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BOUNDARIES FOR ALGEBRAS OF HOLOMORPHIC FUNCTIONS ON BANACH SPACES

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ABSTRACT. We study the relations between boundaries for algebras of holomorphic functions on Banach spaces and complex convexity of their balls. In addition, we show that the Shilov boundary for algebras of holomorphic functions on an order continuous sequence space X is the unit sphere S_X if X is locally c-convex. In particular, it is shown that the unit sphere of the Orlicz-Lorentz sequence space $\lambda_{\varphi,w}$ is the Shilov boundary for algebras of holomorphic functions on $\lambda_{\varphi,w}$ if φ satisfies the δ_2-condition.

1. Introduction and preliminaries

Let X be a complex Banach space and let B_X be the closed unit ball of X. We denote by $H(B_X)$ the set of all holomorphic functions on the interior of B_X, and by $C_b(B_X)$ the Banach algebra of bounded continuous functions on B_X with the sup norm.

Globevnik [10] defined and studied the following analogues of the classical disc algebra:

$$\mathcal{A}_b(B_X) = \{f \in H(B_X) : f \in C_b(B_X)\},$$
$$\mathcal{A}_u(B_X) = \{f \in \mathcal{A}_b(B_X) : f \text{ is uniformly continuous on } B_X\}.$$

It is shown in [3] that $\mathcal{A}_u(B_X)$ is a proper subset of $\mathcal{A}_b(B_X)$ if and only if X is an infinite dimensional Banach space. Then it is easy to see that both $\mathcal{A}_b(B_X)$ and $\mathcal{A}_u(B_X)$ are Banach algebras when given the natural norm

$$\|f\| = \sup\{|f(x)| : x \in B_X\}.$$

Let K be a Hausdorff topological space and $\mathcal{A}$ a *closed function algebra* on K, that is, a closed subalgebra of $C_b(K)$. A subset F of K is called a

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boundary for $\mathcal{A}$ if for all $f \in \mathcal{A}$ we have

$$\|f\| = \sup_{x\in F} |f(x)|.$$

If the intersection of all closed boundaries for $\mathcal{A}$ is again a boundary for $\mathcal{A}$, then it is called the *Shilov boundary* for $\mathcal{A}$, denoted by $\partial\mathcal{A}$. We recall that a function algebra $\mathcal{A}$ is said to be *separating* if (i) for two distinct points x, y in K, there is an element $f \in \mathcal{A}$ such that $f(x) \neq f(y)$ and (ii) for each $t \in K$ there is an $f \in \mathcal{A}$ such that $f(t) \neq 0$. A *uniform algebra* on a compact Hausdorff space K is a closed function algebra which contains constants and separates the points of K.

Given a closed function algebra $\mathcal{A}$ on a metric space K, a set $S \subset K$ is called a *peak set* for $\mathcal{A}$ if there exists $f \in \mathcal{A}$ such that $f(S) = 1$, $|f(x)| < 1$ $(x \in K \setminus S)$. A set $S \subset K$ is called a *strong peak set* for $\mathcal{A}$ if there exists $f \in \mathcal{A}$ such that $f(S) = 1$ and for every $\epsilon > 0$ there exists $\delta > 0$ with $|f(x)| < 1-\delta$ whenever $\mathrm{dist}(x, S) > \epsilon$. If S consists of only one point p and if it is a *peak set* (resp. *strong peak set*) for $\mathcal{A}$, then the point p is called a *peak point* (resp. *strong peak point*) for $\mathcal{A}$.

The set of all peak points for $\mathcal{A}$ is called the *Bishop boundary* for $\mathcal{A}$ and denoted by $\rho\mathcal{A}$. Note that if K is compact, then a peak point $x \in K$ for $\mathcal{A}$ is also a strong peak point for $\mathcal{A}$; hence every closed boundary for $\mathcal{A}$ contains $\rho\mathcal{A}$. Further, Bishop [4, Theorem 1] showed that for any uniform algebra $\mathcal{A}$ on a compact metrizable space K,

$$\partial\mathcal{A} = \overline{\rho\mathcal{A}}.$$

Given a convex set $M \subset X$, a point $x \in M$ is called a *real* (resp. *complex*) *extreme point* of M if for every nonzero $y \in X$, there is a real (resp. complex) number ζ such that $|\zeta| \leq 1$ and $x+\zeta y \notin M$. The set of all real (resp. complex) extreme points of M is denoted by $\mathrm{Ext}_{\mathbb{R}}(M)$ (resp. $\mathrm{Ext}_{\mathbb{C}}(M)$). Let $\mathcal{A}$ be a uniform algebra on a compact Hausdorff space K. Let $\mathcal{A}^*$ be the dual Banach space of $\mathcal{A}$ and let S_1^* be the intersection of the unit sphere $S_{\mathcal{A}^*}$ of $\mathcal{A}^*$ with the hyperplane $\{x^* \in \mathcal{A}^* : x^*(1) = 1\}$. The set $\chi\mathcal{A} = \{x \in K : \delta_x \in \mathrm{Ext}_{\mathbb{R}}(S_1^*)\}$ is called the *Choquet boundary* for $\mathcal{A}$.

It is well-known (see [15, Theorem 9.7.2]) that if $\mathcal{A}$ is a uniform algebra on a compact metrizable space K, then

$$\rho\mathcal{A} = \chi\mathcal{A}. \tag{1.1}$$

Given a convex compact subset K in a complex locally convex space E, Arenson [1] considered the uniform algebra $\mathcal{P}(K)$ generated by the constants and restrictions to K of functions from E^*, and showed that

$$\chi\mathcal{P}(K) = \mathrm{Ext}_{\mathbb{C}}(K). \tag{1.2}$$

In particular, if K is metrizable, we have

$$\rho\mathcal{P}(K) = \chi\mathcal{P}(K) = \mathrm{Ext}_{\mathbb{C}}(K). \tag{1.3}$$

On the other hand, it is shown in [10] that $\rho\mathcal{A}_b(B_X) \subset \mathrm{Ext}_{\mathbb{C}}(B_X)$. We also note that every closed boundary for a function algebra $\mathcal{A}$ must contain the set of all strong peak points for $\mathcal{A}$.

When X is finite dimensional, we get the following observation from the above remark and the inclusion $\rho\mathcal{P}(B_X) \subset \rho\mathcal{A}_b(B_X)$.

PROPOSITION 1.1. *If X is finite dimensional, then*

$$\rho\mathcal{P}(B_X) = \chi\mathcal{P}(B_X) = \mathrm{Ext}_{\mathbb{C}}(B_X) = \rho\mathcal{A}_b(B_X),$$

and

$$\partial\mathcal{A}_b(B_X) = \overline{\mathrm{Ext}_{\mathbb{C}}(B_X)}.$$

The following two observations are easy.

PROPOSITION 1.2. *Let X be a Banach space and suppose that the set of all strong peak points for $\mathcal{A}_u(B_X)$ is dense in the unit sphere S_X. Then the unit sphere is the Shilov boundary for both $\mathcal{A}_u(B_X)$ and $\mathcal{A}_b(B_X)$.*

PROPOSITION 1.3. *Let X be a Banach space and suppose that the unit sphere S_X is the Shilov boundary for $\mathcal{A}_u(B_X)$. Then a subset F of B_X is a boundary for $\mathcal{A}_u(B_X)$ if and only if it is a boundary for $\mathcal{A}_b(B_X)$.*

For each $x \in B_X$, we define the *face* at x as the set

$$\mathcal{F}(x) = \left\{ x + y : y \in X, \quad \sup_{|\zeta|\le 1} \|x + \zeta y\| \le 1 \right\}.$$

Notice that $x \in S_X$ is a complex extreme point of B_X if and only if $\mathcal{F}(x) = \{x\}$. A Banach space X is said to be *strictly c-convex* if every point of S_X is a complex extreme point.

By the maximum modulus theorem, we obtain the following result:

PROPOSITION 1.4. *Let S be a peak set for $\mathcal{A}_b(B_X)$. Then for each $x \in S$, $\mathcal{F}(x)$ is contained in S.*

This shows that every peak point for $\mathcal{A}_b(B_X)$ is a complex extreme point of B_X, which is Theorem 4 in [9].

A point $x \in S_X$ is said to have a *strong face* if for each $\epsilon > 0$, there is $\delta(\epsilon) > 0$ such that if $\mathrm{dist}(\mathcal{F}(x), y) \ge \epsilon$, then

$$\sup_{0\le\theta\le 2\pi} \|x + e^{i\theta}(x - y)\| \ge 1 + \delta(\epsilon).$$

A Banach space X is said to be *locally c-convex* if it is strictly c-convex and every point of the unit sphere S_X has a strong face. The maximum modulus theorem shows that if two elements x, y in a Banach space satisfy $\sup_{0\le\theta\le 2\pi} \|x + e^{i\theta} y\| \le M$, then $\max\{\|x\|, \|y\|\} \le M$.

PROPOSITION 1.5. *Suppose that X is a finite dimensional Banach space. Then every point of S_X has a strong face.*

Proof. Suppose otherwise. Then there exist $x \in S_X$, $\epsilon > 0$, and a sequence $\{y_n\}$ in X such that $\operatorname{dist}(\mathcal{F}(x), y_n) \geq \epsilon$ for each n, but

$$\lim_{n\to\infty} \sup_{0\leq\theta\leq 2\pi} \|x + e^{i\theta}(x - y_n)\| = 1.$$

So we get

$$M := \sup_{n\in\mathbb{N}} \sup_{0\leq\theta\leq 2\pi} \|x + e^{i\theta}(x - y_n)\| < \infty.$$

Hence $\sup_n \|x - y_n\| \leq M$. So we may assume that y_n converges to y. Then $\operatorname{dist}(\mathcal{F}(x), y) \geq \epsilon$. For each $\theta \in \mathbb{R}$,

$$\|x + e^{i\theta}(x - y)\| \leq \|x + e^{i\theta}(x - y_n)\| + \|y_n - y\|.$$

This shows that

$$\sup_{0\leq\theta\leq 2\pi} \|x + e^{i\theta}(x - y)\| \leq \lim_{n} \sup_{0\leq\theta\leq 2\pi} \|x + e^{i\theta}(x - y_n)\| = 1.$$

Therefore, $y = x + (y - x) \in \mathcal{F}(x)$, which contradicts $\operatorname{dist}(\mathcal{F}(x), y) \geq \epsilon$. $\square$

The modulus of complex convexity of a complex Banach space X is defined by

$$H_X(\epsilon) = \inf\left\{ \sup_{0\leq\theta\leq 2\pi} \|x + e^{i\theta}y\| - 1 : x \in S_X, \|y\| \geq \epsilon \right\}$$

for each $\epsilon > 0$. A complex Banach space X is said to be *uniformly c-convex* if $H_X(\epsilon) > 0$ for all $\epsilon > 0$. If X is uniformly c-convex, then every point in S_X has a strong face. A finite dimensional strictly c-convex space is uniformly c-convex.

A sequence $x = \{x(k)\}$ is said to be *positive* if $x(k) \geq 0$ for each $k \in \mathbb{N}$. We define a partial order $x \geq y$ if $x - y$ is positive. The absolute value of x is defined to be $|x| = \{|x(k)|\}$. A *Banach sequence space* $(X, \|\cdot\|)$ is a Banach space consisting of sequences satisfying the following property: if x is a sequence with $|x| \leq |y|$ for some $y \in X$, then $x \in X$ and $\|x\| \leq \|y\|$. A Banach sequence space is said to be *order continuous* if any sequence $\{x_n\}$ in X satisfying

$$0 \leq x_1 \leq x_2 \leq \cdots \leq y \quad \text{for some positive} \quad y \in X,$$

is norm-convergent. The vector e_j is defined to have 1 in the j-th component and zeros in all other components. Note that if a Banach sequence space X is order continuous, then $\{e_n\}$ is a basis of X.

It is known from [16], [17] that a uniformly c-convex sequence space is order continuous.

A Banach sequence space X is said to be *strictly monotone* if for every pair $y \geq x \geq 0$ with $y \neq x$, we have $\|y\| > \|x\|$. Recall also that a Banach sequence space X is said to be *lower (resp. upper) locally uniformly monotone* if for any positive $x \in S_X$ and any $0 < \epsilon < 1$ (resp. $\epsilon > 0$) there is $\delta = \delta(\epsilon, x) > 0$ such that the condition $0 \leq y \leq x$ (resp. $y \geq 0$) and $\|y\| \geq \epsilon$ implies

$$\|x - y\| < 1 - \delta \quad (\text{resp.} \quad \|x + y\| \geq 1 + \delta).$$

A Banach sequence space X is said to be *uniformly monotone* if given $\epsilon > 0$, there is $\delta(\epsilon) > 0$ such that

$$\inf\{\| \, |x| + |y| \, \| : \|y\| \geq \epsilon, x \in S_X\} \geq 1 + \delta.$$

A uniformly monotone Banach sequence space is both lower and upper locally uniformly monotone.

It is shown in [8, Theorem 1] that a Banach sequence space is lower locally uniformly monotone if and only if it is strictly monotone and order continuous. It is also shown in [12], [16], [17] that a Banach sequence space is strictly (resp. uniformly) monotone if and only if it is strictly (resp. uniformly) c-convex.

2. Boundaries of $\mathcal{A}_u(B_X)$ and $\mathcal{A}_b(B_X)$

Proposition 2.1. *Suppose that X is a complex Banach space. Let F be a boundary for $\mathcal{A}_u(B_X)$ and let P be a norm-one projection with a finite dimensional range Y. Then*

$$\mathrm{Ext}_{\mathbb{C}}(B_Y) \subset \overline{P(F)}.$$

Proof. Suppose $x_0 \in \mathrm{Ext}_{\mathbb{C}}(B_Y) \backslash \overline{P(F)}$. Then there exists $\epsilon_0 > 0$ such that $\|P(x) - x_0\| \geq \epsilon_0$ for every $x \in F$. By Proposition 1.1 x_0 is a strong peak point for the algebra $\mathcal{A}_u(B_Y)$, that is, there is a $g \in \mathcal{A}_u(B_Y)$ such that $g(x_0) = 1$ and to every $\epsilon > 0$ corresponds a $\delta(\epsilon) > 0$ satisfying

$$|g(y)| < 1 - \delta(\epsilon),$$

for all $y \in B_Y$ with $\|y - x_0\| \geq \epsilon$. Take $f = g \circ P \in \mathcal{A}_u(B_X)$. Then $f(x_0) = 1$ and for every $x \in F$ we have

$$|f(x)| = |g(P(x))| < 1 - \delta(\epsilon_0).$$

This contradicts the fact that F is a boundary of $\mathcal{A}_u(B_X)$. □

Proposition 2.2. *Suppose that X is a complex Banach space with the following properties: There is a collection $\{P_\alpha\}_{\alpha \in A}$ of projections P_α with finite dimensional ranges Y_α such that $\bigcup_{\alpha \in A} Y_\alpha$ is dense in X, and for each $\alpha \in A$*

$$\sup_{0 \leq \theta \leq 2\pi} \|P_\alpha + e^{i\theta}(I - P_\alpha)\| \leq 1.$$

Then a set $F \subset B_X$ is a boundary for $\mathcal{A}_u(B_X)$ if

$$\mathrm{Ext}_{\mathbb{C}}(B_{Y_\alpha}) \subset \overline{P_\alpha(F)}$$

for every $\alpha \in A$.

Proof. Suppose that F is not a boundary for $\mathcal{A}_u(B_X)$. Then there are $f \in \mathcal{A}_u(B_X)$, $\|f\| = 1$, and $\epsilon > 0$ such that $|f(x)| < 1 - \epsilon$ for every $x \in F$. $\rho\mathcal{A}_u(B_Y)$ is a boundary for $\mathcal{A}_u(B_Y)$ if Y is finite dimensional. Since $\bigcup_{\alpha \in A} B_{Y_\alpha}$ is dense in B_X, it follows from Proposition 1.1 that

$$\|f\| = \sup_{x \in \bigcup_\alpha S_{Y_\alpha}} |f(x)| = \sup_{x \in \bigcup_\alpha \rho\mathcal{A}_u(B_{Y_\alpha})} |f(x)| = \sup_{x \in \bigcup_\alpha \mathrm{Ext}_{\mathbb{C}}(B_{Y_\alpha})} |f(x)|.$$

Hence there is a sequence $\{x_n\}$ such that

$$x_n \in \bigcup_\alpha \mathrm{Ext}_{\mathbb{C}}(B_{Y_\alpha}), \quad \text{and} \quad \lim_n |f(x_n)| = 1.$$

Because f is continuous and $\mathrm{Ext}_{\mathbb{C}}(B_{Y_\alpha}) \subset \overline{P_\alpha(F)}$ for every $\alpha \in A$, there is a sequence $\{u_n\}$ such that

$$\{u_n\} \subset \bigcup_\alpha P_\alpha(F), \quad \text{and} \quad \lim_n |f(u_n)| = 1.$$

Each u_n has the form $u_n = P_{\alpha_n} z_n$, where $z_n \in F$. Set $v_n = (I - P_{\alpha_n})z_n$, and $z_n = u_n + v_n$. By the uniform continuity of f there exists δ, $0 < \delta < 1$, such that if $\|x_1 - x_2\|_X \le \delta$ and $x_1, x_2 \in B_X$, then $|f(x_1) - f(x_2)| < \epsilon/2$. Thus we get for every $n \in \mathbb{N}$,

$$|f(u_n + (1-\delta)v_n) - f(u_n + v_n)| < \frac{\epsilon}{2}.$$

Further, since $z_n = u_n + v_n \in F$, we have $|f(u_n + v_n)| < 1 - \epsilon$, and consequently for each $n \in \mathbb{N}$,

$$|f(u_n + (1-\delta)v_n)| < 1 - \frac{\epsilon}{2}. \tag{2.1}$$

On the other hand, since

$$\|P_{\alpha_n} + e^{i\theta}(I - P_{\alpha_n})\| \le 1$$

for every $\theta \in \mathbb{R}$, the maximum modulus theorem shows that for every $\zeta \in \mathbb{C}$ with $|\zeta| \le 1$,

$$\left\| u_n + \frac{1}{1-\delta}\zeta[(1-\delta)v_n] \right\| \le 1.$$

By [10, Lemma 1.4], there is $C(\epsilon) < \infty$ such that for each $n \in \mathbb{N}$,

$$|f(u_n + (1-\delta)v_n) - f(u_n)| < C(\epsilon)(1 - |f(u_n)|).$$

Since $\lim_n |f(u_n)| = 1$, it follows that $\lim_n |f(u_n + (1-\delta)v_n)| = 1$, which contradicts (2.1). □

Corollary 2.3. *Suppose that X is a complex Banach space with a sequence $\{P_n\}$ of projections with the same properties as in Proposition 2.2. Then a set $F \subset B_X$ is a boundary for $\mathcal{A}_u(B_X)$ if and only if the closure of $P_n(F)$ contains $\mathrm{Ext}_{\mathbb{C}}(B_{Y_n})$ for every positive integer n.*

We remark that an order continuous Banach sequence space has the properties outlined in Proposition 2.2.

Corollary 2.4. *Let X be an order continuous Banach sequence space. Let $F \subset S_X$ and let P_n be a sequence of coordinate projections with finite dimensional range Y_n such that every finite subset of $\mathbb{N}$ is contained in the support of some P_n. If $\mathrm{Ext}_{\mathbb{C}}(B_{Y_n}) \subset \overline{P_n(F)}$ for each $n \in \mathbb{N}$, then F is a boundary for $\mathcal{A}_u(B_X)$.*

Corollary 2.4 extends Theorem 1.5 in [10].

Proposition 2.5. *Suppose that there is a family $\{Y_\alpha\}_{\alpha\in A}$ of finite dimensional subspaces of a Banach space X such that $\bigcup_\alpha B_{Y_\alpha}$ is dense in B_X. Then the set $F = \bigcup_\alpha \mathrm{Ext}_{\mathbb{C}}(B_{Y_\alpha})$ is a boundary for $\mathcal{A}_b(B_X)$.*

3. Shilov boundary for $\mathcal{A}_u(B_X)$ and $\mathcal{A}_b(B_X)$

Proposition 3.1. *Suppose that $x_0 \in S_X$ has a strong face and T is a bounded operator of X into X with $Tx_0 = x_0$ and*

$$\sup_{0\le\theta\le 2\pi} \|T + e^{i\theta}(I - T)\| \le 1.$$

Then for each $\epsilon > 0$, there is $\delta(\epsilon) > 0$ such that whenever $\mathrm{dist}(F(x_0), y) \ge \epsilon$ and $y \in B_X$, we get

$$\|x_0 - Ty\| \ge \delta(\epsilon).$$

Proof. Suppose, on the contrary, that there is $\epsilon_0 > 0$ such that

$$\inf\{\|x_0 - Ty\| : \mathrm{dist}(F(x_0), y) \ge \epsilon_0, \quad y \in B_X\} = 0.$$

Then there is a sequence $\{s_n\}$ in B_X such that $\mathrm{dist}(F(x_0), s_n) \ge \epsilon_0$ and

$$\lim_{n\to\infty} Ts_n = x_0.$$

Since x_0 has a strong face, there are $\delta_1 > 0$ and $\theta_n \in \mathbb{R}$ such that for every n we have

$$\|x_0 + e^{i\theta_n}(s_n - x_0)\| \ge 1 + \delta_1.$$

So

$$\begin{aligned} 1 + \delta_1 &\le \|x_0 + e^{i\theta_n}(s_n - x_0)\| \\ &\le \|Ts_n + e^{i\theta_n}(s_n - x_0)\| + \|x_0 - Ts_n\| \\ &\le \|Ts_n + e^{i\theta_n}(I - T)s_n)\| + 2\|x_0 - Ts_n\|. \end{aligned}$$

This implies that

$$1 + \delta_1 \le \limsup_{n\to\infty} \|Ts_n + e^{i\theta_n}(I - T)s_n)\| \le 1,$$

which is a contradiction. ∎

Proposition 3.2. *Let X be a complex Banach space and let P be a projection of X onto a finite dimensional subspace Y such that*

$$\sup_{0\leq\theta\leq 2\pi} \|P + e^{i\theta}(I-P)\| \leq 1.$$

If $x_0 \in \mathrm{Ext}_{\mathbb{C}}(B_Y)$ and x_0 has a strong face in X, then $\mathcal{F}(x_0)$ is a strong peak set for $\mathcal{A}_u(B_X)$.

Proof. By Proposition 1.1, x_0 is a strong peak point for $\mathcal{A}_b(B_Y)$ since Y is finite dimensional. Hence there is a peak function $g \in \mathcal{A}_b(B_Y)$ such that $g(x_0) = 1$ and for each $\epsilon > 0$ there is $\tilde{\delta}(\epsilon) > 0$ such that for every $\|x_0 - y\| \geq \epsilon$ and $y \in B_Y$, we have

$$|g(y)| < 1 - \tilde{\delta}(\epsilon).$$

By Proposition 3.1 we get $\delta(\epsilon) > 0$ such that if $\mathrm{dist}(\mathcal{F}(x_0), y) \geq \epsilon$

$$\|x_0 - Py\| \geq \delta(\epsilon).$$

Take $f = g \circ P$. Then $f \in \mathcal{A}_u(B_X)$ and for each $z \in B_X$ with $\mathrm{dist}(\mathcal{F}(x_0), z) \geq \epsilon$, we have $f(\mathcal{F}(x_0)) = 1$ by the maximum modulus theorem and $|f(z)| < 1 - \tilde{\delta}(\delta(\epsilon))$. This implies that $\mathcal{F}(x_0)$ is a strong peak set for $\mathcal{A}_u(B_X)$. □

Every element x in the torus in c_0 has a strong face $\mathcal{F}(x)$ and hence $\mathcal{F}(x)$ is a strong peak set for $\mathcal{A}_u(B_X)$. The following result generalizes Theorem 1.9 in [10].

Proposition 3.3. *Let X be a complex Banach space as in Proposition 2.2. Suppose that every point of $\bigcup_\alpha \mathrm{Ext}_{\mathbb{C}}(B_{Y_\alpha})$ has a strong face in X. Then $F \subset B_X$ is a boundary for $\mathcal{A}_u(B_X)$ if and only if $\mathrm{dist}(F, S) = 0$ for each strong peak set S for $\mathcal{A}_u(B_X)$.*

Proof. The necessity is clear. Conversely, suppose that there is a subset $F \subset B_X$ such that $\mathrm{dist}(F, S) = 0$ for each strong peak set S for $\mathcal{A}_u(B_X)$. We shall show that for each α the closure of $P_\alpha(F)$ contains $\mathrm{Ext}_{\mathbb{C}}(B_{Y_\alpha})$. By applying Proposition 2.2, we get the desired result.

Now, let $x_0 \in \mathrm{Ext}_{\mathbb{C}}(B_{Y_\alpha})$. By Proposition 3.2, its face $\mathcal{F}(x_0)$ is a strong peak set for $A_u(B_X)$. Hence $\mathrm{dist}(F, \mathcal{F}(x_0)) = 0$. Then there are sequences $\{x_0 + y_k\}_k$ in $\mathcal{F}(x_0)$ and $\{z_k\}_k$ in F such that

$$\lim_{k\to\infty} \|(x_0 + y_k) - z_k\| = 0. \tag{3.1}$$

Since $x_0 + y_k$ is in $\mathcal{F}(x_0)$ and $\|P_\alpha\| = 1$, we have for each real θ,

$$\|x_0 + e^{i\theta} P_\alpha(y_k)\| \leq \|x_0 + e^{i\theta} y_k\| \leq 1.$$

Since x_0 is a complex extreme point of B_{Y_α}, $P_\alpha(y_k) = 0$, and so (3.1) shows that

$$\limsup_{k\to\infty} \|x_0 - P_\alpha(z_k)\| \leq \limsup_{k\to\infty} \|(x_0 + y_k) - z_k\| = 0.$$

Therefore x_0 is in the closure of $P_\alpha(F)$. □

In the proof of Proposition 3.3, for $F \subset B_X$ to be a boundary for $\mathcal{A}_u(B_X)$ it is sufficient that $\operatorname{dist}(F, \mathcal{F}(x)) = 0$ holds for every $x \in \bigcup_\alpha \operatorname{Ext}_{\mathbb{C}}(B_{Y_\alpha})$.

Corollary 3.4. *Let X be a locally c-convex sequence space. Suppose that $x_0 \in S_X$ is finitely supported. Then x_0 is a strong peak point for $\mathcal{A}_u(B_X)$. In particular, if in addition X is order continuous, then the set of all strong peak points for $\mathcal{A}_u(B_X)$ is dense in S_X.*

Proof. Suppose that $Y = \operatorname{span}\{e_1, \ldots, e_n\}$ contains x_0. Hence x_0 is a complex extreme point of B_Y. Let $P : X \to Y$ be the projection defined by

$$P(x(1), x(2), \ldots) = (x(1), x(2), \ldots, x(n), 0, 0, \ldots).$$

Clearly $\|P + e^{i\theta}(I - P)\| \le 1$ for all $\theta \in \mathbb{R}$. By Proposition 3.2, x_0 is a strong peak point for $\mathcal{A}_u(B_X)$. Notice that if a Banach sequence space is order continuous, then the set of all finitely supported elements in X is dense in X. $\square$

Proposition 1.2 and Corollary 3.4 show the following theorem.

Theorem 3.5. *Let X be an order continuous locally c-convex Banach space. Then S_X is the Shilov boundary for both $\mathcal{A}_u(B_X)$ and $\mathcal{A}_b(B_X)$.*

By [16], [17], every uniformly c-convex sequence space is order continuous.

Proposition 3.6. *A Banach sequence space X is upper locally uniformly monotone if and only if it is locally c-convex.*

Proof. Suppose X is locally c-convex. Then for each positive $x \in S_X$ and $\epsilon > 0$ there is $\delta = \delta(x, \epsilon) > 0$ such that for all $z \in X$ with $\|z\| \ge \epsilon$

$$\sup_{0 \le \theta \le 2\pi} \|x + e^{i\theta} z\| \ge 1 + \delta.$$

Hence we have for every $y \ge 0$ with $\|y\| \ge \epsilon$,

$$\|x + y\| \ge \inf \left\{ \sup_{0 \le \theta \le 2\pi} \|x + e^{i\theta} z\| : \|z\| \ge \epsilon \right\} \ge 1 + \delta.$$

So X is upper locally uniformly monotone.

Conversely, suppose that X is upper locally uniformly monotone. If $x, y \in X$, then by [7, Theorem 7.1],

$$\sup_{0 \le \theta \le 2\pi} \|x + e^{i\theta} y\| \ge \frac{1}{2\pi} \int_0^{2\pi} \|x + e^{i\theta} y\| d\theta \ge \left\| \left(|x|^2 + \frac{1}{2}|y|^2 \right)^{1/2} \right\|.$$

By Lemma 2.3 in [16], for every nonzero pair x, y in X, there exist $\delta_1 = \delta_1(\|x\|, \|y\|) > 0$ and $z \in X$ with $0 \le z \le |y|$ and $\|z\| \ge \|y\|/2$ such that the following holds:

$$\|(|x|^2 + |y|^2)^{1/2}\| \ge \| \, |x| + \delta_1 |z| \, \|.$$

Hence for every $x \in S_X$ and $\epsilon > 0$, we get

$$\inf\left\{\sup_{0\le\theta\le 2\pi}\|x+e^{i\theta}y\| : \|y\|\ge\epsilon\right\}$$
$$\ge \inf\left\{\|\,|x|+|y|\,\| : \|y\| \ge \frac{\epsilon}{2\sqrt{2}}\delta_1\left(1,\frac{\epsilon}{\sqrt{2}}\right)\right\}.$$

Hence the upper local uniform monotonicity implies the local c-convexity. The proof is complete. □

A function $\varphi : \mathbb{R} \to [0,\infty]$ is said to be an *Orlicz function* if φ is even, convex, continuous, and vanishing only at zero. Let $w = \{w(n)\}$ be a *weight sequence*, that is, a non-increasing sequence of positive real numbers satisfying $\sum_{n=1}^{\infty} w(n) = \infty$. Given a sequence x, x^* is the decreasing rearrangement of $|x|$.

The Orlicz-Lorentz sequence space $\lambda_{\varphi,w}$ consists of all sequences $x = \{x(n)\}$ such that for some $\lambda > 0$,

$$\varrho_\varphi(\lambda x) = \sum_{n=1}^{\infty}\varphi(\lambda x^*(n))w(n) < \infty,$$

and equipped with the norm $\|x\| = \inf\{\lambda > 0 : \varrho_\varphi(x/\lambda) \le 1\}$, $\lambda_{\varphi,w}$ is a Banach space. We say an Orlicz function φ satisfies the *condition* δ_2 ($\varphi \in \delta_2$) if there exist $K > 0$, $u_0 > 0$, such that $\varphi(u_0) > 0$ and the inequality

$$\varphi(2u) \le K\varphi(u)$$

holds for $u \in [0, u_0]$.

It was proved in [8, Corollary 4] that the Orlicz-Lorentz sequence space $\lambda_{\varphi,w}$ is strictly monotone if and only if it is both upper and lower locally uniformly monotone. There it was also shown that the strict monotonicity of $\lambda_{\varphi,w}$ is equivalent to the fact that $\varphi \in \delta_2$. In this case, the Orlicz-Lorentz sequence space $\lambda_{\varphi,w}$ is locally c-convex by Proposition 3.6 and order continuous by Theorem 2 and Corollary 4 of [8]. If $\varphi(u) = |u|^p$ for some $1 \le p < \infty$ and if $w \equiv 1$, then $\lambda_{\varphi,w} = \ell_p$. Hence we obtain the following corollary by Theorem 3.5, which extends a result in [2].

Corollary 3.7. *Given an Orlicz function $\varphi \in \delta_2$ and a weight sequence w,*

$$\partial\mathcal{A}_u(B_{\lambda_{\varphi,w}}) = \partial\mathcal{A}_b(B_{\lambda_{\varphi,w}}) = S_{\lambda_{\varphi,w}}.$$

4. Boundaries for $\mathcal{A}_b(B_X)$

Recall that a Banach sequence space X is called *rearrangement invariant* if $y \in X$ and $\|y\| = \|x\|$ whenever y is a sequence with $y^* = x^*$ for some $x \in X$. Let X be a rearrangement invariant Banach sequence space. Given any finite subset M of natural numbers, let $\phi : \mathbb{N} \to \mathbb{N} \setminus M$ be the order preserving bijection and let P_M be the isometry from $\{x \in X : \operatorname{supp}(x) \cap M = \emptyset\}$ onto X given by

$$P_M(x) = \sum_{i=1}^{\infty} \langle x, e_{\phi(i)} \rangle e_i,$$

where the sum is a formal series and $\operatorname{supp}(x) = \{k \in \mathbb{N} : x(k) \neq 0\}$. If $\operatorname{supp}(x)$ is finite, x is called a *finite vector*. Now assume that X has the following additional property:

For each finite vector $x \in B_X$, there exist $\epsilon = \epsilon(x) > 0$ such that for all $y \in B_X$ with $\operatorname{supp}(x) \cap \operatorname{supp}(y) = \emptyset$,

$$\|x + \epsilon y\| \leq 1. \tag{4.1}$$

For each finite vector $x \in S_X$, let $\eta(x) > 0$ be the supremum of the set of all $\epsilon > 0$ satisfying (4.1). Observe that $\eta(x) \leq 1$ and

$$\mathcal{F}(x) \supset \{x + \eta(x)y : y \in B_X, \operatorname{supp}(x) \cap \operatorname{supp}(y) = \emptyset\}.$$

If X is a rearrangement invariant Banach sequence space satisfying the property (4.1) and if $x \in \operatorname{Ext}_{\mathbb{C}}(B_F)$, where F is the subspace spanned by a finite number of $\{e_k\}_k$, then for all $y \in X$ with $x + y \in \mathcal{F}(x)$, we have

$$\operatorname{supp}(x) \cap \operatorname{supp}(y) = \emptyset, \quad \text{and} \quad \|x + \eta(x)y\| \leq 1.$$

Let $S \subset B_X$ and let F be the subspace spanned by a finite number of $\{e_k\}_k$. Given $x \in B_F$ we define

$$S(x) = \{P_{\operatorname{supp}(x)}(y) : y \in B_X,\ x + \eta(x)y \in S,\ \operatorname{supp}(x) \cap \operatorname{supp}(y) = \emptyset\}.$$

Put, for each $0 < \epsilon < 1$,

$$C(S, \epsilon) = \sup\{|f(0)| : f \in \mathcal{A}_b(B_X), \quad \|f\| \leq 1,\ |f(z)| < 1 - \epsilon \text{ for all } z \in S\}.$$

Then S is called a *0-boundary* for $\mathcal{A}_b(B_X)$ if $C(S, \epsilon) < 1$ for every $\epsilon > 0$. A family $\{S_\gamma\}_{\gamma \in \Gamma}$ of subsets of B_X is called a *uniform family of 0-boundaries* for $\mathcal{A}_b(B_X)$ if $\sup_{\gamma \in \Gamma} C(S_\gamma, \epsilon) < 1$ for every $\epsilon > 0$.

THEOREM 4.1. *Let X be a rearrangement invariant Banach sequence space satisfying property (4.1) and let V be a boundary for $\mathcal{A}_b(B_X)$ consisting of norm-one finite vectors. Assume also that $S \subset B_X$ has the property that $\{S(x)\}_{x \in V}$ is a uniform family of 0-boundaries for $\mathcal{A}_b(B_X)$. Then S is a boundary for $\mathcal{A}_b(B_X)$.*

Proof. Suppose S is not a boundary for $\mathcal{A}_b(B_X)$. Then there is $f \in \mathcal{A}_b(B_X)$ with $\|f\| = 1$ and $0 < \delta < 1$ such that $|f(z)| < 1 - \delta$ for all $z \in S$. The assumption on V implies that there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in V such that

$$\lim_{n \to \infty} |f(x_n)| = 1.$$

For each $n \in \mathbb{N}$, $\operatorname{supp}(x_n)$ is finite and there exists $\eta(x_n) > 0$ such that for every $y \in B_X$ with $\operatorname{supp}(x_n) \cap \operatorname{supp}(y) = \emptyset$

$$\|x_n + \eta(x_n)y\| \leq 1.$$

Define ϕ_n on B_X

$$x \mapsto f\left(x_n + \eta(x_n)P^{-1}_{\operatorname{supp}(x_n)}(x)\right),$$

Then $\phi_n \in \mathcal{A}_b(B_X)$ and $\|\phi_n\| \leq 1$ for all n. Moreover, for each $x \in S(x_n)$

$$|\phi_n(x)| < 1 - \delta \quad \text{and} \quad \lim_{n\to\infty} |\phi_n(0)| = 1,$$

and this contradicts the assumption that $\{S(x)\}_{x\in V}$ is a uniform family of 0-boundaries for $\mathcal{A}_b(B_X)$. □

We shall use the following two lemmas, which are proved in [10].

Lemma 4.2 ([10]). *Let $0 \leq r < 1$ and assume $\{S_\gamma\}_{\gamma\in\Gamma}$ is a family of subsets of B_X such that $S_\gamma \cap rB_X \neq \emptyset$ for each $\gamma \in \Gamma$. Then $\{S_\gamma\}_{\gamma\in\Gamma}$ is a uniform family of 0-boundaries for $\mathcal{A}_b(B_X)$.*

Lemma 4.3 ([10]). *Let $\theta_0 > 0$ and let $\{S_\gamma\}_{\gamma\in\Gamma}$ be a family of subsets of B_X with the following property: for each $\gamma \in \Gamma$ there is some $x_\gamma \in S_\gamma$ such that $e^{i\theta}x_\gamma \in S_\gamma$ for every $|\theta| \leq \theta_0$. Then $\{S_\gamma\}_{\gamma\in\Gamma}$ is a uniform family of 0-boundaries for $\mathcal{A}_b(B_X)$.*

Theorem 4.1 and Lemma 4.2 show the following corollary.

Corollary 4.4. *Let X be a rearrangement invariant Banach sequence space satisfying property (4.1) and let V be a boundary for $\mathcal{A}_b(B_X)$ consisting of norm-one finite vectors. Assume that $S \subset B_X$ and that there is $0 \leq r < 1$ such that for each $x \in V$ there exists $y \in X$ such that*

$$\|y\| \leq r\ ,\ \operatorname{supp}(x) \cap \operatorname{supp}(y) = \emptyset \text{ and } x + \eta(x)y \in S.$$

Then S is a boundary for $\mathcal{A}_b(B_X)$.

By Theorem 4.1 and Lemma 4.3, we get the following corollary.

Corollary 4.5. *Let X be a rearrangement invariant Banach sequence space satisfying property (4.1) and let V be a boundary for $\mathcal{A}_b(B_X)$ consisting of norm-one finite vectors. Assume that $S \subset B_X$ and assume that there is $\theta_0 > 0$ such that for each $x \in V$ there exists $y \in B_X$ such that*

$$\operatorname{supp}(x) \cap \operatorname{supp}(y) = \emptyset \text{ and } x + \eta(x)e^{i\theta}y \in S \text{ for all } |\theta| \leq \theta_0.$$

Then S is a boundary for $\mathcal{A}_b(B_X)$.

EXAMPLE 4.6. Assume that $\psi = \{\psi(n)\}$ is a strictly increasing sequence with $\psi(0) = 0$, $\psi(n) > 0$ for $n \in \mathbb{N}$. The *Marcinkiewicz sequence space* m_ψ consists of all sequences $x = \{x(n)\}$ such that

$$\|x\|_{m_\psi} = \sup_{n\in\mathbb{N}} \frac{\sum_{k=1}^{n} x^*(k)}{\psi(n)} < \infty.$$

Let m_Ψ^0 be the closed subspace of m_ψ, equipped with the same norm $\|\cdot\|_{m_\psi}$ consisting of all $x \in m_\psi$ satisfying

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} x^*(k)}{\psi(n)} = 0.$$

Without loss of generality we can add (and we will) in the above definition the assumption that the sequence $\{\frac{\psi(n)}{n}\}_{n=1}^{\infty}$ is decreasing [13]. Notice that if $\psi(n) = n$, then $m_\psi = \ell_\infty$ and $m_\psi^0 = c_0$, and if $\lim_n \psi(n) < \infty$, then $m_\psi^0 = \{0\}$.

It is shown in [14] that if $\lim_n \psi(n) = \infty$, then for each $x \in B_{m_\psi^0}$, there exist $n \in \mathbb{N}$ and $\epsilon > 0$ such that $\|x + \lambda y\| \leq 1$ for all $y \in B_{m_\psi}$ with $y = (0, \cdots, 0, y(n+1), y(n+2), \cdots)$ and all λ with $|\lambda| \leq \epsilon$. Now it is easy to see that m_ψ^0 satisfies (4.1) because m_ψ^0 is a rearrangement invariant sequence space.

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