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WELL-POSEDNESS FOR EQUATIONS OF BENJAMIN-ONO TYPE

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ABSTRACT. The Cauchy problem $u_t - |D|^\alpha u_x + uu_x = 0$ in $(-T,T)\times\mathbb{R}$, $u(0)=u_0$, is studied for $1<\alpha<2$. Using suitable spaces of Bourgain type, local well-posedness for initial data $u_0 \in H^s(\mathbb{R}) \cap \dot{H}^{-\omega}(\mathbb{R})$ for any $s > -\frac{3}{4}(\alpha-1)$, $\omega := 1/\alpha - 1/2$ is shown. This includes existence, uniqueness, persistence, and analytic dependence on the initial data. These results are sharp with respect to the low frequency condition in the sense that if $\omega < 1/\alpha - 1/2$, then the flow map is not C^2 due to the counterexamples previously known. By using a conservation law, these results are extended to global well-posedness in $H^s(\mathbb{R}) \cap \dot{H}^{-\omega}(\mathbb{R})$ for $s \geq 0$, $\omega = 1/\alpha - 1/2$, and real valued initial data.

1. Introduction

We consider the Cauchy problem

$$\begin{aligned} u_t - |D|^\alpha u_x + uu_x &= 0 \quad \text{in } (-T,T)\times\mathbb{R}, \\ u(0) &= u_0, \end{aligned} \tag{1.1}$$

for $1<\alpha<2$ and initial data belonging to some L^2-based Sobolev space. Here $|D|^\alpha$ denotes the Fourier multiplier operator defined via $\mathcal{F}|D|^\alpha v(\xi) = |\xi|^\alpha \mathcal{F}v(\xi)$.

The case $\alpha = 2$ coincides with the Korteweg de Vries equation (KdV) and it is well understood due to the works of Bourgain [2] and Kenig, Ponce and Vega [17], [18], Christ, Colliander and Tao [4], as well as Colliander, Keel, Staffilani, Takaoka and Tao [5] and many others. One feature of this problem is that it is globally well-posed with locally Lipschitz continuous (even analytic) dependence on the initial data $u_0 \in H^s(\mathbb{R})$ for $s > -3/4$; see [18], [5]. The local result is established by the contraction mapping principle.

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The case $\alpha = 1$ corresponds to the Benjamin-Ono equation (BO). This Cauchy problem is differently behaved with respect to the smoothness properties of the flow map. In [20] Koch and Tzvetkov proved that the flow map is not uniformly continuous on bounded subsets of $H^s(\mathbb{R})$ for $s > 0$. On the other hand, the problem is globally well-posed with continuous dependence on the (real valued) data in $H^s(\mathbb{R})$ for any $s \geq 0$ due to a recent result of Ionescu and Kenig [12]. This is established by combining the gauge transformation introduced in the work of Tao [23] with a new bilinear estimate. For some previous results we refer the reader to [19], [14], [23], [3].

In this work we focus on the cases $1 < \alpha < 2$. There are three formally conserved quantities, namely $\int u\,dx$, the L^2 norm, and the Hamiltonian

$$\frac{1}{2}\int \left||D|^{\frac{\alpha}{2}}u\right|^2\,dx - \frac{1}{6}\int u^3\,dx,$$

and, e.g., Ginibre and Velo [9] constructed global weak solutions by compactness arguments. Here, we are interested in low regularity well-posedness results which include existence, uniqueness, persistence, and continuous or smooth dependence on the initial data. It was observed by Molinet, Saut and Tzvetkov [22] that there is a major obstruction in the range $\alpha < 2$. They showed that interactions of linear waves of very low frequency with linear waves of high frequency cannot be controlled by bilinear estimates based on $H^s(\mathbb{R})$ information only and that the flow map for $H^s(\mathbb{R})$ data is not C^2. Therefore, we say that these equations are of Benjamin-Ono type. Nevertheless, there are well-posedness results in $H^s(\mathbb{R})$ (for real valued initial data) for these problems due to Kenig, Ponce and Vega [16] ($s \geq (9-3\alpha)/4$), improved by Kenig and Koenig [14] ($s > 3/2 - 3\alpha/8$). The proofs are based on local smoothing, maximal function, Strichartz and energy type estimates. In [6] Colliander, Kenig and Staffilani[1] proved a local well-posedness result for $s \geq \alpha/2$ by a contraction argument for initial data in a weighted Sobolev space in the range $1 < \alpha < 2$.

In light of the aforementioned examples from [22] it seems natural to impose a low frequency condition on the initial data in terms of a homogeneous Sobolev weight, which is persistent under the time evolution. That this might be useful was also indicated by Kato [13] for the BO case, and for $1 < \alpha < 2$ this was carried out by the author in an earlier version of this paper[2], which turned out to be an improvement of a previous result by Molinet and Ribaud [21]. Recently, a related low frequency condition was also used by Ionescu and Kenig [12] in their main bilinear estimate for the BO case.

In the following, we prove a local well-posedness result for data in $H^s(\mathbb{R}) \cap \dot{H}^{-\omega}(\mathbb{R})$ in the range $1 < \alpha < 2$, which almost closes the gap to the local

[1]Notice that α in the present work plays the role of $a+1$ in [6].

[2]This contained a bilinear estimate and a version of Theorem 2.6 (resp. Theorem 2.11) with the low regularity threshold $s \geq 1 - \alpha/2$ (resp. $s \geq \alpha/2$) and $\omega = 1/\alpha - 1/2$.

well-posedness theory for KdV known so far. This includes local existence, uniqueness, persistence, and the analytic dependence on the data. Moreover, we derive a global result for real valued data, based on the L^2 conservation law. The low frequency condition in these results is shown to be sharp with respect to the C^2 continuity of the flow map. This will be made precise in Theorems 2.6, 2.11 and 6.1.

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2. Notation and main results

We denote by $\mathcal{S}(\mathbb{R}^n)$ the space of Schwartz functions on $\mathbb{R}^n$. Moreover, we use the notation $\langle x\rangle = (1+|x|^2)^{\frac{1}{2}}$ and define the Fourier transform by

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx.$$

The Fourier transform w.r.t. $(t,x) \in \mathbb{R}^2$ will be denoted by $\mathcal{F}$, whereas the Fourier transform w.r.t. $t \in \mathbb{R}$ $(x \in \mathbb{R})$ will be denoted by $\mathcal{F}_t$ $(\mathcal{F}_x = \widehat{\cdot})$.

We start by defining the space of initial data.

Definition 2.1. For $s \in \mathbb{R}$ and $0 \le \omega < \frac{1}{2}$ we define the Sobolev space $H^{(s,\omega)}$ as the completion of $\mathcal{S}(\mathbb{R})$ with respect to the norm

$$\|u\|^2_{H^{(s,\omega)}} := \int_{\mathbb{R}} \langle\xi\rangle^{2s+2\omega}|\xi|^{-2\omega}|\widehat{u}(\xi)|^2\, d\xi. \tag{2.1}$$

Next, we introduce the resolution space which is an adaption of the Bourgain type spaces [2] to our problem.

Definition 2.2. For $0 \le \omega < \frac{1}{2}$ and $s, b \in \mathbb{R}$ we define the space $X_{s,\omega,b}$ as the completion of $\mathcal{S}(\mathbb{R}^2)$ with respect to the norm

$$\|u\|^2_{X_{s,\omega,b}} := \int_{\mathbb{R}^2} |\xi|^{-2\omega}\langle\xi\rangle^{2s-2\alpha\omega}\langle|\tau|+|\xi|^{1+\alpha}\rangle^{2\omega}\langle\tau-\xi|\xi|^{\alpha}\rangle^{2b}|\mathcal{F}u(\tau,\xi)|^2\, d\tau d\xi. \tag{2.2}$$

Moreover, we define for $T > 0$ the restriction norm space

$$X^T_{s,\omega,b} := \{u|_{[-T,T]} \mid u \in X_{s,\omega,b}\}$$

with norm

$$\|u\|_{X^T_{s,\omega,b}} = \inf\left\{\|\widetilde{u}\|_{X_{s,\omega,b}} \mid u = \widetilde{u}|_{[-T,T]},\, \widetilde{u} \in X_{s,\omega,b}\right\}.$$

Remark 2.3. Notice that since $0 \le \omega < \frac{1}{2}$ we have the continuous embedding

$$L^p(\mathbb{R}) \cap H^s(\mathbb{R}) \subset H^{(s,\omega)}$$

for $1 \le p \le 2$ and $\omega < \frac{1}{p} - \frac{1}{2}$ by Hausdorff-Young.

Remark 2.4. The additional elliptic weight $\langle |\tau| + |\xi|^{1+\alpha} \rangle$ will be used to control interactions of low frequency waves with a Fourier transform that is localized far away from the characteristic set

$$P_\alpha := \{(\tau, \xi) \,|\, \tau = \xi|\xi|^\alpha\}$$

with essentially linear waves of high frequency which result in an essentially linear wave of high frequency; see Theorem 4.1. Notice that, informally speaking, for $|\xi| \geq 1$ and close to P_α the space $X_{s,\omega,b}$ corresponds to the $X_{s,b}$ space of Bourgain, whereas far away from P_α the space $X_{s,\omega,b}$ corresponds to $X_{s-(\alpha+1)\omega,b+\omega}$.

A similar weight was used by Bejenaru [1] in the context of certain nonlinear Schrödinger equations. We try to use only a minimal portion of such a weight because we focus on a low regularity threshold, but see Remark 4.2 at the end of Section 4.

Remark 2.5. $X_{s,\omega,b}$ is closed under complex conjugation due to the symmetry of the weights.

Within this framework we now state our main results of this paper.

Theorem 2.6. *Let $1 < \alpha < 2$ and $\omega = \frac{1}{\alpha} - \frac{1}{2}$. Then, for $s \geq s_0 > -\frac{3}{4}(\alpha - 1)$ there exists $b > \frac{1}{2}$ and a non-increasing function $T : (0, \infty) \to (0, \infty)$, such that for any $u_0 \in H^{(s,\omega)}$ and $T = T(\|u_0\|_{H^{(s_0,\omega)}})$, there exists a solution*

$$u \in X^T_{s,\omega,b} \subset C\big([-T, T], H^{(s,\omega)}\big)$$

of the Cauchy problem

$$u_t - |D|^\alpha u_x + uu_x = 0 \quad \text{in } (-T, T) \times \mathbb{R},$$
$$u(0) = u_0,$$

which is unique in the class of $X^T_{s_0,\omega,b}$ solutions. Moreover, for any $r > 0$ there exists $T = T(r)$, such that for $B = \{v_0 \in H^{(s,\omega)} \mid \|v_0\|_{H^{(s_0,\omega)}} \leq r\}$ the flow map

$$F : H^{(s,\omega)} \supset B \to C\big([-T, T], H^{(s,\omega)}\big) \cap X^T_{s,\omega,b}, \quad u_0 \mapsto u$$

is analytic.

Remark 2.7. By a solution we always mean a fixed point of (an extension of) the operator

$$\Phi_T(u)(t) = \psi(t) W_\alpha(t) u_0 - \frac{1}{2}\psi_T(t) \int_0^t W_\alpha(t - t') \partial_x (u^2)(t')\, dt'$$

in $X^T_{s,\omega,b}$. These solutions are solutions in the sense of distributions, at least[3] for $s \geq 0$. Here,

$$W_\alpha(t) : H^{(s,\omega)} \to H^{(s,\omega)}, \quad \widehat{W_\alpha(t)u_0}(\xi) = e^{it\xi|\xi|^\alpha}\widehat{u_0}(\xi)$$

for the solution operator of the linear homogeneous problem, which defines a unitary group on $H^{(s,\omega)}$.

REMARK 2.8. If $u_0 \in H^{(s,\omega)}$ is real valued, the solution stays real valued.

REMARK 2.9. Let $F : X_1 \supset B \to X_2$, where X_i are Banach spaces over the complex (real) numbers and B is open. We say that F is analytic (real analytic) if locally we can expand F into a series of continuous symmetric n-linear maps; see, e.g., [7], Definition 15.1.

REMARK 2.10. If $\alpha \to 1^+$, the lower bound for s tends to 0, and for $\alpha \to 2^-$ the bound converges to $-3/4$. For all admissible values of α our result includes the L^2 case where a conserved quantity is available.

Lemma 5.2 leads to global well-posedness:

THEOREM 2.11. *Let $1 < \alpha < 2$, $\omega = \frac{1}{\alpha} - \frac{1}{2}$ and $s \geq 0$. In the case of real valued initial data in $H^{(s,\omega)}$ the conclusions of Theorem 2.6 remain valid for arbitrarily large $T > 0$ and with real analytic dependence on the data.*

3. Linear estimates and bilinear refinements

In this section we prove estimates for the solutions to the linear homogeneous and inhomogeneous equation and recall the Strichartz estimates for problem (1.1). These are well known for the Bourgain type spaces and we will adjust them to our setting. Finally, inspired by Grünrock's bilinear estimates [10], [11], we will prove a sharp estimate for a bilinear pseudo-differential operator, which provides a highly useful tool for the proof of the main bilinear estimate.

In the following let $\psi \in C_0^\infty([-2,2])$ be a nonnegative, symmetric function with $\psi|_{[-1,1]} \equiv 1$ and let $\psi_T(t) := \psi(t/T)$.

In the case $b > \frac{1}{2}$ we have

$$X_{s,\omega,b} \subset C\big(\mathbb{R}, H^{(s,\omega)}\big)$$

with a continuous embedding by an application of Sobolev's theorem with respect to time.

LEMMA 3.1. *Let $0 \leq \omega < \frac{1}{2}$, $s, b \in \mathbb{R}$. Then,*

$$\|\psi W_\alpha u_0\|_{X_{s,\omega,b}} \leq c\|u_0\|_{H^{(s,\omega)}} \tag{3.1}$$

for all $u_0 \in H^{(s,\omega)}$.

[3] Even for $s < 0$ one can still use some smoothing properties to verify this.

Proof. We may assume $u_0 \in \mathcal{S}(\mathbb{R}) \subset H^{(s,\omega)}$ by density and calculate

$$\mathcal{F}(\psi W_\alpha u_0)(\tau,\xi) = c\mathcal{F}_t\psi(\tau - \xi|\xi|^\alpha)\widehat{u_0}(\xi).$$

Let $N \in \mathbb{N}$ be a positive integer with $b < N-1$. Since $\mathcal{F}_t\psi$ is a Schwartz function, we conclude for $s = \omega = 0$

$$\begin{aligned}\|\psi W_\alpha u_0\|^2_{X_{0,0,b}} &= c\int_{\mathbb{R}^2} \langle \tau - \xi|\xi|^\alpha\rangle^{2b} |\mathcal{F}_t\psi(\tau - \xi|\xi|^\alpha)\widehat{u_0}(\xi)|^2\, d\tau d\xi \\ &\le c\int_{\mathbb{R}}\int_{\mathbb{R}} \langle \tau - \xi|\xi|^\alpha\rangle^{2b}\langle \tau - \xi|\xi|^\alpha\rangle^{-2N}\, d\tau |\widehat{u_0}(\xi)|^2\, d\xi \\ &\le c\|u_0\|^2_{L^2}.\end{aligned}$$

Now let $\omega \ge 0, s, b \in \mathbb{R}$. By using the inequality

$$\langle |\tau| + |\xi|^{1+\alpha}\rangle^\omega \le c(\langle \tau - \xi|\xi|^\alpha\rangle^\omega + \langle\xi\rangle^{(1+\alpha)\omega}) \tag{3.2}$$

we estimate

$$|\xi|^{-\omega}\langle\xi\rangle^{s-\alpha\omega}\langle |\tau| + |\xi|^{1+\alpha}\rangle^\omega |\widehat{u_0}(\xi)| \le c\langle \tau - \xi|\xi|^\alpha\rangle^\omega |\widehat{v_0}(\xi)|,$$

where $\widehat{v_0}(\xi) = |\xi|^{-\omega}\langle\xi\rangle^{s+\omega}\widehat{u_0}(\xi)$. With $b' = b+\omega$ this gives

$$\|\psi W_\alpha u_0\|_{X_{s,\omega,b}} \le c\|\psi W_\alpha v_0\|_{X_{0,0,b'}} \le c\|v_0\|_{L^2} = c\|u_0\|_{H^{(s,\omega)}}. \qquad \square$$

The following lemma contains the estimate for the linear, inhomogeneous problem.

LEMMA 3.2. *Let $0 \le \omega < \frac{1}{2}$, $s \in \mathbb{R}$, and $-\frac{1}{2} < b' \le 0 \le b < b'+1$, as well as $b' \le -\omega$. There exists $\varepsilon > 0$, such that for all $0 < T \le 1$*

$$\left\|\psi_T(t)\int_0^t W_\alpha(t-t')N(t')\,dt'\right\|_{X_{s,\omega,b}} \le cT^\varepsilon\|N\|_{X_{s,\omega,b'}} \tag{3.3}$$

for all $N \in \mathcal{S}(\mathbb{R}^2)$.

Proof. In the case $\omega = 0$ this is a well known estimate; see, e.g., [8], Lemme 3.2. We will reduce (3.3) to this case. Define

$$\widehat{M(t)}(\xi) := |\xi|^{-\omega}\widehat{N(t)}(\xi).$$

Using (3.2) we see that

$$\begin{aligned}\left\|\psi_T\int_0^t W_\alpha(t-t')N(t')\,dt'\right\|_{X_{s,\omega,b}} &\le c\left\|\psi_T\int_0^t W_\alpha(t-t')M(t')\,dt'\right\|_{X_{s-\alpha\omega,0,b+\omega}} \\ &\quad + c\left\|\psi_T\int_0^t W_\alpha(t-t')M(t')\,dt'\right\|_{X_{s+\omega,0,b}},\end{aligned}$$

where on the right hand side the usual Bourgain type norms appear. Thus, by our restrictions on b, b' and ω we may bound this, using the standard estimate, by

$$cT^\varepsilon(\|M\|_{X_{s-\alpha\omega,0,b'+\omega}} + \|M\|_{X_{s+\omega,0,b'}}).$$

Because of the inequalities

$$\langle \tau - \xi|\xi|^{\alpha}\rangle^{b'+\omega} \le \langle \tau - \xi|\xi|^{\alpha}\rangle^{b'}\langle |\tau| + |\xi|^{1+\alpha}\rangle^{\omega},$$
$$\langle \xi\rangle^{s+\omega} \le c\langle \xi\rangle^{s-\alpha\omega}\langle |\tau| + |\xi|^{1+\alpha}\rangle^{\omega}$$

we obtain the upper bound

$$\left\| \psi_T(t)\int_0^t W_\alpha(t-t')N(t')\,dt'\right\|_{X_{s,\omega,b}} \le cT^{\varepsilon}\,\|N\|_{X_{s,\omega,b'}}$$

as desired. □

Next, we recall the $L^4_tL^\infty_x$ Strichartz estimate. Let J^s be the Bessel potential operator defined via $\mathcal{F}J^sf(\xi) = \langle\xi\rangle^s\mathcal{F}f(\xi)$.

LEMMA 3.3. *For $b > \frac{1}{2}$ we have*

$$\|J^{\frac{\alpha-1}{4}}u\|_{L^4_tL^\infty_x} \le c\|u\|_{X_{0,0,b}}. \tag{3.4}$$

Proof. From [15], Theorem 2.1, we know that

$$\||D|^{\frac{\alpha-1}{4}}W_\alpha(t)u_0\|_{L^4_tL^\infty_x} \le c\|u_0\|_{L^2}. \tag{3.5}$$

Now we use a general property of Bourgain spaces $X_{0,0,b}$ with $b > \frac{1}{2}$; see, e.g., [8], Lemme 3.3. By the Fourier inversion formula we may write $u \in X_{0,0,b}$ as

$$u(t) = c\int e^{it\tau}W_\alpha(t)\mathcal{F}_t(W_\alpha(-\cdot)u)(\tau)\,d\tau, \tag{3.6}$$

and this implies

$$\||D|^{\frac{\alpha-1}{4}}u\|_{L^4_tL^\infty_x} \le c\int \||D|^{\frac{\alpha-1}{4}}W_\alpha(t)\mathcal{F}_t(W_\alpha(-\cdot)u)(\tau)\|_{L^4_tL^\infty_x}\,d\tau.$$

Using (3.5) and Cauchy-Schwarz, we arrive at

$$\begin{aligned}\||D|^{\frac{\alpha-1}{4}}u\|_{L^4_tL^\infty_x} &\le c\left(\int\langle\tau\rangle^{-2b}d\tau\right)^{\frac{1}{2}}\left(\int\langle\tau\rangle^{2b}\|\mathcal{F}_t(W_\alpha(-\cdot)u)(\tau)\|^2_{L^2_x}\,d\tau\right)^{\frac{1}{2}}\\ &\le c\left(\iint\langle\tau-\xi|\xi|^{\alpha}\rangle^{2b}|\mathcal{F}u(\tau,\xi)|^2\,d\tau d\xi\right)^{\frac{1}{2}},\end{aligned}$$

since $b > \frac{1}{2}$ and therefore

$$\||D|^{\frac{\alpha-1}{4}}u\|_{L^4_tL^\infty_x} \le c\|u\|_{X_{0,0,b}}. \tag{3.7}$$

By smooth cutoffs in frequency, we split u into a low frequency part u^{low}

$$\mathcal{F}u^{low}(\tau,\xi) = \psi(\xi)\mathcal{F}u(\tau,\xi)$$

and a high frequency part $u^{high} := u - u^{low}$. Then,

$$\|J^{\frac{\alpha-1}{4}}u\|_{L^4_tL^\infty_x} \le \|J^{\frac{\alpha-1}{4}}u^{low}\|_{L^4_tL^\infty_x} + \|J^{\frac{\alpha-1}{4}}u^{high}\|_{L^4_tL^\infty_x}.$$

By an application of the Sobolev inequality, the first part is bounded by

$$c\|J^{\frac{\alpha+1}{4}+\varepsilon}u^{low}\|_{L_t^4L_x^2} \leq c\|u\|_{L_t^4L_x^2} \leq c\|u\|_{X_{0,0,b}},$$

whereas the second part is bounded by

$$c\||D|^{-\frac{\alpha-1}{4}}J^{\frac{\alpha-1}{4}}u^{high}\|_{X_{0,0,b}} \leq c\|u\|_{X_{0,0,b}}$$

due to (3.7). This gives the desired estimate. □

The next lemma is the main tool in the proof of the crucial bilinear estimate (cf. [10], Definition 2.1, and [11], Lemma 1). For $\delta > 0$ we define

$$|x|_\delta := \zeta(x/\delta)|x|$$

with an even function $\zeta \in C^\infty$ with $\zeta|_{[-1,1]} \equiv 0$, $\zeta|_{\mathbb{R}\setminus[-2,2]} \equiv 1$, and $0 \leq \zeta \leq 1$.

LEMMA 3.4. *We define the bilinear operator* I_δ^s *via*

$$\mathcal{F}_x\, I_\delta^s(u_1,u_2)(\xi) = \int_{\xi=\xi_1+\xi_2} \left||\xi_1|^{2s} - |\xi_2|^{2s}\right|_\delta^{\frac{1}{2}} \widehat{u}_1(\xi_1)\widehat{u}_2(\xi_2)\, d\xi_1$$

for all $u_1, u_2 \in \mathcal{S}(\mathbb{R})$. *Then, for all* $\delta > 0$

$$\left\|I_\delta^{\frac{\alpha}{2}}(W_\alpha u_1, W_\alpha u_2)\right\|_{L^2_{xt}} \leq \sqrt{\frac{2}{1+\alpha}}\|u_1\|_{L^2_x}\|u_2\|_{L^2_x}. \tag{3.8}$$

Proof. For fixed $t \in \mathbb{R}$ we use Plancherel in x and calculate

(3.9)

$$\begin{aligned}
&\left\|I_\delta^{\frac{\alpha}{2}}(W_\alpha(t)u_1, W_\alpha(t)u_2)\right\|^2_{L^2_x}\\
&\quad= \frac{1}{2\pi}\int\left|\int_{\xi=\xi_1+\xi_2} ||\xi_1|^\alpha - |\xi_2|^\alpha|_\delta^{\frac{1}{2}}\, e^{it(\xi_1|\xi_1|^\alpha+\xi_2|\xi_2|^\alpha)}\widehat{u}_1(\xi_1)\widehat{u}_2(\xi_2)\,d\xi_1\right|^2 d\xi\\
&\quad= \frac{1}{2\pi}\iiint e^{itP(\xi,\xi_1,\eta_1)}\phi(\xi,\xi_1,\eta_1)\,d\eta_1 d\xi_1 d\xi
\end{aligned}$$

with the phase function

$$P(\xi,\xi_1,\eta_1) = \xi_1|\xi_1|^\alpha + (\xi-\xi_1)|\xi-\xi_1|^\alpha - \eta_1|\eta_1|^\alpha - (\xi-\eta_1)|\xi-\eta_1|^\alpha$$

and

$$\begin{aligned}
&\phi(\xi,\xi_1,\eta_1)\\
&= ||\xi_1|^\alpha - |\xi-\xi_1|^\alpha|_\delta^{\frac{1}{2}}\, ||\eta_1|^\alpha - |\xi-\eta_1|^\alpha|_\delta^{\frac{1}{2}}\, \widehat{u}_1(\xi_1)\widehat{u}_2(\xi-\xi_1)\overline{\widehat{u}_1(\eta_1)\widehat{u}_2(\xi-\eta_1)}.
\end{aligned}$$

For fixed ξ, ξ_1 the function $P_1(\eta_1) = P(\xi,\xi_1,\eta_1)$ has only two simple roots ξ_1, $\xi-\xi_1$ in the support of ϕ. Moreover,

$$|P_1'(\eta_1)| = (1+\alpha)||\xi-\eta_1|^\alpha - |\eta_1|^\alpha| \geq (1+\alpha)\delta \ \text{ in } \ \mathrm{supp}(\phi) \tag{3.10}$$

and

$$|P_1'(\xi_1)| = |P_1'(\xi-\xi_1)| = (1+\alpha)||\xi-\xi_1|^\alpha - |\xi_1|^\alpha|.$$

With the help of the function $g(x) = \pi^{-\frac{1}{2}} e^{-x^2}$ we construct the approximate identity $g_\varepsilon(x) = \varepsilon^{-1} g(\varepsilon^{-1} x)$ with $\mathcal{F} g_\varepsilon \uparrow (2\pi)^{-\frac{1}{2}}$. By Fubini's theorem and the Fourier inversion formula

$$
\begin{aligned}
I(\varepsilon) &:= (2\pi)^{-\frac{1}{2}} \int \mathcal{F} g_\varepsilon(t) \iiint e^{itP(\xi,\xi_1,\eta_1)} \phi(\xi,\xi_1,\eta_1)\, d\eta_1 d\xi_1 d\xi dt \\
&= \iiint g_\varepsilon(P(\xi,\xi_1,\eta_1)) \phi(\xi,\xi_1,\eta_1)\, d\eta_1 d\xi_1 d\xi.
\end{aligned}
$$

Now, because of (3.10) we may use the dominated convergence theorem to conclude

$$
\begin{aligned}
&\lim_{\varepsilon\to 0} I(\varepsilon) = \iint \lim_{\varepsilon \to 0} \int g_\varepsilon(P(\xi,\xi_1,\eta_1)) \phi(\xi,\xi_1,\eta_1)\, d\eta_1 d\xi_1 d\xi \\
&= \iint \frac{\phi(\xi,\xi_1,\xi_1)}{|P_1'(\xi_1)|} + \frac{\phi(\xi,\xi_1,\xi-\xi_1)}{|P_1'(\xi-\xi_1)|}\, d\xi_1 d\xi \\
&\le \frac{1}{1+\alpha} \iint |\widehat{u_1}(\xi_1)|^2 |\widehat{u_2}(\xi-\xi_1)|^2 + |\widehat{u_1}(\xi_1)\widehat{u_2}(\xi_1)||\widehat{u_1}(\xi-\xi_1)\widehat{u_2}(\xi-\xi_1)|\, d\xi_1 d\xi \\
&\le \frac{2}{1+\alpha} \|u_1\|_{L^2_x}^2 \|u_2\|_{L^2_x}^2.
\end{aligned}
$$

On the other hand, by the monotone convergence theorem and (3.9) we see that

$$
\lim_{\varepsilon\to 0} I(\varepsilon) = \left\| I_\delta^{\frac{\alpha}{2}} (W_\alpha u_1, W_\alpha u_2) \right\|_{L^2_{xt}}^2 .
$$

This implies (3.8). □

Remark 3.5. Roughly speaking, the bilinear operator $I_\delta^{\alpha/2}$ controls $\alpha/2$ derivatives on the product of two solutions at different frequency.

Remark 3.6. The proof shows that the above estimate is sharp in the sense that for $u \in \mathcal{S}(\mathbb{R})$

$$
\lim_{\delta\to 0} \left\| I_\delta^{\frac{\alpha}{2}} (W_\alpha u, W_\alpha u) \right\|_{L^2_{xt}} = \sqrt{\frac{2}{1+\alpha}} \|u\|_{L^2}^2.
$$

Corollary 3.7. *For $u_1, u_2 \in \mathcal{S}(\mathbb{R}^2)$ we define the bilinear operator I_*^s via*

$$
\mathcal{F} I_*^s(u_1,u_2)(\tau,\xi) = \int_{\substack{\xi=\xi_1+\xi_2 \\ \tau=\tau_1+\tau_2}} \left| |\xi_1|^{2s} - |\xi_2|^{2s} \right|^{\frac{1}{2}} \mathcal{F} u_1(\tau_1,\xi_1) \mathcal{F} u_2(\tau_2,\xi_2)\, d\tau_1 d\xi_1.
$$

For $b > \frac{1}{2}$ there exists a unique, bilinear extension $I_*^{\frac{\alpha}{2}}$ with

$$
\left\| I_*^{\frac{\alpha}{2}} (u_1,u_2) \right\|_{L^2_{xt}} \le c \|u_1\|_{X_{0,0,b}} \|u_2\|_{X_{0,0,b}}, \quad u_1,u_2 \in X_{0,0,b}. \tag{3.11}
$$

For $u_1, u_2 \in \mathcal{S}(\mathbb{R}^2)$ we define the operator $K_*^{\frac{\alpha}{2}}$ by

$$\mathcal{F}\, K_*^{\frac{\alpha}{2}}(u_1,u_2)(\tau,\xi) = \int_{\substack{\xi=\xi_1+\xi_2\\ \tau=\tau_1+\tau_2}} \big||\xi|^\alpha - |\xi_1|^\alpha\big|^{\frac{1}{2}}\, \mathcal{F}\overline{u}_1(\tau_1,\xi_1)\mathcal{F}u_2(\tau_2,\xi_2)\, d\tau_1 d\xi_1.$$

$K_*^{\frac{\alpha}{2}}$ is the formal adjoint of $u_2 \mapsto I_*^{\frac{\alpha}{2}}(u_1,u_2)$ with respect to L^2_{xt}, and for $b > \frac{1}{2}$ there exists a unique, bilinear extension $K_*^{\frac{\alpha}{2}}$ with

$$\left\|K_*^{\frac{\alpha}{2}}(u_1,u_2)\right\|_{X_{0,0,-b}} \leq c\|u_1\|_{X_{0,0,b}}\|u_2\|_{L^2_{xt}}, \quad u_1 \in X_{0,0,b},\ u_2 \in L^2_{xt}. \tag{3.12}$$

Proof. We may assume that $u_1, u_2, v \in \mathcal{S}(\mathbb{R}^2)$. We write u_1, u_2 as in (3.6) and estimate

$$\begin{aligned}
&\|I_*^{\frac{\alpha}{2}}(u_1,u_2)\|_{L^2_{xt}} = \lim_{\delta\to 0}\|I_\delta^{\frac{\alpha}{2}}(u_1,u_2)\|_{L^2_{xt}}\\
&\leq c\lim_{\delta\to 0}\iint \left\|I_\delta^{\frac{\alpha}{2}}(W_\alpha\mathcal{F}_t(W_\alpha(-\cdot)u_1)(\tau_1), W_\alpha\mathcal{F}_t(W_\alpha(-\cdot)u_2)(\tau_2))\right\|_{L^2_{xt}} d\tau_1 d\tau_2\\
&\leq c\iint \|\mathcal{F}_t(W_\alpha(-\cdot)u_1)(\tau_1)\|_{L^2_x}\|\mathcal{F}_t(W_\alpha(-\cdot)u_2)(\tau_2)\|_{L^2_x}\, d\tau_1 d\tau_2,
\end{aligned}$$

where we used the estimate (3.8) for the last inequality. Next, we insert $\langle\tau_i\rangle^{-2b}\langle\tau_i\rangle^{2b}$ in each integral and use Cauchy-Schwarz to obtain

$$\left\|I_*^{\frac{\alpha}{2}}(u_1,u_2)\right\|_{L^2_{xt}} \leq c\|u_1\|_{X_{0,0,b}}\|u_2\|_{X_{0,0,b}}.$$

Now we calculate the adjoint of $I_*^{\frac{\alpha}{2}}(u_1,\cdot)$ with respect to L^2_{xt} for Schwartz functions. By Plancherel

$$\begin{aligned}
&\left(I_*^{\frac{\alpha}{2}}(u_1,u_2),v\right)_{L^2_{xt}}\\
&=\int \big||\xi_1|^\alpha - |\xi-\xi_1|^\alpha\big|^{\frac{1}{2}}\mathcal{F}u_1(\tau_1,\xi_1)\mathcal{F}u_2(\tau-\tau_1,\xi-\xi_1)\, d\tau_1 d\xi_1\overline{\mathcal{F}v(\tau,\xi)}\, d\tau d\xi\\
&=\int \mathcal{F}u_2(\tau-\tau_1,\xi-\xi_1)\overline{\big||\xi_1|^\alpha - |\xi-\xi_1|^\alpha\big|^{\frac{1}{2}}\mathcal{F}\overline{u}_1(-\tau_1,-\xi_1)\mathcal{F}v(\tau,\xi)}\, d\tau_1 d\xi_1 d\tau d\xi.
\end{aligned}$$

The change of variables $(\tau_1,\xi_1,\tau,\xi) \mapsto (-\tau_1,-\xi_1,\tau-\tau_1,\xi-\xi_1)$ yields

$$\begin{aligned}
&\left(I_*^{\frac{\alpha}{2}}(u_1,u_2),v\right)_{L^2_{xt}}\\
&=\int \mathcal{F}u_2(\tau,\xi)\overline{\int \big||\xi_1|^\alpha - |\xi|^\alpha\big|^{\frac{1}{2}}\mathcal{F}\overline{u}_1(\tau_1,\xi_1)\mathcal{F}v(\tau-\tau_1,\xi-\xi_1)\, d\tau_1 d\xi_1}\, d\tau d\xi\\
&=\left(u_2, K_*^{\frac{\alpha}{2}}(u_1,v)\right)_{L^2_{xt}}
\end{aligned}$$

due to the Plancherel identity. Therefore, (3.12) is dual to (3.11). □

The Bourgain type approach to nonlinear dispersive equations such as (1.1) relies heavily on the specific resonance relation. This is analyzed in the next lemma.

LEMMA 3.8. *Let* $1 < \alpha < 2$. *Define*

$$h(\xi_1, \xi_2, \xi) = \xi|\xi|^\alpha - \xi_1|\xi_1|^\alpha - \xi_2|\xi_2|^\alpha. \tag{3.13}$$

Then, for $\xi = \xi_1 + \xi_2$ *we have*

$$|h(\xi_1, \xi_2, \xi)| \geq c|\xi_{\min}||\xi_{\max}|^\alpha, \tag{3.14}$$

where $|\xi_{\min}| := \min\{|\xi_1|, |\xi_2|, |\xi|\}$ *and* $|\xi_{\max}| := \max\{|\xi_1|, |\xi_2|, |\xi|\}$.

Proof. For $\beta \geq 0$ we define $f(\beta) := (1+\beta)^{1+\alpha} - \beta^{1+\alpha} - 1$. This function satisfies $f(0) = 0$ and $f'(\beta) = (1+\alpha)((1+\beta)^\alpha - \beta^\alpha) > 0$, as well as $f''(\beta) = (1+\alpha)\alpha((1+\beta)^{\alpha-1} - \beta^{\alpha-1}) > 0$ for $\beta > 0$. This implies

$$f(\beta) \geq f'(0)\beta = (1+\alpha)\beta \quad \text{for} \quad \beta \in [0,1].$$

We observe that $f(\beta) = \beta^{1+\alpha} f(1/\beta)$ for all $\beta > 0$, which implies

$$f(\beta) = \beta^{1+\alpha} f(1/\beta) \geq (1+\alpha)\beta^\alpha \quad \text{for } \beta \geq 1.$$

Now we start we the proof of (3.14). We suppose the constraint $\xi = \xi_1 + \xi_2$ holds and consider two cases:

Case 1: $\xi_1\xi_2 > 0$*:* Since h is symmetric with respect to ξ_1 and ξ_2, it suffices to consider $\xi_1 = \beta\xi_2$ with $\beta \geq 1$. Then,

$$|h(\xi_1, \xi_2, \xi)| = f(\beta)|\xi_2|^{1+\alpha} \geq c\beta^\alpha|\xi_2|^{1+\alpha} = c|\xi_1|^\alpha|\xi_2|.$$

Since $|\xi| \leq |\xi_1| + |\xi_2| \leq 2|\xi_1|$, we have $|\xi_1| \geq 1/2|\xi_{\max}|$, and this implies (3.14).

Case 2: $\xi_1\xi_2 < 0$*:* By symmetry we may assume $\xi_2\xi < 0$ and $\xi_1\xi > 0$. Then $\xi_1 = \beta\xi$ for some $\beta > 1$. We calculate

$$\begin{aligned}
|h(\xi_1, \xi_2, \xi)| &= |1 - \beta|\beta|^\alpha - (1-\beta)|1-\beta|^\alpha||\xi|^{1+\alpha} \\
&= f(\beta - 1)|\xi|^{1+\alpha} \\
&\geq c \begin{cases} |\beta - 1||\xi|^{1+\alpha}, & 1 < \beta \leq 2 \\ |\beta - 1|^\alpha|\xi|^{1+\alpha}, & \beta > 2 \end{cases} \\
&= c \begin{cases} |\xi_2||\xi|^\alpha, & 1 < \beta \leq 2 \\ |\xi_2|^\alpha|\xi|, & \beta > 2. \end{cases}
\end{aligned}$$

We have $|\xi_1| = |\xi_{\max}|$. If $1 < \beta \leq 2$, then $2|\xi| \geq |\xi_1|$, which implies (3.14). In the case $\beta > 2$ we use $|\xi_2| = (1 - 1/\beta)|\xi_1| \geq 1/2|\xi_1|$ to conclude (3.14). ☐

4. The bilinear estimate

This section is devoted to a proof of the following theorem.

THEOREM 4.1. *Let $1 < \alpha < 2$, $s \geq s_0 > -\frac{3}{4}(\alpha-1)$, and $\omega = \frac{1}{\alpha} - \frac{1}{2}$. There exists $b' > -\frac{1}{2}$ and $b \in (1/2, b'+1)$ such that*

$$\|\partial_x(u_1u_2)\|_{X_{s,\omega,b'}} \leq c\|u_1\|_{X_{s,\omega,b}}\|u_2\|_{X_{s_0,\omega,b}} + \|u_1\|_{X_{s_0,\omega,b}}\|u_2\|_{X_{s,\omega,b}} \tag{4.1}$$

holds true for all $u_1, u_2 \in \mathcal{S}(\mathbb{R}^2)$.

Proof. Let us fix notation. We define $\sigma = |\tau|+|\xi|^{1+\alpha}$ and $\sigma_i = |\tau_i|+|\xi_i|^{1+\alpha}$, as well as $\lambda = \tau - \xi|\xi|^\alpha$ and $\lambda_i = \tau_i - \xi_i|\xi_i|^\alpha$. Moreover, we set

$$f_i(\tau_i, \xi_i) = |\xi_i|^{-\omega}\langle\xi_i\rangle^{s-\alpha\omega}\langle\lambda_i\rangle^b\langle\sigma_i\rangle^\omega\mathcal{F}u_i(\tau_i,\xi_i)$$

and

$$\mathcal{F}v_i(\tau_i,\xi_i) := f_i(\tau_i,\xi_i)\langle\lambda_i\rangle^{-b}.$$

For brevity we write

$$\int_* g(\tau_1,\xi_1)h(\tau_2,\xi_2) := \int_{\substack{\xi=\xi_1+\xi_2\\ \tau=\tau_1+\tau_2}} g(\tau_1,\xi_1)h(\tau_2,\xi_2)\,d\tau_1 d\xi_1.$$

We first consider the case $s = s_0 = -\frac{3}{4}(\alpha-1)+\varepsilon$ for small $\varepsilon > 0$. Our goal is to bound

$$\|\partial_x(u_1u_2)\|_{X_{s,\omega,b'}} = \left\| |\xi|^{1-\omega}\langle\xi\rangle^{s-\alpha\omega}\langle\lambda\rangle^{b'}\langle\sigma\rangle^\omega \int_* \prod_{i=1}^{2} \frac{|\xi_i|^\omega\langle\xi_i\rangle^{\alpha\omega-s}f_i(\tau_i,\xi_i)}{\langle\lambda_i\rangle^b\langle\sigma_i\rangle^\omega}\right\|_{L^2_{\tau,\xi}}$$

by the product of the L^2 norms of the f_i, where we may assume that $0 \leq f_i \in \mathcal{S}(\mathbb{R}^2)$.

Due to the symmetry in ξ_1, ξ_2 it suffices to consider the subregion of the domain of integration where $|\xi_1| \leq |\xi_2|$. By the convolution constraint $\xi = \xi_1+\xi_2$ we then have $|\xi| \leq 2|\xi_2|$. The associated region is split up as follows:

(1) Region D_1: $4|\xi_1| \leq |\xi_2|$. There, $|\xi_1| \leq \frac{1}{4}|\xi_2| \leq \frac{1}{3}|\xi| \leq \frac{2}{3}|\xi_2|$.
(2) Region D_2: $|\xi_1| \leq |\xi_2| \leq 4|\xi_1|$. There, $|\xi| \leq 2|\xi_2|$, $|\xi| \leq 5|\xi_1|$.

Let A, A_1, A_2 be the subregions of the domain of integration such that in A we have $\langle\lambda\rangle \geq \langle\lambda_1\rangle, \langle\lambda_2\rangle$; in A_1 we have $\langle\lambda_1\rangle \geq \langle\lambda\rangle, \langle\lambda_2\rangle$; and in A_2 the inequalities $\langle\lambda_2\rangle \geq \langle\lambda\rangle, \langle\lambda_1\rangle$ hold.

We first consider the region D_1 and subdivide it into two parts, $D_1 = D_{11} \cup D_{12}$, where in D_{11} we have $|\xi_1| \leq 2$ and in D_{12} we have $|\xi_1| \geq 2$. In D_1 we see by Lemma 3.8

$$|\lambda - \lambda_1 - \lambda_2| = |h(\xi_1,\xi_2,\xi)| \geq c|\xi_1||\xi|^\alpha$$

because $|\xi_1| \leq |\xi|, |\xi_2|$.

Now we start the analysis in the subregion D_{11}. We exploit the inequality

$$|\xi|^{1-\frac{\alpha}{2}} = |\xi|^{\alpha\omega} \leq c|\xi_1|^{-\omega}(\chi_A\langle\lambda\rangle^\omega + \chi_{A_1}\langle\lambda_1\rangle^\omega + \chi_{A_2}\langle\lambda_2\rangle^\omega).$$

Therefore in D_{11} the bilinear estimate follows from

$$\sum_{k=0}^{2} \|J_{11,k}\|_{L^2} \le c \prod_{i=1}^{2} \|f_i\|_{L^2}, \tag{4.2}$$

where

$$J_{11,0} = \int_* \chi_{D_{11}\cap A} |\xi|^{\frac{\alpha}{2}-\omega} \langle \xi \rangle^{s-\alpha\omega} \langle \lambda \rangle^{b'+\omega} \langle \sigma \rangle^{\omega} |\xi_2|^{\omega} \prod_{i=1}^{2} \frac{f_i(\tau_i, \xi_i) \langle \xi_i \rangle^{\alpha\omega - s}}{\langle \lambda_i \rangle^{b} \langle \sigma_i \rangle^{\omega}}$$

and for $k = 1, 2$

$$J_{11,k} = \int_* \chi_{D_{11}\cap A_k} |\xi|^{\frac{\alpha}{2}-\omega} \langle \xi \rangle^{s-\alpha\omega} \langle \lambda \rangle^{b'} \langle \sigma \rangle^{\omega} |\xi_2|^{\omega} \langle \lambda_k \rangle^{\omega} \prod_{i=1}^{2} \frac{f_i(\tau_i, \xi_i) \langle \xi_i \rangle^{\alpha\omega - s}}{\langle \lambda_i \rangle^{b} \langle \sigma_i \rangle^{\omega}}.$$

We observe that in D_{11}

$$\langle \xi_2 \rangle^{\alpha\omega - s} \langle \xi \rangle^{s-\alpha\omega} \le c \text{ and } \langle \xi_1 \rangle^{\alpha\omega - s} \le c. \tag{4.3}$$

In addition, we use $b' + \omega \le 0$ and $|\xi_2|^{\omega} \le c|\xi|^{\omega}$ to show that

$$\|J_{11,0}\|_{L^2} \le c \left\| \int_* \chi_{D_{11}\cap A} |\xi|^{\frac{\alpha}{2}} \langle \sigma \rangle^{\omega} \prod_{i=1}^{2} \frac{f_i(\tau_i, \xi_i)}{\langle \lambda_i \rangle^{b} \langle \sigma_i \rangle^{\omega}} \right\|_{L^2}.$$

Because of the convolution constraint $(\tau, \xi) = (\tau_1, \xi_1) + (\tau_2, \xi_2)$ we also have

$$\frac{\langle \sigma \rangle}{\langle \sigma_1 \rangle \langle \sigma_2 \rangle} \le c \frac{1}{\min_{i=1,2} \langle \sigma_i \rangle} \le c, \tag{4.4}$$

which implies

$$\|J_{11,0}\|_{L^2} \le c \left\| \int_* \chi_{D_{11}\cap A} |\xi|^{\frac{\alpha}{2}} \prod_{i=1}^{2} \frac{f_i(\tau_i, \xi_i)}{\langle \lambda_i \rangle^{b}} \right\|_{L^2}.$$

We observe that in D_{11}

$$|\xi|^{\frac{\alpha}{2}} \le c \big||\xi_2|^{\alpha} - |\xi_1|^{\alpha}\big|^{\frac{1}{2}},$$

such that, with (3.11),

$$\begin{aligned} \|J_{11,0}\|_{L^2} &\le c \left\| \int_* \big||\xi_2|^{\alpha} - |\xi_1|^{\alpha}\big|^{\frac{1}{2}} \prod_{i=1}^{2} \frac{f_i(\tau_i, \xi_i)}{\langle \lambda_i \rangle^{b}} \right\|_{L^2} \\ &\le c \left\| I_*^{\frac{\alpha}{2}}(v_1, v_2) \right\|_{L^2} \le c \prod_{i=1}^{2} \|v_i\|_{X_{0,0,b}} = c \prod_{i=1}^{2} \|f_i\|_{L^2}, \end{aligned}$$

since $b > 1/2$. For $J_{11,1}$ we use (4.3) and (4.4) again and get

$$\|J_{11,1}\|_{L^2} \le c \left\| \int_* \frac{\chi_{D_{11}\cap A_1} |\xi|^{\frac{\alpha}{2}}}{\min_{i=1,2} \langle \sigma_i \rangle^{\omega}} \langle \lambda \rangle^{b'} f_1(\tau_1, \xi_1) \langle \lambda_1 \rangle^{\omega - b} f_2(\tau_2, \xi_2) \langle \lambda_2 \rangle^{-b} \right\|_{L^2}.$$

We may assume that $|\lambda_1| \geq 2|\lambda|$, because otherwise the same argument as for $J_{11,0}$ applies. If $\langle\sigma_1\rangle \leq \langle\sigma_2\rangle$, then $\langle\lambda_1\rangle^\omega \leq \min_{i=1,2}\langle\sigma_i\rangle^\omega$. If $\langle\sigma_2\rangle \leq \langle\sigma_1\rangle$, then

$$|\lambda_1| = |\tau_1 - \xi_1|\xi_1|^\alpha| = |\tau - \tau_2 - \xi|\xi|^\alpha + \xi|\xi|^\alpha - \xi_1|\xi_1|^\alpha| \leq |\lambda| + 16|\sigma_2|$$

since we are in region D_{11}. This implies $\langle\lambda_1\rangle \leq c\langle\sigma_2\rangle$, and we also have

$$\langle\lambda_1\rangle^\omega \leq c \min_{i=1,2}\langle\sigma_i\rangle^\omega.$$

Therefore,

$$\|J_{11,1}\|_{L^2} \leq c\left\|\int_* \chi_{D_{11}\cap A_1}|\xi|^{\frac{\alpha}{2}}\langle\lambda\rangle^{b'} f_1(\tau_1,\xi_1)\langle\lambda_1\rangle^{-b} f_2(\tau_2,\xi_2)\langle\lambda_2\rangle^{-b}\right\|_{L^2}.$$

In D_{11} we have $|\xi|^{\frac{\alpha}{2}} \leq c||\xi_2|^\alpha - |\xi_1|^\alpha|^{\frac{1}{2}}$ and by assumption $b' \leq 0$. Thus we may proceed as above with $J_{11,0}$ and use the estimate (3.11) to conclude

$$\|J_{11,1}\|_{L^2} \leq c\left\|\int_* ||\xi_2|^\alpha - |\xi_1|^\alpha|^{\frac{1}{2}} \prod_{i=1}^2 \frac{f_i(\tau_i,\xi_i)}{\langle\lambda_i\rangle^b}\right\|_{L^2} \leq c\prod_{i=1}^2 \|f_i\|_{L^2}.$$

For $J_{11,2}$, we have by (4.3) and (4.4)

$$\|J_{11,2}\|_{L^2} \leq c\left\|\int_* \chi_{D_{11}\cap A_2}|\xi|^{\frac{\alpha}{2}}\langle\lambda\rangle^{b'} f_1(\tau_1,\xi_1)\langle\lambda_1\rangle^{-b} f_2(\tau_2,\xi_2)\langle\lambda_2\rangle^{\omega-b}\right\|_{L^2}.$$

In $D_{11}\cap A_2$ we have

$$|\xi|^{\frac{\alpha}{2}} \leq c||\xi|^\alpha - |\xi_1|^\alpha|^{\frac{1}{2}} \text{ and } \langle\lambda_2\rangle^{\omega-b} \leq \langle\lambda\rangle^{\omega-b},$$

such that, because of $b'+\omega \leq 0$,

$$\begin{aligned}\|J_{11,2}\|_{L^2} &\leq c\left\|K_*^{\frac{\alpha}{2}}(\overline{v}_1, \mathcal{F}^{-1}f_2)\right\|_{X_{0,0,-b}}\\ &\leq c\|v_1\|_{X_{0,0,b}}\|\mathcal{F}^{-1}f_2\|_{L^2} = c\prod_{i=1}^2\|f_i\|_{L^2}\end{aligned}$$

for $b > 1/2$ by the estimate (3.12).

Let us now consider the region D_{12}. We define the contributions

$$J_{12,0} = \int_* \chi_{D_{12}\cap A}|\xi|^{1-\omega}\langle\xi\rangle^{s-\alpha\omega}\langle\lambda\rangle^{b'}\langle\sigma\rangle^\omega \prod_{i=1}^2 \frac{|\xi_i|^\omega\langle\xi_i\rangle^{\alpha\omega-s}f_i(\tau_i,\xi_i)}{\langle\lambda_i\rangle^b\langle\sigma_i\rangle^\omega}$$

and, for $k=1,2$,

$$J_{12,k} = \int_* \chi_{D_{12}\cap A_k}|\xi|^{1-\omega}\langle\xi\rangle^{s-\alpha\omega}\langle\lambda\rangle^{b'}\langle\sigma\rangle^\omega \prod_{i=1}^2 \frac{|\xi_i|^\omega\langle\xi_i\rangle^{\alpha\omega-s}f_i(\tau_i,\xi_i)}{\langle\lambda_i\rangle^b\langle\sigma_i\rangle^\omega}.$$

In the subregion $D_{12}\cap A$ we use

$$|\xi|^{-\alpha b'}\langle\xi_1\rangle^{-b'} \leq c\langle\lambda\rangle^{-b'}$$

and $\|J_{12,0}\|_{L^2}$ is bounded by

$$\left\| \int_* \chi_{D_{12}\cap A} |\xi|^{1-\omega+\alpha b'} \langle \xi\rangle^{s-\alpha\omega} \langle\sigma\rangle^{\omega} \langle \xi_1\rangle^{b'+\alpha\omega-s} \langle\xi_2\rangle^{\alpha\omega-s} \prod_{i=1}^{2} \frac{f_i(\tau_i,\xi_i)|\xi_i|^{\omega}}{\langle\lambda_i\rangle^b \langle\sigma_i\rangle^{\omega}} \right\|_{L^2}.$$

Using $\langle\xi_2\rangle^{\alpha\omega-s}\langle\xi\rangle^{s-\alpha\omega} \le c$ and (4.4) this is bounded by

$$\left\| \int_* \chi_{D_{12}\cap A} |\xi|^{1+\alpha b'} \frac{\langle \xi_1\rangle^{b'+\alpha\omega-s+\omega}}{\min_{i=1,2}\langle\sigma_i\rangle^{\omega}} \prod_{i=1}^{2} \frac{f_i(\tau_i,\xi_i)}{\langle\lambda_i\rangle^b} \right\|_{L^2}.$$

Now, for $b'+\omega \le 0$ we estimate

$$|\xi|^{1+\alpha b'} \le c|\xi|^{\frac{\alpha}{2}} \langle\xi_1\rangle^{1-\frac{\alpha}{2}+\alpha b'},$$

since $1-\frac{\alpha}{2}+\alpha b' \le 0$. Moreover,

$$1-\frac{\alpha}{2}+\alpha b' + b' + \alpha\omega - s + \omega - (1+\alpha)\omega = 1-\frac{\alpha}{2}+\alpha b' + b' - s,$$

which is negative for

$$s \ge \alpha(-\frac{1}{2}+b') + 1 + b'$$

Therefore, choosing $b' \le \min\{-\omega, -\frac{1}{4}\}$, we continue for $s \ge -\frac{3}{4}(\alpha-1)$ with

$$\left\| \int_* \chi_{D_{12}\cap A} |\xi|^{\frac{\alpha}{2}} \prod_{i=1}^{2} \frac{f_i(\tau_i,\xi_i)}{\langle\lambda_i\rangle^b} \right\|_{L^2} \le c \left\| I_*^{\frac{\alpha}{2}}(v_1,v_2)\right\|_{L^2} \le c \prod_{i=1}^{2} \|f_i\|_{L^2}.$$

Next, we study the contribution of $J_{12,1}$. We may assume that $\langle\lambda_1\rangle \ge 2\langle\lambda\rangle$, because otherwise we can use the same argument as in $D_{12}\cap A$. In $D_{12}\cap A_1$ we exploit the inequality

$$|\xi|\langle\xi_1\rangle^{\frac{1}{\alpha}} \le c\langle\lambda_1\rangle^{\frac{1}{\alpha}}.$$

We observe that

$$|\lambda_1| = |\tau_1 - \xi_1|\xi_1|^{\alpha}| \le |\lambda| + c\langle\sigma_2\rangle \Rightarrow \langle\lambda_1\rangle \le c\langle\sigma_2\rangle$$

and therefore

$$\langle\lambda_1\rangle^{\omega} \le c \min_{i=1,2} \langle\sigma_i\rangle^{\omega}.$$

This shows

$$\|J_{12,1}\|_{L^2} \le c \left\| \int_* \chi_{D_{12}\cap A_1} \langle\lambda\rangle^{b'} \langle\xi_1\rangle^{-\frac{1}{\alpha}+\omega+\alpha\omega-s} \langle\lambda_1\rangle^{\frac{1}{2}-b} \langle\lambda_2\rangle^{-b} \prod_{i=1}^{2} f_i(\tau_i,\xi_i) \right\|_{L^2}.$$

We choose $b > \frac{1}{2}$ and in D_{12} we have $|\xi_1| \le |\xi_2|$. Since we only consider $s \le \frac{1}{2} - \frac{\alpha}{2}$ (which means $\varepsilon \le \frac{\alpha-1}{4}$), we have

$$\|J_{12,1}\|_{L^2} \le c \left\| \int_* \langle\lambda\rangle^{b'} \langle\xi_2\rangle^{\frac{1}{2}-\frac{\alpha}{2}-s} \langle\lambda_2\rangle^{-b} \prod_{i=1}^{2} f_i(\tau_i,\xi_i) \right\|_{L^2}.$$

With $b' \le -\frac{1}{4}$ and using Sobolev in time, we see that

$$\begin{aligned}\|J_{12,1}\|_{L^2} &\le c \left\| \mathcal{F}^{-1} f_1 J^{\frac{1}{2}-\frac{\alpha}{2}-s} v_2 \right\|_{L_t^{4/3} L_x^2} \\ &\le c \|f_1\|_{L_t^2 L_x^2} \|J^{\frac{1}{2}-\frac{\alpha}{2}-s} v_2\|_{L_t^4 L_x^\infty}.\end{aligned}$$

Finally, by (3.4) we have

$$\|J^{\frac{1}{2}-\frac{\alpha}{2}-s} v_2\|_{L_t^4 L_x^\infty} \le c\|v_2\|_{X_{0,0,b}} = \|f_2\|_{L^2}$$

if $\frac{1}{2} - \frac{\alpha}{2} - s \le \frac{\alpha-1}{4}$, which is equivalent to $s \ge -\frac{3}{4}(\alpha-1)$.

Now we turn to the contribution of $D_{12} \cap A_2$, where we use the inequality

$$|\xi|^{-\alpha b'} \langle \xi_1 \rangle^{-b'} \le c \langle \lambda_2 \rangle^{-b'}.$$

It follows that

$$\begin{aligned}&\|J_{12,2}\|_{L^2} \\ &\le c \left\| \int_* \chi_{D_{12} \cap A_2} |\xi|^{1+\alpha b'} \frac{\langle \xi_1 \rangle^{b'+\omega+\alpha\omega-s}}{\min_{i=1,2} \langle \sigma_i \rangle^{\omega}} \langle \lambda \rangle^{b'} \langle \lambda_2 \rangle^{-b'-b} \langle \lambda_1 \rangle^{-b} \prod_{i=1}^{2} f_i(\tau_i, \xi_i) \right\|_{L^2}.\end{aligned}$$

We have

$$\langle \lambda \rangle^{b'} \langle \lambda_2 \rangle^{-b'-b} \le \langle \lambda \rangle^{-b}$$

and

$$\min_{i=1,2} \langle \sigma_i \rangle^{\omega} \ge \langle \xi_1 \rangle^{(1+\alpha)\omega},$$

and if $b' \le -\omega$ we have $1 + \alpha b' - \frac{\alpha}{2} \le 0$ and therefore

$$|\xi|^{1+\alpha b'} \le c|\xi|^{\frac{\alpha}{2}} \langle \xi_1 \rangle^{1+\alpha b' - \frac{\alpha}{2}}.$$

If $b' \le -\frac{1}{4}$ and $s \ge -\frac{3}{4}(\alpha-1)$ we estimate $b' - s + 1 + \alpha b' - \frac{\alpha}{2} \le 0$ and

$$|\xi|^{\frac{\alpha}{2}} \le c||\xi|^{\alpha} - |\xi_1|^{\alpha}|^{\frac{1}{2}}.$$

Therefore, by the dual bilinear Strichartz estimate (3.12),

$$\begin{aligned}\|J_{12,2}\|_{L^2} &\le c \left\| \int_* ||\xi|^{\alpha} - |\xi_1|^{\alpha}|^{\frac{1}{2}} \langle \lambda \rangle^{-b} \langle \lambda_1 \rangle^{-b} \prod_{i=1}^{2} f_i(\tau_i, \xi_i) \right\|_{L^2} \\ &\le c \prod_{i=1}^{2} \|f_i\|_{L^2}.\end{aligned}$$

This completes the discussion of the subregion D_1.

Let us now consider the domain D_2, where $|\xi_1| \le |\xi_2| \le 4|\xi_1|$, $|\xi| \le 2|\xi_2|$ and $|\xi| \le 5|\xi_1|$. We subdivide $D_2 = D_{21} \cup D_{22}$, where D_{21} and D_{22} are defined by

$$D_{21} \, : \, \xi_1 \xi_2 > 0 \text{ or } |\xi| \ge \frac{1}{2}|\xi_1| \text{ or } |\xi_2| \le 1$$

and

$$D_{22}\ :\ \xi_1\xi_2 < 0 \text{ and } |\xi| \leq \frac{1}{2}|\xi_1| \text{ and } |\xi_2| \geq 1.$$

As above, we define for $j = 1, 2$

$$J_{2j,0} = \int_* \chi_{D_{2j}\cap A} |\xi|^{1-\omega}\langle\xi\rangle^{s-\alpha\omega}\langle\lambda\rangle^{b'}\langle\sigma\rangle^{\omega}\prod_{i=1}^{2}\frac{|\xi_i|^{\omega}\langle\xi_i\rangle^{\alpha\omega-s}f_i(\tau_i,\xi_i)}{\langle\lambda_i\rangle^{b}\langle\sigma_i\rangle^{\omega}}$$

and for $k = 1, 2$

$$J_{2j,k} = \int_* \chi_{D_{2j}\cap A_k} |\xi|^{1-\omega}\langle\xi\rangle^{s-\alpha\omega}\langle\lambda\rangle^{b'}\langle\sigma\rangle^{\omega}\prod_{i=1}^{2}\frac{|\xi_i|^{\omega}\langle\xi_i\rangle^{\alpha\omega-s}f_i(\tau_i,\xi_i)}{\langle\lambda_i\rangle^{b}\langle\sigma_i\rangle^{\omega}}.$$

We start with the discussion of D_{21}, where all frequencies are of a size comparable to or smaller than a constant. This shows that

$$\frac{|\xi|^{1-\omega}\langle\xi\rangle^{s-\alpha\omega}|\xi_1|^{\omega}|\xi_2|^{\omega}\langle\sigma\rangle^{\omega}}{\langle\xi_1\rangle^{s-\alpha\omega}\langle\xi_2\rangle^{s-\alpha\omega}\langle\sigma_1\rangle^{\omega}\langle\sigma_2\rangle^{\omega}} \leq c\langle\xi\rangle^{1-s}.$$

Therefore,

$$\|J_{21,0}\|_{L^2} \leq c\left\|\int_* \chi_{D_{21}\cap A}\langle\xi\rangle^{1-s}\langle\lambda\rangle^{b'}\prod_{i=1}^{2}\frac{f_i(\tau_i,\xi_i)}{\langle\lambda_i\rangle^{b}}\right\|_{L^2}.$$

In A we have

$$\langle\xi\rangle^{-b'(1+\alpha)} \leq c\langle\lambda\rangle^{-b'},$$

and using the Strichartz estimate (3.4) we conclude

$$\begin{aligned}\|J_{21,0}\|_{L^2} &\leq c\left\|\int_* \langle\xi\rangle^{1-s+b'(1+\alpha)-\frac{\alpha-1}{4}}\langle\xi_1\rangle^{\frac{\alpha-1}{4}}\mathcal{F}v_1(\tau_1,\xi_1)\mathcal{F}v_2(\tau_2,\xi_2)\right\|_{L^2}\\ &\leq c\|J^{\frac{\alpha-1}{4}}v_1\|_{L^4_tL^\infty_x}\|v_2\|_{L^4_tL^2_x} \leq c\prod_{i=1}^{2}\|f_i\|_{L^2},\end{aligned}$$

since $1-s+b'(1+\alpha)-\frac{\alpha-1}{4} \leq 0$, which is equivalent to $\frac{5}{4}+b'-\frac{\alpha}{4}+\alpha b' \leq s$. This is fulfilled for $b' \leq -\frac{1}{2}+\frac{\varepsilon}{3}$. In A_1 we have

$$\langle\xi\rangle^{b(1+\alpha)} \leq c\langle\lambda_1\rangle^{b},$$

and using Sobolev in time and the Strichartz estimate (3.4), we conclude for $b' \leq -\frac{1}{4}$

$$\begin{aligned}\|J_{21,1}\|_{L^2} &\leq c\left\|\int_* \langle\xi\rangle^{1-s-b(1+\alpha)-\frac{\alpha-1}{4}}\langle\lambda\rangle^{b'}f_1(\tau_1,\xi_1)\langle\xi_2\rangle^{\frac{\alpha-1}{4}}\mathcal{F}v_2(\tau_2,\xi_2)\right\|_{L^2}\\ &\leq c\|\mathcal{F}^{-1}f_1J^{\frac{\alpha-1}{4}}v_2\|_{L^{4/3}_tL^2_x} \leq c\|f_1\|_{L^2_{tx}}\|J^{\frac{\alpha-1}{4}}v_2\|_{L^4_tL^\infty_x}\\ &\leq c\prod_{i=1}^{2}\|f_i\|_{L^2}.\end{aligned}$$

The same argument applies to $J_{21,2}$ by interchanging the roles of f_1, f_2.

Finally, we turn to the contributions from the region D_{22}. Here, we have $\xi_1\xi_2 < 0$. Therefore, we may write $\xi_1 = \beta\xi_2$ for $\beta \in [-1, -\frac{1}{4}]$. By the mean value theorem, this shows

$$||\xi_1|^\alpha - |\xi_2|^\alpha|^{\frac{1}{2}} = ||\beta|^\alpha - 1|^{\frac{1}{2}} |\xi_2|^{\frac{\alpha}{2}} \geq \frac{1}{2}||\beta| - 1|^{\frac{1}{2}}|\xi_2|^{\frac{\alpha}{2}} = \frac{1}{2}|\xi|^{\frac{1}{2}}|\xi_2|^{\frac{\alpha-1}{2}}. \tag{4.5}$$

Let us start with the subregion A. We have

$$\langle\sigma\rangle^\omega \leq c\langle\lambda\rangle^\omega + c\chi_{|\xi|\geq 1}\langle\xi\rangle^{\omega+\alpha\omega},$$

which implies

$$\begin{aligned}\|J_{22,0}\|_{L^2} \leq c &\left\| \int_* \chi_{D_{22}\cap A}|\xi|^{1-\omega}\langle\xi\rangle^{s-\alpha\omega}\langle\lambda\rangle^{b'+\omega} \prod_{i=1}^2 \frac{|\xi_i|^\omega\langle\xi_i\rangle^{\alpha\omega-s} f_i(\tau_i,\xi_i)}{\langle\lambda_i\rangle^b\langle\sigma_i\rangle^\omega}\right\|_{L^2}\\ &+ c\left\| \int_* \chi_{D_{22}\cap A}\chi_{|\xi|\geq 1}\langle\xi\rangle^{1+s}\langle\lambda\rangle^{b'} \prod_{i=1}^2 \frac{|\xi_i|^\omega\langle\xi_i\rangle^{\alpha\omega-s} f_i(\tau_i,\xi_i)}{\langle\lambda_i\rangle^b\langle\sigma_i\rangle^\omega}\right\|_{L^2}.\end{aligned}$$

Using

$$|\xi|^{-b'-\omega}\langle\xi_2\rangle^{-\alpha b'-\alpha\omega} \leq c\langle\lambda\rangle^{-b'-\omega},$$

and (4.5) we see that the first term is bounded by

$$\begin{aligned}&\left\| \int_* \chi_{D_{22}\cap A}|\xi|^{1+b'}\langle\xi\rangle^{s-\alpha\omega}\langle\xi_2\rangle^{-2s+\alpha b'+\alpha\omega} \prod_{i=1}^2 \frac{f_i(\tau_i,\xi_i)}{\langle\lambda_i\rangle^b}\right\|_{L^2}\\ &\leq c\left\| \int_* \langle\xi\rangle^{\frac{1}{2}+b'+s-\alpha\omega}\langle\xi_2\rangle^{-2s+\alpha b'+\alpha\omega-\frac{\alpha-1}{2}}\,||\xi_1|^\alpha - |\xi_2|^\alpha|^{\frac{1}{2}} \prod_{i=1}^2 \frac{f_i(\tau_i,\xi_i)}{\langle\lambda_i\rangle^b}\right\|_{L^2}.\end{aligned}$$

If $b' \leq -\frac{1}{4}$ and $\varepsilon \leq \frac{1}{4}$, then $\frac{1}{2} + b' + s - \alpha\omega \leq 0$. Moreover, for $b' \leq -\frac{1}{2} + \varepsilon$, we have $-2s + \alpha b' + \alpha\omega - \frac{\alpha-1}{2} \leq 0$. Therefore, by the bilinear Strichartz estimate (3.11), this is bounded by

$$\cdots \leq c\left\| \int_* ||\xi_1|^\alpha - |\xi_2|^\alpha|^{\frac{1}{2}} \prod_{i=1}^2 \frac{f_i(\tau_i,\xi_i)}{\langle\lambda_i\rangle^b}\right\|_{L^2} \leq c\prod_{i=1}^2 \|f_i\|_{L^2}.$$

For the second term we use

$$|\xi|^{-b'}\langle\xi_2\rangle^{-\alpha b'} \leq c\langle\lambda\rangle^{-b'},$$

and with (4.5) obtain the bound

$$\begin{aligned}\cdots &\leq c\left\| \int_* \chi_{D_{22}\cap A}\chi_{|\xi|\geq 1}\langle\xi\rangle^{s+1+b'}\langle\xi_2\rangle^{\alpha b'-2s} \prod_{i=1}^2 \frac{f_i(\tau_i,\xi_i)}{\langle\lambda_i\rangle^b}\right\|_{L^2}\\ &\leq c\left\| \int_* \langle\xi\rangle^{\frac{1}{2}+s+b'}\langle\xi_2\rangle^{\alpha b'-2s-\frac{\alpha-1}{2}}\,||\xi_1|^\alpha - |\xi_2|^\alpha|^{\frac{1}{2}} \prod_{i=1}^2 \frac{f_i(\tau_i,\xi_i)}{\langle\lambda_i\rangle^b}\right\|_{L^2}.\end{aligned}$$

We only consider $\varepsilon < \frac{3}{4}(\alpha-1)$. Then, for $b' \le -\frac{1}{2} + \frac{3}{4}(\alpha-1) - \varepsilon$ we observe that $\frac{1}{2} + s + b' \le 0$. Moreover, $\alpha b' - 2s - \frac{\alpha-1}{2} \le 0$ for $b' \le -\frac{1}{2} + \varepsilon$. Using the bilinear Strichartz estimate (3.11), we arrive at

$$\|J_{22,0}\|_{L^2} \le c \prod_{i=1}^{2} \|f_i\|_{L^2}.$$

Next, we consider the subregion A_1. We have

$$\langle\sigma\rangle^{\omega} \le c\langle\lambda_1\rangle^{\omega} + c\chi_{|\xi|\ge 1}\langle\xi\rangle^{\omega+\alpha\omega},$$

which shows that $\|J_{22,1}\|_{L^2}$ is bounded by

$$\left\| \int_* \chi_{D_{22}\cap A_1} |\xi|^{1-\omega} \langle\xi\rangle^{s-\alpha\omega} \langle\xi_2\rangle^{-2s} \langle\lambda\rangle^{b'} \langle\lambda_1\rangle^{-b+\omega} \langle\lambda_2\rangle^{-b} \prod_{i=1}^{2} f_i(\tau_i,\xi_i) \right\|_{L^2}$$
$$+ \left\| \int_* \chi_{D_{22}\cap A_1} \chi_{|\xi|\ge 1} \langle\xi\rangle^{1+s} \langle\lambda\rangle^{b'} \langle\lambda_1\rangle^{-b} \langle\lambda_2\rangle^{-b} \langle\xi_2\rangle^{-2s} \prod_{i=1}^{2} f_i(\tau_i,\xi_i) \right\|_{L^2}.$$

As above, using

$$|\xi|^{-b'-\omega} \langle\xi_2\rangle^{-\alpha b'-\alpha\omega} \le c\langle\lambda_1\rangle^{-b'-\omega}$$

we see that the first term is bounded by

$$\left\| \int_* \chi_{D_{22}\cap A} |\xi|^{1+b'} \langle\xi\rangle^{s-\alpha\omega} \langle\xi_2\rangle^{-2s+\alpha b'+\alpha\omega} \langle\lambda\rangle^{-b} \langle\lambda_2\rangle^{-b} \prod_{i=1}^{2} f_i(\tau_i,\xi_i) \right\|_{L^2},$$

which in turn is controlled by

$$\left\| \int_* \langle\xi\rangle^{1+b'+s-\alpha\omega} \langle\xi_2\rangle^{-2s+\alpha b'+\alpha\omega-\frac{\alpha}{2}} \left||\xi|^{\alpha} - |\xi_2|^{\alpha}\right|^{\frac{1}{2}} \langle\lambda\rangle^{-b} \langle\lambda_2\rangle^{-b} \prod_{i=1}^{2} f_i(\tau_i,\xi_i) \right\|_{L^2}.$$

Here, we used that due to the inequalities $|\xi| \le \frac{3}{4}|\xi_2|$ and $|\xi_2| \ge 1$ we have

$$\left||\xi|^{\alpha} - |\xi_2|^{\alpha}\right|^{\frac{1}{2}} \ge c\langle\xi_2\rangle^{\frac{\alpha}{2}}.$$

By estimating $\langle\xi\rangle^{\frac{1}{2}} \le \langle\xi_2\rangle^{\frac{1}{2}}$ and with the same restrictions on s, b' as above we may apply the dual bilinear Strichartz estimate (3.12) and get

$$\cdots \le c \left\| \int_* \left||\xi|^{\alpha} - |\xi_2|^{\alpha}\right|^{\frac{1}{2}} \langle\lambda\rangle^{-b} \langle\lambda_2\rangle^{-b} \prod_{i=1}^{2} f_i(\tau_i,\xi_i) \right\|_{L^2} \le c \prod_{i=1}^{2} \|f_i\|_{L^2}.$$

For the second term we use

$$|\xi|^{-b'} \langle\xi_2\rangle^{-\alpha b'} \le c\langle\lambda_1\rangle^{-b'},$$

and obtain

$$\cdots \le c \left\| \int_* \langle\xi\rangle^{1+s+b'} \langle\xi_2\rangle^{\alpha b'-2s-\frac{\alpha}{2}} \left||\xi|^\alpha - |\xi_2|^\alpha\right|^{\frac{1}{2}} \langle\lambda\rangle^{-b}\langle\lambda_2\rangle^{-b} \prod_{i=1}^{2} f_i(\tau_i,\xi_i) \right\|_{L^2}$$
$$\le c \prod_{i=1}^{2} \|f_i\|_{L^2}$$

by (3.12) with the same restrictions on s, b', b as in the region A, since $\langle\xi\rangle^{\frac{1}{2}} \le \langle\xi_2\rangle^{\frac{1}{2}}$.

Finally, we turn to the region A_2. In D_{22} the frequencies ξ_1 and ξ_2 are of comparable size and due to the inequalities $|\xi| \le \frac{1}{2}|\xi_1|$ and $|\xi_1| \ge \frac{1}{4}|\xi_2| \ge \frac{1}{4}$ we have

$$\left||\xi|^\alpha - |\xi_1|^\alpha\right|^{\frac{1}{2}} \ge c\langle\xi_1\rangle^{\frac{\alpha}{2}}.$$

Now we use the same argument as for A_1 with the roles of f_1, f_2 interchanged.

This finishes the proof of the bilinear estimate for $s = s_0 = -\frac{3}{4}(\alpha-1)+\varepsilon$, for $\varepsilon \le \frac{\alpha-1}{4}$. The restrictions on b' can be summarized by

$$b' \le \min\left\{-\frac{1}{4}, -\omega, -\frac{1}{2}+\frac{\varepsilon}{3}, -\frac{1}{2}+\frac{3}{4}(\alpha-1)-\varepsilon\right\}.$$

For b we assumed $\frac{1}{2} < b < b'+1$. Now we turn to the case $s > s_0 = -\frac{3}{4}(\alpha-1)+\varepsilon$. Let $\rho = s - s_0$. Because of the inequality

$$\langle\xi\rangle^\rho \le c\langle\xi_1\rangle^\rho + c\langle\xi_2\rangle^\rho \tag{4.6}$$

we see that

$$\begin{aligned}\|\partial_x(u_1u_2)\|_{X_{s,\omega,b'}} &\le c\|\partial_x(J^\rho u_1u_2)\|_{X_{s_0,\omega,b'}} + \|\partial_x(u_1J^\rho u_2)\|_{X_{s_0,\omega,b'}}\\ &\le c\|u_1\|_{X_{s,\omega,b}}\|u_2\|_{X_{s_0,\omega,b}} + \|u_1\|_{X_{s_0,\omega,b}}\|u_2\|_{X_{s,\omega,b}}.\end{aligned}$$

This proves that for all $s \ge s_0 > -\frac{3}{4}(\alpha-1)$ we can find suitable numbers $b' \in (-\frac{1}{2}, 0)$ and $b \in (\frac{1}{2}, b'+1)$ such that the bilinear estimate holds true. □

Remark 4.2. Instead of (4.6) one could also perform a similar argument with an elliptic weight in order to increase regularity in t and x simultaneously.

5. Proof of the main results

In this section we prove Theorems 2.6 and 2.11. The proofs will be a standard application of methods which are well known from the literature. Throughout this section let $1 < \alpha < 2$, $s \ge s_0 > -\frac{3}{4}(\alpha-1)$, and $\omega = 1/\alpha - 1/2$. Moreover, we fix b', b according to Theorem 4.1. We may restrict ourselves to $0 < T \le 1$, since the same arguments apply on any compact time interval. For $u \in \mathcal{S}(\mathbb{R}^2)$ we define

$$\Phi_T(u)(t) := -\frac{1}{2}\psi_T(t)\int_0^t W_\alpha(t-t')\partial_x(u^2)(t')\,dt'.$$

An application of Lemma 3.2 and the bilinear estimate (4.1) allows us to extend Φ_T uniquely to

$$\Phi_T : X_{s,\omega,b} \to X_{s,\omega,b},$$

such that

$$\|\Phi_T(u) - \Phi_T(v)\|_{X_{s,\omega,b}} \leq cT^\varepsilon(\|u\|_{X_{s,\omega,b}} + \|v\|_{X_{s,\omega,b}})\|u - v\|_{X_{s,\omega,b}} \tag{5.1}$$

holds true. We can also define

$$\Phi_T \mid_{[-T,T]}: X^T_{s,\omega,b} \to X^T_{s,\omega,b},$$

since $\Phi_T(u) \mid_{[-T,T]}$ only depends on $u \mid_{[-T,T]}$.

Definition 5.1. We say that $u \in X^T_{s,\omega,b} \subset C([-T,T], H^{(s,\omega)})$ is a solution of our problem (1.1) on $[-T,T]$, if

$$u(t) = W_\alpha(t)u_0 + \Phi_T(u)(t), \quad \text{for } t \in [-T,T]. \tag{5.2}$$

We divide the proof of Theorems 2.6 and 2.11 into four parts.

Proof of Theorem 2.6: local existence and analytic dependence. For $0 < T \leq 1$ we define

$$\Lambda_T : H^{(s,\omega)} \times X_{s,\omega,b} \to X_{s,\omega,b},$$
$$\Lambda_T(u_0, u) := \psi W_\alpha u_0 + \Phi_T(u).$$

Obviously, Λ_T is an analytic map (see [7], Definition 15.1), since it is a composition of bounded linear and bilinear maps. Let $u_0 \in H^{(s,\omega)}$ with $\|u_0\|_{H^{(s,\omega)}} \leq r$ and $u \in X_{s,\omega,b}$ with $\|u\|_{X_{s,\omega,b}} \leq R$. Then, by (3.1) and the estimate (5.1),

$$\|\Lambda_T(u_0, u)\|_{X_{s,\omega,b}} \leq cr + RcT^\varepsilon\|u\|_{X_{s,\omega,b}} < R$$

for $R = 2cr$ and $T^\varepsilon = (8c^2r)^{-1}$. With these choices of R and T we restrict Λ_T to closed balls $\overline{B}_r \times \overline{B}_R \subset H^{(s,\omega)} \times X_{s,\omega,b}$. The bilinear estimate (5.1) shows that

$$\Lambda_T(u_0, \cdot) : \overline{B}_R \to \overline{B}_R$$

is a strict contraction, uniformly in $u_0 \in \overline{B}_r$. Therefore we obtain

$$F : H^{(s,w)} \supset B_r \to B_R \subset X_{s,\omega,b}$$

with

$$\Lambda_T(u_0, u) = u \in B_R \iff u = F(u_0)$$

for all $u_0 \in B_r$. An application of the implicit function theorem (see [7], Theorem 15.3) to $\mathrm{Id} - \Lambda_T$ yields the analyticity of F and also that of $F\big|_{[-T,T]} : H^{(s,w)} \to X^T_{s,\omega,b}$. Moreover, the functions $F(u_0)\big|_{[-T,T]} \in X^T_{s,\omega,b}$ are solutions of (5.2). □

Proof of Theorem 2.6: persistence and uniqueness. The persistence property follows from the embedding $X_{s,\omega,b} \subset C(\mathbb{R}, H^{(s,\omega)})$. Assume that $u, v \in X^T_{s_0,\omega,b}$ are two solutions of (5.2) with extensions $\tilde{u}, \tilde{v} \in X_{s_0,\omega,b}$, such that

$$T' := \sup\{t \in [0,T] \mid u(t) = v(t)\} < T.$$

Define $u^*(t) := \tilde{u}(t+T')$, $v^*(t) := \tilde{v}(t+T')$ for $-T' \leq t \leq T - T'$. Because both u and v are solutions of (5.2), we see by approximation with smooth functions that

$$u^*(t) - v^*(t) = \Phi_T(u^*)(t) - \Phi_T(v^*)(t) \tag{5.3}$$

for $-T' \leq t \leq T - T'$. Therefore, for small $\delta > 0$ we have

$$\|\psi_\delta(u^* - v^*)\|_{X_{s_0,\omega,b}} \leq c\delta^\varepsilon \, \|\psi_\delta(u^* - v^*)\|_{X_{s_0,\omega,b}} \left(\|u^*\|_{X_{s_0,\omega,b}} + \|v^*\|_{X_{s_0,\omega,b}}\right).$$

By choosing δ small enough we conclude $u^*(t) = v^*(t)$ for $|t| \leq \delta$, which implies $u(t+T') = v(t+T')$ for $|t| \leq \delta$. This contradicts the definition of T'. If u, v did not coincide on $[-T, 0]$, we would obtain a similar contradiction. □

LEMMA 5.2. *Let $s \geq 0$. There exists $C > 0$, such that for all smooth, real valued solutions u of* (1.1), *we have*

$$\sup_{t \in [-T,T]} \|u(t)\|_{H^{(0,\omega)}} \leq C\|u(0)\|_{H^{(0,\omega)}} + CT\|u(0)\|^2_{H^{(0,\omega)}}. \tag{5.4}$$

Proof. We easily verify the conservation law

$$\|u(t)\|^2_{L^2} = \|u(0)\|^2_{L^2}, \quad t \in (-T,T).$$

Therefore it suffices to prove an a priori estimate for the low frequency part in $\dot{H}^{-\omega}$. Let $\psi \in C_0^\infty([-2,2])$ be nonnegative and symmetric with $\psi|_{[-1,1]} \equiv 1$. We define

$$\mathcal{F}_x v(t)(\xi) = \psi(\xi)|\xi|^{-\omega}\mathcal{F}_x u(t)(\xi).$$

The function v solves the equation

$$\begin{aligned} v_t - |D|^\alpha v_x &= f \ \text{ in } (-T,T) \times \mathbb{R}, \\ v(0) &= v_0, \end{aligned}$$

where v_0, f are defined via $\mathcal{F}_x v_0(\xi) = \psi(\xi)|\xi|^{-\omega}\mathcal{F}_x u(0)(\xi)$ and $\mathcal{F}_x f(t)(\xi) = -\frac{i}{2}\psi(\xi)\xi|\xi|^{-\omega}\mathcal{F}_x u^2(t)(\xi)$, respectively. For fixed t we estimate

$$\begin{aligned} \|f(t)\|_{L^2_x} &\leq c\|\psi(\xi)\mathcal{F}_x u^2(t)(\xi)\|_{L^2_\xi} \leq c\|\mathcal{F}_x u^2(t)\|_{L^\infty_\xi} \\ &\leq c\|u^2(t)\|_{L^1_x} \leq c\|u(t)\|^2_{L^2_x}. \end{aligned}$$

This shows that

$$\begin{aligned} \|v\|_{L^\infty_T L^2_x} &\leq c\|v_0\|_{L^2_x} + c\|f\|_{L^1_T L^2_x} \leq c\|u(0)\|_{H^{(0,\omega)}} + cT\|u\|^2_{L^\infty_T L^2_x} \\ &\leq c\|u(0)\|_{H^{(0,\omega)}} + cT\|u(0)\|^2_{H^{(0,\omega)}}, \end{aligned}$$

as desired. □

Proof of Theorem 2.6: time of existence. We fix $s \geq s_0$ and a ball

$$B_{r,r_s} = \{v_0 \in H^{(s,\omega)} \mid \|v_0\|_{H^{(s_0,\omega)}} \leq r \text{ and } \|v_0\|_{H^{(s,\omega)}} \leq r_s\}$$

and define T_s as the supremum of all $T \in [0,1]$ such that the following statement is true: There exists an analytic map $F : B_{r,r_s} \to X_{s,\omega,b}$, such that

$$\Lambda_T(v_0, F(v_0)) = F(v_0),$$

and if $u \in X^T_{s_0,\omega,b}$ is a solution of (5.2), then

$$u\big|_{[-T,T]} = F(v_0)\big|_{[-T,T]}.$$

Parts 1 and 2 of the proof show that $T_s > 0$. Let $v = F(v_0) \in X_{s,\omega,b}$. If $T_s^\varepsilon \leq (8c_{s_0}^2 r)^{-1} < 1$, we see from the proof of part 1 that $\|v\|_{X_{s_0,\omega,b}} \leq 2c_{s_0} r$. An application of our bilinear estimate (4.1) together with (3.1), (3.3) gives

$$\|v\|_{X_{s,\omega,b}} \leq c_s r_s + c_s T_s^\varepsilon \|v\|_{X_{s_0,\omega,b}} \|v\|_{X_{s,\omega,b}}.$$

Therefore,

$$\|v\|_{X_{s,\omega,b}} \leq c_s r_s + 2c_s c_{s_0} r T_s^\varepsilon \|v\|_{X_{s,\omega,b}},$$

and if additionally $T_s^\varepsilon \leq (4c_{s_0} c_s r)^{-1}$, then we conclude

$$\sup_{|t| \leq T_s} \|v(t)\|_{H^{(s,\omega)}} \leq c\|v\|_{X_{s,\omega,b}} < C_s r_s. \tag{5.5}$$

If these assumptions about T_s hold, we can apply parts 1 and 2 of the proof. We find a $\delta > 0$ and an analytic map $G : H^{(s,\omega)} \supset B_{C_s r_s} \to X_{s,\omega,b}$ such that

$$\Lambda_{2\delta}(w_0, G(w_0)) = G(w_0),$$

and if $u \in X^{2\delta}_{s_0,\omega,b}$ is a solution of (5.2) with initial datum $w_0 \in B_{C_s r_s}$, then

$$u\big|_{[-2\delta,2\delta]} = G(w_0)\big|_{[-2\delta,2\delta]}.$$

Define

$$H : v_0 \mapsto \chi_\delta F(v_0) + \chi_\delta^+ G(F(v_0)(T_s))(\cdot - T_s) + \chi_\delta^- G(F(v_0)(-T_s))(\cdot + T_s)$$

as a map from B_{r,r_s} to $X_{s,\omega,b}$ with smooth cutoff functions $\chi_\delta, \chi_\delta^+, \chi_\delta^-$, such that $\chi_\delta + \chi_\delta^+ + \chi_\delta^- = 1$ on $[-T_s - \delta, T_s + \delta]$ with

$$\operatorname{supp}(\chi_\delta) \subset [-T_s + \delta, T_s - \delta], \quad \operatorname{supp}(\chi_\delta^\pm) \subset [\pm T_s - 2\delta, \pm T_s + 2\delta].$$

It is not hard to verify that H is analytic, since it is a composition of analytic maps, and that H satisfies

$$\Lambda_{T_s+\delta}(v_0, H(v_0)) = H(v_0).$$

If $u \in X^{T_s+\delta}_{s_0,\omega,b}$ is a solution of (5.2), then part 2 of the proof also gives

$$u\big|_{[-T_s-\delta,T_s+\delta]} = H(v_0)\big|_{[-T_s-\delta,T_s+\delta]},$$

which contradicts the definition of T_s, and we conclude that

$$T_s^\varepsilon \geq \min\{(4c_{s_0} c_s r)^{-1}, (8c_{s_0}^2 r)^{-1}\}.$$

This lower bound shows that if $T_s < 1$, then

$$\lim_{t \uparrow T_s} \|u(t)\|_{H^{(s_0,\omega)}} = \infty \tag{5.6}$$

because otherwise we could iterate the argument above. □

Proof of Theorem 2.11. The same proof as above applies in the closed subspaces of real valued functions in $H^{(s,\omega)}$, $X_{s,\omega,b}$ and $X^T_{s,\omega,b}$. We regard these as Hilbert spaces over the real numbers and the analytic flow maps as real analytic. Using (5.6) and the a priori bound (5.4) this proves that $T_s = 1$ for all $s \geq 0$. As already mentioned at the beginning of this section, the same arguments may be applied to any compact time interval. □

6. Sharpness of the low frequency condition

In this section we modify the counterexamples from Molinet, Saut and Tzvetkov [22] (which imply that the flow map is not C^2 without any low frequency condition) in order to prove the sharpness of our choice of ω. Here, we also include the Benjamin-Ono case ($\alpha = 1$).

THEOREM 6.1. *Let $1 \leq \alpha < 2$ and suppose that $\omega < \frac{1}{\alpha} - \frac{1}{2}$. For any $s \in \mathbb{R}$, there does not exist $T > 0$, such that the flow map of* (1.1)

$$F : H^s(\mathbb{R}) \cap \dot{H}^{-\omega}(\mathbb{R}) \to C([0,T], H^s(\mathbb{R})), \quad u_0 \mapsto u,$$

if it exists, is C^2 (i.e., two times Fréchet-differentiable) at the origin. In particular, the bilinear estimate corresponding to (4.1) *fails.*

Proof. We only describe the modifications of the argument of Molinet, Saut and Tzvetkov. For the details of the calculation[4] we refer the reader to the original work [22]. Define a sequence of initial data via

$$\widehat{\phi}_N := N^{(\alpha+\varepsilon)(\frac{1}{2}-\omega)} \chi_1 + N^{\frac{\alpha+\varepsilon}{2}-s} \chi_2,$$

where

$$\chi_1(\xi) = \chi_{\frac{1}{2}N^{-\alpha-\varepsilon} \leq \xi \leq N^{-\alpha-\varepsilon}}, \quad \chi_2(\xi) = \chi_{N \leq \xi \leq N + N^{-\alpha-\varepsilon}}.$$

Notice that $\|\phi_N\|_{H^{(s,\omega)}} \leq 2$. As shown in [22] we have

$$\left\| \int_0^t W_\alpha(t-t') \partial_x (W_\alpha(t')\phi_N)^2 \, dt' \right\|_{H^s} \geq c \|F\|_{H^s}$$

with

$$\widehat{F(t)}(\xi) = \frac{N^{\alpha+\varepsilon} \xi e^{it\xi|\xi|^\alpha}}{N^{s+(\alpha+\varepsilon)\omega}} \int (\chi_1(\xi_1)\chi_2(\xi-\xi_1) + \chi_2(\xi_1)\chi_1(\xi-\xi_1)) \frac{e^{itr(\xi_1,\xi)} - 1}{r(\xi_1,\xi)} \, d\xi_1,$$

---

[4] Notice that our notation slightly differs from [22].

where

$$r(\xi_1, \xi) = \xi_1|\xi_1|^\alpha + (\xi - \xi_1)|\xi - \xi_1|^\alpha - \xi|\xi|^\alpha$$

is essentially our resonance function from Lemma 3.8. In the domain of integration we have $|r(\xi_1, \xi)| \leq cN^{-\alpha-\varepsilon}N^\alpha = cN^{-\varepsilon}$. A Taylor expansion shows that

$$\left| \frac{e^{itr(\xi_1,\xi)} - 1}{r(\xi_1, \xi)} \right| = |t| + \mathcal{O}(N^{-\varepsilon}),$$

which implies

$$\|F\|_{H^s} \geq cNN^sN^{-\alpha-\varepsilon}N^{-\frac{\alpha+\varepsilon}{2}}N^{\alpha+\varepsilon}N^{-s-(\alpha+\varepsilon)\omega}.$$

This tends to infinity if $1 - \frac{\alpha+\varepsilon}{2} - (\alpha + \varepsilon)\omega > 0$, which is equivalent to

$$\omega < \frac{1}{\alpha + \varepsilon} - \frac{1}{2} \to \frac{1}{\alpha} - \frac{1}{2} \quad (\varepsilon \to 0).$$

This calculation implies that the bilinear expression, which corresponds to a second derivative at the origin in direction ϕ_N, is unbounded as $N \to \infty$, but on the other hand $\|\phi_N\|_{H^{(s,\omega)}} \leq 2$. This contradicts the C^2 regularity of the flow and the bilinear estimate. □

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