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A CLASS OF MÖBIUS INVARIANT FUNCTION SPACES

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Abstract. We introduce a class of Möbius invariant spaces of analytic functions in the unit disk, characterize these spaces in terms of Carleson type measures, and obtain a necessary and sufficient condition for a lacunary series to be in such a space. Special cases of this class include the Bloch space, the diagonal Besov spaces, BMOA, and the so-called Q_p spaces that have attracted much attention lately.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and let $\mathrm{Aut}(\mathbb{D})$ denote the group of all Möbius maps of the disk. For any $a \in \mathbb{D}$ the function

$$\varphi_a(z) = \frac{a-z}{1-\overline{a}z}, \qquad z \in \mathbb{D},$$

is a Möbius map that interchanges the points a and 0.

For $0 < p < \infty$, $-1 < \alpha < \infty$, and n a positive integer, we let $Q(n,p,\alpha)$ denote the space of analytic functions f in $\mathbb{D}$ with the property that

$$\|f\|_{n,p,\alpha}^p = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |(f\circ\varphi_a)^{(n)}(z)|^p (1-|z|^2)^\alpha \, dA(z) < \infty,$$

where dA is the area measure on $\mathbb{D}$, normalized so that the unit disk has area equal to 1.

Since every $\varphi \in \mathrm{Aut}(\mathbb{D})$ is of the form

$$\varphi(z) = \varphi_a(e^{it}z), \qquad z \in \mathbb{D},$$

where $a \in \mathbb{D}$ and t is real, we see that

$$\|f\|_{n,p,\alpha}^p = \sup_{\varphi\in\mathrm{Aut}(\mathbb{D})} \int_{\mathbb{D}} |(f\circ\varphi)^{(n)}(z)|^p (1-|z|^2)^\alpha \, dA(z).$$

Thus the space $Q(n,p,\alpha)$ is Möbius invariant, in the sense that an analytic function f in $\mathbb{D}$ belongs to $Q(n,p,\alpha)$ if and only if $f\circ\varphi$ belongs to $Q(n,p,\alpha)$

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for every (or some) Möbius map φ. Moreover,

$$\|f\circ\varphi\|_{n,p,\alpha}=\|f\|_{n,p,\alpha},\qquad f\in Q(n,p,\alpha),\varphi\in \mathrm{Aut}(\mathbb{D}).$$

It is clear that each space $Q(n,p,\alpha)$ contains all constant functions. We say that $Q(n,p,\alpha)$ is trivial if its only members are the constant functions. It is also clear that

$$\|f\|=|f(0)|+\|f\|_{n,p,\alpha}$$

defines a complete norm on $Q(n,p,\alpha)$ whenever $p\geq 1$. Thus $Q(n,p,\alpha)$ is a Banach space of analytic functions when $p\geq 1$. See [2] for general properties of Möbius invariant Banach spaces.

When $0<p<1$, the space $Q(n,p,\alpha)$ is not necessarily a Banach space, but is always a complete metric space. However, we will not hesitate to use the phrase "semi-norm" for $\|f\|_{n,p,\alpha}$ and use the word "norm" for $\|f\|$ even in the case $0<p<1$.

With definitions of weighted Bergman spaces, Besov spaces, and the Bloch space deferred to the next section, we can state our main results as Theorems A, B, C, and D below.

THEOREM A. *The space $Q(n,p,\alpha)$ is trivial when $np>\alpha+2$, it contains all polynomials when $np\leq\alpha+2$, and it coincides with the Besov space B_p when $np=\alpha+2$.*

It turns out that the most interesting case for us is when the parameters satisfy

$$\alpha+1\leq pn\leq\alpha+2.$$

When np falls below $\alpha+1$, $Q(n,p,\alpha)$ is just the Bloch space (see Proposition 7); and when np rises above $\alpha+2$, $Q(n,p,\alpha)$ becomes trivial.

THEOREM B. *If $\gamma=(\alpha+2)-np>0$, then an analytic function f in $\mathbb{D}$ belongs to $Q(n,p,\alpha)$ if and only if the measure*

$$|f^{(n)}(z)|^p(1-|z|^2)^\alpha\,dA(z)$$

is γ-Carleson.

Here we say that a positive Borel measure μ on $\mathbb{D}$ is a γ-Carleson measure if there exists a positive constant C such that $\mu(S_h)\leq Ch^\gamma$, where S_h is any Carleson square with side width h.

THEOREM C. *Suppose $\alpha+1\leq pn\leq\alpha+2$ and*

$$f(z)=\sum_{k=0}^\infty a_kz^{n_k}$$

is a lacunary series in $\mathbb{D}$. Then the following conditions are equivalent.

(a) *The function f is in $Q(n,p,\alpha)$.*

(b) *The function* f *satisfies*

$$\int_{\mathbb{D}} |f^{(n)}(z)|^p (1-|z|^2)^\alpha \, dA(z) < \infty.$$

(c) *The Taylor coefficients of* f *satisfy*

$$\sum_{k=0}^{\infty} \frac{|a_k|^p}{n_k^{\alpha+1-np}} < \infty.$$

Note that replacing f by its nth anti-derivative in (b) and (c) above gives a characterization of lacunary series in weighted Bergman spaces; see Theorem 8 in Section 5. We also prove an optimal pointwise estimate for lacunary series in weighted Bergman spaces.

Theorem D. *If* f *is a lacunary series satisfying*

$$\int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha \, dA(z) < \infty,$$

then

$$\lim_{|z|\to 1^-} (1-|z|^2)^{\alpha+1} |f(z)|^p = 0.$$

Note that if we drop the assumption that f be lacunary, then the best we can expect is

$$\lim_{|z|\to 1^-} (1-|z|^2)^{\alpha+2} |f(z)|^p = 0.$$

See Lemma 3.2 of [7] and the comments following it.

The papers [9] and [12] study a similar class of function spaces $F(p,q,s)$, where $p > 0$, $q > -2$, and $s \geq 0$. It is easy to see that the two classes have a nontrivial intersection, but neither contains the other. For example, the class $F(p,q,s)$ contains spaces that are not Möbius invariant, while the class $Q(n,p,\alpha)$ contains Besov spaces B_p, $0 < p \leq 1$, that are not in the class $F(p,q,s)$.

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2. Preliminaries

We begin with two elementary identities that will be needed several times later.

Lemma 1. *Suppose f is analytic in $\mathbb{D}$, $a \in \mathbb{D}$, and n is a positive integer. Then*

$$(f \circ \varphi_a)^{(n)}(z) = \sum_{k=1}^{n} c_k f^{(k)}(\varphi_a(z)) \frac{(1-|a|^2)^k}{(1-\overline{a}z)^{n+k}}, \tag{1}$$

and

$$f^{(n)}(\varphi_a(z)) \frac{(1-|a|^2)^n}{(1-\overline{a}z)^{2n}} = \sum_{k=1}^{n} \frac{d_k}{(1-\overline{a}z)^{n-k}} (f \circ \varphi_a)^{(k)}(z), \tag{2}$$

where c_k and d_k are polynomials of $\overline{a}$.

Proof. It is obvious that (1) and (2) both hold when $n = 1$.

Assume that (1) and (2) both hold for $n = m$. We proceed to show that they also hold for $n = m + 1$.

First, differentiating (1) with $n = m$ gives

$$\begin{aligned}
(f \circ \varphi_a)^{(m+1)}(z) &= -\sum_{k=1}^{m} c_k f^{(k+1)}(\varphi_a(z)) \frac{(1-|a|^2)^{k+1}}{(1-\overline{a}z)^{m+k+2}} \\
&\quad + \sum_{k=1}^{m} c_k (m+k)\overline{a} f^{(k)}(\varphi_a(z)) \frac{(1-|a|^2)^k}{(1-\overline{a}z)^{m+k+1}} \\
&= -\sum_{k=2}^{m+1} c_{k-1} f^{(k)}(\varphi_a(z)) \frac{(1-|a|^2)^k}{(1-\overline{a}z)^{m+1+k}} \\
&\quad + \sum_{k=1}^{m} c_k (m+k)\overline{a} f^{(k)}(\varphi_a(z)) \frac{(1-|a|^2)^k}{(1-\overline{a}z)^{m+1+k}} \\
&= \sum_{k=1}^{m+1} c'_k f^{(k)}(\varphi_a(z)) \frac{(1-|a|^2)^k}{(1-\overline{a}z)^{m+1+k}},
\end{aligned}$$

that is, (1) holds for $n = m + 1$.

Next, differentiating (2) with $n = m$ shows that

$$-f^{(m+1)}(\varphi_a(z)) \frac{(1-|a|^2)^{m+1}}{(1-\overline{a}z)^{2(m+1)}} + 2m\overline{a} f^{(m)}(\varphi_a(z)) \frac{(1-|a|^2)^m}{(1-\overline{a}z)^{2m+1}} \tag{3}$$

is equal to

$$\sum_{k=1}^{m} \left[\frac{(m-k)d_k\overline{a}}{(1-\overline{a}z)^{m-k+1}} (f \circ \varphi_a)^{(k)}(z) + \frac{d_k}{(1-\overline{a}z)^{m-k}} (f \circ \varphi_a)^{(k+1)}(z) \right].$$

Applying (2) with $n = m$ to the second term in (3), we obtain

$$f^{(m+1)}(\varphi_a(z))\frac{(1-|a|^2)^{(m+1)}}{(1-\overline{a}z)^{2(m+1)}} = 2m\overline{a}\sum_{k=1}^{m}\frac{d_k}{(1-\overline{a}z)^{m+1-k}}(f\circ\varphi_a)^{(k)}(z) - \sum_{k=1}^{m}\frac{(m-k)d_k\overline{a}}{(1-\overline{a}z)^{m+1-k}}(f\circ\varphi_a)^{(k)}(z) - \sum_{k=1}^{m}\frac{d_k}{(1-\overline{a}z)^{m-k}}(f\circ\varphi_a)^{(k+1)}(z).$$

The last sum above is the same as

$$\sum_{k=2}^{m+1}\frac{d_{k-1}}{(1-\overline{a}z)^{m+1-k}}(f\circ\varphi_a)^{(k)}(z).$$

Therefore,

$$f^{(m+1)}(\varphi_a(z))\frac{(1-|a|^2)^{(m+1)}}{(1-\overline{a}z)^{2(m+1)}} = \sum_{k=1}^{m+1}\frac{d_k'}{(1-\overline{a}z)^{m+1-k}}(f\circ\varphi_a)^{(k)}(z),$$

namely, (2) holds for $n = m+1$.

The proof of the lemma is complete by induction. □

Several classical function spaces appear in various places of the paper. We give their definitions here.

For $0 < p < \infty$ the Hardy space H^p consists of analytic functions f in $\mathbb{D}$ such that

$$\|f\|_{H^p}^p = \sup_{0<r<1}\frac{1}{2\pi}\int_0^{2\pi}|f(re^{it})|^p\,dt < \infty.$$

It is well known that every function $f \in H^p$ has radial limit, denoted by $f(e^{it})$, at almost every point e^{it} on the unit circle. Moreover,

$$\|f\|_{H^p} = \left[\frac{1}{2\pi}\int_0^{2\pi}|f(e^{it})|^p\,dt\right]^{1/p}$$

for every $f \in H^p$. If f is represented as a power series

$$f(z) = \sum_{k=0}^{\infty}a_k z^k,$$

then it is easy to see that

$$\|f\|_{H^2}^2 = \sum_{k=0}^{\infty}|a_k|^2$$

for every $f \in H^p$.

BMOA is the space of functions $f \in H^2$ with the property that

$$\|f\|_{BMO} = \sup_{a\in\mathbb{D}} \|f \circ \varphi_a - f(a)\|_{H^2} < \infty.$$

See [5] for basic properties of Hardy spaces and BMOA.

For $0 < p < \infty$ and $-1 < \alpha < \infty$ the weighted Bergman space A^p_α consists of analytic functions f in $\mathbb{D}$ with

$$\|f\|^p_{p,\alpha} = (\alpha+1) \int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha \, dA(z) < \infty.$$

If a_k are the Taylor coefficients of f at $z = 0$, then it is easy to see that

$$\|f\|^2_{2,\alpha} = \sum_{k=0}^\infty \frac{k!\,\Gamma(2+\alpha)}{\Gamma(k+2+\alpha)} |a_k|^2.$$

By Stirling's formula, the above sum is comparable to

$$\sum_{k=0}^\infty \frac{|a_k|^2}{(k+1)^{\alpha+1}}.$$

See [7] for the modern theory of Bergman spaces.

The following result about Bergman spaces will be important for us later.

LEMMA 2. *Suppose n is a positive integer, $\alpha > -1$, and $p > 0$. Then the integral*

$$\int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha \, dA(z)$$

is comparable to

$$\sum_{k=0}^{n-1} |f^{(k)}(0)|^p + \int_{\mathbb{D}} |f^{(n)}(z)|^p (1-|z|^2)^{np+\alpha} \, dA(z),$$

where f is any analytic function in $\mathbb{D}$.

Proof. See Theorem 2.17 of [14]. □

An analytic function f in $\mathbb{D}$ belongs to the Bloch space $\mathcal{B}$ if

$$\sup_{a\in\mathbb{D}} \|f \circ \varphi_a - f(a)\|_{p,\alpha} < \infty.$$

It is well known that this definition of $\mathcal{B}$ is independent of the choice of p and α. In fact, it can be shown that $f \in \mathcal{B}$ if and only if

$$\|f\|_{\mathcal{B}} = \sup_{z\in\mathbb{D}} (1-|z|^2)|f'(z)| < \infty.$$

See [4].

We are going to need the following characterizations of the Bloch space in terms of higher order derivatives.

Lemma 3. *Suppose n is any positive integer. Then the following are equivalent for an analytic function f in $\mathbb{D}$.*

(a) *f belongs to the Bloch space.*

(b) *f satisfies the condition*

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^n|f^{(n)}(z)|<\infty.$$

(c) *f satisfies the condition*

$$\sup_{a\in\mathbb{D}}|(f\circ\varphi_a)^{(n)}(0)|<\infty.$$

Proof. See Theorem 5.15 of [13] for the equivalence of (a) and (b). It is clear that the set of functions satisfying the condition in (c) is a Möbius invariant Banach space. It follows from the maximality of the Bloch space among Möbius invariant Banach spaces (see [10]) that (c) implies (a). According to Lemma 1,

$$(f\circ\varphi_a)^{(n)}(0)=\sum_{k=1}^{n}c_k(\bar{a})(1-|a|^2)^k f^{(k)}(a),$$

where each $c_k(\bar{a})$ is a polynomial in $\bar{a}$, so the equivalence of (a) and (b) shows that (a) implies (c). □

Suppose $0<p<\infty$ and n is a positive integer satisfying $np>1$. The (diagonal) Besov space B_p consists of analytic functions f in $\mathbb{D}$ such that

$$\int_{\mathbb{D}}|f^{(n)}(z)|^p(1-|z|^2)^{np-2}\,dA(z)<\infty.$$

It is well known that the definition is independent of the choice of n; see [14]. In particular, for $p>1$, we have $f\in B_p$ if and only if

$$\int_{\mathbb{D}}|f'(z)|^p(1-|z|^2)^p\,d\lambda(z)<\infty,$$

where

$$d\lambda(z)=\frac{dA(z)}{(1-|z|^2)^2}$$

is the Möbius invariant measure on $\mathbb{D}$.

The following estimate will play a crucial role in our analysis.

Lemma 4. *Suppose $\alpha>-1$ and t is real. Then the integral*

$$I(a)=\int_{\mathbb{D}}\frac{(1-|z|^2)^\alpha\,dA(z)}{|1-\overline{a}z|^{2+\alpha+t}}$$

has the following properties:

(a) *If $t<0$, $I(a)$ is comparable to* 1.

(b) *If $t=0$, $I(a)$ is comparable to* $\log(2/(1-|a|^2))$.

(c) *If* $t>0$, $I(a)$ *is comparable to* $1/(1-|a|^2)^t$.

Proof. See Lemma 4.2.2 of [13]. □

We can now determine exactly when the space $Q(n,p,\alpha)$ is nontrivial.

THEOREM 5. *The following conditions are equivalent.*

(a) *The space* $Q(n,p,\alpha)$ *is nontrivial.*
(b) *The space* $Q(n,p,\alpha)$ *contains all polynomials.*
(c) *The parameters satisfy the condition* $pn\le\alpha+2$.

Proof. It is trivial that (b) implies (a).

For any analytic function f in $\mathbb{D}$ we consider the integral

$$I_a=\int_{\mathbb{D}}|(f\circ\varphi_a)^{(n)}(z)|^p(1-|z|^2)^\alpha\,dA(z).$$

By (1) and a change of variables,

$$I_a=(1-|a|^2)^{\alpha+2-np}\int_{\mathbb{D}}\left|\sum_{k=1}^n c_kf^{(k)}(z)(1-\overline{a}z)^{n+k}\right|^p\frac{(1-|z|^2)^\alpha\,dA(z)}{|1-\overline{a}z|^{4+2\alpha}}.$$

If f is a polynomial, then each $f^{(k)}$ is bounded. After we factor out $(1-\overline{a}z)^{n+1}$ from every term in the above sum, we find a constant $C>0$, independent of a, such that

$$I_a\le C(1-|a|^2)^{\alpha+2-np}\int_{\mathbb{D}}\frac{(1-|z|^2)^\alpha\,dA(z)}{|1-\overline{a}z|^{4+2\alpha-(n+1)p}}.$$

It follows from Lemma 4 that I_a is bounded for $a\in\mathbb{D}$ when $np\le\alpha+2$. This proves that (c) implies (b).

Working with the integral I_a from the preceding paragraph, we have

$$\int_{\mathbb{D}}\left|\sum_{k=1}^n c_kf^{(k)}(z)(1-\overline{a}z)^{n+k}\right|^p\frac{(1-|z|^2)^\alpha\,dA(z)}{|1-\overline{a}z|^{4+2\alpha}}=(1-|a|^2)^{np-(\alpha+2)}I_a.$$

Since $|1-\overline{a}z|\le 2$, we have

$$\int_{\mathbb{D}}\left|\sum_{k=1}^n c_kf^{(k)}(z)(1-\overline{a}z)^{n+k}\right|^p(1-|z|^2)^\alpha\,dA(z)\le C(1-|a|^2)^{np-(\alpha+2)}I_a,$$

where $C=2^{4+2\alpha}$. Now if $np>\alpha+2$ and $f\in Q(n,p,\alpha)$, we can let a approach the unit circle and use Fatou's lemma to conclude that

$$\int_{\mathbb{D}}\left|\sum_{k=1}^n c_kf^{(k)}(z)(1-\overline{a}z)^{n+k}\right|^p(1-|z|^2)^\alpha\,dA(z)=0$$

whenever $|a| = 1$. But the integrand above is a polynomial of $\overline{a}$, so we must have

$$\sum_{k=1}^{n} c_k f^{(k)}(z)(1 - \overline{a}z)^{n+k} = 0$$

for all $a \in \mathbb{D}$, and hence $I_a = 0$ for all $a \in \mathbb{D}$. This can happen only when f is constant. Therefore, we see that (a) implies (c), and the proof of the theorem is complete. □

The Bloch space $\mathcal{B}$ is maximal among all Möbius invariant Banach spaces (see [10]), so $Q(n, p, \alpha) \subset \mathcal{B}$ when $p \geq 1$. We show that this is also true for $0 < p < 1$, although in this case $Q(n, p, \alpha)$ is not necessarily a Banach space.

Lemma 6. *The space $Q(n, p, \alpha)$ is always contained in the Bloch space.*

Proof. It follows from the subharmonicity of $|f|^p$ that $|f(0)| \leq \|f\|_{p,\alpha}$, where f is analytic in $\mathbb{D}$ and $\| \; \|_{p,\alpha}$ is the norm in the weighted Bergman space A^p_α. Replacing f by $(f \circ \varphi_a)^{(n)}$, we obtain

$$|(f \circ \varphi_a)^{(n)}(0)| \leq (\alpha + 1)\|f\|_{n,p,\alpha}, \qquad f \in Q(n, p, \alpha).$$

By condition (c) in Lemma 3, every function in $Q(n, p, \alpha)$ belongs to the Bloch space. □

As a consequence of Lemmas 2, 3, and 6, we see that

$$Q(n, p, \alpha) = Q(n + 1, p, \alpha + p) \tag{4}$$

whenever $\alpha > -1$, $p > 0$, and $n \geq 1$. This shows that the class $Q(n, p, \alpha)$ depends on only two parameters. In fact, if for $0 < p < \infty$ and β real we define

$$Q'(p, \beta) = Q(n, p, (n - 1)p + \beta),$$

where n is large enough so that $\alpha = (n - 1)p + \beta > -1$, then (4) shows that the definition of $Q'(p, \beta)$ is independent of the choice of n and the classes $Q(n, p, \alpha)$ and $Q'(p, \beta)$ are the same.

Alternatively, the class $Q(n, p, \alpha)$ depends on the parameters p and $\gamma = \alpha + 2 - np$. Several results of the paper can be stated more simply in terms of these two parameters.

3. Several special cases

We now identify several special cases of the spaces $Q(n, p, \alpha)$.

When $n = 1$ and $p = 2$, the integral

$$\int_{\mathbb{D}} |(f \circ \varphi_a)'(z)|^p (1 - |z|^2)^\alpha \, dA(z)$$

can be rewritten as

$$\int_{\mathbb{D}} |f'(z)|^2 (1 - |\varphi_a(z)|^2)^\alpha \, dA(z)$$

via a change of variables. Therefore, the resulting spaces $Q(n,p,\alpha)$ become the so-called Q_α spaces. More generally, if $2n < \alpha+3$, then $Q(n,2,\alpha) = Q_\beta$, where $\beta = \alpha - 2(n-1)$. This follows easily from Lemma 2. The book [11] is a good source of information for the spaces Q_α.

Although the Q_α spaces cover both BMOA and the Bloch space, we single out these two important cases to show their relative location in the scale $Q(n,p,\alpha)$.

Proposition 7. *If $np < \alpha+1$, then $Q(n,p,\alpha) = \mathcal{B}$.*

Proof. Recall from Lemma 6 that $Q(n,p,\alpha) \subset \mathcal{B}$. To prove the other direction, we fix some $f \in \mathcal{B}$. If $np < \alpha+1$, we can write $\alpha = np+\beta$, where $\beta > -1$. By Lemmas 2 and 3, the integral

$$\int_{\mathbb{D}} |(f\circ\varphi_a)^{(n)}(z)|^p (1-|z|^2)^\alpha \, dA(z)$$

is bounded for $a \in \mathbb{D}$ if and only if the integral

$$\int_{\mathbb{D}} |f\circ\varphi_a(z) - f(a)|^p (1-|z|^2)^\beta \, dA(z)$$

is bounded for $a \in \mathbb{D}$. Since the latter condition is satisfied by every Bloch function, the proof is complete. □

Theorem 8. *If $np = \alpha+2$, we have $Q(n,p,\alpha) = B_p$.*

Proof. Setting $a = 0$ in the integral

$$I_a = \int_{\mathbb{D}} |(f\circ\varphi_a)^{(n)}(z)|^p (1-|z|^2)^\alpha \, dA(z)$$

shows that $Q(n,p,\alpha) \subset B_p$ for $np = \alpha+2$.

We proceed to show that $B_p \subset Q(n,p,\alpha)$ when $np = \alpha+2$.

If $p > 1$, the Besov space B_p is Möbius invariant with the following semi-norm:

$$\|f\|_{B_p} = \left[\int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2} \, dA(z)\right]^{1/p}.$$

If n is any positive integer and $\alpha = np-2 > -1$, then by Lemma 2 there exists a constant $C > 0$, depending on p and n, such that

$$\int_{\mathbb{D}} |f^{(n)}(z)|^p (1-|z|^2)^{np-2} \, dA(z) \le C\|f\|^p_{B_p}$$

for all $f \in B_p$. Replacing f by $f\circ\varphi_a$ and using the Möbius invariance of the semi-norm $\|\ \|_{B_p}$, we conclude that

$$\sup\{I_a : a \in \mathbb{D}\} < \infty$$

whenever $f \in B_p$. This shows that $B_p \subset Q(n,p,\alpha)$ when $p > 1$ and $np = \alpha+2$.

A similar argument works for $p = 1$. As a matter of fact, B_1 admits a Möbius invariant norm (not just a semi-norm) $\|f\|_m$; see [2]. If $n > 1$ is an integer, then

$$\|f\|_n = \sum_{k=0}^{n-1} |f^{(k)}(0)| + \int_{\mathbb{D}} |f^{(n)}(z)|(1-|z|^2)^{n-2}\, dA(z)$$

defines a norm on B_1 that is equivalent to $\|f\|_m$. Therefore, we can find a constant $C > 0$, independent of f and a, such that

$$\|f \circ \varphi_a\|_n \le C\|f \circ \varphi_a\|_m = C\|f\|_m$$

for all $f \in B_1$ and $a \in \mathbb{D}$. This shows that $f \in B_1$ implies the integral

$$\int_{\mathbb{D}} |(f \circ \varphi_a)^{(n)}|(1-|z|^2)^{n-2}\, dA(z)$$

is bounded for $a \in \mathbb{D}$, or equivalently, $B_1 \subset Q(n, 1, np-2)$.

We prove the case $0 < p < 1$ using a version of atomic decomposition for the space B_p. By Theorem 6.6 of [14], if $0 < p < 1$ and $f \in B_p$, there exists a sequence $\{a_k\}$ in $\mathbb{D}$ such that

$$f(z) = \sum_{k=1}^{\infty} c_k \frac{1-|a_k|^2}{1-\overline{a}_k z},$$

where

$$\sum_{k=1}^{\infty} |c_k|^p < \infty.$$

Let

$$f_k(z) = \frac{1-|a_k|^2}{1-\overline{a}_k z}, \qquad 1 \le k < \infty.$$

Then by Hölder's inequality, the integral

$$\int_{\mathbb{D}} |(f \circ \varphi_a)^{(n)}(z)|^p (1-|z|^2)^{\alpha}(z)$$

is less than or equal to

$$\sum_{k=1}^{\infty} |c_k|^p \int_{\mathbb{D}} |(f_k \circ \varphi_a)^{(n)}(z)|^p (1-|z|^2)^{\alpha}\, dA(z).$$

Since

$$f_k(z) = 1 - \overline{a}_k \varphi_{a_k}(z),$$

we have

$$f_k(\varphi_a(z)) = 1 - \overline{a}_k \varphi_{a_k} \circ \varphi_a(z) = 1 - \overline{a}_k e^{it_k} \varphi_{\lambda_k}(z),$$

where t_k is a real number and $\lambda_k = \varphi_a(a_k)$. It follows that

$$(f_k \circ \varphi_a)^{(n)}(z) = \frac{A_k(1-|\lambda_k|^2)}{(1-\overline{\lambda}_k z)^{n+1}},$$

where $A_k = n!\overline{a}_k e^{it_k}\overline{\lambda}_k^{n-1}$. Therefore, the integral

$$\int_{\mathbb{D}} |(f_k \circ \varphi_a)^{(n)}(z)|^p (1-|z|^2)^\alpha \, dA(z)$$

does not exceed $n!$ times

$$(1-|\lambda_k|^2)^p \int_{\mathbb{D}} \frac{(1-|z|^2)^\alpha \, dA(z)}{|1-\overline{\lambda}_k z|^{(n+1)p}} = (1-|\lambda_k|^2)^p \int_{\mathbb{D}} \frac{(1-|z|^2)^\alpha \, dA(z)}{|1-\overline{\lambda}_k z|^{\alpha+2+p}}.$$

By Lemma 4, there exists a constant $C > 0$, independent of k and a, such that

$$\int_{\mathbb{D}} |(f_k \circ \varphi_a)^{(n)}(z)|^p (1-|z|^2)^\alpha \, dA(z) \le C$$

for all $k \ge 1$ and all $a \in \mathbb{D}$. It follows that $f \in Q(n,p,\alpha)$, and the proof of the theorem is complete. □

Proposition 9. *If $p = 2$ and $\alpha = 2n-1$, then $Q(n,p,\alpha) = \mathrm{BMOA}$.*

Proof. If $f \in \mathcal{B}$, then Lemmas 2 and 3 show that the integral

$$\int_{\mathbb{D}} |(f \circ \varphi_a)^{(n)}(z)|^2 (1-|z|^2)^{2n-1} \, dA(z)$$

is bounded in a if and only if the integral

$$\int_{\mathbb{D}} |(f \circ \varphi_a)'(z)|^2 (1-|z|^2) \, dA(z)$$

is bounded in a. The latter integral, by a classical identity of Littlewood and Paley (see page 236 of [5] or Theorem 8.1.9 of [13]), is comparable to

$$\|f \circ \varphi_a - f(a)\|_{H^2}^2.$$

This proves the desired result. □

Finally in this section, we mention that in studying the spaces $Q(n,p,\alpha)$, we may as well assume that $-1 < \alpha \le p-1$. Otherwise, we can write $\alpha = p + \alpha'$ with $\alpha' > -1$. Then the integral

$$\int_{\mathbb{D}} |(f \circ \varphi_a)^{(n)}(z)|^p (1-|z|^2)^\alpha \, dA(z),$$

is comparable to

$$|(f \circ \varphi_a)^{(n-1)}(0)|^p + \int_{\mathbb{D}} |(f \circ \varphi_a)^{(n-1)}(z)|^p (1-|z|^2)^{\alpha'} \, dA(z)$$

when $n > 1$, and is comparable to

$$\int_{\mathbb{D}} |f \circ \varphi_a(z) - f(a)|^p (1-|z|^2)^{\alpha'} \, dA(z)$$

when $n = 1$. Therefore, either $Q(n,p,\alpha) = Q(n-1,p,\alpha')$ or $Q(n,p,\alpha) = \mathcal{B}$. Continuing this process, the space $Q(n,p,\alpha)$ is either equal to some $Q(m,p,\beta)$ with $\beta \le p-1$ or equal to the Bloch space.

4. Characterization in terms of Carleson-type measures

In this section we are going to characterize the spaces $Q(n,p,\alpha)$ in terms of Carleson type measures. We begin with the following elementary inequality.

Lemma 10. *For any $p>0$ and complex numbers z_k we have*

$$|z_1+\cdots+z_n|^p \le C(|z_1|^p+\cdots+|z_n|^p), \tag{5}$$

where $C=1$ if $0<p\le 1$ and $C=n^{p-1}$ when $p>1$.

Proof. This is a direct consequence of Hölder's inequality. □

To simplify the presentation for the next two lemmas, we introduce the expressions

$$M(f,n,a)=\int_{\mathbb{D}} |(f\circ\varphi_a)^{(n)}(z)|^p(1-|z|^2)^\alpha\, dA(z)$$

and

$$N(f,n,a)=\int_{\mathbb{D}} \left| f^{(n)}(\varphi_a(z))\left(\frac{1-|a|^2}{(1-\overline{a}z)^2}\right)^n\right|^p (1-|z|^2)^\alpha\, dA(z).$$

By a change of variables, we can write

$$N(f,n,a)=\int_{\mathbb{D}} |f^{(n)}(z)|^p\left(\frac{1-|a|^2}{|1-\overline{a}z|^2}\right)^{\alpha+2-np}(1-|z|^2)^\alpha\, dA(z).$$

We will also need the following notation.

$$P(f,n)=\sum_{k=1}^n \sup_{a\in\mathbb{D}}(1-|a|^2)^{kp}|f^{(k)}(a)|^p,$$

and

$$Q(f,n)=\sum_{k=1}^n \sup\{|(f\circ\varphi_a)^{(k)}(0)|^p : a\in\mathbb{D}\}.$$

According to Lemma 3, $P(f,n)<\infty$ if and only if $f\in\mathcal{B}$, and $Q(f,n)<\infty$ if and only if $f\in\mathcal{B}$.

Lemma 11. *If $np<\alpha+2$, then there exists a constant $C>0$, independent of f and a, such that*

$$M(f,n,a)\le C\,[N(f,n,a)+P(f,n)]$$

for all analytic f and $a\in\mathbb{D}$.

Proof. We prove the inequality by induction on n.

It is clear that $M(f,n,a)=N(f,n,a)$ when $n=1$. So we assume that the inequality holds for n and consider the expression $M(f,n+1,a)$ under the condition that $(n+1)p<\alpha+2$.

Fix $a \in \mathbb{D}$ and observe that

$$(f \circ \varphi_a)^{(n+1)}(z) = -(g \circ \varphi_a)^{(n)}(z),$$

where

$$g(z) = \frac{(1-\overline{a}z)^2}{1-|a|^2} f'(z).$$

By the product rule, we have

$$g^{(m)}(z) = \frac{(1-\overline{a}z)^2}{1-|a|^2} f^{(m+1)}(z) - 2m\overline{a}\frac{1-\overline{a}z}{1-|a|^2} f^{(m)}(z) + \frac{m(m-1)\overline{a}^2}{1-|a|^2} f^{(m-1)}(z) \tag{6}$$

for all $m \geq 1$. In particular,

$$(1-|a|^2)^m g^{(m)}(a) = (1-|a|^2)^{m+1} f^{(m+1)}(a) - 2m\overline{a}(1-|a|^2)^m f^{(m)}(a) + m(m-1)\overline{a}^2(1-|a|^2)^{m-1} f^{(m-1)}(a)$$

for $m \geq 1$ and

$$g^{(n)}(\varphi_a(z)) = \frac{1-|a|^2}{(1-\overline{a}z)^2} f^{(n+1)}(\varphi_a(z)) - \frac{2n\overline{a}}{1-\overline{a}z} f^{(n)}(\varphi_a(z)) + \frac{n(n-1)\overline{a}^2}{1-|a|^2} f^{(n-1)}(\varphi_a(z)). \tag{7}$$

It follows from this and the induction hypothesis (note that the condition $(n+1)p < \alpha + 2$ implies $np < \alpha + 2$) that there exist positive constants C_1 and C_2, both independent of f and a, such that

$$\begin{aligned} M(f, n+1, a) = M(g, n, a) &\leq C_1 \left[N(g, n, a) + P(g, n)\right] \\ &\leq C_2 \left[N(g, n, a) + P(f, n+1)\right]. \end{aligned}$$

By equation (7) and inequality (5), we can find another constant $C_3 > 0$, independent of f and a, such that

$$N(g, n, a) \leq C_3(I_1 + I_2 + I_3),$$

where

$$I_1 = N(f, n+1, a),$$

and

$$I_2 = \int_{\mathbb{D}} \left| f^{(n)}(\varphi_a(z)) \frac{(1-|a|^2)^n}{(1-\overline{a}z)^{2n+1}} \right|^p (1-|z|^2)^\alpha \, dA(z),$$

and

$$I_3 = \int_{\mathbb{D}} \left| f^{(n-1)}(\varphi_a(z)) \frac{(1-|a|^2)^{n-1}}{(1-\overline{a}z)^{2n}} \right|^p (1-|z|^2)^\alpha \, dA(z).$$

By Lemma 2 and inequality (5), there exists a constant $C_4 > 0$ such that

$$I_2 \le C_4(1-|a|^2)^{np}|f^{(n)}(a)|^p \tag{8}$$

$$+ C_4 \int_{\mathbb{D}} \left| f^{(n+1)}(\varphi_a(z)) \frac{(1-|a|^2)^{n+1}}{(1-\overline{a}z)^{2n+3}} \right|^p (1-|z|^2)^{p+\alpha}\, dA(z) \tag{9}$$

$$+ C_4 \int_{\mathbb{D}} \left| f^{(n)}(\varphi_a(z)) \frac{(1-|a|^2)^{n}}{(1-\overline{a}z)^{2n+2}} \right|^p (1-|z|^2)^{p+\alpha}\, dA(z). \tag{10}$$

Since

$$(1-|z|^2)^p \le 2^p|1-\overline{a}z|^p,$$

the integral in (9) is less than or equal to $2^pN(f, n+1, a)$. The integral in (10) can be estimated using Lemma 2 again. After this process is repeated n times, we find a constant $C_5 > 0$, independent of f and a, such that

$$\begin{aligned} I_2 &\le C_5 \left[P(f,n) + N(f,n+1,a)\right] \\ &\quad + C_5 \int_{\mathbb{D}} \left| f^{(n)}(\varphi_a(z)) \frac{(1-|a|^2)^{n}}{(1-\overline{a}z)^{2n+1+n}} \right|^p (1-|z|^2)^{np+\alpha}\, dA(z). \end{aligned}$$

First using

$$(1-|\varphi_a(z)|^2)^n|f^{(n)}(\varphi_a(z))| \le P(f,n),$$

then applying Lemma 4 with the condition $(n+1)p < \alpha+2$, we find a constant $C_6 > 0$, independent of f and a, such that

$$I_2 \le C_6 \left[N(f,n+1,a) + P(f,n)\right].$$

After we estimate the integral I_3 in a similar way, we obtain a constant $C > 0$, independent of f and a, such that

$$M(f,n+1,a) \le C \left[N(f,n+1,a) + P(f,n+1)\right].$$

This completes the proof of the lemma. □

We now show that the inequality in Lemma 11 can essentially be reversed.

Lemma 12. *If $np < \alpha + 2$, there exists a constant $C > 0$, independent of f and a, such that*

$$N(f,n,a) \le C \left[M(f,n,a) + Q(f,n)\right]$$

for all analytic f and $a \in \mathbb{D}$.

Proof. By equation (2) and the elementary inequality (5), we can find a constant $C_1 > 0$, independent of f and a, such that

$$N(f,n,a) \le C_1 \sum_{k=1}^{n} I_k(f,n,a),$$

where

$$I_k(f,n,a)=\int_{\mathbb{D}}\left|\frac{1}{(1-\overline{a}z)^{n-k}}(f\circ\varphi_a)^{(k)}(z)\right|^p(1-|z|^2)^\alpha\,dA(z).$$

We are going to use backward induction on k to show that

$$I_k(f,n,a)\le M_k[M(f,n,a)+Q(f,n)],\qquad 1\le k\le n, \tag{11}$$

where each M_k is a positive constant independent of f and a.

It is clear that $I_n(f,n,a)=M(f,n,a)$, so the inequality in (11) holds for $k=n$.

Next we assume that the inequality in (11) holds for $I_{k+1}(f,n,a)$ and proceed to show that the same inequality also holds for $I_k(f,n,a)$. Since

$$\frac{d}{dz}\left[\frac{1}{(1-\overline{a}z)^{n-k}}(f\circ\varphi_a)^{(k)}(z)\right]$$

equals

$$\frac{(n-k)\overline{a}}{(1-\overline{a}z)^{n-k+1}}(f\circ\varphi_a)^{(k)}(z)+\frac{1}{(1-\overline{a}z)^{n-k}}(f\circ\varphi_a)^{(k+1)}(z),$$

we can use Lemma 2 and (5) to find a constant $C_2>0$, independent of f and a, such that $I_k(f,n,a)$ is less than or equal to $C_2|(f\circ\varphi_a)^{(k)}(0)|^p$ plus

$$C_2\int_{\mathbb{D}}\left|\frac{1}{(1-\overline{a}z)^{n-k+1}}(f\circ\varphi_a)^{(k)}(z)\right|^p(1-|z|^2)^{p+\alpha}\,dA(z) \tag{12}$$

plus

$$C_2\int_{\mathbb{D}}\left|\frac{1}{(1-\overline{a}z)^{n-k}}(f\circ\varphi_a)^{(k+1)}(z)\right|^p(1-|z|^2)^{p+\alpha}\,dA(z). \tag{13}$$

The integral in (13) can be estimated by the elementary inequality

$$(1-|z|^2)^p\le 2^p|1-\overline{a}z|^p$$

followed by the induction hypothesis, while the integral in (12) can be estimated by Lemma 2 again. This process can be repeated. After a repetition of k steps, we obtain a constant $C_3>0$, independent of f and a, such that $I_k(f,n,a)$ is less than or equal to

$$C_3\,[M(f,n,a)+Q(f,n)]$$

plus

$$C_3\int_{\mathbb{D}}\left|\frac{1}{(1-\overline{a}z)^{n}}(f\circ\varphi_a)^{(k)}(z)\right|^p(1-|z|^2)^{kp+\alpha}\,dA(z). \tag{14}$$

Since the Bloch space is Möbius invariant, we can find a constant $C_4>0$, independent of f and a, such that

$$\sup_{z\in\mathbb{D}}|(f\circ\varphi_a)^{(k)}(z)|(1-|z|^2)^k\le C_4Q(f,n).$$

We now estimate the integral in (14) first using this, and then using part (a) of Lemma 4 together with the assumption that $np < \alpha + 2$. The result is that

$$I_k(f, n, a) \le M_k\left[M(f, n, a) + Q(f, n)\right].$$

This shows that (11) holds for all $k = 1, 2, \dots, n$, and completes the proof of the lemma. □

Note that by using (2) and arguments similar to those used in the proof of Lemma 12, we can construct a different proof for Lemma 11.

We now state the main result of the section.

Theorem 13. *If $np \le \alpha + 2$, then an analytic function f in $\mathbb{D}$ belongs to $Q(n, p, \alpha)$ if and only if*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f^{(n)}(z)|^p \frac{(1-|a|^2)^{\alpha+2-np}}{|1-\overline{a}z|^{2(\alpha+2-np)}} (1-|z|^2)^\alpha \, dA(z) < \infty. \tag{15}$$

Proof. If $np = \alpha + 2$, the desired result is just Theorem 8.

We already know that $Q(n, p, \alpha)$ is contained in the Bloch space. Using the very first definition of $N(f, n, a)$ and the obvious estimate

$$|g(0)|^p \le (\alpha + 1) \int_{\mathbb{D}} |g(z)|^p (1-|z|^2)^\alpha \, dA(z),$$

we see that condition (15) also implies that $f \in \mathcal{B}$ (see also Lemma 3). The desired result for $np < \alpha + 2$ is then a consequence of Lemmas 11 and 12. □

For any arc I of the unit circle $\partial\mathbb{D}$, we let S_I denote the classical Carleson square in $\mathbb{D}$ generated by I. Suppose $\gamma > 0$ and μ is a positive Borel measure on $\mathbb{D}$. We say that μ is γ-Carleson if there exists a constant $C > 0$ such that

$$\mu(S_I) \le C|I|^\gamma$$

for all I, where $|I|$ denotes the length of I.

Theorem 14. *Suppose $\gamma = \alpha + 2 - np > 0$. Then an analytic function f in $\mathbb{D}$ belongs to $Q(n, p, \alpha)$ if and only if the measure*

$$d\mu(z) = |f^{(n)}(z)|^p (1-|z|^2)^\alpha \, dA(z)$$

is γ-Carleson.

Proof. This follows from Theorem 13 and Lemma 4.1.1 of [11]. □

Corollary 15. *Suppose $p > 0$, $\gamma > 0$, $\alpha > -1$, n is a positive integer, m is a nonnegative integer, and f is analytic in $\mathbb{D}$. Then the measure*

$$|f^{(n)}(z)|^p (1-|z|^2)^\alpha \, dA(z)$$

is γ-Carleson if and only if the measure

$$|f^{(m+n)}(z)|^p(1-|z|^2)^{mp+\alpha}\,dA(z)$$

is γ-Carleson.

Proof. This is a consequence of Theorem 14 and equation (4). □

Replacing f by its nth anti-derivative, we conclude that

$$|f(z)|^p(1-|z|^2)^{\alpha}\,dA(z)$$

is γ-Carleson if and only if

$$|f^{(m)}(z)|^p(1-|z|^2)^{mp+\alpha}\,dA(z)$$

is γ-Carleson.

5. Lacunary series in Bergman type spaces

In this section we characterize lacunary series in Bergman-type spaces. We are going to need two classical results concerning lacunary series in Hardy type spaces.

LEMMA 16. *Suppose $0<p<\infty$ and $1<\lambda<\infty$. There exists a constant $C>0$, depending only on p and λ, such that*

$$C^{-1}\|f\|_{H^2}\le\|f\|_{H^p}\le C\|f\|_{H^2}$$

for all lacunary series

$$f(z)=\sum_{k=0}^{\infty}a_kz^{n_k}$$

with $n_{k+1}/n_k\ge\lambda$ for all k.

Proof. See page 213 of [15]. □

A consequence of the above lemma is that if a lacunary series belongs to some Hardy space, then it belongs to all Hardy spaces. Actually, a lacunary series belongs to a Hardy space if and only if it belongs to BMOA; see [6].

LEMMA 17. *Suppose $0<p<\infty$ and $-1<\alpha<\infty$. There exists a constant $C>0$, depending only on p and α, such that*

$$\frac{1}{C}\sum_{n=0}^{\infty}\frac{t_n^p}{2^{n(\alpha+1)}}\le\int_0^1 f(x)^p(1-x)^{\alpha}\,dx\le C\sum_{n=0}^{\infty}\frac{t_n^p}{2^{n(\alpha+1)}}$$

for all power series

$$f(x)=\sum_{k=0}^{\infty}a_kx^k$$

with nonnegative coefficients, where

$$t_n = \sum_{k\in I_n} a_k$$

and

$$I_0 = \{0,1\}, \qquad I_n = \{k : 2^n \le k < 2^{n+1}\}, \quad 1 \le n < \infty.$$

Proof. See [8]. □

We now characterize lacunary series in the weighted Bergman spaces A^p_α.

THEOREM 18. *Suppose* $0 < p < \infty$, $-1 < \alpha < \infty$, *and* $1 < \lambda < \infty$. *There exists a constant* $C > 0$, *depending only on* p, α *and* λ, *such that*

$$\frac{1}{C}\sum_{k=0}^{\infty}\frac{|a_k|^p}{n_k^{\alpha+1}} \le \int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha \, dA(z) \le C\sum_{k=0}^{\infty}\frac{|a_k|^p}{n_k^{\alpha+1}}$$

for all lacunary series

$$f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}$$

with $n_{k+1}/n_k \ge \lambda$ *for all* k.

Proof. In polar coordinates the integral

$$I = \int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha \, dA(z)$$

can be written as

$$I = \frac{1}{\pi}\int_0^1 r(1-r^2)^\alpha \int_0^{2\pi} \left|\sum_{k=0}^{\infty} a_k r^{n_k} e^{in_k t}\right|^p \, dt.$$

By Lemma 16, the integral I is comparable to

$$2\int_0^1 r(1-r^2)^\alpha \left(\sum_{k=0}^{\infty} |a_k|^2 r^{2n_k}\right)^{p/2} \, dr,$$

which is the same as

$$\int_0^1 \left(\sum_{k=0}^{\infty} |a_k|^2 x^{n_k}\right)^{p/2} (1-x)^\alpha \, dx.$$

Combining this with Lemma 17, we conclude that the integral I is comparable to

$$\sum_{n=0}^{\infty} 2^{-n(\alpha+1)} \left(\sum_{n_k\in I_n} |a_k|^2\right)^{p/2}.$$

Let $N = [\log_\lambda 2] + 1$. Then for each n there are at most N of n_k in I_n. In fact, if

$$2^n \le n_k < n_{k+1} < \cdots < n_{k+m} < 2^{n+1},$$

then

$$\lambda^m \le \frac{n_{k+m}}{n_k} < 2$$

and so $m < \log_\lambda 2$. Therefore,

$$\begin{aligned}\left(\sum_{n_k\in I_n} |a_k|^2\right)^{p/2} &\le \left(N \max_{n_k\in I_n} |a_k|^2\right)^{p/2}\\ &= N^{p/2} \max_{n_k\in I_n} |a_k|^p\\ &\le N^{p/2} \sum_{n_k\in I_n} |a_k|^p.\end{aligned}$$

Similarly,

$$\begin{aligned}\sum_{n_k\in I_n} |a_k|^p &\le N \max_{n_k\in I_n} |a_k|^p\\ &= N\left(\max_{n_k\in I_n} |a_k|^2\right)^{p/2}\\ &\le N\left(\sum_{n_k\in I_n} |a_k|^2\right)^{p/2}.\end{aligned}$$

Combining the results of the last two paragraphs, we see that the integral I is comparable to

$$\sum_{n=0}^\infty 2^{-n(\alpha+1)} \sum_{n_k\in I_n} |a_k|^p.$$

Since n_k is comparable to 2^n for $n_k \in I_n$, we conclude that the integral I is comparable to

$$\sum_{n=0}^\infty \sum_{n_k\in I_n} \frac{|a_k|^p}{n_k^{\alpha+1}} = \sum_{k=0}^\infty \frac{|a_k|^p}{n_k^{\alpha+1}}.$$

This completes the proof of the theorem. □

Corollary 19. *Suppose $0 < p < \infty$, $-1 < \alpha < \infty$, and n is a positive integer. Then a lacunary series*

$$f(z) = \sum_{k=0}^\infty a_k z^{n_k}$$

satisfies

$$\int_{\mathbb{D}} |f^{(n)}(z)|^p (1 - |z|^2)^\alpha \, dA(z) < \infty$$

if and only if

$$\sum_{k=0}^{\infty} \frac{|a_k|^p}{n_k^{\alpha+1-pn}} < \infty.$$

Proof. If the Taylor series of $f(z)$ at $z = 0$ is lacunary, then so is some tail of the Taylor series of $f^{(n)}(z)$. The desired result then follows from Theorem 18. □

Note that lacunary series in B_p are characterized in [3] when $p > 1$. Our approach here is similar to that in [3]. The above corollary covers all Besov spaces B_p, $0 < p < \infty$: simply take $\alpha = np - 2$, where n is any positive integer greater than $1/p$.

Any function $f \in A^p_\alpha$ satisfies the pointwise estimate

$$|f(z)| \le \frac{\|f\|_{p,\alpha}}{(1-|z|^2)^{(\alpha+2)/p}}, \qquad z \in \mathbb{D},$$

and the exponent $(\alpha + 2)/p$ is best possible for general functions. See Lemma 3.2 of [7]. The following result shows that lacunary series in A^p_α grow more slowly near the boundary than a general function does.

Theorem 20. *If $f(z)$ is defined by a lacunary series in $\mathbb{D}$ and belongs to A^p_α, then there exists a constant $C > 0$, depending on f, such that*

$$|f(z)| \le \frac{C}{(1-|z|^2)^{(\alpha+1)/p}}, \qquad z \in \mathbb{D}.$$

Moreover, the exponent $(\alpha + 1)/p$ cannot be improved.

Proof. Suppose $f \in A^p_\alpha$ and

$$f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}$$

is a lacunary series with $n_{k+1}/n_k \ge \lambda > 1$ for all k. By Theorem 18,

$$a_k = o\left(n_k^{(\alpha+1)/p}\right), \qquad k \to \infty.$$

In particular, there exists a constant $C_1 > 0$ such that

$$|a_k| \le C_1 n_k^{(\alpha+1)/p}, \qquad k \ge 0,$$

so

$$|f(z)| \le C_1 \sum_{n=0}^{\infty} \sum_{n_k \in I_n} n_k^{(\alpha+1)/p} |z|^{n_k}.$$

Let $N = [\log_\lambda 2] + 1$ as in the proof of Theorem 18. Then

$$\sum_{n_k \in I_n} n_k^{(\alpha+1)/p} |z|^{n_k} \le N 2^{(n+1)(\alpha+1)/p} |z|^{2^n}.$$

It is clear that

$$2^{n-1}|z|^{2^n} \le \sum_{k\in I_{n-1}} |z|^k.$$

Since 2^{n-1}, 2^n, and 2^{n+1} are all comparable to k for $k \in I_n$ or for $k \in I_{n-1}$, we can find another constant $C_2 > 0$ such that

$$|f(z)| \le C_2 \sum_{k=0}^{\infty} (k+1)^{(\alpha+1)/p-1}|z|^k.$$

It is well known (see page 54 of [13] for example) that the series above is comparable to $(1-|z|^2)^{-(\alpha+1)/p}$. This proves the desired estimate for $f(z)$.

To show that the exponent $(\alpha+1)/p$ is best possible, we assume that there exists some $q > p$ such that for every lacunary series $f \in A^p_\alpha$ there is a positive constant $C_f > 0$ with

$$|f(z)| \le \frac{C_f}{(1-|z|^2)^{(\alpha+1)/q}}, \qquad z \in \mathbb{D}.$$

This would imply that every lacunary series $f \in A^p_\alpha$ also belongs to A^r_α, where $r < q$. Fix some $r \in (p,q)$ and choose σ such that

$$\frac{\alpha+1}{r} < \sigma < \frac{\alpha+1}{p}.$$

By Theorem 18, the lacunary series

$$f(z) = \sum_{k=0}^{\infty} 2^{\sigma k} z^{2^k}$$

belongs to A^p_α but does not belong to A^r_α. This contradiction completes the proof of the theorem. □

We mention that another class of functions in A^p_α enjoy the estimate in Theorem 20, namely, the so-called A^p_α-inner functions. See [7]. Although the exponent $(\alpha+1)/p$ in the preceding theorem cannot be decreased, we can use a standard approximation argument, or refine the argument in the proof above, to improve the result as follows. If f is a lacunary series in A^p_α, then

$$f(z) = o\left(\frac{1}{(1-|z|^2)^{(\alpha+1)/p}}\right)$$

as $|z| \to 1^-$. We omit the routine details.

6. Lacunary series in $Q(n,p,\alpha)$

It is well known that a lacunary series

$$f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}$$

belongs to the Bloch space if and only if its Taylor coefficients a_k are bounded; see [1].

In this section we characterize the lacunary series in $Q(n,p,\alpha)$. Our main result is the following.

THEOREM 21. *Suppose $\alpha+1 \le np \le \alpha+2$. Then the following conditions are equivalent for a lacunary series*

$$f(z)=\sum_{k=0}^{\infty} a_k z^{n_k}.$$

(a) $f \in Q(n,p,\alpha)$.
(b) *f satisfies the condition*

$$\int_{\mathbb{D}} |f^{(n)}(z)|^p (1-|z|^2)^\alpha \, dA(z) < \infty.$$

(c) *The Taylor coefficients of f satisfy the condition*

$$\sum_{k=0}^{\infty} \frac{|a_k|^p}{n_k^{\alpha+1-np}} < \infty.$$

Proof. Choosing $a=0$ in the definition of the semi-norm $\|f\|_{n,p,\alpha}$ shows that (a) implies (b). It follows from Corollary 19 that (b) implies (c).

To prove the remaining implication, we fix a lacunary series

$$f(z)=\sum_{k=0}^{\infty} a_k z^{n_k}$$

and consider the integral

$$N(f,n,a)=\int_{\mathbb{D}} |f^{(n)}(z)|^p \left(\frac{1-|a|^2}{|1-\overline{a}z|^2}\right)^{\alpha+2-np} (1-|z|^2)^\alpha \, dA(z).$$

By Theorem 13, it suffices to show that the condition in (c) implies that the integral $N(f,n,a)$ is bounded in a.

We write

$$N(f,n,a)=\int_{\mathbb{D}} |f^{(n)}(z)|^p (1-|z|^2)^{np-2} (1-|\varphi_a(z)|^2)^{\alpha+2-np} \, dA(z)$$

and

$$f^{(n)}(z)=\sum_{k=0}^{\infty} b_k z^{m_k}.$$

By dropping the first few terms if necessary, we may, without loss of generality, that $f^{(n)}(z)$ is still a lacunary series. It is clear that, as $k\to\infty$, $|b_k|$ is comparable to $|a_k| n_k^n$.

In polar coordinates, the integral $N(f,n,a)$ can be written as

$$\frac{1}{\pi}\int_0^1 r(1-r^2)^{np-2}\,dr\int_0^{2\pi}\left|\sum_{k=0}^\infty b_k r^{m_k}e^{im_k t}\right|^p(1-|\varphi_a(re^{it})|^2)^{\alpha+2-np}\,dt.$$

By the triangle inequality, $N(f,n,a)$ is less than or equal to

$$C_1\int_0^1\left(\sum_{k=0}^\infty |b_k|r^{m_k}\right)^p(1-r)^{np-2}\,dr\frac{1}{2\pi}\int_0^{2\pi}(1-|\varphi_a(re^{it})|^2)^{\alpha+2-np}\,dt,$$

where $C_1 = 2^{np-1}$. Because $0\le\alpha+2-np\le 1$, Hölder's inequality implies that the inner integral above is less than or equal to

$$\left(\frac{1}{2\pi}\int_0^{2\pi}(1-|\varphi_a(re^{it})|^2)\,dt\right)^{\alpha+2-np}=\left[\frac{(1-|a|^2)(1-r^2)}{1-r^2|a|^2}\right]^{\alpha+2-np},$$

which is obviously less than $(1-r^2)^{\alpha+2-np}$. Therefore, there exists a constant $C_2>0$ such that

$$N(f,n,a)\le C_2\int_0^1\left(\sum_{k=0}^\infty |b_k|r^{m_k}\right)^p(1-r)^\alpha\,dr.$$

By Lemma 17, there exists $C_3>0$ such that

$$N(f,n,a)\le C_3\sum_{n=0}^\infty\frac{t_n^p}{2^{n(\alpha+1)}},$$

where

$$t_n=\sum_{m_k\in I_n}|b_k|,\qquad 0\le n<\infty.$$

By the proof of Theorem 18, t_n^p is comparable to

$$\sum_{m_k\in I_n}|b_k|^p.$$

Since $|b_k|$ is comparable to $n_k^n|a_k|$ and 2^n is comparable to $m_k\in I_n$, we conclude that there exists a constant $C_4>0$, independent of a, such that

$$N(f,n,a)\le C_4\sum_{k=0}^\infty\frac{|a_k|^p}{n_k^{\alpha+2-pn}}.$$

This completes the proof of the theorem. □

This result can be used to tell the differences among the spaces $Q(n,p,\alpha)$.

Suppose $\alpha+1\le pn\le\alpha+2$ and let $Q_0(n,p,\alpha)$ be the closure in $Q(n,p,\alpha)$ of the set of polynomials. The above theorem shows that a lacunary series belongs to $Q(n,p,\alpha)$ if and only if it belongs to $Q_0(n,p,\alpha)$. Note that the space $Q(n,p,\alpha)$ is nonseparable for some parameters, for example, when $Q(n,p,\alpha)=\mathrm{BMOA}$. But $Q(n,p,\alpha)$ is separable for some other parameters, for example, when $Q(n,p,\alpha)=B_p$.

7. Other generalizations

It is clear that the nth derivative used in the definition of $Q(n,p,\alpha)$ can be replaced by any reasonable "fractional derivative", for example, the radial fractional derivatives introduced in [14] work perfectly here.

To go even further, we can start out with an arbitrary Banach space $(X, \|\ \|)$ of analytic functions in $\mathbb{D}$ and define $Q(X)$ as the space of analytic functions f in $\mathbb{D}$ with the property that

$$\|f\|_Q = \sup_{\varphi\in\mathrm{Aut}(\mathbb{D})} \|f\circ\varphi\| < \infty.$$

This clearly gives rise to a Möbius invariant space Q_X if it is nontrivial. If X contains all constants, we may also want to use the condition

$$\|f\|_Q = \sup_{\varphi\in\mathrm{Aut}(\mathbb{D})} \|f\circ\varphi - f(\varphi(0))\| < \infty$$

instead. This construction gives rise to all Möbius invariant Banach spaces on $\mathbb{D}$. In fact, if X is Möbius invariant, then $X = Q_X$.

There are many problems concerning the spaces $Q(n,p,\alpha)$ that one may want to study, for example, inner and outer functions in $Q(n,p,\alpha)$, composition operators on $Q(n,p,\alpha)$, and atomic decomposition for $Q(n,p,\alpha)$. We will study such topics in subsequent papers.

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