a $(r+1)(k-1)+1$ stable family of degree 1 morphisms to $\mathbb{P}^1$.*
(2) *$\mathcal{L}$ is a line bundle on $\mathcal{C}$.*
(3) *$e : \mathcal{O}_{\mathcal{C}}^{r+1} \to \mathcal{L}$ is a morphism of sheaves with $\pi_* e$ nowhere zero and that, via the natural isomorphism $H^0(\mathbb{P}^n, \mathcal{O}(1)) \cong H^0(S \times \mathbb{P}^1, \mathcal{O}^{r+1}_{S \times \mathbb{P}^1})$ we have*
$$(e(\bar{t}_i) = 0) = \sum_{j=1}^{d} q_{i,j}.$$

There is a smooth moduli space $\mathbb{P}^r_d(\bar{t}, k)$ for these families that is a torus bundle over an open subset of $\mathbb{P}^1[(r+1)d, (r+1)(k-1)+1]$.

Finally, [13] creates global objects factoring φ in the same way as $\overline{M}_{0,0}(\mathbb{P}^r, d)$ was constructed in [3].

Definition/Theorem 4.3 ([13, p. 23]). *A (d,k)-acceptable family of morphisms is given by the following data:*

$$(\pi : \mathcal{C} \to S, \mu = (\mu_1, \mu_2) : C \to \mathbb{P}^r \times \mathbb{P}^1, \mathcal{L}, e),$$

where:

(1) *$\mathcal{L}$ is a line bundle on $\mathcal{C}$ which, together with the morphism*

$$e : \mathcal{O}_{\mathcal{C}}^{r+1} \to \mathcal{L}$$

determines the rational map $\mu_1 : \mathcal{C} \to \mathbb{P}^r$.

(2) *For any $s \in S$ and any irreducible component C' of C_s, the restriction $e_{C'} : \mathcal{O}_{\mathcal{C}'}^{r+1} \to \mathcal{L}_{C'}$ is non-zero.*

(3) *For any $s \in S$, $\deg \mathcal{L}_{C_s} = d$ and the image $e_{C_s}(H) \in H^0(\mathcal{C}_s, \mathcal{L}_{\mathcal{C}_s})$ of a generic section $H \in H^0(\mathcal{C}_s, \mathcal{O}_{\mathcal{C}_s}^{r+1})$ determines the structure of a $(r+1)(k-1)+1$ stable morphism on $\mu_2 : \mathcal{C}_s \to \mathbb{P}^1$.*

There is a projective coarse moduli space $\mathbb{P}_d^r(k)$ for these objects.

It is exactly these objects of which we take the quotient by $\mathrm{Aut}(\mathbb{P}^1)$. After taking the quotient, there is no longer a parametrized component to refer to in the above definitions. However, when d is odd, there is a unique component of the domain curve that will play this role.

Proposition 4.4. *Let C be a connected genus-0 curve such that each edge is labeled with a number d. If $\sum d_i$ is odd, then there is a unique irreducible component $\bar{C}$ such that if C is a comb with handle $\bar{C}$, no tooth has sum of degrees $> d/2$*

Proof. Let $\{C\}_{d/2}$ be the set of all connected subcurves of degree $\geq d/2$. Intersect all such subcurves. There is a unique component in the intersection that will be $\bar{C}$. $\square$

Definition 4.5. A $(d,k)^*$ acceptable morphism is given by the following data:

$$(\pi : \mathcal{C} \to S, \mu : \mathcal{C} \to \mathbb{P}^r, \mathcal{L}, e),$$

where:

(1) $\mathcal{L}$ is a line bundle on $\mathcal{C}$ which, together with the morphism

$$e : \mathcal{O}_{\mathcal{C}}^{r+1} \to \mathcal{L}$$

determines the rational map $\mu : \mathcal{C} \to \mathbb{P}^r$.

(2) For any $s \in S$ and any irreducible component C' of $\mathcal{C}_s$, the restriction $e_{C'} : \mathcal{O}_{\mathcal{C}'}^{r+1} \to \mathcal{L}_{C'}$ is non-zero.

(3) For any $s \in S$, $\deg \mathcal{L}_{C_s} = d$ and the image $e_{C_s}(H) \in H^0(\mathcal{C}_s, \mathcal{L}_{\mathcal{C}_s})$ of a generic section $H \in H^0(\mathcal{C}_s, \mathcal{O}_{\mathcal{C}_s}^{r+1})$ determines the structure of a $(r+1)(k-1)+1$-stable rigid morphism, where $\bar{C}_s$ plays the role of the parametrized component in the definition of an n-stable degree 1 morphism above.

COROLLARY 4.6. *There is a projective coarse moduli space* $\overline{M}_{0,0}(\mathbb{P}^r, d, k)$ *for families of* $(d,k)^*$ *acceptable morphisms.*

Proof. We show that

$$\overline{M}_{0,0}(\mathbb{P}^r, d, k) := (\mathbb{P}^r_d(k))^s / \operatorname{Aut}(\mathbb{P}^1)$$

satisfies the properties of a coarse moduli space. This quotient is constructed identically to that from Theorem 0.1. We pull back the stable locus from $\mathbb{P}^r_d$ by 2.8. Again, the stable locus in $\mathbb{P}^r_d(k)$ will be those (d,k)-acceptable maps such that no tooth has degree $> d/2$. The universal properties of this space are inherited from the universal properties of $\mathbb{P}^r_d(k)$ as well as the universal properties of a categorical quotient.

First we need to show that there is a natural transformation of functors

$$\phi : \overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d, k) \to \operatorname{Hom}_{Sch}(*, \overline{M}_{0,0}(\mathbb{P}^r, d, k)),$$

where $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d, k)$ is the obvious moduli functor $\{schemes\} \to \{sets\}$. Given a family of $(d,k)^*$-acceptable morphisms

$$(\pi : \mathcal{C} \to S, \mu : \mathcal{C} \to \mathbb{P}^r, \mathcal{L}, e)$$

we can get a (d,k)-acceptable morphism

$$(\pi : \mathcal{C} \to S, \mu = (\mu_1, \mu_2) : C \to \mathbb{P}^r \times \mathbb{P}^1, \mathcal{L}, e)$$

by taking $\mu_2 : \mathcal{C} \to \mathbb{P}^1$ to be identity on $\bar{C}_s$ and constant on the other components. This will lie in the stable locus by construction, and thus gives a map $S \to (\mathbb{P}^r_d(k))^s$. Composing with the quotient gives an element of $\operatorname{Hom}_{Sch}(S, \overline{M}_{0,0}(\mathbb{P}^r, d, k))$.

We need to show that, given a scheme Z and a natural transformation of functors $\psi : \overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d, k) \to \operatorname{Hom}_{Sch}(*, Z)$, there exists a unique morphism of schemes

$$\gamma : \overline{M}_{0,0}(\mathbb{P}^r, d, k) \to Z$$

such that $\psi = \tilde{\gamma} \circ \phi$. By the above argument, we have a functor

$$\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d, k) \to (\mathcal{P}^r_d(k))^s.$$

Thus we get a functor

$$\bar{\psi} : (\mathcal{P}^r_d(k))^s \to \operatorname{Hom}_{Sch}(*, Z),$$

which by representability gives a map

$$\bar{\gamma} : (\mathbb{P}^r_d(k))^s \to Z.$$

This map is G equivariant by construction, and hence factors though the quotient
$$\gamma : \overline{M}_{0,0}(\mathbb{P}^r, d, k) \to Z. \qquad \square$$

We can sum up this corollary with the following figure:

$$\begin{tikzcd}
\mathbb{P}^1[N] = \mathbb{P}^1(N, N-1) \arrow[d] & G(\bar{t}) \arrow[l, "\mathbb{C}^*"'] \arrow[r, "S_d^r"] \arrow[d] & \overline{M}_{0,0}(\mathbb{P}^r \times \mathbb{P}^1, (d,1)) \arrow[d] \arrow[dr] & \\
\mathbb{P}^1[N, (r+1)(d-2)+1] \arrow[d] & \mathbb{P}_d^r(\bar{t}, d-1) \arrow[l, "\mathbb{C}^*"'] \arrow[r, "S_d^r"] \arrow[d] & \mathbb{P}_d^r(d-1) \arrow[d] \arrow[dr] & \overline{M}_{0,0}(\mathbb{P}^r, d) \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \overline{M}_{0,0}(\mathbb{P}^r, d, d-1) \arrow[d] \\
\mathbb{P}^1[N, 1] \arrow[d] & \mathbb{P}_d^r(\bar{t}, 1) \arrow[l, "\mathbb{C}^*"'] \arrow[r, "S_d^r"] \arrow[d] & \mathbb{P}_d^r(1) \arrow[d] \arrow[dr] & \vdots \arrow[d] \\
(\mathbb{P}^1)^N = \mathbb{P}^1[N, 0] & \mathbb{P}_d^r(\bar{t}) \arrow[l, "\mathbb{C}^*"'] \arrow[r, "S_d^r"] & \mathbb{P}_d^r \arrow[dr] & \overline{M}_{0,0}(\mathbb{P}^r, d, 1) \arrow[d] \\
& & & \mathbb{P}_d^r//G
\end{tikzcd}$$

Notice that $\mathbb{P}_d^r/G = \overline{M}_{0,0}(\mathbb{P}^r, d, 1) = \cdots = \overline{M}_{0,0}(\mathbb{P}^r, d, \frac{d+1}{2})$. This is because up to that point, the exceptional loci of the blow-ups will lie outside the stable locus. For example, the exceptional divisor of $\mathbb{P}_d^r(\bar{t}, 1) \to \mathbb{P}_d^r(\bar{t})$ corresponds to a curve with two components. One component is parametrized, and the other has all $d(r+1)$ points on it.

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