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ON DOMINATION OF INESSENTIAL ELEMENTS IN ORDERED BANACH ALGEBRAS

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ABSTRACT. If A is an ordered Banach algebra ordered by an algebra cone C, then we reference the following problem as the 'domination problem': If $0 \leq a \leq b$ and b has a certain property, then does a inherit this property? We extend the analysis of this problem in the setting of radical elements and introduce it for inessential, rank one and finite elements. We also introduce the class of r-inessential operators on Banach lattices and prove that if S and T are operators on a Banach lattice E such that $0 \leq S \leq T$ and T is r-inessential then S is also r-inessential.

1. Introduction

Throughout this paper we will consider A to be a complex Banach algebra with identity 1 satisfying the minimum requirement that A be *semiprime*, that is, $xAx = \{0\}$ implies $x = 0$ holds for all $x \in A$. The *spectrum* of an element a in A will be denoted by $\sigma(a, A)$ and the *spectral radius* by $r(a, A)$. Whenever the meaning is clear, we will drop the A from $\sigma(a)$ and $r(a)$. We denote the set of *quasinilpotent* elements in A by $\mathrm{QN}(A)$ and the *radical* of A by $\mathrm{Rad}(A)$. Recall that $\mathrm{Rad}(A) = \{a \in A \mid aA \subset \mathrm{QN}(A)\}$. A Banach algebra is called *semisimple* if its radical consists of zero only. It is not difficult to show that if a Banach algebra is semisimple then it is semiprime. In [10, Definition 2.2] J. Puhl defines a nonzero element u of a semiprime Banach algebra to be *rank one* if $uAu \subset \mathbb{C}A$. As various authors expanded on Puhl's work numerous other definitions and characterisations of rank one elements surfaced, deeming it necessary to rename Puhl's definition as *spatially rank one* elements. In [7] R. Harte defines a nonzero element u of a semiprime Banach algebra to be *spectrally rank one* if $\#\sigma(xu)\backslash\{0\} \leq 1$ for all x in A where $\#$ denotes the number of elements in a set. Moreover, he shows that every spatially rank one element is spectrally rank one. The converse is true when A is semisimple. For our means, we will term a nonzero element of a semiprime Banach algebra

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rank one if it satisfies the definition of J. Puhl. We shall denote the set of rank one elements by $\mathcal{F}_1$. For properties of these elements we refer to [7], [10]. An element of a semiprime Banach algebra is termed a *finite* element if it can be written as a finite sum of rank one elements. The set of these elements will be denoted by $\mathcal{F}$. By an ideal in a Banach algebra we mean a two sided ideal. If F is an ideal in a Banach algebra A then an element a in A is called *Riesz* relative to F if $a+\bar{F} \in \mathrm{QN}(A/\bar{F})$. The set of these elements will be denoted by $\mathcal{R}(F)$. The *inessential* elements in A relative to F is the set $\{a \in A \mid a+\bar{F} \in \mathrm{Rad}(A/\bar{F})\}$. These elements will be denoted by $\mathrm{kh}(F)$. It is clear that

$$\mathcal{F}_1 \subset \mathcal{F} \subset \mathrm{kh}(\mathcal{F}) \subset \mathcal{R}(\mathcal{F}). \tag{1}$$

Recall the construction of an ordered Banach algebra (OBA) (A, C) in [9], [12] ordered by an algebra cone C. Let (A, C) be an OBA. If $0 \leq a \leq b$ relative to C implies that $r(a) \leq r(b)$ then we say that the spectral radius is *monotone* w.r.t. C. If F is a closed ideal in A and if $\pi : A \to A/F$ is the canonical homomorphism, then $(A/F, \pi C)$ is an OBA. The spectral radius in $(A/F, \pi C)$ is monotone if $F \leq a+F \leq b+F$ in A/F relative to πC implies that $r(a+F, A/F) \leq r(b+F, A/F)$. An algebra cone C in an OBA (A, C) is termed *generating* if every element in A can be written as a linear combination over $\mathbb{C}$ of positive elements, i.e., $A = \operatorname{span} C$. Moreover, it is easy to see that if F is a closed ideal of A then the algebra cone πC in the OBA $(A/F, \pi C)$ is generating when the algebra cone C is generating in (A, C).

Let X be a Banach space. An operator T in the Banach algebra $\mathcal{L}(X)$ of bounded linear operators on X is called a *Riesz* operator if $T + \mathcal{K}(X)$ is quasinilpotent in the quotient algebra $\mathcal{L}(X)/\mathcal{K}(X)$, where $\mathcal{K}(X)$ is the closed ideal of compact operators on X. We denote the set of Riesz operators on X by $\mathcal{R}(X)$. An operator S in $\mathcal{L}(X)$ is called *inessential* if $S + \mathcal{K}(X)$ is in the radical of the quotient algebra $\mathcal{L}(X)/\mathcal{K}(X)$. Let E be a Banach lattice. An operator $T : E \to E$ is termed *regular* if it can be written as a linear combination over $\mathbb{C}$ of positive operators. The space of regular operators on E is denoted by $\mathcal{L}^r(E)$ and it is a subspace of $\mathcal{L}(E)$. If $\mathcal{L}^r(E)$ is provided with the r-norm

$$\|T\|_r = \inf\{\|S\| \mid 0 \leq S \in \mathcal{L}(E),\ |Tx| \leq S|x| \text{ for all } 0 \leq x \in E\}$$

it becomes a Banach algebra which contains the identity of $\mathcal{L}(E)$. Moreover, if E is Dedekind complete, then $\mathcal{L}^r(E)$ is a Banach lattice under the r-norm $\|T\|_r = \|\,|T|\,\|$. An operator T in $\mathcal{L}^r(E)$ is called *r-compact* if it can be approximated in the r-norm by operators of finite rank [2]. This set is denoted by $\mathcal{K}^r(E)$. Since $\mathcal{K}^r(E)$ is a closed ideal in $\mathcal{L}^r(E)$, $\mathcal{L}^r(E)/\mathcal{K}^r(E)$ is a Banach algebra under the quotient norm. An operator $T \in \mathcal{L}^r(E)$ is called *r-asymptotically quasi finite rank* if $T + \mathcal{K}^r(E)$ is quasinilpotent in the quotient algebra $\mathcal{L}^r(E)/\mathcal{K}^r(E)$; see [11]. We denote this set by $\mathcal{R}_0(E)$. An operator $T \in \mathcal{L}^r(E)$ is called *r-inessential* if $T + \mathcal{K}^r(E)$ is in the radical of the quotient algebra $\mathcal{L}^r(E)/\mathcal{K}^r(E)$. The set of these operators will be denoted by $\mathrm{kh}(\mathcal{K}^r(E))$. If the set of finite rank operators on E is denoted by $\mathcal{F}(E)$, then one can show that

$$\mathcal{F}(E) \subset \mathcal{K}^r(E) \subset \mathrm{kh}(\mathcal{K}^r(E)) \subset \mathcal{R}_0(E) \subset \mathcal{R}(E). \tag{2}$$

2. A perturbation result

Recall that a subset I of a Banach algebra A is called a *multiplicative ideal* if $IA \subset I$ and $AI \subset I$. For examples of multiplicative ideals in Banach algebras which are not ideals we refer to [13]. In this section we will prove that if $0 \leq a, b$ in an OBA (A, C) with C closed and if b belongs to some multiplicative ideal, then there exists a positive multiple ac of a such that $a - ac$ belongs to the multiplicative ideal. For $b \in A$ and $0 \neq \lambda \notin \sigma(b)$ the element $(\lambda - b)^{-1}$ exists and for all $p \in \mathbb{N}$

$$\lambda(\lambda - b)^{-1} = 1 + \frac{b}{\lambda} + \ldots + \left(\frac{b}{\lambda}\right)^p \lambda(\lambda - b)^{-1}. \tag{3}$$

Moreover, if $|\lambda| > r(a)$ then

$$(\lambda - b)^{-1} = \sum_{n=0}^{\infty} \frac{b^n}{\lambda^{n+1}} \tag{4}$$

is called the *Neumann* series of $(\lambda - b)^{-1}$.

Theorem 2.1. *Let (A, C) be an OBA with C closed and let I be a non-trivial multiplicative ideal in A. For every $a \in A$ there exists $1 \neq c \in A$ such that $ac - a \in I$. If $a \in C$ and $I \cap C \neq \{0\}$ there is $1 \neq c \in A$ for which $a \leq ac$ with $a - ac \in I$.*

Proof. Let $a \in A$. With no positivity, if $0 \neq b \in I$ and $\lambda \notin \sigma(b)$ take

$$c = \lambda(\lambda - b)^{-1}$$

and find $ac - a = a(\lambda - b)^{-1}b \in I$. If in addition a and b are positive and $r(b) < \lambda \in \mathbb{R}$, then in view of C being closed and (4) $ac - a$ is positive. □

It is obvious that the applicability lies in the fact that $\mathrm{Rad}(A)$, $\mathcal{F}_1$ and $\mathcal{F}$ are all multiplicative ideals.

3. The radical

The essential work analyzing the domination problem pertaining to the property of being a radical element is [8]. Two of their main results are:

Theorem 3.1 [8, Theorem 4.6]. *Let (A, C) be an OBA such that the spectral radius is monotone relative to C and let $0 \leq a \leq b$ w.r.t. C with $b \in \mathrm{Rad}(A)$. If C is generating then $a \in \mathrm{Rad}(A)$.*

THEOREM 3.2 [8, THEOREM 4.10]. *Let (A, C) be an OBA such that the spectral radius is monotone relative to C and let $0 \leq a \leq b$ w.r.t. C with $b \in \mathrm{Rad}(A)$. If $\mathrm{span}(C)$ contains an interior point then $a \in \mathrm{Rad}(A)$.*

Note, however, that the proof of Theorem 4.10 in [8] is redundant since it is rather easy to prove that in an OBA (A, C) $\mathrm{span}(C)$ contains an interior point if and only if the algebra cone C is generating.

In striving to discover other conditions that enforce a positive result to our domination problem, we consider a polynomial of an element to have a certain property in the endeavor of showing that the element itself has that property.

THEOREM 3.3. *Let (A, C) be an OBA such that the spectral radius is monotone relative to C. Suppose $a, b \in A$ such that $0 \leq a \leq b$ w.r.t. C and $b \in \mathrm{QN}(A)$. If $g(a) \in \mathrm{Rad}(A)$ for some polynomial g in a with $k \in \mathbb{N}$ the smallest nonzero power of a in $g(a)$, then $a^k \in \mathrm{Rad}(A)$.*

Proof. Since the spectral radius is monotone w.r.t. C, $0 \leq r(a) \leq r(b)$. As $b \in \mathrm{QN}(A)$, it follows that $r(a) = 0$, i.e., $a \in \mathrm{QN}(A)$. This together with $g(a) \in \mathrm{Rad}(A)$ and the spectral mapping theorem implies that $g(a)$ can be written as $a^k(\lambda_k + \ldots + \lambda_n a^{n-k})$ with $\lambda_k, \ldots, \lambda_n \in \mathbb{C}$ and $\lambda_k \neq 0$. Again by employing the spectral mapping theorem and remembering that $a \in \mathrm{QN}(A)$ we obtain $\sigma(\lambda_k + \ldots + \lambda_n a^{n-k}) = \{\lambda_k\}$. Thus $\lambda_k + \ldots + \lambda_n a^{n-k}$ is invertible in A and so

$$a^k = g(a)(\lambda_k + \ldots + \lambda_n a^{n-k})^{-1} \in \mathrm{Rad}(A). \qquad \square$$

COROLLARY 3.4. *Let (A, C) be an OBA such that the spectral radius is monotone relative to C. Suppose $a, b \in A$ such that $0 \leq a \leq b$ w.r.t. C and $b \in \mathrm{QN}(A)$. If $a + a^2 \in \mathrm{Rad}(A)$, then $a \in \mathrm{Rad}(A)$.*

4. Inessential elements

In [12, Theorem 6.2] it is shown that under some natural imposed conditions, a positive element a is Riesz relative to a closed ideal F when it is dominated by an element b that is Riesz relative to F. We pose the question: Is this the case when we work with inessential elements? We will demonstrate that under some natural imposed conditions this is true.

THEOREM 4.1. *Let (A, C) be an OBA and F a closed ideal in A. Suppose $a, b \in A$ with $0 \leq a \leq b$ relative to C and b inessential relative to F. Let the spectral radius in the OBA $(A/F, \pi C)$ be monotone.*

(i) *Then a is Riesz relative to F.*
(ii) *If a is in the center of A then a is inessential relative to F.*
(iii) *If the algebra cone C is generating then a is inessential relative to F.*

Proof. (i) This follows from the fact that b is Riesz relative to F and that the spectral radius in the quotient algebra $(A/F, \pi C)$ is monotone.

(ii) This follows from [12, Theorem 4.1 4].

(iii) Since C is generating in A, πC is generating in the quotient algebra A/F. Employing Theorem 3.1, we obtain $a + F \in \mathrm{Rad}(A/F)$. □

In view of [12, Theorem 6.1] and Theorem 4.1 we have:

Corollary 4.2. *Let (A, C) be an OBA and F a closed ideal in A. Suppose the spectral radius in the OBA $(A/F, \pi C)$ is monotone and the algebra cone C is generating. Then the algebra cone $C + \mathrm{kh}(F)$ in the quotient algebra $(A/\mathrm{kh}(F), C + \mathrm{kh}(F))$ is proper.*

Corollary 4.3. *Let (A, C) be an OBA and F a closed ideal in A such that $\mathrm{kh}(F)$ is a proper ideal in A. Suppose $a, b \in A$ with $0 \leq a \leq b$ and b is inessential relative to F. If the spectral radius in the OBA $(A/F, \pi C)$ is monotone and C is generating, then a cannot be invertible.*

Proof. In view of (iii) and the fact that a proper ideal cannot contain invertible elements this is clear. □

In [12] the solution to the domination problem in the setting of Riesz elements, namely [12, Theorem 6.2], is illustrated in the setting of C^*-algebras, [12, Proposition 6.4 and Theorem 6.5]. These results can be adapted in the setting of inessential elements to illustrate Theorem 4.1.

Proposition 4.4. *Let A be a commutative C^*-algebra with*

$$C = \{x \in A \mid x = x^* \text{ and } \sigma(x, A) \subset [0, \infty]\}$$

and $a, b \in A$ such that $0 \leq a \leq b$ w.r.t. C. If F is a closed ideal of A such that b is inessential relative to F, then a is inessential relative to F.

The adaption of [12, Theorem 6.5] to the setting of inessential elements is not straightforward since unlike in the case of Riesz elements, if B is a closed subalgebra of A with $1 \in B \subset A$ and I is a closed inessential ideal in A then in general it is not the case that $\mathrm{kh}(A, I) \cap B = \mathrm{kh}(B, I \cap B)$. Nonetheless, it can be shown that

$$\mathrm{kh}(A, I) \cap B \subset \mathrm{kh}(B, I \cap B).$$

However, in view of an inessential element relative to an inessential ideal being Riesz relative to the inessential ideal and [12, Theorem 6.5] we have:

Corollary 4.5. *Let A be a C^*-algebra with*

$$C = \{x \in A \mid x = x^* \text{ and } \sigma(x, A) \subset [0, \infty]\}$$

and $a, b \in A$ such that $ab = ba$ and $0 \leq a \leq b$ w.r.t. C. If F is a closed ideal of A such that b is inessential relative to F, then a is Riesz relative to F.

If in the above corollary we replace the condition $ab = ba$ with A/F being commutative, then it follows that if b is inessential relative to F then a is inessential relative to F. In view of the remark following Corollary 3.4, Theorem 3.3 in the setting of inessential elements takes the following form:

THEOREM 4.6. *Let (A, C) be an OBA and F a closed ideal in A such that the spectral radius relative to πC in $(A/F, \pi C)$ is monotone. Let $a, b \in A$ such that $0 \leq a \leq b$ relative to C and let b be Riesz relative to F. If $g(a)$ is inessential relative to F for some polynomial g in a with $k \in \mathbb{N}$ the smallest nonzero power of a in $g(a)$, then a^k is inessential relative to F.*

5. Rank one and finite elements

In this section we investigate the domination problem in the setting of rank one and finite elements. We provide examples where the domination problem holds and examples where it does not hold.

EXAMPLE 5.1. Let K be a completely regular Hausdorff space and let $C_b(K)$ be the Banach algebra of all complex valued bounded continuous functions on K with the supremum norm. It is noted by J. Puhl [10, p. 658] that the rank one elements are of the form:

$$\delta_s(t) = \begin{cases} \beta & \text{if } t = s, \\ 0 & \text{if } t \neq s, \end{cases}$$

where $\beta \in \mathbb{C}$ is fixed and s is an isolated point of K. The cone $C = \{f \in C_b(K) \mid f(k) \geq 0 \text{ for all } k \in K\}$ is normal. Moreover, if we let $0 \leq f \leq g$ w.r.t. C then it is obvious that if g is rank one then so is f. $\square$

A counter example to the domination problem in the context of rank one elements comes in the form of the matrix algebras.

EXAMPLE 5.2. Let $M_{2\times 2}$ denote the Banach algebra of all complex 2×2 matrices with standard addition, multiplication and norm. Together with the normal algebra cone C of all 2×2 matrices with positive real entries, $(M_{2\times 2}, C)$ forms an OBA. Because $M_{2\times 2}$ is semisimple it is semiprime. A simple argument shows that in general a rank one element cannot be invertible. In this OBA the converse is also true (this is not normally the case). Thus, the rank one elements coincide with the non invertible matrices.

When we consider

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

it is easy to see that $0 \leq A \leq B$ with B rank one and A invertible and hence not rank one.

It is a simple exercise to extend this to the $n \times n$ case for $n \in \mathbb{N}$. By choosing

$$B = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 1 & \cdots & 1 & 0 \\ 1 & \cdots & 1 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \cdots & 1 & 1 \end{pmatrix}$$

we find $0 \leq A \leq B$ with B rank one and A not. $\square$

We provide another counter example to the domination problem. It is taken from [1], and adapted for our situation.

EXAMPLE 5.3. Let E denote the Banach lattice of bounded linear operators from ℓ^1 to ℓ^∞. Let $S, T \in E$ be defined by $Sx = x$ and $Tx = (\sum_{n=1}^{\infty} x_n) \cdot e$ for $x = (x_n)$ and e the constant sequence 1. Then $0 \leq S \leq T$ and it is clear that T is a rank one operator while S is not. Let $G = \ell^1 \oplus \ell^\infty$ and $\mathcal{L}(G)$ denote the Banach algebra of bounded linear operators on G. If

$$A = \begin{pmatrix} 0 & 0 \\ S & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix}$$

then $A, B \in \mathcal{L}(G)$ with $0 \leq A \leq B$, B rank one and A not finite rank. $\square$

Although Examples 5.2 and 5.3 are counter examples to the domination problem, we can nonetheless in view of (1) ascertain the following:

THEOREM 5.4. *Let (A, C) be an OBA with A semiprime such the spectral radius relative to πC in the OBA $(A/\bar{\mathcal{F}}, \pi C)$ is monotone. Suppose $0 \leq a \leq b$ in A w.r.t. C with b a finite element in A.*

(i) *If C is generating then a is inessential relative to $\mathcal{F}$.*
(ii) *If a is in the center of A then a is inessential relative to $\mathcal{F}$.*

In either of these cases if $A/\bar{\mathcal{F}}$ is semisimple, $a \in \bar{\mathcal{F}}$.

The proof follows directly from the proof of Theorem 4.1 if we take the closed ideal F to be the ideal $\bar{\mathcal{F}}$.

In [4, Theorem 2.10] it is shown that in a semiprime Banach algebra that is not semisimple,

$$\bar{\mathcal{F}}_1 \cap \operatorname{Rad} A = \{0\} \quad \text{and} \quad \bar{\mathcal{F}}_1 \cdot \operatorname{Rad} A = \{0\} = \operatorname{Rad} A \cdot \bar{\mathcal{F}}_1. \tag{5}$$

Using the second fact and noting that every finite element is a finite sum of rank one elements yields

$$\mathcal{F} \cdot \operatorname{Rad} A = \operatorname{Rad} A \cdot \mathcal{F} = \{0\}. \tag{6}$$

These comments prompt us to consider the following:

Theorem 5.5. *Let (A, C) be an OBA with A semiprime but not semisimple and let the algebra cone C be proper. Suppose that $a, b \in A$ such that $0 \leq a \leq b$ w.r.t. C.*

(i) *If b is a finite element and $a \in \operatorname{Rad} A$ then $a^2 = 0$.*
(ii) *If $b \in \operatorname{Rad} A$ and a is a finite element then $a^2 = 0$.*
(iii) *If C is generating, the spectral radius is monotone relative to C, $b \in \operatorname{Rad} A$ and a is rank one then $a = 0$.*

Proof. (i) If $0 \leq a \leq b$ in A with b a finite element and $a \in \operatorname{Rad} A$ then in view of C being an algebra cone, $0 \leq a^2 \leq ab$. By (6) $ab = 0$ and C being proper implies $a^2 = 0$.

(ii) Follows that of (i).

(iii) By Theorem 3.1 and (5) this is apparent. □

Theorem 5.6. *Let (A, C) be an OBA with A semiprime and C proper. Suppose $b \geq 0$ is a finite element and $C \cap \operatorname{Rad} A \neq \{0\}$. Then there does not exist an invertible a such that $0 \leq a \leq b$.*

Proof. Let $0 \neq c \in C \cap \operatorname{Rad} A$. Suppose there exists an invertible a with $0 \leq a \leq b$. In view of C being an algebra cone and (6), $0 \leq ac \leq 0$. As C is proper we obtain $ac = 0$. This is impossible as the invertibility of a implies $c = 0$. □

Corollary 5.7. *Let (A, C) be an OBA with A semiprime and C proper. If $C \cap \operatorname{Rad} A \neq \{0\}$ there does not exist finite elements b in A such that $0 \leq 1 \leq b$.*

6. r-inessential operators

Let E be a Banach lattice and suppose operators S and T on E satisfy $0 \leq S \leq T$. From this we can deduce that $\sigma(S, \mathcal{L}(E)) \subset B(0, r(T))$. In this section we are going to illustrate the results in the previous section in the context of operators on Banach lattices. The domination problem mentioned earlier takes the form: If E is a Dedekind complete Banach lattice and if S and T are operators on E such that $0 \leq S \leq T$, then is S r-inessential when T is r-inessential? In view of (2) and [12, Corollary 6.3] we can deduce at least that S is r-asymptotically quasi finite rank. In view of $\mathcal{K}^r(E)$ being an inessential ideal in $\mathcal{L}^r(E)$ this means by [3, Corollary 5.7.5] that the spectrum of S is either a finite set or a sequence converging to zero. However, we can improve on this:

Theorem 6.1. *Let E be a Dedekind complete Banach lattice and suppose S and T are operators on E satisfying $0 \leq S \leq T$. If T is r-inessential then S is r-inessential.*

Proof. Since the algebra cone $C = \{T \in \mathcal{L}(E) \mid Tx \geq 0 \text{ if } 0 \leq x \in E\}$ in the Banach algebra $\mathcal{L}^r(E)$ is generating, the algebra cone $C + \mathcal{K}^r(E)$ in the OBA $\mathcal{L}^r(E)/\mathcal{K}^r(E)$ is generating. In view of the spectral radius in the OBA $\mathcal{L}^r(E)/\mathcal{K}^r(E)$ being monotone [5, Theorem 2.8] and Theorem 3.1 we deduce that $S+\mathcal{K}^r(E) \in \operatorname{Rad}(\mathcal{L}^r(E)/\mathcal{K}^r(E))$ if $T+\mathcal{K}^r(E) \in \operatorname{Rad}(\mathcal{L}^r(E)/\mathcal{K}^r(E))$. □

Suppose positive operators S and T on a Dedekind complete Banach lattice satisfy $0 \leq S \leq T$. If T is either rank one, finite rank or an r-compact operator, then in general the operator S is not finite rank or r-compact; see Examples 5.2 and 5.3 and [5, Lemma 2.7]. By [5, Lemma 2.7] the element $S + \mathcal{K}^r(E)$ is nilpotent in the algebra $\mathcal{L}^r(E)/\mathcal{K}^r(E)$. However, by Theorem 6.1 the operator S is r-inessential. By imposing conditions on the operator S or on the space E we can deduce more.

Corollary 6.2. *Let E be a Dedekind complete Banach lattice and suppose S and T are operators on E satisfying $0 \leq S \leq T$. Suppose T is a finite rank or r-compact operator on E. If S is a projection then S is a finite rank operator.*

Proof. If T is a finite rank or an r-compact operator on E then by Theorem 6.1 S is r-inessential. This together with $S + \mathcal{K}^r(E)$ being an idempotent in $\mathcal{L}^r(E)/\mathcal{K}^r(E)$ implies that $S \in \mathcal{K}^r(E)$. Hence S is a compact projection and so a finite rank operator. □

Note that one can also prove Corollary 6.2 by employing [5, Lemma 2.7].

Corollary 6.3. *Let E be a Dedekind complete Banach lattice and suppose S and T are operators on E satisfying $0 \leq S \leq T$. Suppose T is a finite rank or r-compact operator on E. If $S + \mathcal{K}^r(E)$ is a finite element in the algebra $\mathcal{L}^r(E)/\mathcal{K}^r(E)$ then S is r-compact.*

Proof. If T is a finite rank or an r-compact operator on E then by Theorem 6.1 S is r-inessential. This together with $S + \mathcal{K}^r(E)$ being a finite element in $\mathcal{L}^r(E)/\mathcal{K}^r(E)$ and [6, Lemma 4(i)] gives $S \in \mathcal{K}^r(E)$. □

If in Corollary 6.3 E is a Dedekind complete Banach lattice such that $\mathcal{L}^r(E)/\mathcal{K}^r(E)$ has finite dimensional radical then $S+\mathcal{K}^r(E)$ is a finite element in $\mathcal{L}^r(E)/\mathcal{K}^r(E)$.

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References

[1] C. D. Aliprantis and O. Burkinshaw, *Positive compact operators on Banach lattices*, Math. Z. **174** (1980), 289–298. MR 593826 (81m:47053)

[2] W. Arendt, *On the* o*-spectrum of regular operators and the spectrum of measures*, Math. Z. **178** (1981), 271–287. MR 631633 (83h:47027)

[3] B. Aupetit, *A primer on spectral theory*, Universitext, Springer-Verlag, New York, 1991. MR 1083349 (92c:46001)

[4] R. M. Brits, L. Lindeboom, and H. Raubenheimer, *On the structure of rank one elements in Banach algebras*, Extracta Math. **18** (2003), 297–309. MR 2056597 (2006a:46058)

[5] J. Martínez and J. M. Mazón, *Quasi-compactness of dominated positive operators and* C_0*-semigroups*, Math. Z. **207** (1991), 109–120. MR 1106817 (92g:47048)

[6] S. Giotopoulos and M. Roumeliotis, *Algebraic ideals of semiprime Banach algebras*, Glasgow Math. J. **33** (1991), 359–363. MR 1127528 (92g:46060)

[7] R. Harte, *On rank one elements*, Studia Math. **117** (1995), 73–77. MR 1367694 (96i:46055)

[8] H. d. T. Mouton and S. Mouton, *Domination properties in ordered Banach algebras*, Studia Math. **149** (2002), 63–73. MR 1881716 (2003e:46076)

[9] S. Mouton and H. Raubenheimer, *More spectral theory in ordered Banach algebras*, Positivity **1** (1997), 305–317. MR 1660397 (2000a:46070)

[10] J. Puhl, *The trace of finite and nuclear elements in Banach algebras*, Czechoslovak Math. J. **28(103)** (1978), 656–676. MR 506439 (81a:47024)

[11] H. Raubenheimer, r*-asymptotically quasifinite rank operators and the spectrum of measures*, Quaestiones Math. **10** (1986), 97–111. MR 857772 (87m:47090)

[12] H. Raubenheimer and S. Rode, *Cones in Banach algebras*, Indag. Math. (N.S.) **7** (1996), 489–502. MR 1620116 (99i:46035)

[13] H. Raubenheimer and T. J. D. Wilkins, *On a spectral condition in Banach algebras*, Bull. London Math. Soc. **28** (1996), 305–310. MR 1374410 (96k:46084)

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