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INEQUALITIES AND ASYMPTOTICS FOR A TERMINATING ${}_4F_3$ SERIES

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ABSTRACT. In this paper we give upper bounds for a certain terminating ${}_4F_3$ series. Our estimates confirm special cases of a conjecture of Kresch and Tamvakis. We also give asymptotic estimates when the parameters in the ${}_4F_3$ series are large, and they confirm the same conjecture.

1. Introduction

We first introduce the needed terminology. For a complex number a and an integer n, the shifted factorial $(a)_n$ is defined by

$$(a)_n := \prod_{j=1}^{n}(a+j-1) = \Gamma(a+n)/\Gamma(a).$$

We set $(a)_0 := 1$ if $a \neq 0$. Next, for an integer n and complex numbers a, b, c, d, e, f, and z, such that $\{d,\, e,\, f\} \cap \{-n+1, -n+2, \dots, -1, 0\} = \emptyset$, the terminating ${}_4F_3$ hypergeometric series is defined by

$$(1.1) \qquad {}_4F_3\left(\begin{matrix} -n,\ a,\ b,\ c \\ d,\ e,\ f \end{matrix} \,\middle|\, z \right) := \sum_{k=0}^{n} \frac{(-n)_k (a)_k (b)_k (c)_k}{(d)_k (e)_k (f)_k k!} z^k.$$

Let $Q > s, n \geq 1$ be integers. We define

$$(1.2) \qquad R(n,s,Q) := {}_4F_3\left(\begin{matrix} -n,\ n+1,\ -s,\ s+1 \\ 1+Q,\ 1,\ 1-Q \end{matrix} \,\middle|\, 1 \right).$$

The series defining R has at most $n+1$ terms. In this paper we study the following conjecture:

CONJECTURE 1.1. *The terminating ${}_4F_3$ series $R(n,s,Q)$ defined with* (1.2) *satisfies*

$$(1.3) \qquad |R(n,s,Q)| \leq 1$$

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for all integer numbers $Q > s \geq n \geq 1$.

This inequality was conjectured by A. Kresch and H. Tamvakis in [7]. Extensive numerical evaluations provided overwhelming evidence supporting this conjecture. The expression $R(n, s, Q)$ is the special case $\alpha = \beta = \gamma = 0$ of the Racah polynomials considered by Dunkl in [4].

The Racah polynomials [1], [2], [6], are defined by

$$R_n(\lambda(x); \alpha, \beta, \gamma, \delta) = {}_4F_3\left(\begin{matrix} -n,\ n+\alpha+\beta+1,\ -x,\ x+\gamma+\delta+1 \\ \alpha+1,\ \beta+\delta+1,\ \gamma+1 \end{matrix} \,\middle|\, 1\right), \tag{1.4}$$

for $n = 0, 1, \ldots, N$, where $\lambda(x) = x(x+\gamma+\delta+1)$ and $\alpha + 1 = -N$ or $\beta + \delta + 1 = -N$ or $\gamma + 1 = -N$. Selecting $\alpha = \beta = 0$, $\gamma = -N - 1$, and $\delta = N + 1$ we obtain

$$R_n(x(x+1); 0, 0, -N-1, N+1) = R(n, x, N+1). \tag{1.5}$$

The conjecture of Kresch and Tamvakis states that the absolute value of a Racah polynomial is bounded by its value at $x = 0$.

Following the ideas of [5] one can establish the generating function (see [6])

$$\sum_{n=0}^{N} \frac{(N+2)_n(-N)_n}{n!^2} R(n, x, N+1) t^n = {}_2F_1\left(\begin{matrix} -x,\ -x \\ 1 \end{matrix} \,\middle|\, t\right) {}_2F_1\left(\begin{matrix} x-N,\ x+N+2 \\ 1 \end{matrix} \,\middle|\, t\right). \tag{1.6}$$

We will use the Whipple transform [6]: If $n \in \mathbf{N}$ and $a + b + c + 1 = d + e + f + n$, then

$$ {}_4F_3\left(\begin{matrix} -n,\ a,\ b,\ c \\ d,\ e,\ f \end{matrix} \,\middle|\, 1\right) = \frac{(e-a)_n(f-a)_n}{(e)_n(f)_n} \times {}_4F_3\left(\begin{matrix} -n,\ a,\ d-b,\ d-c \\ d,\ a-e-n+1,\ a-f-n+1 \end{matrix} \,\middle|\, 1\right), \tag{1.7}$$

the Pfaff-Saalschutz formula [6]:

$$ {}_3F_2\left(\begin{matrix} -n,\ a,\ b \\ c,\ 1+a+b-c-n \end{matrix} \,\middle|\, 1\right) = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n}, \tag{1.8}$$

and the Pfaff-Kummer transform [6]:

$$ {}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix} \,\middle|\, z\right) = (1-z)^{-a} {}_2F_1\left(\begin{matrix} a,\ c-b \\ c \end{matrix} \,\middle|\, \frac{z}{z-1}\right). \tag{1.9}$$

In Section 2 we verify the conjecture in several special cases. In Section 3 we use an integral representation based on the generating function (1.6), and the methods of Darboux and Laplace to obtain asymptotic estimates of

$R(n,x,N+1)$ when x is fixed, and $R(n,\lambda n,\gamma n+1)$ with fixed $\lambda>0$ and $\gamma>1$. These asymptotic estimates also confirm the conjecture.

2. Some special cases

We set

$$R_{2n}(x):=R(n,x,N+1)={}_4F_3\left(\begin{array}{c}-n,\ n+1,\ -x,\ x+1\\ 1,\ N+2,\ -N\end{array}\middle|\ 1\right), \tag{2.1}$$

$n,x=0,1,\dots,N$. Note that $R_{2n}(x)$ is the Racah polynomial in (1.5). These Racah polynomials are discrete orthogonal polynomials and their orthogonality relation is

$$\sum_{x=0}^{N}(2x+1)R_{2n}(x)^2=\frac{(N+1)^2}{2n+1}, \tag{2.2}$$

(see [1], [6]). From (2.2) it follows that

$$|R_{2n}(x)|\le\frac{N+1}{\sqrt{(2n+1)(2x+1)}}. \tag{2.3}$$

Hence, $|R_{2n}(x)|\le 1$ when $2N+1\ge 2x+1\ge (N+1)^2/(2n+1)$. This leads to the following lemma.

LEMMA 2.1. *The inequality $|R_{2n}(x)|\le 1$ holds for every n and x such that $n\ge N^2/(4N+2)$ and $x\ge((N+1)^2/(2n+1)-1)/2$. Furthermore, if $N/n\to\gamma\ge 1$ and $x/n\to\lambda>0$, then*

$$\limsup_{n\to\infty}|R_{2n}(x)|\le\frac{\gamma}{2\sqrt{\lambda}}. \tag{2.4}$$

Next, we consider the special cases $x=0,1,2$, and $x=N$.

LEMMA 2.2. *The inequality $|R_{2n}(x)|\le 1$ holds for $x=0,1,2$, and $x=N$.*

Proof. The cases $x=0$ and $x=1$ are trivial since $R_{2n}(0)=1$ and

$$R_{2n}(1)=1-\frac{2n(n+1)}{N(N+2)}.$$

Now let $x=2$. From (2.1) we have

$$\begin{aligned}R_{2n}(2)&=1-\frac{6n(n+1)}{N(N+2)}+\frac{6(n-1)n(n+1)(n+2)}{(N-1)N(N+2)(N+3)}\\&=1-\frac{6n(n+1)(N(N+2)-1-n(n+1))}{N(N+2)(N(N+2)-3)}.\end{aligned}$$

It is clear that $R_{2n}(2) \le 1$. Furthermore, since $t(N(N+2)-1-t) \le (N(N+2)-1)^2/4$ when t is between 0 and $N(N+1)$, we get

$$R_{2n}(2) \ge 1 - \frac{3(N(N+2)-1)^2}{2N(N+2)(N(N+2)-3)} > -1.$$

To verify the last inequality we set $A = N(N+2)-1$. We have to show that $3A^2 < 4(A+1)(A-2)$, which is equivalent to $(A-2)^2 - 12 > 0$. This is true since $A \ge 7$ when $N \ge 2$.

At $x = N$, from (2.1) and (1.8) we obtain

$$\begin{aligned} R_{2n}(N) &= {}_3F_2\left(\begin{matrix} -n,\ n+1,\ N+1 \\ 1,\ N+2 \end{matrix} \,\middle|\, 1 \right) = \frac{(-n)_n(-N)_n}{(1)_n(-n-N-1)_n} \\ &= (-1)^n \frac{N!(N+1)!}{(N-n)!(N+n+1)!} = (-1)^n \prod_{j=1}^{n} \frac{N-n+j}{N+1+j}, \end{aligned}$$

where we applied (1.8). Thus, $|R_{2n}(N)| \le 1$. □

Lemma 2.3. *The inequality* $|R_{2n}(N-1)| \le 1$ *holds for every* $N \ge 6$.

Proof. Applying (1.7) to $R_{2n}(x)$ with $a = n+1$ and $d = -N$ we obtain

$$R_{2n}(x) = (-1)^n \frac{(N-n+1)_n}{(N+2)_n} {}_4F_3\left(\begin{matrix} -n,\ n+1,\ -N+x,\ -N-x-1 \\ -N,\ 1,\ -N \end{matrix} \,\middle|\, 1 \right).$$

In particular,

$$|R_{2n}(N-1)| = \frac{(N-n+1)_n}{(N+2)_n} \frac{|2n(n+1)-N|}{N}.$$

Clearly, $|R_{2n}(N-1)| \le 1$ when $n(n+1) \le N$. So assume that $n(n+1) > N$. We have

$$\begin{aligned} \frac{(N-n+1)_n}{(N+2)_n} &= \prod_{j=0}^{n-1} \frac{N-n+1+j}{N+2+j} = \exp\left(\sum_{j=0}^{n-1} \log\left(1 - \frac{n+1}{N+2+j}\right)\right) \\ &\le \exp\left(-\sum_{j=0}^{n-1} \frac{n+1}{N+2+j} \right) \le \exp\left(-(n+1) \int_{N+2}^{N+n+2} \frac{1}{u}\, du \right) \\ &= \exp\left(-(n+1) \log \frac{N+n+2}{N+2} \right) = \left(1 - \frac{n}{N+n+2}\right)^{n+1} \\ &\le e^{-n(n+1)/(N+n+2)}, \end{aligned}$$

where we used the inequalities $\log(1-t) \le -t$ and $1 - t \le e^{-t}$ for $t \in [0,1)$. Thus, it is enough to show that

$$e^{-n(n+1)/(N+n+2)}(2n(n+1)-N)/N \le 1,$$

or equivalently,

$$-\frac{n(n+1)}{N+n+2}+\log\left(\frac{2n(n+1)}{N}-1\right)\leq 0. \tag{2.5}$$

In view of Lemma 2.1 we may assume that $n \leq N/3-1$. Then, $N+n+2 \leq 3N/2$ and it is sufficient to verify the inequality

$$-\frac{2n(n+1)}{3N}+\log\left(\frac{2n(n+1)}{N}-1\right)\leq 0. \tag{2.6}$$

Set $h(t) = -t/3+\log(t-1)$ with $t = 2n(n+1)/N \geq 2$. We have $h'(t) = (4-t)/(3(t-1))$, hence $h(t) \leq h(4) = \log 3 - 4/3 < 0$ for $t \geq 2$, and (2.6) follows from here. □

3. Asymptotic estimates

Since $R(n,x,N+1) = R(x,n,N+1)$ we may assume that $x \leq n$. Integrating the generating function (1.6) we obtain

$$\begin{aligned} R(n,x,N+1) = {} & \frac{n!^2}{(N+2)_n(-N)_n}\frac{1}{2\pi i}\int_\Gamma {}_2F_1\left(\begin{matrix}-x,\ -x\\ 1\end{matrix}\,\middle|\, t\right) \\ & \times {}_2F_1\left(\begin{matrix}-(N-x),\ N+x+2\\ 1\end{matrix}\,\middle|\, t\right) t^{-n-1}\,dt, \end{aligned} \tag{3.1}$$

where Γ is a simple closed contour containing 0 in its interior. The ${}_2F_1$ functions can be expressed in terms of the Jacobi polynomials

$$p_n^{(\alpha,\beta)}(t) = \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n,\ n+\alpha+\beta+1\\ \alpha+1\end{matrix}\,\middle|\, \frac{1-t}{2}\right). \tag{3.2}$$

From (1.9) we have

$$\begin{aligned} {}_2F_1\left(\begin{matrix}-x,\ -x\\ 1\end{matrix}\,\middle|\, t\right) &= (1-t)^x\,{}_2F_1\left(\begin{matrix}-x,\ x+1\\ 1\end{matrix}\,\middle|\, \frac{t}{t-1}\right) \\ &= (1-t)^x P_x\left(\frac{1+t}{1-t}\right), \end{aligned} \tag{3.3}$$

where $P_x = p_x^{(0,0)}$ denotes the Legendre polynomial of degree x. The second ${}_2F_1$ becomes

$${}_2F_1\left(\begin{matrix}-(N-x),\ N+x+2\\ 1\end{matrix}\,\middle|\, t\right) = p_{N-x}^{(0,2x+1)}(1-2t). \tag{3.4}$$

1. Asymptotic estimate for fixed x. Let x be fixed and $N/n = \gamma_n \to \gamma \geq 1$, as $n \to \infty$. Taking a limit in (2.1) as $n \to \infty$ we obtain

$$\lim_{n\to\infty} R(n,x,N+1) = \lim_{n\to\infty} \sum_{k=0}^{x} \frac{(-x)_k(x+1)_k}{k!^2} \frac{(-n)_k(n+1)_k}{(-N)_k(N+2)_k}$$
$$= \sum_{k=0}^{x} \frac{(-x)_k(x+1)_k}{k!^2}\gamma^{-2k} = {}_2F_1\left(\begin{matrix} -x, \; x+1 \\ 1 \end{matrix} \,\middle|\, \gamma^{-2} \right) = P_x(1-2\gamma^{-2}).$$

The above limit belongs to the interval $[-1,1]$ since the Legendre polynomials P_x satisfy $|P_x(t)| \leq 1$ for $-1 \leq t \leq 1$, (see [12, Section 7.21]).

2. Asymptotic estimate for large x. Let $x/n = \lambda \in (0,1]$ and $N/n = \gamma > 1$ be fixed rational numbers. From [12, Theorem 8.21.7] and [12, Theorem 8.21.9], we have the asymptotic formula

(3.5)
$$P_x(w) = (2\pi x)^{-1/2} \left\{ \frac{(w+(w^2-1)^{1/2})^{1/2}}{(w^2-1)^{1/4}} + O(x^{-1}) \right\} (w+(w^2-1)^{1/2})^x,$$

uniformly on compact subsets of $\mathbf{C} \setminus [-1,1]$. Furthermore, by the Bernstein-Walsh lemma [13], $|P_x(w)| \leq (w+(w^2-1)^{1/2})^x$ for every $w \in \mathbf{C}$. Here we use the branch of the logarithmic function defined by $\log z = \log|z| + i\arg(z)$ with $\arg(z) \in (-\pi,\pi)$, $z \in \mathbf{C} \setminus (-\infty, 0]$.

An asymptotic formula for the polynomials $p_{N-x}^{(0,2x+1)}(1-2t)$ can be derived using the method of Darboux. We will use the generating function [10]

$$g(w) := \sum_{n=0}^{\infty} p_n^{(\alpha_0+\alpha n, \beta_0+\beta n)}(z) w^n = \frac{(1+\xi)^{\alpha_0+1}(1+\eta)^{\beta_0+1}}{1-\alpha\xi-\beta\eta-(1+\alpha+\beta)\xi\eta}, \tag{3.6}$$

where

$$2\xi = (z+1)w(1+\xi)^{1+\alpha}(1+\eta)^{1+\beta}, \quad 2\eta = (z-1)w(1+\xi)^{1+\alpha}(1+\eta)^{1+\beta}, \tag{3.7}$$

and $\alpha > -1$, $\beta > -1$, α_0, and β_0 are real constants. This generating function was used in [3] to determine the strong asymptotics of the above Jacobi polynomials on the interval $[-1,1]$.

The generating function in (3.6) has a singularity when

$$D(\xi) := 1-\alpha\xi-\beta\eta-(1+\alpha+\beta)\xi\eta = 0. \tag{3.8}$$

From (3.7) we get $\eta = (z-1)\xi/(z+1)$ and (3.8) takes the form

$$(1+\alpha+\beta)(1-z)\xi^2 - (\alpha(z+1)+\beta(z-1))\xi + (z+1) = 0. \tag{3.9}$$

If $(1+\alpha+\beta)(1-z) \neq 0$, the roots of (3.9) are

$$\xi_\pm = \frac{(\alpha+\beta)z+(\alpha-\beta) \pm \sqrt{\triangle}}{2(1+\alpha+\beta)(1-z)}, \tag{3.10}$$

where

$$\triangle = ((\alpha+\beta)z + (\alpha-\beta))^2 - 4(\alpha+\beta+1)(1-z^2). \tag{3.11}$$

The corresponding w-values are obtained from (3.7):

$$w_\pm = 2\xi_\pm(1+\xi_\pm)^{-\alpha-1}(1+\eta_\pm)^{-\beta-1}/(z+1). \tag{3.12}$$

Now we study the behavior of $g(w)$ near its singularities. From (3.7) we obtain

$$\xi(1+\xi)(1+\eta)\frac{dw}{d\xi} = wD(\xi), \tag{3.13}$$

hence $dw/d\xi = 0$ at $\xi = \xi_\pm$. Differentiating (3.13) with respect to ξ at $\xi = \xi_\pm$ we obtain

$$2A_\pm := \left.\frac{d^2w}{d\xi^2}\right|_{\xi_\pm} = \frac{w_\pm D'(\xi_\pm)}{\xi_\pm(1+\xi_\pm)(1+\eta_\pm)} = \frac{\pm w_\pm\sqrt{\triangle}/(z+1)}{\xi_\pm(1+\xi_\pm)(1+\eta_\pm)}. \tag{3.14}$$

Thus, $w - w_\pm = (A_\pm + O(\xi-\xi_\pm))(\xi-\xi_\pm)^2$ as $\xi \to \xi_\pm$, and therefore,

$$\xi - \xi_\pm = (w-w_\pm)^{1/2}(A_\pm + O((w-w_\pm)^{1/2}))^{-1/2}, \quad w \to w_\pm. \tag{3.15}$$

From (3.15) it follows that $w_+ = w_-$ if and only if $\xi_+ = \xi_-$. Indeed, if $w_+ = w_-$, (3.15) implies $\xi \to \xi_+$ as $w \to w_+$, and $\xi \to \xi_-$ as $w \to w_- = w_+$, hence $\xi_+ = \xi_-$, which is equivalent to $\triangle = 0$.

Assume first that $\triangle \neq 0$ and set $B_\pm := \lim_{w\to w_\pm}(w-w_\pm)^{1/2}g(w)$. From (3.6) and (3.7) it follows that $B_\pm \neq 0$. Then, we define $w_0 = w_+$ and $B_0 = B_+$ if $|w_+| \le |w_-|$, and $w_0 = w_-$ and $B_0 = B_-$ if $|w_+| > |w_-|$. The function $g(w)$ is analytic in $|w| < |w_0|$, and in a neighborhood of $w_\pm$,

$$g(w) = \sum_{n=0}^{\infty} g_{n,\pm}(w-w_\pm)^{n-1/2},$$

where $g_{n,\pm} = ((w-w_\pm)^{1/2}g(w))^{(n)}|_{w_\pm}/n!$. Consider the function H defined by

$$\begin{aligned} H(w) := g(w) &- g_{0,+}(w-w_+)^{-1/2} - g_{1,+}(w-w_+)^{1/2} \\ &- g_{0,-}(w-w_-)^{-1/2} - g_{1,-}(w-w_-)^{1/2}. \end{aligned} \tag{3.16}$$

It has a continuous first derivative $h(w) = H'(w)$ in $|w| \le |w_0|$. Let $H(w) = \sum_{n=0}^{\infty} h_n w^n$ be the power series expansion of H around $w = 0$. Using that $h(w)$ is continuous in $|w| \le |w_0|$ we obtain

$$\begin{aligned} (n+1)h_{n+1} &= \lim_{\rho\to|w_0|,\rho<|w_0|} \frac{1}{2\pi i}\int_{|w|=\rho} \frac{h(w)}{w^{n+1}}\,dw \\ &= \frac{1}{2\pi|w_0|^n}\int_0^{2\pi} h(|w_0|e^{i\theta})e^{-in\theta}\,d\theta. \end{aligned}$$

For a fixed z, $nh_n|w_0|^n \to 0$ as $n \to \infty$ by the Riemann-Lebesgue lemma. This convergence is uniform with respect to z. Indeed, let E be a compact set. Note that $\tilde{h}(z,\theta) := h(|w_0|e^{i\theta})$ is continuous and therefore uniformly continuous on the compact set $E \times [0,2\pi]$. Let $\epsilon > 0$ and choose $\delta > 0$ so that $|\tilde{h}(z_1,\theta_1) - \tilde{h}(z_2,\theta_2)| < \epsilon/2\pi$ whenever $|z_1 - z_2| + |\theta_1 - \theta_2| < \delta$, $z_{1,2} \in E$, $\theta_{1,2} \in [0,2\pi]$. Let $\{z_i\}_{i=1}^k \subset E$ be such that for every $z \in E$ there exists z_i such that $|z - z_i| < \delta$. Finally, for each $i = 1,\ldots,k$, let $s_i(\theta)$ be a step-function with p_i steps, such that $||\tilde{h}(z_i,\theta) - s_i(\theta)||_{[0,2\pi]} < \epsilon/2\pi$. Then,

$$\begin{aligned}\left|\int_0^{2\pi} \tilde{h}(z,\theta)e^{-in\theta}\,d\theta\right| &\le \left|\int_0^{2\pi} s_j(\theta)e^{-in\theta}\,d\theta\right| \\ &+ \left|\int_0^{2\pi} (\tilde{h}(z_j,\theta) - s_j(\theta))e^{-in\theta}\,d\theta\right| + \left|\int_0^{2\pi} (\tilde{h}(z,\theta) - \tilde{h}(z_j,\theta))e^{-in\theta}\,d\theta\right| \\ &\le \frac{2\max\{p_i\}_{i=1}^k \max\{||s_i||_{[0,2\pi]}\}_{i=1}^k}{n} + 2\epsilon < 3\epsilon,\end{aligned}$$

if n is large enough. We have shown that $h_n = o(n^{-1}|w_0|^{-n})$ uniformly on compact sets of the variable z. Since $\binom{\nu-1/2}{n} = O(n^{-\nu-1/2})$ and $g_{1,\pm}(z)$ are bounded on compact sets we obtain

$$\begin{aligned}(3.17)\quad & p_n^{(\alpha_0+\alpha n,\beta_0+\beta n)}(z) \\ &= -i\left|\binom{-1/2}{n}\right| w_0^{-n-1/2}\left(B_0 + B_1\left(\frac{w_0}{w_1}\right)^{n+1/2}\right) + o(n^{-1}|w_0|^{-n}),\end{aligned}$$

where $B_1 = (B_+ + B_-) - B_0$ and $w_1 = (w_+ + w_-) - w_0$. Formula (3.17) holds uniformly on compact sets of the variable z.

Similarly, if $\triangle = 0$, then $\xi_+ = \xi_-$. At $\xi = \xi_+$, $d^2w/d\xi^2 = 0$ and from (3.13) we get

$$(3.18)\qquad \left.\frac{d^3w}{d\xi^3}\right|_{\xi_+} = \frac{2(1+\alpha+\beta)(1-z)w_+}{(z+1)\xi_+(1+\xi_+)(1+\eta_+)}.$$

Hence, $w - w_+ = O((\xi - \xi_+)^3)$ and $\xi - \xi_+ = O((w - w_+)^{1/3})$, $w \to w_+$. We set $w_0 = w_+ = w_-$. Then, $g(w) = \sum_{n=0}^\infty g_{n,0}(w - w_0)^{n-2/3}$, $w \to w_0$, where $g_{n,0} = ((w - w_0)^{2/3}g(w))^{(n)}|_{w_0}/n!$. Using the function

$$H(w) := g(w) - g_{0,0}(w - w_0)^{-2/3} - g_{1,0}(w - w_0)^{1/3}$$

and the above argument we can show that in the case $\triangle = 0$,

$$(3.19)\quad p_n^{(\alpha_0+\alpha n,\beta_0+\beta n)}(z) = e^{-2\pi i/3}g_{0,0}\left|\binom{-2/3}{n}\right| w_0^{-n-2/3} + o(n^{-1}|w_0|^{-n}).$$

The factor w_0 in (3.17) or in (3.19) (the n-th root asymptotics) can be found using the asymptotic zero distribution of the polynomials $p_n^{(\alpha_0+\alpha n,\beta_0+\beta n)}$.

For $\alpha \ge 0$ and $\beta \ge 0$ the Jacobi weight $w_{\alpha,\beta}$ is defined by

$$w_{\alpha,\beta}(x) := (1-x)^\alpha(1+x)^\beta, \quad x \in [-1,1].$$

The corresponding extremal measure $\mu_{\alpha,\beta}$ has probability density ([11, Section IV.5])

$$v_{\alpha,\beta}(t) = \frac{(1+\alpha+\beta)}{\pi}\frac{\sqrt{(t-a)(b-t)}}{1-t^2}, \quad t \in S_{\alpha,\beta}, \tag{3.20}$$

where ([11, Section IV.1]) $S_{\alpha,\beta}$ denotes the interval

$$[a,b] = [\lambda_2^2 - \lambda_1^2 - D^{1/2}, \lambda_2^2 - \lambda_1^2 + D^{1/2}], \tag{3.21}$$

with $\lambda_1 = \alpha/(1+\alpha+\beta)$, $\lambda_2 = \beta/(1+\alpha+\beta)$, and $D = (1-(\lambda_1+\lambda_2)^2)(1-(\lambda_1-\lambda_2)^2)$. In particular, $ab = 2(\lambda_1^2+\lambda_2^2)-1$ and $a+b = 2(\lambda_2^2-\lambda_1^2)$, which yield the identities

$$\sqrt{(1-a)(1-b)} = \frac{2\alpha}{1+\alpha+\beta}, \quad \sqrt{(1+a)(1+b)} = \frac{2\beta}{1+\alpha+\beta}. \tag{3.22}$$

The Jacobi polynomials $\{p_n^{(\alpha,\beta)}\}$ are orthogonal with respect to $w_{\alpha,\beta}$ on $[-1,1]$. The normalized zero-counting measure $\nu_{n,\alpha,\beta}$ associated with $p_n^{(\alpha,\beta)}$ is the discrete probability measure having mass $1/n$ at each zero of $p_n^{(\alpha,\beta)}$. Let $\gamma_n^{(\alpha,\beta)}$ denote the leading coefficient of $p_n^{(\alpha,\beta)}$.

THEOREM 3.1. *Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences of nonnegative numbers satisfying $\alpha_n/n \to 2\alpha \geq 0$ and $\beta_n/n \to 2\beta \geq 0$ as $n \to \infty$. Then,*

$$\nu_{n,\alpha_n,\beta_n} \to \mu_{\alpha,\beta}, \quad n \to \infty \tag{3.23}$$

in the weak-star topology of measures, and

$$\lim_{n\to\infty} p_n^{(\alpha_n,\beta_n)}(z)^{1/n} = c_{\alpha,\beta} e^{-u_{\alpha,\beta}(z)}, \tag{3.24}$$

uniformly on compact subsets of $\mathbf{C} \setminus S_{\alpha,\beta}$, where

$$c_{\alpha,\beta} = \lim_{n\to\infty}\left(\gamma_n^{(\alpha_n,\beta_n)}\right)^{1/n} = \frac{(2\alpha+2\beta+2)^{2\alpha+2\beta+2}}{2(2\alpha+2\beta+1)^{2\alpha+2\beta+1}} \tag{3.25}$$

and $u_{\alpha,\beta}(z)$ is the complex logarithmic potential defined by

$$u_{\alpha,\beta}(z) := \int_{S_{\alpha,\beta}} \log\frac{1}{z-t} v_{\alpha,\beta}(t)\,dt, \quad z \in \mathbf{C} \setminus (-\infty, b]. \tag{3.26}$$

A proof of Theorem 3.1 can be found in [9]. Let $U^\mu(z) := \int \log 1/|z-t|\,d\mu(t)$ denote the logarithmic potential of a measure μ. We define

$$\tilde{u}_{\alpha,\beta}(z) := \begin{cases} u_{\alpha,\beta}(z), & z \in \mathbf{C} \setminus (-\infty, b], \\ U^{\mu_{\alpha,\beta}}(z), & z \in (-\infty, a). \end{cases} \tag{3.27}$$

LEMMA 3.2. *Let $z \in \mathbf{C} \setminus S_{\alpha/2,\beta/2}$. Then, $|w_0| = e^{U^{\mu_{\alpha/2,\beta/2}}(z)}/c_{\alpha/2,\beta/2}$. Furthermore, if $|w_0| < |w_1|$ or $w_0 = w_1$, then $w_0 = e^{u_{\alpha/2,\beta/2}(z)}/c_{\alpha/2,\beta/2}$.*

Proof. If $w_0 = w_1$, the statement follows from (3.19) and (3.24). If $w_0 \neq w_1$, from (3.17) and (3.24) it follows that the limit

$$L := \lim_{n\to\infty} \left(1 + \frac{B_1}{B_0}\left(\frac{w_0}{w_1}\right)^{n+1/2}\right)^{1/n} = w_0 c_{\alpha/2,\beta/2} e^{-u_{\alpha/2,\beta/2}(z)} \tag{3.28}$$

exists for every $z \in \mathbf{C} \setminus S_{\alpha/2,\beta/2}$. In particular, (3.24) shows that $w_0 \neq 0$, $z \in \mathbf{C} \setminus S_{\alpha/2,\beta/2}$. Note that $L = L(z)$. We will show that $|L| = 1$.

From (3.28) and $|w_0| \leq |w_1|$ it follows that $0 \leq |L| \leq 1$ and since $w_0 \neq 0$, $|L| > 0$. Assuming that $|L| < 1$ for some $z \in \mathbf{C} \setminus S_{\alpha/2,\beta/2}$, from (3.28) we get

$$\lim_{n\to\infty} (w_0/w_1)^{n+1/2} = \lim_{n\to\infty} B_0/B_1(-1 + (L + o(1))^n) = -B_0/B_1 \neq 0,$$

and therefore, $|w_0| = |w_1|$ and $|B_0| = |B_1|$. Setting $w_0/w_1 = e^{i\theta}$ with $\theta \in [0, 2\pi)$ we obtain

$$\lim_{n\to\infty} e^{in\theta} = -e^{-i\theta/2} B_0/B_1,$$

which is possible if and only if $\theta = 0$. Then, $w_0 = w_1$, which is a contradiction. Thus, $|L| = 1$, that is,

$$c_{\alpha/2,\beta/2}|w_0| = |\exp(u_{\alpha/2,\beta/2}(z))| = \exp(U^{\mu_{\alpha/2,\beta/2}}(z)).$$

If $|w_0| < |w_1|$, (3.28) yields

$$L = \lim_{n\to\infty} \exp\left(\frac{1}{n}\log\left(1 + \frac{B_0}{B_1}(w_0/w_1)^{n+1/2}\right)\right) = 1.$$

The lemma is proved. □

We recall that $x = \lambda n$ and $N = \gamma n$ with $\lambda \in (0, 1]$ and $\gamma > 1$. Without loss of generality we may assume that $\triangle \neq 0$. Indeed, in what follows $\alpha = 0$ and $z = 1 - 2t$. In view of (3.11) the solutions of $\triangle = 0$ are $t = 0$ and $t = 1 - \beta^2/(\beta + 2)^2$. The contours Γ in (3.1) will be selected so that the size of $|R(n, \lambda n, \gamma n + 1)|$ will be determined at a value t_1 defined by (3.40) that is either a complex number, or a real number larger than $1/\lambda^2 > 1$.

From (3.1), (3.3), (3.4), (3.5), (3.17), and Lemma 3.2 we obtain:

Lemma 3.3. *The ${}_4F_3$ expression $R(n, \lambda n, \gamma n + 1)$ has the representation*

$$R(n, \lambda n, \gamma n + 1) = \frac{n!^2}{(N + 2)_n(-N)_n} \frac{1}{2\pi i} \int_\Gamma A_n(t) \exp(nf(t))\, dt, \tag{3.29}$$

where

$$A_n(t) = \frac{1}{2t\sqrt{\pi x}}\left(\frac{1+\sqrt{t}}{\sqrt[4]{t}} + O(x^{-1})\right)$$
$$\times\left(-i\left|\binom{-1/2}{N-x}\right|\left(B_0 + B_1\left(\frac{w_0(1-2t)}{w_1(1-2t)}\right)^{N-x+1/2}\right) + o((N-x)^{-1})\right)$$
$$\times\left(\frac{\exp(\tilde{u}_{0,\lambda/(\gamma-\lambda)}(1-2t))}{w_0(1-2t)}\right)^{N-x} w_0(1-2t)^{-1/2},$$

$$f(t) = -\log t + 2\lambda\log(1+\sqrt{t}) - (\gamma-\lambda)\tilde{u}_{0,\lambda/(\gamma-\lambda)}(1-2t), \tag{3.30}$$

and Γ *is a simple closed contour containing* 0 *in its interior.*

We shall write $A(x) \sim B(x)$ if $A(x)/B(x) \to 1$ as $x \to \infty$. Using the asymptotic formula $\Gamma(x+1) \sim (x/e)^x\sqrt{2\pi x}$ we derive

$$\frac{n!^2}{(N+2)_n(-N)_n} \sim (-1)^n \frac{2\pi\gamma}{\gamma+1}\left(\frac{\gamma-1}{\gamma+1}\right)^{1/2}\left(\frac{(\gamma-1)^{\gamma-1}}{(\gamma+1)^{\gamma+1}}\right)^n n. \tag{3.31}$$

From (3.31) and Lemma 3.3 we obtain

$$|R(n,\lambda n,\gamma n+1)| \le c\left(c_{0,\lambda/(\gamma-\lambda)}^{\gamma-\lambda}\frac{(\gamma-1)^{\gamma-1}}{(\gamma+1)^{\gamma+1}}\right)^n \int_\Gamma |A(t)|e^{n\operatorname{Re} f(t)}\,|dt|, \tag{3.32}$$

where

$$A(t) = \frac{1}{|t|\sqrt{|w_0(1-2t)|}}\left(\frac{(1+|\sqrt{t}|)}{|t|^{1/4}}(|B_0(t)|+|B_1(t)|)+1\right) \tag{3.33}$$

and

$$c = c(\lambda,\gamma) = \frac{1}{\sqrt{\lambda(\gamma-\lambda)}}\frac{\gamma}{\gamma+1}\left(\frac{\gamma-1}{\gamma+1}\right)^{1/2}.$$

In (3.32) we used that $\left|\binom{-1/2}{x}\right| \sim 1/\sqrt{\pi x}$ for large x.

From [11, Section IV.5] we have the formula

$$\frac{d}{dz}\tilde{u}_{\alpha,\beta}(z) = (1+\alpha+\beta)\frac{\sqrt{(z-a)(z-b)}}{1-z^2} - \frac{\alpha}{1-z} + \frac{\beta}{1+z}, \tag{3.34}$$

for $z \in \mathbf{C}\setminus[a,b]$, $z \ne \pm 1$, where on $(-\infty,a)$ this is the real derivative of $U^{\mu_{\alpha,\beta}}$ restricted on $(-\infty,a)$.

When $\alpha = 0$ and $\beta = \lambda/(\gamma-\lambda)$, from (3.22) we get $(1-a)(1-b) = 0$ and $(1+a)(1+b) = 4\lambda^2/\gamma^2$. Since $a < b \le 1$, it follows that $b = 1$ and $a = 2\lambda^2/\gamma^2 - 1$. Formula (3.34) implies

$$\frac{d}{dz}\tilde{u}_{0,\lambda/(\gamma-\lambda)}(z)|_{z=1-2t} = -\frac{\gamma}{\gamma-\lambda}\frac{\sqrt{t(t-b^*)}}{2t(1-t)} + \frac{\lambda/(\gamma-\lambda)}{2(1-t)}, \tag{3.35}$$

where $b^* := 1 - \lambda^2/\gamma^2$. We compute $f'(t)$ using (3.30) and (3.35):

$$f'(t) = -\frac{1}{t} + \frac{\lambda}{(1+\sqrt{t})\sqrt{t}} + 2(\gamma - \lambda)\frac{d}{dz}\tilde{u}_{0,\lambda/(\gamma-\lambda)}(z)|_{z=1-2t} \tag{3.36}$$
$$= \frac{t - 1 + \lambda\sqrt{t} - \gamma\sqrt{t(t-b^*)}}{t(1-t)} = \frac{1}{t}\left(-1 + \frac{(\lambda - \gamma\sqrt{t-b^*})\sqrt{t}}{1-t}\right)$$
$$= \frac{1}{t}\left(-1 + \frac{\gamma^2\sqrt{t}}{\lambda + \gamma\sqrt{t-b^*}}\right).$$

The solutions of the equation $f'(t) = 0$ will be used to determine the asymptotics of the integral in (3.32). From (3.36) it follows that $f'(t) = 0$ is equivalent to

$$\lambda + \gamma\sqrt{t-b^*} = \gamma^2\sqrt{t}, \tag{3.37}$$

which implies

$$\gamma^2(t - b^*) = (\gamma^2\sqrt{t} - \lambda)^2, \tag{3.38}$$

and then,

$$(\gamma^2 - 1)t - 2\lambda\sqrt{t} + 1 = 0. \tag{3.39}$$

The solutions of (3.39) are

$$t_{1,2} = \left(\frac{\lambda \pm \sqrt{\lambda^2 - \gamma^2 + 1}}{\gamma^2 - 1}\right)^2. \tag{3.40}$$

If t_1 and t_2 are complex numbers, at $t = t_{1,2}$, $\mathrm{Re}\sqrt{t-b^*} > 0$ by the choice of the square root branch, and $\mathrm{Re}(\gamma^2\sqrt{t}-\lambda) = \lambda/(\gamma^2-1) > 0$. Thus, $t_{1,2}$ are the solutions of (3.37) and the equation $f'(t) = 0$ in this case. If t_1 and t_2 are real, (3.38) implies $t_{1,2} \geq b^*$. Since $(\gamma^2\sqrt{t_1} - \lambda) + (\gamma^2\sqrt{t_2} - \lambda) = 2\lambda/(\gamma^2 - 1) > 0$, we get $\gamma^2\sqrt{t_1} - \lambda > 0$, and by (3.37) and (3.36), $f'(t_1) = 0$. Next,

$$(\gamma^2\sqrt{t_1} - \lambda)(\gamma^2\sqrt{t_2} - \lambda) = \gamma^4\sqrt{t_1t_2} - \gamma^2\lambda(\sqrt{t_1} + \sqrt{t_2}) + \lambda^2 = \frac{\gamma^4 - (\gamma^2+1)\lambda^2}{\gamma^2 - 1},$$

which shows that in this case $f'(t_2) = 0$ if and only if $\lambda \leq \gamma^2/\sqrt{\gamma^2+1}$.

We will use the following formula for $u_{\alpha,\beta}$ from [11, Section IV.5]:

$$u_{\alpha,\beta}(z) = -\alpha\log\left(\frac{\zeta - \zeta_+}{\zeta_+\zeta - 1}\right) - \beta\log\left(\frac{\zeta - \zeta_-}{\zeta_-\zeta - 1}\right) \tag{3.41}$$
$$- (\alpha + \beta + 1)\log\zeta + \alpha\log(1 - z) + \beta\log(1 + z) + F_{\alpha,\beta},$$

where

$$\zeta = \phi(z) = \frac{2z - a - b + 2\sqrt{(z-a)(z-b)}}{b - a} = \frac{(\sqrt{z-a} + \sqrt{z-b})^2}{b - a}, \tag{3.42}$$

$\zeta_+ = \phi(1)$, $\zeta_- = \phi(-1)$, a and b are defined with (3.21), and $F_{\alpha,\beta}$ is a real constant. Note that $u_{\alpha,\beta}(z) \sim -\log z$ and $\zeta \sim 4z/(b-a)$ as $z \to \infty$. Thus, taking real parts in (3.41) and then letting $z \to \infty$ we get

$$F_{\alpha,\beta} = -\alpha \log|\zeta_+| - \beta \log|\zeta_-| + (\alpha+\beta+1)\log(4/(b-a)). \tag{3.43}$$

From (3.42) and (3.43) we obtain

$$\begin{aligned} F_{\alpha,\beta} &= (\alpha+\beta+1)\log 4 - \log(b-a) \\ &\quad - 2\alpha \log\left|\sqrt{1-a}+\sqrt{1-b}\right| - 2\beta \log\left|\sqrt{1+a}+\sqrt{1+b}\right|. \end{aligned} \tag{3.44}$$

When $\alpha = 0$ and $\beta = \lambda/(\gamma-\lambda)$, we have $a = 2\lambda^2/\gamma^2 - 1$, $b = 1$, and then,

$$\begin{aligned} F_1 : &= e^{-(\gamma-\lambda)F_{0,\lambda/(\gamma-\lambda)}} = 4^{-\gamma}(2(1-\lambda^2/\gamma^2))^{\gamma-\lambda}(\sqrt{2}(1+\lambda/\gamma))^{2\lambda} \\ &= 2^{-\gamma}(1-\lambda/\gamma)^{\gamma-\lambda}(1+\lambda/\gamma)^{\gamma+\lambda}. \end{aligned} \tag{3.45}$$

From (3.42) we get

$$\zeta_- = -\frac{(1+\lambda/\gamma)^2}{(1-\lambda^2/\gamma^2)}. \tag{3.46}$$

Furthermore, (3.41) and the identity ([11, Section IV.5])

$$(\zeta-\zeta_\pm)(\zeta_\pm\zeta - 1) = 4(z\mp 1)\zeta_\pm\zeta/(b-a)$$

yield

$$\begin{aligned} e^{-(\gamma-\lambda)u_{0,\lambda/(\gamma-\lambda)}(z)} &= F_1(1+z)^{-\lambda}\left(\frac{\zeta-\zeta_-}{\zeta_-\zeta-1}\right)^{\lambda}\zeta^{\gamma} \\ &= F_1((b-a)/4)^{\lambda}\zeta_-^{-\lambda}\left(\frac{\zeta-\zeta_-}{1+z}\right)^{2\lambda}\zeta^{\gamma-\lambda}. \end{aligned} \tag{3.47}$$

Hence, from (3.30), (3.47), and (3.46) with $z = 1-2t$ it follows that

$$e^{f(t)} = F_1(-2)^{-\lambda}(1-\lambda/\gamma)^{2\lambda}\frac{1}{t}\left(\frac{\zeta-\zeta_-}{2(1-\sqrt{t})}\right)^{2\lambda}\zeta^{\gamma-\lambda}. \tag{3.48}$$

Note that by (3.25),

$$c_{0,\lambda/(\gamma-\lambda)}^{\gamma-\lambda} = \frac{2^{\gamma+\lambda}\gamma^{2\gamma}}{(\gamma-\lambda)^{\gamma-\lambda}(\gamma+\lambda)^{\gamma+\lambda}}. \tag{3.49}$$

The product of the constant factors that are raised to power n in (3.32) can be computed using (3.45), (3.48), and (3.49):

$$F := c_{0,\lambda/(\gamma-\lambda)}^{\gamma-\lambda}\frac{(\gamma-1)^{\gamma-1}}{(\gamma+1)^{\gamma+1}}F_1 2^{-\lambda}(1-\lambda/\gamma)^{2\lambda} = \frac{(\gamma-1)^{\gamma-1}}{(\gamma+1)^{\gamma+1}}(1-\lambda/\gamma)^{2\lambda}. \tag{3.50}$$

LEMMA 3.4. *The function* $F(t) := c_{0,\lambda/(\gamma-\lambda)}^{\gamma-\lambda}\frac{(\gamma-1)^{\gamma-1}}{(\gamma+1)^{\gamma+1}}e^{f(t)}$ *satisfies*

$$|F(t_1)F(t_2)| = 1. \tag{3.51}$$

Proof. From (3.39) we obtain the identities

$$\sqrt{t_1}+\sqrt{t_2}=2\lambda/(\gamma^2-1),\quad \sqrt{t_1t_2}=1/(\gamma^2-1),$$
$$t_1+t_2=2(2\lambda^2-\gamma^2+1)/(\gamma^2-1)^2. \tag{3.52}$$

From (3.42) with $z=1-2t$, $a=2\lambda^2/\gamma^2-1$, and $b=1$ we get

$$\zeta=\frac{1-2t-\lambda^2/\gamma^2-2\sqrt{t(t-1+\lambda^2/\gamma^2)}}{1-\lambda^2/\gamma^2}. \tag{3.53}$$

In particular, at $t=t_{1,2}$, equations (3.53), (3.36), and (3.39) yield

$$\begin{aligned}\zeta(t)&=\frac{1-2t-\lambda^2/\gamma^2-2(t-1+\lambda\sqrt{t})/\gamma}{1-\lambda^2/\gamma^2}\\&=\frac{\gamma^2-\lambda^2-\gamma((\gamma+1)^2t-1)}{\gamma^2-\lambda^2}.\end{aligned} \tag{3.54}$$

Furthermore, using (3.54) and (3.46), at $t=t_{1,2}$ we obtain

$$\begin{aligned}r(t)&:=\frac{\zeta-\zeta_-}{2(1-\sqrt{t})}\\&=\frac{1-2t-\lambda^2/\gamma^2-2(t-1+\lambda\sqrt{t})/\gamma+(1+\lambda/\gamma)^2}{2(1-\lambda^2/\gamma^2)(1-\sqrt{t})}\\&=\frac{(1+1/\gamma)(1-t)+\lambda(1-\sqrt{t})/\gamma}{(1-\lambda^2/\gamma^2)(1-\sqrt{t})}=\frac{(\gamma+1)(1+\sqrt{t})+\lambda}{\gamma(1-\lambda^2/\gamma^2)}.\end{aligned} \tag{3.55}$$

We evaluate the product $\zeta(t_1)\zeta(t_2)$ using (3.54) and (3.52):

$$\begin{aligned}(\gamma^2-\lambda^2)^2\zeta(t_1)\zeta(t_2)&=(\gamma^2-\lambda^2)^2-\gamma(\gamma^2-\lambda^2)[(\gamma+1)^2(t_1+t_2)-2]\\&\quad+\gamma^2[(\gamma+1)^4t_1t_2-(\gamma+1)^2(t_1+t_2)+1]\\&=(\gamma^2-\lambda^2)^2-4\gamma(\gamma^2-\lambda^2)(\lambda^2-\gamma^2+\gamma)/(\gamma-1)^2\\&\quad+4\gamma^2(\gamma^2-\lambda^2)/(\gamma-1)^2\\&=(\gamma^2-\lambda^2)^2(1+4\gamma/(\gamma-1)^2).\end{aligned}$$

Thus,

$$\zeta(t_1)\zeta(t_2)=\frac{(\gamma+1)^2}{(\gamma-1)^2}. \tag{3.56}$$

Next, we evaluate the product $r(t_1)r(t_2)$ using (3.55) and (3.52):

$$\begin{aligned}
&(\gamma^2-\lambda^2)^2 r(t_1)r(t_2)/\gamma^2\\
&\quad=(\gamma+\lambda+1)^2+(\gamma+\lambda+1)(\gamma+1)(\sqrt{t_1}+\sqrt{t_2})+(\gamma+1)^2\sqrt{t_1t_2}\\
&\quad=[(\gamma-1)(\gamma+\lambda+1)^2+2\lambda(\gamma+\lambda+1)+\gamma+1]/(\gamma-1)\\
&\quad=[(\gamma-1)((\gamma+1)^2+2\lambda(\gamma+1)+\lambda^2)+(2\lambda+1)(\gamma+1)+2\lambda^2]/(\gamma-1)\\
&\quad=(\gamma+1)[(\gamma^2-1)+2\lambda(\gamma-1)+\lambda^2+2\lambda+1]/(\gamma-1)\\
&\quad=(\gamma+\lambda)^2(\gamma+1)/(\gamma-1).
\end{aligned}$$

Therefore,

$$r(t_1)r(t_2)=\frac{\gamma^2(\gamma+1)}{(\gamma-\lambda)^2(\gamma-1)}. \tag{3.57}$$

Finally, (3.48), (3.50), (3.52), (3.56), and (3.57) yield

$$\begin{aligned}
F(t_1)F(t_2)&=(-1)^{-2\lambda}F^2(t_1t_2)^{-1}(r(t_1)r(t_2))^{2\lambda}(\zeta(t_1)\zeta(t_2))^{\gamma-\lambda}\\
&=(-1)^{-2\lambda}\frac{(\gamma-1)^{2(\gamma-1)}}{(\gamma+1)^{2(\gamma+1)}}\frac{(\gamma-\lambda)^{4\lambda}}{\gamma^{4\lambda}}(\gamma^2-1)^2\\
&\qquad\times\frac{\gamma^{4\lambda}(\gamma+1)^{2\lambda}}{(\gamma-\lambda)^{4\lambda}(\gamma-1)^{2\lambda}}\frac{(\gamma+1)^{2(\gamma-\lambda)}}{(\gamma-1)^{2(\gamma-\lambda)}}\\
&=(-1)^{-2\lambda},
\end{aligned}$$

and (3.51) follows. □

In the proof of our main result below we will use the following lemma.

LEMMA 3.5. *Let f be analytic function in a domain D, $u=\operatorname{Re}(f)$, and $z=re^{i\theta}$. Then,*

$$\frac{\partial u}{\partial r}=\operatorname{Re}(zf'(z))/r,\quad \frac{\partial u}{\partial\theta}=-\operatorname{Im}(zf'(z)). \tag{3.58}$$

Proof. Let $f=u+iv$ and $z=e^{i\theta}=x+iy$. Then, with $u_x=\partial u/\partial x$ and $u_y=\partial u/\partial y$ we have

$$\frac{\partial u}{\partial r}=u_x\cos\theta+u_y\sin\theta=(xu_x+yu_y)/r=\operatorname{Re}((x+iy)(u_x-iu_y))/r,$$

and

$$\frac{\partial u}{\partial\theta}=-u_x(r\sin\theta)+u_y(r\cos\theta)=-yu_x+xu_y=-\operatorname{Im}((x+iy)(u_x-iu_y)).$$

Then (3.58) follows since $u_x-iu_y=u_x+iv_x=f'(z)$ by the Cauchy-Riemann equations. □

We will also use a theorem based on the Laplace method from [8, Theorem 3.7.1].

THEOREM 3.6. *Let $p(\tau)$ and $q(\tau)$ be functions defined on an interval (a,b) that satisfy the following conditions:*

(a) *$p(\tau) < p(a)$ when $\tau \in (a,b)$, and for every $c \in (a,b)$ the infimum of $p(a) - p(\tau)$ in $[c,b)$ is positive.*

(b) *$p'(\tau)$ and $q(\tau)$ are continuous in a neighborhood of a, except possibly at a.*

(c) *As $\tau \to a$ from the right,*

$$p(\tau) - p(a) \sim P(\tau - a)^{\nu}, \quad q(\tau) \sim Q,$$

and the first of these relations is differentiable. Here $P < 0$ and $\nu > 0$ are constants.

(d) *The integral*

$$I(n) = \int_a^b q(\tau) e^{np(\tau)}\, d\tau$$

converges absolutely throughout its range for all sufficiently large n.

Then,

$$I(n) \sim \frac{Q}{\nu}\Gamma\left(\frac{1}{\nu}\right)\frac{e^{np(a)}}{(-Pn)^{1/\nu}}, \quad n \to \infty.$$

Next, we determine the set $\{t : \operatorname{Im}(tf'(t)) = 0\}$ for the function $f(t)$ defined with (3.30). For $t \in \mathbf{C} \setminus (-\infty, b^*]$ we set

$$t = J(w) := \frac{b^*}{4}\left(w + \frac{1}{w}\right)^2, \quad w = Re^{i\theta}, \quad R > 1, \quad \theta \in (-\pi/2, \pi/2). \tag{3.59}$$

Then, $t - b^* = (b^*/4)(w - 1/w)^2$. Substituting in (3.36) we obtain

$$\begin{aligned} tf'(t) &= -1 + \frac{\gamma^2\sqrt{b^*}(w^2+1)}{2\lambda w + \gamma\sqrt{b^*}(w^2-1)} = -1 + \frac{\gamma(w^2+1)}{w^2+\delta w - 1} \\ &= -1 + \gamma + \frac{\gamma(2-\delta w)}{w^2+\delta w-1} = -1 + \gamma\left(1 + \frac{w_1}{w-w_1} + \frac{w_2}{w-w_2}\right) \\ &= -1 + \gamma\left(1 + \frac{w_1(\bar{w}-w_1)}{|w-w_1|^2} + \frac{w_2(\bar{w}-w_2)}{|w-w_2|^2}\right), \end{aligned} \tag{3.60}$$

where $\delta := 2\lambda/(\gamma\sqrt{b^*}) = 2\lambda/\sqrt{\gamma^2-\lambda^2}$ and $w^2 + \delta w - 1 = (w-w_1)(w-w_2)$. Note that the numbers $w_{1,2} = (-\lambda \pm \gamma)/\sqrt{\gamma^2 - \lambda^2}$ are real. From (3.60) it follows that

$$\operatorname{Im}(tf'(t)) = -\gamma\left(\frac{w_1}{|w-w_1|^2} + \frac{w_2}{|w-w_2|^2}\right) R\sin\theta.$$

Thus, $\operatorname{Im}(tf'(t)) = 0$ is equivalent to $\sin\theta = 0$, that is, $t \in (b^*, \infty)$, or

$$w_1|w-w_2|^2 + w_2|w-w_1|^2 = 0.$$

Since $w_1 + w_2 = -\delta$ and $w_1 w_2 = -1$, the last equation becomes

$$\begin{aligned} w_1(R^2 + w_2^2 - 2Rw_2\cos\theta) + w_2(R^2 + w_1^2 - 2Rw_1\cos\theta) \\ = -\delta R^2 + 4R\cos\theta + \delta = 0, \end{aligned} \tag{3.61}$$

which represents a circle $\tilde{C}$ with center $2/\delta$ and radius $\tilde{r} = \sqrt{4/\delta^2 + 1} = \gamma/\lambda$. Setting $\tilde{C}_+ := \{w \in \tilde{C} \,:\, \mathrm{Re}(w) > 0\}$ we obtain:

Lemma 3.7. *The zero set of* $\mathrm{Im}(tf'(t))$ *is the set* $(b^*, \infty) \cup J(\tilde{C}_+)$.

The set $J(\tilde{C}_+)$ has an interesting property. If $t = t(\theta) = J(w)$, where $w = Re^{i\theta}$ and $R = R(\theta) > 1$ is the solution of (3.61), then $|t(\theta)|$ decreases as $|\theta| \in [0, \pi/2)$ increases. This can be seen as follows: For $t \in J(\tilde{C}_+)$,

$$t(\theta) = J\left(Re^{i\theta}\right) = \frac{b^*}{4}\left((R + 1/R)\cos\theta + i(R - 1/R)\sin\theta\right)^2,$$

and from (3.61) we have $R - 1/R = (R^2 - 1)/R = 4\cos\theta/\delta$. Therefore,

$$\begin{aligned} (4/b^*)|t(\theta)| &= (R + 1/R)^2\cos^2\theta + (R - 1/R)^2\sin^2\theta \\ &= R^2 + 1/R^2 + 2\cos 2\theta = (16/\delta^2)\cos^2\theta + 2 + 2\cos 2\theta \\ &= 4(4/\delta^2 + 1)\cos^2\theta = 4(\gamma^2/\lambda^2)\cos^2\theta, \end{aligned} \tag{3.62}$$

which is a decreasing function of $|\theta| \in [0, \pi/2)$. In particular, since the set $J(\tilde{C}_+)$ is symmetric about the real line, every circle C_r with center at the origin and radius $r > 0$ intersects that set at most twice.

The main result of this paper is the following theorem:

Theorem 3.8. *Let* $\lambda \in (0, 1]$ *and* $\gamma > 1$ *be fixed rational numbers. Then,* $R(n, \lambda n, \gamma n + 1) \to 0$ *as* $n \to \infty$.

Proof. From (3.36) we obtain

$$(tf'(t))' = f'(t) + tf''(t) = \frac{\gamma^2\left(\frac{\lambda + \gamma\sqrt{t - b^*}}{2\sqrt{t}} - \frac{\gamma\sqrt{t}}{2\sqrt{t - b^*}}\right)}{(\lambda + \gamma\sqrt{t - b^*})^2}, \quad t \notin (-\infty, b^*]. \tag{3.63}$$

At $t = t_{1,2}$, $f'(t) = 0$ and (3.63) and (3.37) yield

$$2t^2 f''(t) = 1 - \frac{\sqrt{t}}{\gamma^2\sqrt{t} - \lambda} = \frac{(\gamma^2 - 1)\sqrt{t} - \lambda}{\gamma^2\sqrt{t} - \lambda}, \quad t = t_{1,2}. \tag{3.64}$$

We consider three cases separately.

Case 1. The numbers $t_{1,2}$ *in (3.40) are complex.* In this case $D := \lambda^2 + 1 - \gamma^2 < 0$ and $\sqrt{t_{1,2}} = (\lambda \pm i\sqrt{|D|})/(\gamma^2 - 1)$. We set $t = t_1 e^{i\tau}$. Then, as $\tau \to 0$, $t - t_1 = t_1(i\tau - \tau^2/2 + O(\tau^3))$ and

$$\begin{aligned} f(t) &= f(t_1) + (t - t_1)^2 f''(t_1)/2 + O\left((t - t_1)^3\right) \\ &= f(t_1) + (-\tau^2 - i\tau^3 + O(\tau^4))t_1^2 f''(t_1)/2 + O(\tau^3). \end{aligned}$$

Therefore,

$$\operatorname{Re}(f(t)-f(t_1)) = -\tau^2 \operatorname{Re}(t_1^2 f''(t_1))/2 + O(\tau^3), \quad \tau \to 0. \tag{3.65}$$

Now from (3.64) we get

$$2t_1^2 f''(t_1) = \frac{i\sqrt{|D|}}{\gamma^2(\lambda + i\sqrt{|D|})/(\gamma^2-1) - \lambda} = \frac{i(\gamma^2-1)\sqrt{|D|}}{\lambda + i\gamma^2\sqrt{|D|}},$$

and (3.65) becomes

$$\operatorname{Re}(f(t)-f(t_1)) = -\frac{(\gamma^2-1)\gamma^2|D|}{4(\lambda^2+\gamma^4|D|)}\tau^2 + O(\tau^3), \quad \tau \to 0. \tag{3.66}$$

Hence, $\operatorname{Re}(f(t_1))$ is a local maximum of $\operatorname{Re}(f(t))$ on the circle $C_{|t_1|}$. Note that in this case $t_2 = \bar{t}_1$ and by (3.64) the real parts of $t_1^2 f''(t_1)$ and $t_2^2 f''(t_2)$ are the same. Since $C_{|t_1|} \cap J(\tilde{C}_+) = \{t_1, t_2\}$, using Lemmas 3.5 and 3.7 we obtain $\operatorname{Re}(f(t)) < \operatorname{Re}(f(t_1))$ for every $t \in C_{|t_1|}$, $t \neq t_{1,2}$. We choose the contour of integration in (3.32) to be $\Gamma = C_{|t_1|}$ in this case. To apply Theorem 3.6 we set (for all cases)

$$p(\tau) := \log|F(t(\tau))|, \quad q(\tau) := |A(t(\tau))|,$$

where $F(t)$ is the function defined in Lemma 3.4 and $t = t(\tau)$ is a suitable parametrization of the contour Γ or part of Γ. In this case it is enough to consider $t(\tau) = t_1 e^{i\tau}$ with $\tau \in [0, \pi)$. Then, $P = -\operatorname{Re}(t_1^2 f''(t_1))/2 < 0$, $\nu = 2$, and $Q = |A(t_1)|$. It is clear that all conditions of Theorem 3.6 are satisfied, including (d), which follows from Lemma 3.4 and the choice of the contour Γ. From (3.32), Theorem 3.6, and Lemma 3.4 we get

$$|R(n, \lambda n, \gamma n + 1)| = O\left(\frac{|A(t_1)| \cdot |F(t_1)|^n}{\sqrt{n}}\right) = O\left(\frac{1}{\sqrt{n}}\right). \tag{3.67}$$

Case 2. The numbers $t_{1,2}$ are real and $t_1 \neq t_2$. Now we have $D > 0$, $\sqrt{t_{1,2}} = (\lambda \pm \sqrt{D})/(\gamma^2-1)$, and by (3.38), $t_1 > t_2 > b^*$. From (3.64), at $t = t_{1,2}$ we get

$$2t^2 f''(t) = \frac{\pm(\gamma^2-1)\sqrt{D}}{\lambda \pm \gamma^2\sqrt{D}}. \tag{3.68}$$

In particular, $f''(t_1) > 0$. Furthermore, if $f'(t_2) = 0$, then $\gamma^4 \geq (\gamma^2+1)\lambda^2$, which implies $\lambda^2 \geq \gamma^4 D$ and by (3.68), $f''(t_2) < 0$ or it is undefined. Then, $f(t_1) < f(t_2)$ and by Lemma 3.4, $F(t_1) < 1$. We set $t = t_1 e^{i\tau}$ and using that $f'(t) = (t-t_1)f''(t_1) + O((t-t_1)^2)$ as $\tau \to 0$, $\tau > 0$ and Lemma 3.5 we obtain

$$\begin{aligned} \frac{d}{d\tau}\operatorname{Re}(f(t)) &= -\operatorname{Im}(t_1^2(e^{2i\tau} - e^{i\tau})f''(t_1)) + O(\tau^2) \\ &= -\frac{(\gamma^2-1)\sqrt{D}}{2(\lambda+\gamma^2\sqrt{D})}\tau + O(\tau^2), \quad \tau \to 0, \quad \tau > 0. \end{aligned} \tag{3.69}$$

Thus, $\mathrm{Re}(f(t))$ has a local maximum at t_1 on the circle $C_{|t_1|}$. Furthermore, by (3.62),

$$t^* := \max\{|t| \,:\, t \in J(\tilde{C}_+)\} = \frac{\gamma^2 - \lambda^2}{\lambda^2} < \frac{1}{\lambda^2} < \frac{1}{\gamma^2 - 1} = \sqrt{t_1 t_2} < t_1 \tag{3.70}$$

in this case, and therefore, $C_{|t_1|} \cap J(\tilde{C}_+) = \emptyset$. From Lemmas 3.5 and 3.7 it follows that $\mathrm{Re}(f(t)) < \mathrm{Re}(f(t_1))$ for every $t \in C_{|t_1|}$, $t \neq t_1$, and we again select the contour Γ in (3.32) to be the circle $C_{|t_1|}$. As in Case 1, we use the parametrization $t(\tau) = t_1 e^{i\tau}$, $\tau \in [0, \pi)$, and the same P, ν, and Q. From (3.32), Theorem 3.6, and Lemma 3.4 it follows that

$$|R(n, \lambda n, \gamma n + 1)| = O\left(\frac{F(t_1)^n}{\sqrt{n}}\right). \tag{3.71}$$

Case 3. The numbers $t_{1,2}$ are equal. In this case $\gamma^2 = \lambda^2 + 1$ and $t_1 = t_2 = 1/\lambda^2$. From (3.36) we have $f'(t) = N(t)/S(t)$ with

$$N(t) := \gamma^2\sqrt{t} - \gamma\sqrt{t - b^*} - \lambda, \quad S(t) := t(\lambda + \gamma\sqrt{t - b^*}).$$

Differentiating the equation $Sf' = N$ twice we get $S''f' + 2S'f'' + Sf''' = N''$. At $t = t_1$ we have $f'(t) = 0$, $f''(t) = 0$, $\sqrt{t - b^*} = 1/(\gamma\lambda)$, and $S(t) = \gamma^2/\lambda^3$. Therefore, at $t = t_1$ we obtain

$$\begin{aligned} f'''(t) &= \frac{N''(t)}{S(t)} = \left(-\frac{\gamma^2}{4t^{3/2}} + \frac{\gamma}{4(t - b^*)^{3/2}}\right)\frac{1}{S(t)} \\ &= -(\gamma/4)(\lambda^3\gamma - \lambda^3\gamma^3)\lambda^3/\gamma^2 = \lambda^8/4. \end{aligned} \tag{3.72}$$

By Taylor's theorem,

$$f(t) = f(t_1) + (t - t_1)^3 f'''(t_1)/6 + O((t - t_1)^4), \quad t \to t_1, \quad t \in \mathbf{R}, \tag{3.73}$$

and since $f'''(t_1) > 0$, it follows that on the interval (t_1, ∞), $f(t)$ is increasing.

Setting $t = t_1 + se^{i\pi/3}$ with $s > 0$ in the Taylor series for f' we obtain

$$f'(t) = (t - t_1)^2 f'''(t_1)/2 + O((t - t_1)^3) = s^2 e^{2i\pi/3}\lambda^8/8 + O(s^3), \quad s \to 0,$$

and then,

$$\begin{aligned} \frac{d}{ds}\mathrm{Re}(f(t_1 + se^{i\pi/3})) &= \mathrm{Re}(\frac{d}{ds} f(t_1 + se^{i\pi/3})) = \mathrm{Re}(f'(t)dt/ds) \\ &= \mathrm{Re}(s^2 e^{i\pi}\lambda^8/8 + O(s^3)) = -s^2\lambda^8/8 + O(s^3), \quad s \to 0, \end{aligned}$$

which shows that $\mathrm{Re}(f(t_1 + se^{i\pi/3}))$ is decreasing on an interval $(0, h)$ for some $h > 0$. We set

$$r := \left|t_1 + he^{i\pi/3}\right| = \sqrt{t_1^2 + h^2 + t_1 h} > t_1 = t^*,$$

where t^* is the number defined with (3.70). By the definitions of r and t^* it follows that $C_r \cap J(\tilde{C}_+) = \emptyset$, and therefore, $\mathrm{Re}(f(t))$ is monotone on each of the semicircles $C_r^{\pm} = \{re^{\pm i\theta} : \theta \in (0, \pi)\}$. Since

$$\mathrm{Re}(f(r)) > \mathrm{Re}(f(t_1)) > \mathrm{Re}(f(t_1 + he^{i\pi/3}))$$

and $\mathrm{Re}(f(\bar{t})) = \mathrm{Re}(f(t))$, the functions $\mathrm{Re}(f(re^{\pm i\theta}))$ are decreasing on $(0, \pi)$.

In Case 3 we choose the contour Γ in (3.32) to be the union of the arc $\{t \in C_r : \mathrm{Re}(t) \leq t_1 + h/2\}$ and the line segments $\{t = t_1 + se^{\pm i\pi/3} : s \in [0, h]\}$. It is sufficient to apply Theorem 3.6 only on one of the line segments: $t(\tau) = t_1 + \tau e^{i\pi/3}$, $\tau \in [0, h]$. In this case $P = -f'''(t_1)/6 = -\lambda^8/24 < 0$, $\nu = 3$, and $Q = |A(t_1)|$. From (3.32), Theorem 3.6, and Lemma 3.4 we get $F(t_1) = 1$ and

$$|R(n, \lambda n, \gamma n + 1)| = O\left(|A(t_1)| \cdot F(t_1)^n n^{-1/3}\right) = \left(n^{-1/3}\right). \tag{3.74}$$

This completes the proof of Theorem 3.8. □

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