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ON THE SECOND EIGENVALUE OF THE LAPLACIAN IN AN ANNULUS

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Abstract. It is shown that the second eigenvalue of the Laplacian with either Dirichlet or Neumann boundary conditions in an annulus in a Euclidean space, or in a sphere, or in a hyperbolic space of dimension $n > 1$ has multiplicity n.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^2$ and consider the corresponding Dirichlet boundary problem (DBP)

$$\begin{cases} \triangle u + \lambda u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with eigenfunctions $u_i \in H_0^1(\Omega)$, and with eigenvalues $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$. Let us consider a solution u_i of the DBP on Ω. We denote by $\mathcal{N}(u_i)$ the nodal set of u_i, that is, $\mathcal{N}(u_i) = \{x \in \Omega : u_i(x) = 0\}$. The nodal domains of u_i are the connected components of $\Omega \backslash \mathcal{N}(u_i)$.

In 1967 Payne conjectured in [16] that u_2 cannot have a closed nodal line in Ω, and in 1982 Yau [18] asked the same question for more restrictive convex domains in $\mathbb{R}^2$. In 1992 Melas [15] answered positively Yau's question; see also a more recent treatment of Liboff [13]. In 1997 a beautiful counterexample to the original conjecture of Payne was given by M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof and N. Nadirashvili [9], who constructed a multi-connected domain in $\mathbb{R}^2$ such that the corresponding second eigenfunction u_2 has a closed nodal line. Even though, it is widely believed [8], [10] that Payne's conjecture holds true for general simply-connected domains in the plane.

Motivated by this question, the author tried to determine $\mathcal{N}(u_2)$ when the domain is a Euclidean annulus, since annuli are the simplest multi-connected regions in the Euclidean space. We can also ask similar questions in curved

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spaces. Recently Shieh [17] proved that the second eigenvalue of the DBP on a spherical band in the unit sphere $\mathbb{S}^2$ has multiplicity 2; as a natural corollary, Payne's conjecture holds true for that spherical band.

The main purpose of this paper is to extend the earlier work of Shieh and the author. Denote by $\mathbb{M}_k^n$ the n-dimensional ($n > 1$) simply-connected complete Riemannian manifold with constant sectional curvature k, and by $B_R \subset \mathbb{M}_k^n$ a geodesic ball of radius R centered at a fixed point $p \in \mathbb{M}_k^n$. For two concentric balls $B_{R_1} \subsetneqq B_{R_2}$ (if $k > 0$, we further require $R_2 < \pi/\sqrt{k}$), we denote by $\mathcal{W}_{R_1,R_2}$ the annular domain $B_{R_2} \backslash \overline{B_{R_1}}$. First we will establish:

THEOREM 1.1. *The second eigenvalue of the Dirichlet Laplacian in $\mathcal{W}_{R_1,R_2}$ has multiplicity n.*

Next we consider the Neumann boundary problem (NBP) on $\mathcal{W}_{R_1,R_2}$

$$\begin{cases} \triangle v + \mu v = 0 & \text{in } \mathcal{W}_{R_1,R_2}, \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\mathcal{W}_{R_1,R_2}, \end{cases}$$

with eigenfunctions $v_i \in H^1(\Omega)$ and eigenvalues $\mu_1 < \mu_2 \leq \mu_3 \leq \cdots$, where ν denotes the unit outward normal vector on $\partial\mathcal{W}_{R_1,R_2}$. Corresponding to Theorem 1.1 we have:

THEOREM 1.2. *The second eigenvalue of the Neumann Laplacian in $\mathcal{W}_{R_1,R_2}$ has multiplicity n.*

The main source of inspiration for the analysis in this paper are the papers [7], [14], [19] concerning the multiplicity of the second eigenvalue of the Dirichlet Laplacian. This paper is organized as follows. In Section 2 we briefly describe how to study the second eigenvalue of the Laplacian in an annulus and we state two propositions to be proved in Section 4. The main results of this paper are proved in Section 3.

2. Candidates for the second eigenvalue

2.1. Candidates for the second Dirichlet eigenvalue. Let $\exp : M_p \mapsto \mathbb{M}_k^n$ be the exponential map from M_p, the tangent space of $\mathbb{M}_k^n$ at p, onto $\mathbb{M}_k^n$. Let $\mathbb{S}_p^{n-1}$ be the unit sphere in M_p and $\Box$ be the Laplacian acting on functions on $\mathbb{S}_p^{n-1}$. For a function $F : \mathcal{W}_{R_1,R_2} \mapsto \mathbb{R}$ of the form

$$F(\exp r\eta) = T(r)G(\eta),$$

where $R_1 < r < R_2$, $\eta \in \mathbb{S}_p^{n-1}$, one has by a standard calculation [3, pp. 38–41] that F is an eigenfunction of the DBP on $\mathcal{W}_{R_1,R_2}$ with eigenvalue λ if and only if (i) G is an eigenfunction of $-\Box$, and (ii) T satisfies the following

Sturm-Liouville problem

$$\begin{cases} (S^{n-1}T')' + \{\lambda - \gamma S^{-2}\}S^{n-1}T = 0, \\ T(R_1) = T(R_2) = 0, \end{cases} \tag{2.1}$$

where γ is the eigenvalue of G and $S : (R_1, R_2) \mapsto \mathbb{R}$ is of the form

$$S(r) = \begin{cases} \sin(\sqrt{k}r)/\sqrt{k}, & k > 0, \\ r, & k = 0, \\ \sinh(\sqrt{-k}r)/\sqrt{-k}, & k < 0. \end{cases}$$

Standard theory for the Sturm-Liouville problem [12, Theorem 1.3.3] asserts that (2.1) has eigenvalues $\lambda_{\gamma,1} < \lambda_{\gamma,2} < \lambda_{\gamma,3} < \cdots$, and the eigenfunction $T_{\gamma,s}$ corresponding to $\lambda_{\gamma,s}$ has $s-1$ zeros over (R_1, R_2) for all $s \in \mathbb{N}$. To emphasize the dependence on the region, we shall also write $\lambda_{\gamma,s} = \lambda_{\gamma,s}(\mathcal{W}_{R_1,R_2})$.

Next we describe some spectral properties of $-\Box$. Since $\mathbb{S}_p^{n-1}$ is isometric to $\mathbb{S}^{n-1}$, the unit sphere in $\mathbb{R}^n$, we can view $\mathbb{S}_p^{n-1}$ as simply $\mathbb{S}^{n-1}$. Let $\widetilde{P}$ be any homogeneous harmonic polynomial of degree j ($j \geq 0$) on $\mathbb{R}^n$, and let P be its restriction to $\mathbb{S}^{n-1}$. Then $-\Box P = j(j + n - 2)P$. In this way we obtain all the eigenfunctions of $-\Box$ (cf., e.g., [2], [3]). In particular, the first eigenfunction is of a constant value, while the second one with multiplicity n can be written in the form

$$\theta_1 x_1 + \cdots + \theta_n x_n|_{\mathbb{S}^{n-1}}, \tag{2.2}$$

where $(\theta_1, \ldots, \theta_n) \in \mathbb{R}^n \backslash \{0\}$.

Let u_2 be a second eigenfunction of the DBP on $\mathcal{W}_{R_1,R_2}$. According to Courant's nodal domain theorem [4], u_2 has exactly two nodal domains. Hence $\lambda_2(\mathcal{W}_{R_1,R_2})$ must be either $\lambda_{0,2}(\mathcal{W}_{R_1,R_2})$ or $\lambda_{n-1,1}(\mathcal{W}_{R_1,R_2})$. Considering $\lambda_1(\mathcal{W}_{R_1,R_2}) = \lambda_{0,1}(\mathcal{W}_{R_1,R_2})$, we further have

$$\lambda_2(\mathcal{W}_{R_1,R_2}) = \min\{\lambda_{0,2}(\mathcal{W}_{R_1,R_2}),\ \lambda_{n-1,1}(\mathcal{W}_{R_1,R_2})\}. \tag{2.3}$$

2.2. Candidates for the second Neumann eigenvalue. Similarly, for a function $\widetilde{F} : \mathcal{W}_{R_1,R_2} \mapsto \mathbb{R}$ of the form

$$\widetilde{F}(\exp r\eta) = \widetilde{T}(r)\widetilde{G}(\eta),$$

where $R_1 < r < R_2$, $\eta \in \mathbb{S}_p^{n-1}$, one has by a standard calculation that $\widetilde{F}$ is an eigenfunction of the NBP on $\mathcal{W}_{R_1,R_2}$ with eigenvalue μ if and only if (i) $\widetilde{G}$ is an eigenfunction of $-\Box$, and (ii) $\widetilde{T}$ satisfies the Sturm-Liouville problem

$$\begin{cases} (S^{n-1}\widetilde{T}')' + \{\mu - \delta S^{-2}\}S^{n-1}\widetilde{T} = 0, \\ \widetilde{T}'(R_1) = \widetilde{T}'(R_2) = 0, \end{cases} \tag{2.4}$$

where δ is the eigenvalue of $\widetilde{G}$. (2.4) has eigenvalues $\mu_{\delta,1} < \mu_{\delta,2} < \mu_{\delta,3} < \cdots$, and the eigenfunction $\widetilde{T}_{\delta,s}$ corresponding to $\mu_{\delta,s}$ has $s - 1$ zeros over (R_1, R_2)

for all $s \in \mathbb{N}$. We shall also write $\mu_{\delta,s} = \mu_{\delta,s}(\mathcal{W}_{R_1,R_2})$ to emphasize the dependence on the region. Similar to (2.3) one has

$$\mu_2(\mathcal{W}_{R_1,R_2}) = \min\{\mu_{0,2}(\mathcal{W}_{R_1,R_2}),\ \mu_{n-1,1}(\mathcal{W}_{R_1,R_2})\}. \tag{2.5}$$

2.3. Two useful propositions. Let $\Phi_t : \mathbb{M}_k^n \mapsto \mathbb{M}_k^n$ $(t \in \mathbb{R})$ be a nontrivial isometric transformation group and let X be its infinitesimal generator. Since $\mathbb{M}_k^n$ is homogeneous and isotropic, and by rescaling the isometric group if necessary, for fixed $\theta \in M_p \backslash \{0\}$ one can assume $X(p) = \theta$ without loss of generality. Let $\omega(\exp(r\eta)) = r$ be the distance function from p and $\varphi(\exp(r\eta)) = \langle \eta, \theta \rangle$ be the angle function with respect to θ. We first claim that

Proposition 2.1. *One always has*

$$X\omega = \varphi. \tag{2.6}$$

Note that the geodesic hyperplane $\{\exp(r\eta) : r \geq 0, \eta \in \mathbb{S}_p^{n-1}, \eta \perp \theta\}$ divides $\partial\mathcal{W}_{R_1,R_2}$ into four parts, namely $\partial(R_i, \pm) \doteq \{\exp(r\eta) : r = R_i, \pm\langle \eta, \theta \rangle > 0\}$. Remembering that ν denotes the unit outward normal vector on $\partial\mathcal{W}_{R_1,R_2}$, we next claim:

Proposition 2.2. *One always has*

$$\begin{cases} X \cdot \nu > 0, & \text{on } \partial(R_1, -) \cup \partial(R_2, +); \\ X \cdot \nu < 0, & \text{on } \partial(R_1, +) \cup \partial(R_2, -). \end{cases} \tag{2.7}$$

Propositions 2.1 and 2.2 are trivial for the case when the annulus lies in a Euclidean space as the interested reader can easily verify. The proofs of both propositions will be given later in Section 4.

3. The second eigenvalue of the Laplacian

3.1. The second Dirichlet eigenvalue.

Proposition 3.1. $\lambda_{0,2}(\mathcal{W}_{R_1,R_2}) \neq \lambda_{n-1,1}(\mathcal{W}_{R_1,R_2})$.

Proof. We argue by contradiction and suppose that $\lambda_{0,2} = \lambda_{n-1,1}$. Hence there exist second eigenfunctions ψ and ϕ with corresponding eigenvalues $\lambda_{0,2}$ and $\lambda_{n-1,1}$, respectively, and $\theta \in \mathbb{S}_p^{n-1}$, such that (i) ψ is negative near $\partial(R_1,+) \cup \partial(R_1,-)$ and positive near $\partial(R_2,+) \cup \partial(R_2,-)$, and (ii) ϕ is positive near $\partial(R_1,+) \cup \partial(R_2,+)$ and negative near $\partial(R_1,-) \cup \partial(R_2,-)$. Here by saying that a function f is positive/negative near $\partial(R_i,\pm)$ we mean that for any $x \in \partial(R_i,\pm)$, there exists an open set U containing x in $\mathbb{M}_k^n$ such that f is positive/negative in $\mathcal{W}_{R_1,R_2} \cap \{U \backslash \partial(R_i,\pm)\}$. Note that (see Section 2.3 for the notation X)

$$\begin{aligned} X\phi \cdot \triangle\psi - \triangle X\phi \cdot \psi &= X\phi \cdot \triangle\psi - X\triangle\phi \cdot \psi \\ &= -\lambda_{0,2} X\phi \cdot \psi + \lambda_{n-1,1} X\phi \cdot \psi \\ &= 0. \end{aligned}$$

Thus by Green's second formula,

$$\int_{\partial \mathcal{W}_{R_1,R_2}} X\phi \cdot \frac{\partial\psi}{\partial\nu} = 0. \tag{3.1}$$

But by Hopf's lemma and Proposition 2.2, $X\phi = \partial\phi/\partial\nu\langle X, \nu\rangle$ and $\partial\psi/\partial\nu$ are both negative on $\partial(R_2,+)\cup\partial(R_2,-)$ and positive on $\partial(R_1,+)\cup\partial(R_1,-)$. These facts contradict (3.1) and consequently the proposition follows. □

Proposition 3.2. *Suppose $\mathcal{W}_{R_1,R_2} \subset \mathbb{R}^n$. Then when R_1 is sufficiently small, one has $\lambda_{n-1,1}(\mathcal{W}_{R_1,R_2}) < \lambda_{0,2}(\mathcal{W}_{R_1,R_2})$.*

Proof. Let $u(x) = \widetilde{u}(|x|)$ be the radial solution to the equation

$$\begin{cases} \triangle u + \lambda^* u = 0 & \text{in } B_{R_2}, \\ u = 0 & \text{on } \partial B_{R_2}, \end{cases}$$

such that $\widetilde{u}$ has exactly one zero over $(0, R_2)$. By the monotonicity of the Dirichlet eigenvalues with respect to domains [6], one has

$$\lambda^*(B_{R_2}) < \lambda_{0,2}(\mathcal{W}_{R_1,R_2}).$$

On the other hand, by the continuity of the Dirichlet eigenvalues under continuous deformations of domains [5], one has

$$\lim_{R_1\to 0} \lambda_{n-1,1}(\mathcal{W}_{R_1,R_2}) = \lambda_2(B_{R_2}).$$

In addition, since $B_{R_2} \subset \mathbb{R}^n$ one can easily deduce

$$\lambda_2(B_{R_2}) < \lambda^*(B_{R_2}).$$

Combining these facts yields the desired result. □

Proof of Theorem 1.1. We first tackle the case when $\mathcal{W}_{R_1,R_2} \subset \mathbb{R}^n$. Define a continuous function Π on $(0, R_2)$ by

$$\Pi(r) = \lambda_{0,2}(\mathcal{W}_{r,R_2}) - \lambda_{n-1,1}(\mathcal{W}_{r,R_2}).$$

By Propositions 3.1 and 3.2, Π remains positive over $(0, R_2)$.

In general, suppose $\mathcal{W}_{R_1,R_2} \subset \mathbb{M}^n_k$. We shall also write $\mathcal{W}^n_{R_1,R_2,k}$ for $\mathcal{W}_{R_1,R_2}$ to emphasize the dimension and the curvature. Let $\widetilde{\Pi}$ be the continuous function on $(-\infty, \max\{0,k\}]$ defined by

$$\widetilde{\Pi} : s \mapsto \lambda_{0,2}(\mathcal{W}^n_{R_1,R_2,s}) - \lambda_{n-1,1}(\mathcal{W}^n_{R_1,R_2,s}).$$

By Proposition 3.1, $\widetilde{\Pi}(0)\widetilde{\Pi}(k) > 0$. Since we have already shown that $\widetilde{\Pi}(0) > 0$, it follows that $\widetilde{\Pi}(k) > 0$. Thus by (2.3), $\lambda_2(\mathcal{W}_{R_1,R_2}) = \lambda_{n-1,1}(\mathcal{W}_{R_1,R_2})$, and consequently by (2.2), $\lambda_2(\mathcal{W}_{R_1,R_2})$ has multiplicity n. $\square$

3.2. The second Neumann eigenvalue.

PROPOSITION 3.3. $\mu_{0,2}(\mathcal{W}_{R_1,R_2}) \neq \mu_{n-1,1}(\mathcal{W}_{R_1,R_2})$.

Proof. We argue by contradiction and suppose that $\mu_{0,2} = \mu_{n-1,1}$. Hence there exist eigenfunctions $\psi(x) = \widetilde{T}_{0,2}(\omega(x))$ and $\phi(x) = \widetilde{T}_{n-1,1}(\omega(x))\varphi(x)$ (see Section 2.3 for the notations of ω and φ) with corresponding eigenvalues $\mu_{0,2}$ and $\mu_{n-1,1}$, respectively. Since $\widetilde{T}_{0,2}$ has a unique zero over (R_1, R_2) and either $\widetilde{T}_{0,2}(R_1) = 0$ or $\widetilde{T}_{0,2}(R_2) = 0$ implies $\psi \equiv 0$, we may further assume that $\widetilde{T}_{0,2}(R_1) < 0 < \widetilde{T}_{0,2}(R_2)$; else we can choose $-\psi$ for the purpose. Note that

$$\begin{aligned} X\psi \cdot \triangle\phi - \triangle X\psi \cdot \phi &= X\psi \cdot \triangle\phi - X\triangle\psi \cdot \phi \\ &= -\mu_{n-1,1} X\psi \cdot \phi + \mu_{0,2} X\psi \cdot \phi \\ &= 0. \end{aligned}$$

By Green's second formula,

$$\int_{\partial\mathcal{W}_{R_1,R_2}} \phi \cdot \frac{\partial}{\partial\nu} X\psi = 0. \tag{3.2}$$

By (2.4), $\widetilde{T}'_{0,2}(R_i) = 0$ and $\widetilde{T}''_{0,2}(R_i) = -\mu_{0,2}\widetilde{T}_{0,2}(R_i)$ for $i = 1, 2$. Hence by Proposition 2.1,

$$\begin{aligned} \int_{\partial\mathcal{W}_{R_1,R_2}} \phi \cdot \frac{\partial}{\partial\nu} X\psi &= \int_{\partial\mathcal{W}_{R_1,R_2}} \widetilde{T}_{n-1,1}(\omega)\varphi \cdot \frac{\partial}{\partial\nu}\{\widetilde{T}'_{0,2}(\omega)X\omega\} \\ &= \int_{\partial\mathcal{W}_{R_1,R_2}} \widetilde{T}_{n-1,1}(\omega)\varphi \cdot \frac{\partial}{\partial\nu}\{\widetilde{T}'_{0,2}(\omega)\varphi\} \\ &= \int_{\partial B_{R_2}} \widetilde{T}_{n-1,1}(\omega)\varphi^2 \cdot \widetilde{T}''_{0,2}(\omega) \\ &\quad - \int_{\partial B_{R_1}} \widetilde{T}_{n-1,1}(\omega)\varphi^2 \cdot \widetilde{T}''_{0,2}(\omega) \\ &= -\mu_{0,2} \int_{\partial B_{R_2}} \widetilde{T}_{n-1,1}(R_2)\varphi^2 \cdot \widetilde{T}_{0,2}(R_2) \\ &\quad + \mu_{0,2} \int_{\partial B_{R_1}} \widetilde{T}_{n-1,1}(R_1)\varphi^2 \cdot \widetilde{T}_{0,2}(R_1). \end{aligned}$$

Also by (2.4), $\widetilde{T}_{n-1,1}(R_1) \neq 0$ and $\widetilde{T}_{n-1,1}(R_2) \neq 0$. Since $\widetilde{T}_{n-1,1}$ has no zero over (R_1, R_2), we may assume that $\widetilde{T}_{n-1,1}(R_1) > 0, \widetilde{T}_{n-1,1}(R_2) > 0$; else we can choose $-\widetilde{T}_{n-1,1}$ for the purpose. With this choice, we have

$$\int_{\partial \mathcal{W}_{R_1,R_2}} \phi \cdot \frac{\partial}{\partial \nu} X\psi < 0, \tag{3.3}$$

since $\widetilde{T}_{0,2}(R_1) < 0 < \widetilde{T}_{0,2}(R_2)$. (3.3) contradicts (3.2) and consequently the proposition follows. □

Proposition 3.4. *Suppose $\mathcal{W}_{R_1,R_2} \subset \mathbb{R}^n$. Then $\mu_2(\mathcal{W}_{R_1,R_2}) \leq \frac{2n+2}{R_1^2}$.*

Proof. Set $r = \sqrt{x_1^2 + \cdots + x_n^2}$. By the max-min principle (cf. [2], [3]),

$$\begin{aligned}
\mu_2(\mathcal{W}_{R_1,R_2}) &\leq \frac{\int_{\mathcal{W}_{R_1,R_2}} |\nabla \frac{x_1}{r}|^2 dx}{\int_{\mathcal{W}_{R_1,R_2}} |\frac{x_1}{r}|^2 dx} \\
&\leq 2\frac{\int_{\mathcal{W}_{R_1,R_2}} \frac{1}{r^2}|\nabla x_1|^2 dx}{\frac{1}{n}\int_{\mathcal{W}_{R_1,R_2}} dx} + 2\frac{\int_{\mathcal{W}_{R_1,R_2}} x_1^2|\nabla \frac{1}{r}|^2 dx}{\frac{1}{n}\int_{\mathcal{W}_{R_1,R_2}} dx} \\
&= 2n\frac{\int_{\mathcal{W}_{R_1,R_2}} \frac{1}{r^2} dx}{\int_{\mathcal{W}_{R_1,R_2}} dx} + 2\frac{\int_{\mathcal{W}_{R_1,R_2}} r^2 \cdot \frac{1}{r^4} dx}{\int_{\mathcal{W}_{R_1,R_2}} dx} \\
&\leq \frac{2n+2}{R_1^2}.
\end{aligned}$$

□

Proof of Theorem 1.2. Similar to the proof of Theorem 1.1, we need only show that there exists a Euclidean annulus $\mathcal{W}_{R_1,R_2} \subset \mathbb{R}^n$ such that

$$\mu_{n-1,1}(\mathcal{W}_{R_1,R_2}) < \mu_{0,2}(\mathcal{W}_{R_1,R_2}).$$

Denote by J the Bessel function of order $n/2-1$, and by J_i the i-th positive zero of J. It is well-known (cf. [1]) that $J_i \to \infty$ as $i \to \infty$ and for $x \in \mathbb{R}^n \backslash \{0\}$

$$\triangle\{|x|^{1-n/2} J(|x|)\} + |x|^{1-n/2} J(|x|) = 0.$$

Since $J(J_i) = J(J_{i+1}) = J(J_{i+2}) = 0$, there exist $J_i < \xi_i < J_{i+1} < \zeta_i < J_{i+2}$ such that the derivative of $r^{1-n/2}J(r)$ vanishes at both ξ_i and ζ_i, which implies $\mu_{0,2}(\mathcal{W}_{\xi_i,\zeta_i}) = 1$. By Proposition 3.4, $\mu_2(\mathcal{W}_{\xi_i,\zeta_i}) \leq 2n + 2/\xi_i^2 < 2n + 2/J_i^2$, which means that for sufficiently large i, say $i > N_1$, $\mu_2(\mathcal{W}_{\xi_i,\zeta_i}) < \mu_{0,2}(\mathcal{W}_{\xi_i,\zeta_i})$. Consequently by (2.5),

$$\mu_{n-1,1}(\mathcal{W}_{\xi_i,\zeta_i}) < \mu_{0,2}(\mathcal{W}_{\xi_i,\zeta_i}) \quad \text{for } i > N_1.$$

This concludes the proof of Theorem 1.2. □

4. Proofs of Propositions 2.1 and 2.2

In this section we prove Propositions 2.1 and 2.2.

Case A: $k > 0$: Set $R = 1/\sqrt{k}$ and denote by $\mathbb{S}^n_R$ the sphere of radius R centered at the origin in $\mathbb{R}^{n+1}$, with the metric induced by the Euclidean metric on $\mathbb{R}^{n+1}$. Obviously $\mathbb{S}^n_R$ is a model for $\mathbb{M}^n_k$. Let $\Phi_t : \mathbb{S}^n_R \mapsto \mathbb{S}^n_R$ $(t \in \mathbb{R})$ be the isometric transformation group defined by

$$(x_1, x_2, x_3, \ldots, x_{n+1}) \mapsto (x_1 \cos t - x_2 \sin t, x_1 \sin t + x_2 \cos t, x_3, \ldots, x_{n+1}).$$

This isometric group is generated by the vector field $X : \mathbb{S}^n_R \mapsto \mathbb{R}^{n+1}$ defined by

$$X(x_1, x_2, x_3, \ldots, x_{n+1}) = (-x_2, x_1, 0, \ldots, 0). \tag{4.1}$$

Now we choose $p = (R, 0, 0, \ldots, 0)$ with $\theta = X(p) = (0, R, 0, \ldots, 0) \in M_p \backslash \{0\}$. With this choice,

$$\mathcal{W}_{R_1,R_2} = \left\{ (x_1, x_2, \ldots, x_{n+1}) \in \mathbb{S}^n_R : R \cos \frac{R_2}{R} < x_1 < R \cos \frac{R_1}{R} \right\}$$

and

$$\partial(R_i, \pm) = \left\{ (x_1, x_2, \ldots, x_{n+1}) \in \mathbb{S}^n_R : x_1 = R \cos \frac{R_i}{R}, \pm x_2 > 0 \right\}.$$

Suppose $x = (x_1, x_2, \ldots, x_{n+1}) \in \partial(R_i, \pm)$. Write $l(x)$ for the line $\{x + s\nu(x) : s \in \mathbb{R}\}$ and l for the line $\{sp : s \in \mathbb{R}\}$.

Case A1: $x_1 \neq 0$: Suppose $l(x)$ touches l at $(a(x), 0, \ldots, 0)$ for some $a(x) \neq 0$. Then

$$\nu(x) = \pm \frac{(x_1 - a(x), x_2, \ldots, x_{n+1})}{\sqrt{(x_1 - a(x))^2 + x_2^2 + \cdots + x_{n+1}^2}}. \tag{4.2}$$

Consequently by (4.1) and (4.2),

$$(X \cdot \nu)(x) = \pm \frac{a(x) x_2}{\sqrt{(x_1 - a(x))^2 + x_2^2 + \cdots + x_{n+1}^2}} \neq 0.$$

Case A2: $x_1 = 0$: Note that in this case $\nu(x) = \pm(1, 0, \ldots, 0)$. Thus by (4.1),

$$(X \cdot \nu)(x) = \mp x_2 \neq 0.$$

In either case $X \cdot \nu$ is of constant sign on $\partial(R_i, \pm)$. Let $p_{i,\pm} = (R \cos R_i/R, \pm R \sin R_i/R, 0, \ldots, 0) \in \partial(R_i, \pm)$. Then it is straightforward to verify that $(X \cdot \nu)(p_{1,+}) < 0$, $(X \cdot \nu)(p_{2,+}) > 0$, $(X \cdot \nu)(p_{1,-}) > 0$ and $(X \cdot \nu)(p_{2,-}) < 0$. Hence (2.7) follows by continuity.

Note that in coordinates $x = (x_1, x_2, \ldots, x_{n+1})$,

$$\omega(x) = R \left\{ \frac{\pi}{2} - \arcsin \frac{x_1}{R} \right\}, \quad \varphi(x) = \frac{R x_2}{\sqrt{x_2^2 + \cdots + x_{n+1}^2}}.$$

Hence

$$X\omega(x) = \lim_{t\to 0} \frac{\omega(\Phi_t(x)) - \omega(x)}{t} = \frac{Rx_2}{\sqrt{x_2^2 + \cdots + x_{n+1}^2}} = \varphi(x).$$

Case B: $k = 0$: Let $\Phi_t : \mathbb{R}^n \mapsto \mathbb{R}^n$ $(t \in \mathbb{R})$ be the isometric transformation group defined by $(x_1, x_2, \dots, x_n) \mapsto (x_1 + t, x_2, \dots, x_n)$. This isometric group is generated by the constant-valued vector field $X : \mathbb{R}^n \mapsto \mathbb{R}^n$ defined by $X(x) \equiv (1, 0, \dots, 0)$. The verification of (2.7) is trivial and we omit the details. Let p be the origin of $\mathbb{R}^n$. Then $\omega(x) = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$ and

$$\varphi(x) = \frac{x_1}{\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}}.$$

Hence

$$X\omega(x) = \lim_{t\to 0} \frac{\omega(\Phi_t(x)) - \omega(x)}{t} = \frac{x_1}{\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}} = \varphi(x).$$

Case C: $k < 0$: Set $R = 1/\sqrt{-k}$ and denote by $\mathbb{U}_R^n$ the upper half-space in $\mathbb{R}^n$ defined in coordinates $(x_1, \dots, x_{n-1}, x_n)$ by $\{x_n > 0\}$, with the metric

$$R^2 \frac{dx_1^2 + \cdots + dx_{n-1}^2 + dx_n^2}{x_n^2}.$$

Obviously $\mathbb{U}_R^n$ is a model for $\mathbb{M}_k^n$. Let $\Phi_t : \mathbb{U}_R^n \mapsto \mathbb{U}_R^n$ $(t \in \mathbb{R})$ be the isometric transformation group defined by

$$(x_1, x_2, \dots, x_n) \mapsto (x_1 + t, x_2, \dots, x_n).$$

This isometric group is generated by the vector field $X : \mathbb{U}_R^n \mapsto \mathbb{R}^n$ defined by $X(x_1, x_2, \dots, x_n) = (1, 0, \dots, 0)$. Now we choose $p = (0, \dots, 0, R)$ with $\theta = X(p) = (1, 0, \dots, 0) \in M_p \backslash \{0\}$.

For any $x \in \partial(R_i, \pm)$, we show that $(X \cdot \nu)(x) \neq 0$. We argue by contradiction. Suppose $(X \cdot \nu)(x^*) = 0$ for some $x^* \in \partial(R_i, \pm)$. By definition, there exists $\eta \in \mathbb{S}_p^{n-1}$, $\pm\langle \eta, \theta \rangle > 0$, such that $x^* = \exp(R_i \eta)$. Since $\langle \eta, \theta \rangle \neq 0$, according to the standard structure of the geodesic line in hyperbolic spaces [11, pp. 83–86], there exist $\xi = (\xi_1, \xi_2, \dots, \xi_{n-1}, 0) \in \mathbb{R}^n$ and $r^* \in \mathbb{R}^+$ such that $\{\exp r\eta : R_1 < r < R_2\}$ lies entirely in the semicircle $\mathcal{F} \subset \mathbb{R}^n$ of radius r^* centered at ξ passing through both x^* and p. Note that by the assumption $(X \cdot \nu)(x^*) = 0$, it follows immediately that $\mathcal{F} \subset \{(x_1, x_2, \dots, x_n) \in \mathbb{U}_R^n : x_1 = \xi_1\}$. Since $p \in \mathcal{F}$, $\xi_1 = 0$. Consequently one has $\{\exp r\eta : R_1 < r < R_2\} \subset \{(x_1, x_2, \ldots, x_n) \in \mathbb{U}_R^n : x_1 = 0\}$. This obviously implies $\langle \eta, \theta \rangle = 0$. Hence we get a contradiction.

Thus in each of the cases, $X \cdot \nu$ is of constant sign on $\partial(R_i, \pm)$. Let $p_{i,\pm} = \exp(\pm R_i\theta/\|\theta\|)$. Then it is straightforward to verify that $(X \cdot \nu)(p_{1,+}) < 0$, $(X \cdot \nu)(p_{2,+}) > 0$, $(X \cdot \nu)(p_{1,-}) > 0$ and $(X \cdot \nu)(p_{2,-}) < 0$. Hence (2.7) follows again by continuity.

Denote by $\mathbb{B}_R^n$ the ball of radius R in $\mathbb{R}^n$ centered at the origin with the metric given in coordinates $(y_1, y_2, \ldots, y_n)$ by

$$4R^4 \frac{dy_1^2 + dy_2^2 + \cdots + dy_n^2}{(R^2 - |y|^2)^2}.$$

Denote by $\Psi : \mathbb{B}_R^n \mapsto \mathbb{U}_R^n$ the generalized Cayley transform given in coordinates $(z, v) \in \mathbb{R}^{n-1} \times \mathbb{R}$ by

$$\Psi(z, v) = \left(\frac{2R^2 z}{|z|^2 + (v - R)^2},\ R\frac{R^2 - |z|^2 - v^2}{|z|^2 + (v - R)^2} \right). \tag{4.3}$$

It is well-known [11, p. 38] that $\mathbb{U}_R^n \cong \mathbb{B}_R^n$ and that Ψ is an isometric map between $\mathbb{B}_R^n$ and $\mathbb{U}_R^n$. Note that the inverse of Ψ can be written as

$$\Psi^{-1}(z, v) = \left(\frac{2R^2 z}{|z|^2 + (v + R)^2},\ R\frac{|z|^2 + v^2 - R^2}{|z|^2 + (v + R)^2} \right). \tag{4.4}$$

Note that $\Theta_t \doteq \Psi^{-1}\Phi_t\Psi : \mathbb{B}_R^n \mapsto \mathbb{B}_R^n$ $(t \in \mathbb{R})$ is an isometric transformation group on $\mathbb{B}_R^n$. Suppose $\{\Theta_t\}_{t\in\mathbb{R}}$ is generated by the vector field $\widetilde{X}$. Set $\widetilde{p} = \Psi^{-1}(p) = (0, 0, \ldots, 0) \in \mathbb{B}_R^n$. Since isometric transformations preserves the lengths of tangent vectors under transformations, it follows easily from $X(p) = (1, 0, \ldots, 0, 0) \in \mathbb{S}_p^{n-1}$ that

$$\widetilde{X}(\widetilde{p}) = \left(\frac{1}{2}, 0, \ldots, 0, 0 \right) \in \mathbb{S}_{\widetilde{p}}^{n-1}.$$

Hence in coordinates $y = (y_1, y_2, \ldots, y_n) \in \mathbb{B}_R^n$ one has

$$\varphi(y) = \frac{y_1}{\sqrt{y_1^2 + y_2^2 + \cdots + y_n^2}}.$$

Note also that

$$\omega(y) = \int_0^{|y|} \frac{2R^2}{R^2 - s^2} ds.$$

Consequently, by a long calculation (see below) one has

$$\begin{aligned} \widetilde{X}\omega(y) &= \lim_{t\to 0} \frac{\omega(\Theta_t(y)) - \omega(y)}{t} \\ &= \frac{y_1}{\sqrt{y_1^2 + y_2^2 + \cdots + y_n^2}} \\ &= \varphi(y). \end{aligned} \tag{4.5}$$

Verification of (4.5). We first observe that

$$
\begin{aligned}
(4.6) \qquad \widetilde{X}\omega(y) &= \lim_{t\to 0}\frac{\omega(\Theta_t(y))-\omega(y)}{t}\\
&= \lim_{t\to 0}\frac{\omega(\Theta_t(y))-\omega(y)}{|\Theta_t(y)|-|y|}\cdot\frac{|\Psi^{-1}\Phi_t\Psi(y)|-|\Psi^{-1}\Psi y|}{t}\\
&= \frac{2R^2}{R^2-|y|^2}\cdot\{\partial_1|\Psi^{-1}|\}(\Psi(y))\\
&= \frac{2R^2}{R^2-|y|^2}\cdot\left\{\sum_{i=1}^{n}\frac{\Psi_i^{-1}\partial_1\Psi_i^{-1}}{|\Psi^{-1}|}\right\}(\Psi(y))\\
&= \frac{2R^2}{R^2-|y|^2}\cdot\sum_{i=1}^{n}\frac{y_i}{|y|}\{\partial_1\Psi_i^{-1}\}(\Psi(y)).
\end{aligned}
$$

By (4.4), if $i=1$, then

$$\left\{\partial_1\Psi_i^{-1}\right\}(z,v)=\frac{2R^2z_1}{|z|^2+(v+R)^2}\cdot\frac{1}{z_1}-\left\{\frac{2R^2z_1}{|z|^2+(v+R)^2}\right\}^2\cdot\frac{1}{R^2};$$

if $2\le i\le n-1$, then

$$\{\partial_1\Psi_i^{-1}\}(z,v)=\frac{2R^2z_i}{|z|^2+(v+R)^2}\cdot\frac{2R^2z_1}{|z|^2+(v+R)^2}\cdot\frac{-1}{R^2};$$

and if $i=n$, then

$$
\begin{aligned}
\{\partial_1\Psi_i^{-1}\}(z,v) = {} & \frac{2R^2z_1}{|z|^2+(v+R)^2}\cdot\frac{1}{R}+\\
& R\cdot\frac{|z|^2+v^2-R^2}{|z|^2+(v+R)^2}\cdot\frac{2R^2z_1}{|z|^2+(v+R)^2}\cdot\frac{-1}{R^2}.
\end{aligned}
$$

Consequently, for $i=1$ (note (4.3)),

$$(4.7) \qquad \left\{\partial_1\Psi_i^{-1}\right\}(\Psi(y))=y_1\frac{\sum_{j=1}^{n-1}y_j^2+(y_n-R)^2}{2R^2y_1}-\frac{y_1^2}{R^2};$$

for $i=2,\dots,n-1$,

$$(4.8) \qquad \left\{\partial_1\Psi_i^{-1}\right\}(\Psi(y))=-\frac{y_iy_1}{R^2};$$

and for $i=n$,

$$(4.9) \qquad \left\{\partial_1\Psi_i^{-1}\right\}(\Psi(y))=\frac{y_1}{R}-\frac{y_ny_1}{R^2}.$$

Combining (4.7)–(4.9) yields

$$\text{(4.10)}\quad \sum_{i=1}^{n} y_i\{\partial_1\Psi_i^{-1}\}(\Psi(y)) = \frac{y_1\{|y|^2 - y_n^2 + (y_n - R)^2\}}{2R^2} - \frac{y_1|y|^2}{R^2} + \frac{y_1 y_n}{R} = \frac{y_1\{R^2 - |y|^2\}}{2R^2}.$$

Hence (4.5) follows from (4.6) and (4.10).

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