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GOTZMANN MONOMIAL IDEALS

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ABSTRACT. A Gotzmann monomial ideal of a polynomial ring is a monomial ideal which is generated in one degree and which satisfies Gotzmann's persistence theorem. Let $R = K[x_1, \ldots, x_n]$ denote the polynomial ring in n variables over a field K and M^d the set of monomials of R of degree d. A subset $V \subset M^d$ is said to be a Gotzmann subset if the ideal generated by V is a Gotzmann monomial ideal. In the present paper, we find all integers $a > 0$ such that every Gotzmann subset $V \subset M^d$ with $|V| = a$ is lexsegment (up to the permutations of the variables). In addition, we classify all Gotzmann subsets of $K[x_1, x_2, x_3]$.

0. Introduction

Let K be an arbitrary field and $R = K[x_1, x_2, \ldots, x_n]$ the polynomial ring with $\deg(x_i) = 1$ for $i = 1, 2, \ldots, n$. Let $I = \oplus_{d=0}^{\infty} I_d$ be a homogeneous ideal of R. We denote the Hilbert function of I by $H(I, d)$, i.e., $H(I, d) = \dim_K I_d$.

The minimal growth of Hilbert functions of homogeneous ideals was determined by Macaulay. Gotzmann's persistence theorem [6] states that if an ideal has no generator of degree $i > d$ and if the growth of the d-th Hilbert function is minimal, then the growth of the k-th Hilbert function is also minimal for $k > d$. In the following we explain in more detail Gotzmann's persistence theorem.

Let n and h be positive integers. Then h can be written uniquely in the following form, called the *n-th binomial representation of h*:

$$h = \binom{h(n)+n}{n} + \binom{h(n-1)+n-1}{n-1} + \cdots + \binom{h(i)+i}{i},$$

where $h(n) \geq h(n-1) \geq \cdots \geq h(i) \geq 0,\ i \geq 1$; see [3, Lemma 4.2.6]. Given this representation of h, we define

$$h^{<n>} = \binom{h(n)+n+1}{n} + \binom{h(n-1)+n}{n-1} + \cdots + \binom{h(i)+i+1}{i},$$

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$$h_{\ll n \gg} = \binom{h(n)+n-1}{n-1} + \binom{h(n-1)+n-2}{n-2} + \cdots + \binom{h(i)+i-1}{i-1}.$$

THEOREM (Minimal growth of Hilbert function). *Let I be a homogeneous ideal of $R = K[x_1, x_2, \ldots, x_n]$. Then one has*

$$H(I, d+1) \geq H(I, d)^{<n-1>}. \tag{1}$$

This theorem was proved by Macaulay. We refer the reader to [3, §4.2] for further information.

THEOREM (Gotzmann's Persistence Theorem [6]). *Let $R = K[x_1, \ldots, x_n]$ and I be a homogeneous ideal of R generated in degree $\leq d$. If $H(I, d+1) = H(I, d)^{<n-1>}$, then $H(I, k+1) = H(I, k)^{<n-1>}$ for all $k \geq d$.*

A monomial ideal $I \subset R$ is called a *Gotzmann monomial ideal* if I is generated in one degree d and if I satisfies $H(I, d)^{<n-1>} = H(I, d+1)$. Instead of discussing the ideal itself, we consider its minimal set of monomial generators.

Let $A = (a_1, a_2, \ldots, a_n)$ and $B = (b_1, b_2, \ldots, b_n)$ be elements of $\mathbb{Z}^n_{\geq 0}$. The *lexicographic order* on $\mathbb{Z}^n_{\geq 0}$ is defined by $A < B$ if the leftmost nonzero entry of $B - A$ is positive. Moreover, the lexicographic order on monomials of the same degree is defined by ${x_1}^{a_1}{x_2}^{a_2}\ldots{x_n}^{a_n} < {x_1}^{b_1}{x_2}^{b_2}\ldots{x_n}^{b_n}$ if $A < B$ on $\mathbb{Z}^n_{\geq 0}$.

Let M denote the set of variables $\{x_1, x_2, \ldots, x_n\}$ and M^d the set of all monomials of degree d, where $M^0 = \{1\}$. For a finite set $V \subset M^d$, we write $MV = \{x_i v \mid v \in V,\ i = 1, 2, \ldots, n\}$ and write $|V|$ for the number of elements of V.

(i) $V \subset M^d$ is called a *lexsegment set* if V is the set of the first $|V|$ monomials with respect to the lexicographic order. Denote the lexsegment set V of $K[x_1, \ldots, x_n]$ in degree d with $|V| = a$ by $\mathrm{Lex}(n, d, a)$.

(ii) $V \subset M^d$ is called a *Gotzmann set* if the ideal I which is generated by V satisfies $H(I, d+1) = H(I, d)^{<n-1>}$, where $I = \{0\}$ if $V = \emptyset$. In other words, V is a Gotzmann set if $|MV| = |V|^{<n-1>}$.

(iii) V is called *strongly stable* if, for any monomial $u \in V$, one has $\frac{x_i}{x_j}u \in V$ for all i and j with $i < j$ and with $x_j | u$.

A lexsegment set is Gotzmann and strongly stable. In general, however, a Gotzmann set is not necessarily lexsegment. We define $V \sim V'$ if we can obtain V' from V by a permutation of variables. In other words, there exists a permutation π of $\{1, 2, \ldots, n\}$ such that $\pi(V) = V'$, where for the permutation $\pi = (\pi(1), \ldots, \pi(n))$ of $\{1, 2, \ldots, n\}$, we define $\pi(x_1^{a_1} \ldots x_n^{a_n}) = x_{\pi(1)}^{a_1} \ldots x_{\pi(n)}^{a_n}$ and $\pi(V) = \{\pi(u) \mid u \in V\}$.

The main result of the present paper determines all integers $a > 0$ such that every Gotzmann set V of degree d with $|V| = a$ and with $\gcd(V) = 1$

satisfies $V \sim \mathrm{Lex}(n,d,a)$, where $\gcd(V)$ is the greatest common divisor of the monomials belonging to V.

THEOREM 1. *Let $R = K[x_1,\ldots,x_n]$ be the polynomial ring in n variables and $a = \sum_{j=p}^{n-1} \binom{a(j)+j}{j}$ the $(n-1)$th binomial representation of $a > 0$. Then the following conditions are equivalent:*

(i) $a(n-1) = a(p)$.

(ii) *If $V \subset M^d$ is a Gotzmann set with $|V| = a$ and $\gcd(V) = 1$, then d is determined by a and $V \sim \mathrm{Lex}(n,d,a)$.*

(iii) *If $V \subset M^d$ is a Gotzmann set with $|V| = a$ and $\gcd(V) = 1$, then d is determined by a and $V \sim V'$ for some strongly stable set V' consisting of monomials of R.*

We also classify all Gotzmann sets of $K[x_1,x_2,x_3]$; see Proposition 8.

Aramova, Herzog and Hibi [2] considered Gotzmann theorems for the exterior algebras. Furthermore, Gasharov [5] generalized the persistence theorem to finitely generated modules over the polynomial ring and to exterior algebras. Results related to Theorem 1 have been obtain by Füredi and Griggs [4]. They determined all integers $a > 0$ such that every squarefree Gotzmann set V with $|V| = a$ is unique up to the permutation of variables.

1. Proof of Theorem 1

Let K be an arbitrary field and $R = K[x_1,\ldots,x_n]$ the polynomial ring in n variables over K. For a monomial $u \in R$ and a subset $V \subset M^d$, we write $uV = \{uv \mid v \in V\}$.

LEMMA 2. *Let V be a set of monomials of the same degree, $u = \gcd(V)$ and $uV' = V$. Then V is a Gotzmann set if and only if V' is a Gotzmann set.*

Proof. By construction, we have $|V| = |V'|$ and $|MV| = |MV'|$. Thus the relevant conditions are equivalent. □

We can obtain a Gotzmann set which is not a lexsegment set by multiplying a lexsegment set by a monomial. For example, if V is a lexsegment set, then x_1x_2V is not a lexsegment set, but a Gotzmann set. But this is essentially the same as a lexsegment set. Therefore we often assume $\gcd(V) = 1$.

Let V be a set of monomials of degree d and $u = \gcd(V)$. If $|V| > 1$, we define $V_{0,i} = \{v \in M^d \mid \ x_iu \text{ divides } v\}$ and $V_{d,i} = V \setminus V_{0,i}$ for $i = 1,2,\ldots,n$. If $|V| = 1$, then we define $V_{0,i} = V$ and $V_{d,i} = \emptyset$. Note that if $|V| > 1$ then $V_{d,i} \neq \emptyset$.

To prove the main theorem, we need some lemmas from [8].

Lemma 3 ([8, Lemma 1.5]). *Let a, b and n be positive integers. One has*

$$a^{<n>} + b^{<n>} > (a+b)^{<n>}.$$

Let h be a positive integer and $h = \sum_{j=i}^{n} \binom{h(j)+j}{j}$ the nth binomial representation of h. Let $\alpha = \max\{0, \max\{k \in \mathbb{Z} \mid h - \binom{k+n}{n} > 0\}\}$. We denote $h - \binom{\alpha+n}{n}$ by $\bar{h}^{(n)}$. In other words:

(i) If $h = 1$, then $\bar{h}^{(n)} = 0$.
(ii) If $h > 1$ and $i = n$, then $\bar{h}^{(n)} = \binom{h(n)+n-1}{n-1}$.
(iii) If $h > 1$ and $i < n$, then $\bar{h}^{(n)} = \sum_{j=i}^{n-1} \binom{h(j)+j}{j}$.

Lemma 4 ([8, Lemma 2.2]). *Let V be a Gotzmann set of monomials of degree d and $h = |V|$. Then, for $i = 1, 2, \ldots, n$, we have*

$$\overline{h}^{(n-1)} \leq |V_{d,i}| \leq h_{\ll n-1 \gg}. \tag{2}$$

Lemma 5 ([8, Lemma 2.3]). *Let V be a Gotzmann set of monomials of degree d with $\gcd(V) = 1$ and with $V \neq M^d$. Let $\overline{M_i} = M \setminus \{x_i\}$. Then there exists an integer $1 \leq i \leq n$ such that:*

(i) *$V_{d,i}$ is a Gotzmann set of $K[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$, $V_{0,i}$ is a Gotzmann set of $K[x_1, \ldots, x_n]$ and $|V_{d,i}| < |V|_{\ll n-1 \gg}$.*
(ii) *$x_i V_{d,i} \subset \overline{M_i} V_{0,i}$.*

Lemma 6. *Let V be a Gotzmann set of monomials of degree d. If $\gcd(V) = 1$ and $\binom{\alpha+n-1}{n-1} < |V| \leq \binom{\alpha+1+n-1}{n-1}$, then we have $d = \alpha+1$ and $\gcd(V_{0,i}) = x_i$ if $|V| > 1$.*

Proof. We use induction on $|V|$. In the case when $|V| = 1$ we have $V = \{1\} = M^0$. Thus we may assume $|V| > 1$ and $n > 1$. By Lemma 5 we may take $V_{0,i}$ as a Gotzmann set. Since $V = V_{0,i} \cup V_{d,i}$ and $V_{d,i} \neq \emptyset$, we can use induction.

Let $u = \gcd(V_{0,i})$ and $V' = \frac{1}{u} V_{0,i}$. Lemma 4 says

$$\binom{\alpha - 1 + n - 1}{n-1} < |V_{0,i}| = |V| - |V_{d,i}| \leq \binom{\alpha + n - 1}{n-1}.$$

Thus, by induction, we have $V' \subset M^{\alpha}$. On the other hand, we have $MV_{0,i} \supset x_i V_{d,i}$ by Lemma 5. Thus, for any $v \in V_{d,i}$, we have $x_i v \in MV_{0,i} = uMV'$. Therefore u divides $x_i v$. By the definition of $V_{0,i}$ it follows that x_i divides u. Thus u/x_i divides v and all elements of V are divisible by u/x_i. Since $\gcd(V) = 1$, we have $u = x_i$. Thus $V_{0,i} = x_i V' \subset M^{\alpha+1}$. Therefore we have $d = \alpha + 1$. □

Lemma 6 says that the degree of the generators of a Gotzmann set V with $\gcd(V) = 1$ is determined by $|V|$. Furthermore, if V is a Gotzmann set with $|V| = \binom{\alpha+n}{n-1}$ and $\gcd(V) = 1$, then $V = M^{\alpha+1}$.

Proof of Theorem 1. (i)⇒(ii)*:* Let $a = a(n-1) = a(n-2) = \cdots = a(p)$. We use induction on $|V|$. If $|V| = 1$, then $V = \{1\}$ since $\gcd(V) = 1$. Thus V is a lexsegment set. If $n-1 = p$, then $V = M^d$ by Lemma 6. Thus we may assume $p < n-1$ and $|V| > 1$. By Lemma 5, we may assume $|V_{d,i}| < |V|_{\ll n-1 \gg}$ and that $V_{d,i}$ is a Gotzmann set. Hence, by Lemma 4, $|V_{0,i}|$ and $|V_{d,i}|$ are of the form

$$|V_{d,i}| = \sum_{j=p}^{n-2} \binom{a+j}{j} + b \quad \text{and} \quad |V_{0,i}| = \sum_{j=p+1}^{n-1} \binom{a-1+j}{j} + \binom{a+p-1}{p} + c$$

with $0 \le b < \binom{a+p-1}{p-1}$ and $0 < c \le \binom{a+p-1}{p-1}$.

Since Lemma 5 (ii) says $MV_{0,i} \supset x_i V_{d,i}$, we have $MV = MV_{0,i} \cup \overline{M}_i V_{d,i}$. Also, since this union is disjoint, by Lemmas 3 and 5, we have

$$\begin{aligned} |MV| &= |MV_{0,i}| + |\overline{M}_i V_{d,i}| \\ &= \left\{ \sum_{j=p}^{n-2} \binom{a+j}{j} \right\}^{[+1]} + \left\{ \sum_{j=p}^{n-1} \binom{a-1+j}{j} \right\}^{[+1]} + b^{<p-1>} + c^{<p-1>} \\ &\le \sum_{j=p+1}^{n-1} \binom{a+j}{j} + \binom{a+p}{p} + \{b+c\}^{<p-1>} \\ &= |V|^{<n-1>}. \end{aligned}$$

Lemma 3 implies that the above inequality becomes an equality if and only if $b = 0$ or $c = 0$. Thus $b = 0$.

Therefore we have $|V_{0,i}| = \binom{a+n-1}{n-1}$. Thus $V_{0,i} = x_i M^{d-1}$. Moreover, $|V_{d,i}| = \sum_{j=p}^{n-2} \binom{a(j)+j}{j}$ and $V_{d,i}$ is a Gotzmann set of $n-1$ variables. Then, by induction, $V_{d,i}$ is a lexsegment set after a proper permutation of variables. We may assume that $V_{d,i}$ is a lexsegment set of $K[x_2, \ldots, x_n]$ and $i = 1$. Since $V_{d,1}$ is a lexsegment set of $K[x_2, \ldots, x_n]$, $V = x_1 V_{0,1} \cup V_{d,1} = x_1 M^{d-1} \cup V_{d,1}$ is a lexsegment set.

(ii)⇒(iii)*:* Since lexsegment sets are strongly stable, the direction (ii)⇒(iii) is obvious.

(iii)⇒(i)*:* In the case when $a(n-1) > a(p)$, we will construct a Gotzmann set that is not strongly stable. By assumption, we have $a(n-1) > a(p)$. Thus there exists $1 \le k \le n-2$ such that $a(k+1) > a(k)$. Let $V_{n-1} = x_n M^{a(n-1)} = u_{n-1} M^{a(n-1)}$. Denote $\{x_1, x_2, \ldots, x_j\}$ by $\overline{M}_{\le j}$. Inductively we define V_j as follows:

- If $j \ne k$, then $V_j = u_j \overline{M}^{a(j)}_{\le j+1}$, where $u_j = u_{j+1} \frac{{x_{j+1}}^{1+a(j+1)-a(j)}}{x_{j+2}}$.
- If $j = k$, then $V_k = u_k \overline{M}^{a(k)}_{\le k+1}$, where $u_k = u_{k+1} x_1 \frac{{x_{k+1}}^{a(k+1)-a(k)}}{x_{k+2}}$.

Let $V = \bigcup_{j=p}^{n-1} V_j$. If $i > j$, then we have $V_j \cap V_i = \emptyset$ since V_j has no element that is divisible by u_i. Thus $V = \bigcup_{j=p}^{n-1} V_j$ is a disjoint union. Therefore, $|V| = \sum_{j=p}^{n-1} |V_j| = \sum_{j=p}^{n-1} \binom{a(j)+j}{j} = a$. Moreover, since $u_{j+1} | \frac{x_{j+2}}{x_{j+1}} u_j$, we have $u_i | \frac{x_{i+1}}{x_{j+1}} u_j$ for $i > j$. Since $u_i \in K[x_1, x_{i+1}, x_{i+2}, \ldots, x_n]$, for $i > j$ we have

$$x_{i+1} V_j = x_{i+1} u_j \overline{M}_{\leq j+1}^{a(j)} \subset u_i \overline{M}_{\leq i+1}^{a(i)+1} \subset \overline{M}_{\leq i+1} V_i.$$

Hence, we have $MV = \bigcup_{j=p}^{n-1} MV_j = \bigcup_{j=p}^{n-1} \overline{M}_{\leq j+1} V_j$. This union is also disjoint. Thus we have $|MV| = \sum_{j=p}^{n-1} \binom{a(j)+1+j}{j} = a^{<n-1>}$ and V is a Gotzmann set.

Next, we will prove that V is not strongly stable. Let $u' = u_{k+1}/x_{k+2} \in K[x_{k+2}, \ldots, x_n]$. Since $a(k+1) > a(k)$, we have $\deg(u') = \deg(u_k) - 1 - \{a(k+1) - a(k)\} \leq d - 2$. Let $d_0 = \deg(u')$. We will prove that $u' {x_1}^{d-d_0}$ and $u' {x_{k+1}}^{d-d_0}$ do not belong to V, i.e., that $u' x_1^{d-d_0} \notin V_j$ and $u' {x_{k+1}}^{d-d_0} \notin V_j$ for all j.

- (i) For $j = k$, since $V_k = x_1 x_{k+1}^{a(k+1)-a(k)} u' \overline{M}_{\leq k+1}^{a(k)}$, we have $u' x_1^{d-d_0} \notin V_k$ and $u' {x_{k+1}}^{d-d_0} \notin V_k$.
- (ii) For any $j < k$, x_{j+1} divides every monomial $u \in V_j$. Since $u' \in K[x_{k+1}, \ldots, x_n]$, it follows that $u' x_1^{d-d_0} \notin V_j$ and $u' {x_{k+1}}^{d-d_0} \notin V_j$.
- (iii) For any $j > k$, u_j does not divide u'. Since $u_j \in K[x_{j+1}, \ldots, x_n]$, u_j does not divide $u' {x_1}^{d-d_0}$ and $u' {x_{k+1}}^{d-d_0}$. Thus $u' x_1^{d-d_0} \notin V_j$ and $u' {x_{k+1}}^{d-d_0} \notin V_j$.

However, if there is a strongly stable set V' with $V' \sim V$, then either $x_1^{d-d_0} u' \in V$ or $x_{k+1}^{d-d_0} u' \in V$ must be satisfied since $x_1 {x_{k+1}}^{d-d_0-1} u' \in V_k \subset V$. Thus V is not strongly stable. □

Definition 7. Let V be a Gotzmann set and $|V| = \sum_{j=p}^{n-1} \binom{a(j)+j}{j}$ the $(n-1)$th binomial representation. By Theorem 1, if $a(p) = a(n-1)$, then V must be a lexsegment set. We call $|V|$ an nth *lexnumber*, or simply a *lexnumber*, if $a(p) = a(n-1)$.

Example 1. Here are some lexnumbers for $n = 3, 4, 5$.

$n = 3$: $1, 2, 3, 5, 6, 9, 10, 14, 15, 20, 21, 27, 28, 35, 36, 44, 45, 54, 55, 65, 66, 77, 78, 90, 91, 104, 105, \ldots$
$n = 4$: $1, 2, 3, 4, 7, 9, 10, 16, 19, 20, 30, 34, 35, 50, 55, 56, 77, 83, 84, 112, \ldots,$
$n = 5$: $1, 2, 3, 4, 5, 9, 12, 14, 15, 25, 31, 34, 35, 55, 65, 104, 105, \ldots,$

For fixed d, there are only $\{d(n-1)+1\}$ lexnumbers, since there are $(n-1)$ lexnumbers between $\binom{t+n-1}{n-1}$ and $\binom{t+n}{n-1}$.

2. Gotzmann sets in three variables

In this section we consider Gotzmann sets with a few variables. If $n = 1$, then all sets V are Gotzmann sets. If $n = 2$, we can easily show that V is a Gotzmann set if and only if $V = \emptyset$ or $V = M^d$, provided we assume that $\gcd(V) = 1$. We consider the case $n = 3$ in Proposition 8.

We define a map $\pi_i : \bigoplus_{d=0}^{\infty} M^d \to \mathbb{Z}_{\geq 0}^{n-1}$ by setting

$$\pi_i(x_1^{a_1} \dots x_n^{a_n}) = (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n).$$

It follows that $\pi_i|_{M^d}$ is injective.

Let V be a set of monomials of degree d and let $u = x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}$ be a monomial of degree d. We say that a monomial $v = x_1^{b_1} x_2^{b_2} \dots x_n^{b_n}$ with degree d is *under* u *for* i if $b_j \leq a_j$ for all $j \neq i$. We call u a *fixed empty element of* V *for* i if $u \notin V$ and any monomial which is under u for i does not belong to V.

Note that u is a fixed empty element of V for i if and only if $\pi_i(u) \notin \pi_i(M^t V)$ for $t \geq 0$. Also, if u is a fixed empty element of V for i, then any monomial v which is under u for i is also a fixed empty element of V for i.

Proposition 8. *Let $V \subset R$ be a set of monomials of degree d with $\gcd(V) = 1$. If V is a Gotzmann set, then any monomial $v \notin V$ is a fixed empty element of V for some i and $|V| > \binom{d-1+n-1}{n-1}$. Moreover, if $n = 3$, then these conditions are equivalent.*

Proof. The inequality $|V| > \binom{d-1+n-1}{n-1}$ follows from Lemma 6.

We use induction on $|V|$. If $|V| = 1$, then $V = \{1\}$. Thus, in this case, the conditions are satisfied. Hence we may assume $|V| > 1$. By Lemma 5, there exists i such that $V_{0,i}$ and $V_{d,i}$ are Gotzmann sets and $\overline{M_i} V_{0,i} \supset x_i V_{d,i}$.

Let $w = x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}$ be a monomial of degree d. We will prove that if $w \notin V$ then w is a fixed empty element of V for some j. We consider two cases:

Case I: If x_i divides w, then $w \in V_{0,i}$. By induction, there exists j such that w is a fixed empty element of $V_{0,i}$ for j. Thus for any $v \neq w$ which is under w for j, we have $v \notin V_{0,i}$. Hence we have to prove that $v \notin V_{d,i}$.

If $j = i$ then $x_i | v$, and thus $v \notin V_{d,i}$.

If $j \neq i$, we may assume that x_i does not divide v since if $x_i | v$ then $v \notin V_{d,i}$. Let $v = x_1^{b_1} x_2^{b_2} \dots x_n^{b_n} \neq w$. Since $x_i | w$ and $x_i \nmid v$, we have $b_i + 1 \leq a_i$. Thus for any $x_k | v$, $\frac{x_i}{x_k} v$ is under w for j. Hence we have $\frac{x_i}{x_k} v \notin V_{0,i}$ and $x_i v \notin M V_{0,i}$. Since $\overline{M_i} V_{0,i} \supset x_i V_{d,i}$, we have $v \notin V_{d,i}$.

Case II: If x_i does not divide w, then by induction there exists $j \neq i$ such that w is a fixed empty element of $V_{d,i}$ for j. Since $j \neq i$, for any v which is under w for i, v cannot be divided by x_i. Thus we have $v \notin V_{0,i} \cup V_{d,i} = V$.

Next, in the case when $n = 3$, we will prove that these conditions are equivalent. Let $u = x_1^{a_1}x_2^{a_2}x_3^{a_3}$ be a fixed empty element of V for i. Assume that $i = 1$. Since $x_1^{a_1+1}x_2^{a_2-1}x_3^{a_3}$ and $x_1^{a_1+1}x_2^{a_2}x_3^{a_3-1}$ are under u for 1 and $u \notin V$, we have $x_1u = x_1^{a_1+1}x_2^{a_2}x_3^{a_3} \notin MV$. By the same reasoning, for each $1 \le i \le 3$, if u is a fixed empty element of V for i, then x_iu is a fixed empty element of MV for i. Let

$$U_i(V) = \{u \notin V : u \text{ is a fixed empty element of } V \text{ for } i\}.$$

Then any element $v \in x_iU_i(V)$ is also a fixed empty element of MV for i.

We will show $x_iU_i(V) \cap x_jU_j(V) = \emptyset$ if $i \neq j$. If $x_iu = x_ju' \in x_iU_i(V) \cap x_jU_j(V)$, then the monomials

$$x_i^{a_i}x_j^{a_j}x_k^{a_k},\ x_i^{a_i+1}x_j^{a_j-1}x_k^{a_k},\ \dots,\ x_i^{a_i+a_j}x_j^{0}x_k^{a_k}$$

are under u for i. Also, the monomials

$$x_i^{a_i-1}x_j^{a_j+1}x_k^{a_k},\ x_i^{a_i-2}x_j^{a_j+2}x_k^{a_k},\ \dots,\ x_j^{a_j+a_i}x_k^{a_k},\ x_j^{a_j+a_i+1}x_k^{a_k-1},\ \dots,\ x_j^{d}$$

are under u' for j. Hence we can take $\{(a_j+1)+(d-a_j)\}$ monomials which do not belong to V. Thus we have $|V| \le |M^d| - (d+1) = \binom{d+1}{2}$. However the assumption says $|V| > \binom{d+1}{2}$. This is a contradiction. Thus we have $x_iU_i(V) \cap x_jU_j(V) = \emptyset$.

Hence, if V has l fixed empty elements, then MV has at least l fixed empty elements. Thus we have

$$|MV| \le \binom{d+3}{2} - l = \binom{d+2}{2} + \binom{d+2-l}{1}.$$

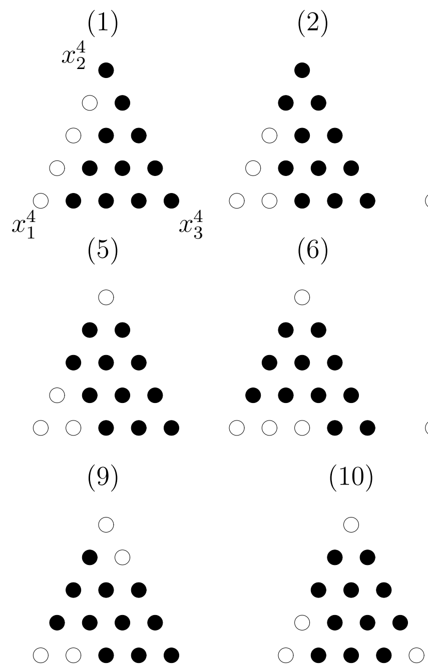
Moreover, by the minimal growth of the Hilbert function (1), we have

$$|MV| \ge |V|^{<2>} = \left\{\binom{d+2}{2} - l\right\}^{<2>} = \left\{\binom{d+1}{2} + \binom{d+1-l}{1}\right\}^{<2>}.$$

Therefore $|MV| = |V|^{<2>}$. Thus V is a Gotzmann set. □

Example 2. To understand the meaning of Proposition 8, drawing a picture of the monomials is useful. (A similar idea can be found in [7].) In the picture below, all monomials of degree 4 in $K[x_1, x_2, x_3]$ are displayed. The monomial x_1^4 is in the lower left corner, x_3^4 is in the lower right corner, and x_2^4 is at the top. The black dots denote monomials in V and the empty circles denote monomials which are missing from V. For example, figure (1) means that x_1^4, $x_1^3x_2$, $x_1^2x_2^2$ and $x_1x_2^3$ are missing. In the picture below, we classify all Gotzmann sets V in $K[x_1, x_2, x_3]$ with $\gcd(V) = 1$ and $|V| = \binom{4+2}{2} - 4 = 11$ up to permutations.

By Proposition 8, each connected component of empty circles must be at a corner. Also, the numbers of empty circles must be equal to or less than the degree of the elements of V.



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