THE SPECTRUM OF THE AVERAGING OPERATOR ON A NETWORK (METRIC GRAPH)

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ABSTRACT. A network is a countable, connected graph X viewed as a one-complex, where each edge $[x, y] = [y, x]$ ($x, y \in X^0$, the vertex set) is a copy of the unit interval within the graph's one-skeleton X^1 and is assigned a positive conductance $\mathsf{c}(xy)$. A reference "Lebesgue" measure X^1 is built up by using Lebesgue measure with total mass $\mathsf{c}(xy)$ on each edge xy. There are three natural operators on X: the transition operator P acting on functions on X^0 (the reversible Markov chain associated with c), the averaging operator A over spheres of radius 1 on X^1, and the Laplace operator Δ on X^1 (with Kirchhoff conditions weighted by c at the vertices). The relation between the ℓ^2-spectrum of P and the H^2-spectrum of Δ was described by Cattaneo [4]. In this note we describe the relation between the ℓ^2-spectrum of P and the L^2-spectrum of A.

1. Introduction

Let X be a countable, connected graph with symmetric neighbourhood relation $\sim$ and without loops and multiple edges. We shall view it as a one-complex, where each edge is a (homeomorphic) copy of the unit interval and edges are glued together at common endpoints (vertices). We write X^0 for the vertex set and X^1 for the one-skeleton of X. Every point in X^1 is of the form (xy, t), the point at distance t from x on the non-oriented edge $[x, y] = [y, x]$, where $0 \le t \le 1$, and $x, y \in X^0$, $x \sim y$. Thus, $(xy, 0) = x$ and $(xy, t) = (yx, 1-t)$. In this way, the discrete graph metric $d(\cdot, \cdot)$ on the vertex set (minimal length=number of edges of a connecting path) has a natural extension to X^1.

We equip each edge $[x, y]$ with a positive *conductance* $\mathsf{c}(xy) = \mathsf{c}(yx)$. On X^0, we consider the discrete measure m^0, where $\mathsf{m}^0(x) = \sum_{y:y \sim x} \mathsf{c}(xy)$. Our *basic assumption* is that $\mathsf{m}^0(x) < \infty$ for all $x \in X^0$. On X^1, we introduce the continuous weighted "Lebesgue" measure m^1 which at the point (xy, t) is given by $\mathsf{c}(xy) \cdot dt$, if $0 < t < 1$ (the vertex set has m^1-measure 0). The pair (X, c), together with these measures, is called a *network*, or—in the recent literature—also *metric graph* or *quantum graph*.

Associated with a network, there are three natural operators.

The first is the *transition operator* P acting on functions $g : X^0 \to \mathbb{C}$ by

$$(1.1) \qquad Pg(x) = \frac{1}{\mathsf{m}^0(x)} \sum_{y:y \sim x} \mathsf{c}(xy) \, g(y) \,.$$

The second is the *Laplace operator* Δ. It can be defined via Dirichlet form theory, or by considering the space of all continuous functions $F : X^1 \to \mathbb{C}$ which are twice differentiable in the interior of each edge and satisfy the Kirchhoff equations

$$\sum_{y:y \sim x} \mathsf{c}(xy) \, F'(xy, 0+) = 0 \quad \text{for all } x \in X^0 \,.$$

We then have

$$\Delta F(xy, t) = F''(xy, t) \,,$$

the 2nd derivative with respect to $t \in (0, 1)$, and Δ has to be closed suitably. See, e.g., Cattaneo [4], Solomyak [18] or Eells and Fuglede [7] for precise details. (The paper [4] seems to have escaped the attention of many people working on metric graphs.)

The third operator is the *averaging operator* A over balls of radius 1. It acts on locally integrable functions $F : X^1 \to \mathbb{C}$ by

$$(1.2) \qquad AF(xy, t) = \frac{1}{\mathsf{m}^0(x)} \sum_{u \sim x} \mathsf{c}(xu) \int_0^{1-t} F(xu, s) \, ds$$

$$+ \frac{1}{\mathsf{m}^0(y)} \sum_{v \sim y} \mathsf{c}(yv) \int_0^t F(yv, s) \, ds \,.$$

In the *regular* case, i.e., when $\mathsf{m}^0(\cdot)$ is constant, this is just the m^1-average of F over the ball with radius 1 centered at (xy, t).

Each of the three operators gives rise to a Markov process. For P, this is the *random walk* (reversible Markov chain) with discrete time and state space X^0 whose transition probabilities are $p(x, y) = \mathsf{c}(xy)/\mathsf{m}^0(x)$, if $y \sim x$, and $p(x, y) = 0$, otherwise.

The Laplace operator Δ is the infinitesimal generator of *Brownian motion* on the network.

The stochastic interpretation of A is more similar to that of P. Namely, A governs the *random walk* with discrete time and state space X^1, where at

any time n, if the current position is (xy, t), the next step goes to a random point in X^1 at distance at most 1. The random choice depends on t, the conductance of $[x, y]$ and the edges incident with $[x, y]$.

These stochastic aspects are not at the heart of the present paper. What we are interested in here is the relation between the spectra of the operators A and P. Cattaneo [4] has given a complete description of the H^2-spectrum of Δ in terms of the ℓ^2-spectrum of P. Our plan is to describe the L^2-spectrum of A in terms of the ℓ^2-spectrum of P.

This refers to the (complex) Hilbert spaces $L^2(X^1, \mathsf{m}^1)$ and $\ell^2(X^0, \mathsf{m}^0)$. The inner product of the latter is given by

$$\langle g_1, g_2 \rangle = \sum_{x \in X^0} g_1(x) \overline{g_2(x)}\, \mathsf{m}^0(x)\,,$$

and it is well known and easy to check that P is self-adjoint with $\|P\| \leq 1$ on this space. Analogously, the inner product on $L^2(X^1, \mathsf{m}^1)$ is

$$\langle F_1, F_2 \rangle = \frac{1}{2} \sum_{x \in X^0} \sum_{y \in X^0: y \sim x} \mathsf{c}(xy) \int_0^1 F_1(xy, t) \overline{F_2(xy, t)}\, dt\,.$$

The factor $\frac{1}{2}$ occurs because corresponding to each edge $[x, y]$, we get two equal terms on the right, namely $\int_0^1 F_1(xy, t) \overline{F_2(xy, t)}\, dt$ and $\int_0^1 F_1(yx, t) \overline{F_2(yx, t)}\, dt$. Again, it is straightforward to check that A is self-adjoint with norm bounded by 1 on $L^2(X^1, \mathsf{m}^1)$.

There is a large body of literature on the spectrum of transition (resp. adjacency and discrete Laplace) operators on finite graphs; see, e.g., the books by Biggs [2], Cvetković, Doob and Sachs [6] and Chung [5]. Transition operators on infinite graphs are also very well studied objects; see, e.g., the books by Soardi [17] and Woess [19]. A lot is known about the ℓ^2-spectrum of transition operators on various classes of infinite graphs, see, e.g., Mohar and Woess [14] for a general survey (up to 1989), and the many more recent papers, mostly embedded into the context of Markovian convolution operators on groups, of which we quote here only a few: de la Harpe, Robertson, and Vallette [10], [11], Cartwright [3], Grigorchuk and Żuk [8], [9], Bartholdi and Woess [1].

On the other hand, not much work has been done regarding the spectra of averaging operators on networks, whence it appears to be useful to have a recipe for translating the spectrum of P into the spectrum of A. Our main result is the following.

THEOREM 1.3.   *The spectrum of A is*

$$\mathsf{spec}(A) = \{0\} \cup \left\{ \frac{\sin \omega}{\omega} : \omega \in \mathbb{R} \setminus \{0\}\,,\ \cos \omega \in \mathsf{spec}(P) \right\} \cup \{1 : 1 \in \mathsf{spec}(P)\}\,.$$

Here, by "$\cup\{1 : 1 \in \mathsf{spec}(P)\}$" we mean that 1 is included in $\mathsf{spec}(A)$ if and only if $1 \in \mathsf{spec}(P)$. This theorem has the following obvious consequence.

COROLLARY 1.4.   *Let $\rho = \rho(P)$ denote the spectral radius of P. Then the spectral radius of A is*

$$\rho(A) = \begin{cases} 1\,, & if \quad \rho = 1\,, \\ \sqrt{1-\rho^2}\big/\arccos(\rho)\,, & if \quad \rho < 1\,. \end{cases}$$

Let $\mathsf{spec}_p(P)$ denote the point spectrum of P, i.e., the set of $\ell^2(X^0, \mathsf{m}^0)$-eigenvalues of P.

THEOREM 1.5.   *We have*

$$\mathsf{spec}_p(A)\backslash\{0\} = \big\{1 : 1 \in \mathsf{spec}_p(P)\big\} \cup \left\{\frac{\sin \omega}{\omega} : \omega \in \mathbb{R} \setminus \{0\}, \cos \omega \in \mathsf{spec}_p(P)\right\}.$$

Moreover, $0 \in \mathsf{spec}_p(A)$ unless $\mathsf{m}^0(X^0) = \infty$ and X is a tree with the property that after removal of any edge, at least one of the two connected components is recurrent.

For the precise meaning of this last condition, see Definition 3.8 and Proposition 3.9 below.

The structure of this paper is as follows. In §2, we set up the basic tools for relating P and A. In §3, we study the contribution to the kernel of A that comes from flows in the network. In §4, we prove the above two main theorems, and we also specify for finite graphs how one can obtain an orthonormal basis of $L^2(X^1, \mathsf{m}^1)$ consisting of eigenvectors (-functions) of A. In §5, we exhibit several examples.

2. Interpolation of functions on the vertex set

For $g \in \ell^2(X^0, \mathsf{m}^0)$ and $u \in L^2[0,1]$, define a function $F_{g,u}$ on X^1 by

$$(2.1) \qquad\qquad F_{g,u}(xy, t) = u(1-t)g(x) + u(t)g(y).$$

Recall that $(xy, t) = (yx, 1-t)$; the definition of $F_{g,u}$ is compatible with this parametrization. It is easy to check that $F_{g,u} \in L^2(X^1, \mathsf{m}^1)$. In fact, it is routine to calculate, for $g_1, g_2 \in \ell^2(X^0, \mathsf{m}^0)$ and $u_1, u_2 \in L^2[0,1]$, that

$$(2.2) \qquad \langle F_{g_1, u_1}, F_{g_2, u_2} \rangle = \langle g_1, g_2 \rangle \langle u_1, u_2 \rangle + \langle P g_1, g_2 \rangle \langle u_1, S u_2 \rangle,$$

where S is defined in (2.4) below, and $\langle u_1, u_2 \rangle = \int_0^1 u_1(t)\overline{u_2(t)}\, dt$ is the standard inner product on $L^2[0,1]$, while the inner products on $L^2(X^1, \mathsf{m}^1)$ and $\ell^2(X^0, \mathsf{m}^0)$ are those defined in the introduction.

LEMMA 2.3.   *The action of A on a function $F_{g,u}$ is given by*

$$AF_{g,u} = F_{g, JSu} + F_{Pg, Ju}\,,$$

where the operators S and J are given by

$$(2.4) \qquad Su(t) = u(1-t) \quad and \quad Ju(t) = \int_0^t u(s) \, ds \, .$$

DEFINITION 2.5. Denote by $\mathcal{M}_0$ the linear span of the functions $F_{g,u}$, where $g \in \ell^2(X^0, \mathsf{m}^0)$, and where $u \in L^2[0,1]$. Let $\mathcal{M}$ denote the closure of $\mathcal{M}_0$ in $L^2(X^1, \mathsf{m}^1)$.

Lemma 2.3 shows that $\mathcal{M}_0$ is invariant under A, and therefore $\mathcal{M}$ is too.

LEMMA 2.6. *The orthogonal complement of $\mathcal{M}$ in $L^2(X^1, \mathsf{m}^1)$ consists of the (equivalence classes of) square integrable functions $F : X^1 \to \mathbb{C}$ such that for each $x \in X^0$,*

$$(2.7) \qquad \sum_{y:y\sim x} \mathsf{c}(xy) \, F(xy,t) = 0 \quad for \ almost \ all \quad t \in [0,1].$$

Proof. Suppose that $F \in \mathcal{M}^\perp$. Then in particular, $\langle F_{g,u}, F \rangle = 0$ for each $u \in L^2[0,1]$ and for $g = \delta_x$, and for this g, one calculates that

$$\langle F_{g,u}, F \rangle = \int_0^1 u(1-t) \, \overline{\sum_{y:y\sim x} \mathsf{c}(xy) \, F(xy,t)} \, dt.$$

Since $u \in L^2[0,1]$ is arbitrary, (2.7) holds.

Conversely if (2.7) holds, then $\langle F_{g,u}, F \rangle = 0$ for all $g \in \ell^2(X^0, \mathsf{m}^0)$ of the form $g = \delta_x$. By linearity, $\langle F_{g,u}, F \rangle = 0$ if g is finitely supported, and using finite approximations, this implies that $\langle F_{g,u}, F \rangle = 0$ for all $g \in \ell^2(X^0, \mathsf{m}^0)$ and $u \in L^2[0,1]$. $\qquad \square$

COROLLARY 2.8. *The operator A leaves $\mathcal{M}$ and $\mathcal{M}^\perp$ invariant, and is identically zero on $\mathcal{M}^\perp$.*

Proof. If $F \in \mathcal{M}^\perp$, then $AF = 0$, as is immediate from the definition of A and Lemma 2.6. $\qquad \square$

In the next Section 3, we shall give a complete description of $\mathcal{M}^\perp$ in terms of *flows* in the network, characterising, in particular, those networks for which $\mathcal{M}^\perp = \{0\}$.

Let us record some elementary properties of the operators J and S arising in (2.4).

LEMMA 2.9. *The operator S satisfies $S^* = S$ and $S^2 = I$. If $v, w \in L^2[0,1]$, with $Sv = v$ (in which case we say that v is* even*) and $Sw = -w$ (in which case w is called* odd*), then $\langle v, w \rangle = 0$. For any $u \in L^2[0,1]$, we can write*

$$(2.10) \qquad u = v + w, \quad where \quad v = \frac{u+Su}{2} \quad and \quad w = \frac{u-Su}{2} \, ,$$

and then $Sv = v$, $Sw = -w$ (and so $\langle v, w \rangle = 0$).

LEMMA 2.11.   *The operator J is compact, but not normal. Moreover, $J^* = SJS$, so that JS and SJ are self-adjoint operators on $L^2[0,1]$. For any $u \in L^2[0,1]$ we have*

$$(2.12) \qquad\qquad JSu + SJu = \langle u, \mathbf{1} \rangle \, \mathbf{1} \,,$$

where $\mathbf{1}$ is the function taking the constant value 1 on $[0,1]$.

Proof. The operator J is Hilbert-Schmidt (and therefore compact), since its kernel belongs to the space $L^2((0,1) \times (0,1))$; see Meise and Vogt [13, Proposition 16.12 and Lemma 16.7(1)] or Pedersen [15, Proposition 3.4.16 and Lemma 3.4.5]. The other assertions are easily checked. $\square$

LEMMA 2.13.   *Let $-1 \le \lambda \le 1$. For $u, v \in L^2[0,1]$, write*

$$\langle u, v \rangle_\lambda = \langle u, v \rangle + \lambda \langle u, Sv \rangle.$$

Then

$$(2.14) \qquad\qquad (1 - |\lambda|)\langle u, u \rangle \le \langle u, u \rangle_\lambda \le (1 + |\lambda|)\langle u, u \rangle \,.$$

If $-1 < \lambda < 1$ then $\langle \cdot, \cdot \rangle_\lambda$ is an inner product on $L^2[0,1]$.
In the degenerate cases $\lambda = \pm 1$, we have $\langle u, u \rangle_1 = 0 \iff Su = -u$, and $\langle u, u \rangle_{-1} = 0 \iff Su = u$, respectively.

Proof. This is routine, using $\langle u, Sv \rangle = \langle Su, v \rangle$. $\square$

For $-1 < \lambda < 1$, we shall denote by L^2_λ the space $L^2[0,1]$ endowed with the inner product $\langle \cdot, \cdot \rangle_\lambda$. By (2.14), it is a Hilbert space.

LEMMA 2.15.   *Let $-1 < \lambda < 1$. Then the operator $J_\lambda = JS + \lambda J$ is compact and self-adjoint on the Hilbert space L^2_λ.*

Proof. If $u, v \in L^2[0,1]$, then using $J^* = SJS$ and $(JS)^* = JS$,

$$\langle J_\lambda u, v \rangle_\lambda = \langle (JS + \lambda J)\, u, v \rangle + \lambda \, \langle (JS + \lambda J)\, u, Sv \rangle$$
$$= \langle JSu, v \rangle + \lambda \, (\langle Ju, v \rangle + \langle JSu, Sv \rangle) + \lambda^2 \langle Ju, Sv \rangle$$
$$= \langle u, JSv \rangle + \lambda \, (\langle u, SJSv \rangle + \langle u, Jv \rangle) + \lambda^2 \langle u, SJv \rangle$$
$$= \langle u, J_\lambda v \rangle_\lambda.$$

The compactness of J_λ follows from the compactness of J on $L^2[0,1]$, plus the fact that the norms of $L^2[0,1]$ and L^2_λ are equivalent. $\square$

As for any compact self-adjoint operator, the space L^2_λ has an orthonormal basis consisting of eigenfunctions for J_λ; see [13, Proposition 16.2] or [15, Theorem 3.3.8]. More explicitly:

LEMMA 2.16.   *Let* $-1 < \lambda < 1$ *and set* $\omega = \arccos \lambda \in (0, \pi)$. *Then the functions*

$$u_{\lambda,n}(t) = \frac{\sqrt{2}}{\sin \omega} \sin\left((\omega + 2\pi n)\, t\right), \quad n \in \mathbb{Z},$$

form a complete orthonormal basis of L_λ^2 *consisting of eigenvectors of* J_λ. *In fact,* $u_{\lambda,n}$ *is an eigenfunction for the eigenvalue*

(2.17) $$\mu_{\lambda,n} = \frac{\sin \omega}{\omega + 2\pi n}.$$

Proof. Setting $u(t) = \sin(\vartheta t)$, where $\vartheta \neq 0$, we compute

$$J_\lambda u(t) = \int_0^t \sin(\vartheta(1 - s))\, ds + \lambda \int_0^t \sin(\vartheta s)\, ds$$

$$= \frac{\sin \vartheta}{\vartheta} \sin(\vartheta t) + \frac{\lambda - \cos \vartheta}{\vartheta} \left(1 - \cos(\vartheta t)\right).$$

So if ϑ is such that $\cos \vartheta = \lambda$, then $J_\lambda u = \mu\, u$, where $\mu = \sin \vartheta / \vartheta$. Taking $\vartheta = \omega + 2\pi n$, we see that $u_{\lambda,n}$ is an eigenfunction for the eigenvalue $\mu_{\lambda,n}$. Since these eigenvalues are distinct for distinct n's, the $u_{\lambda,n}$'s are orthogonal in L_λ^2. It is routine to check that they are in fact orthonormal.

Suppose that $u \in L_\lambda^2$ and that $\langle u, u_{\lambda,n} \rangle_\lambda = 0$ for all $n \in \mathbb{Z}$. We claim that $u = 0$. Taking $v(t) = \sin(\vartheta t)$, where $\cos \vartheta = \cos \omega$, we find that $(v + (\cos \omega)\, Sv)\, (t) = \sin(\vartheta) \cos(\vartheta(1 - t))$. So from $\langle u, u_{\lambda,n} + \lambda\, Su_{\lambda,n} \rangle = 0$ for all n we find that

(2.18) $$\int_0^1 u(1 - t)\, \cos\left((\omega + 2\pi n)t\right)\, dt = 0$$

for all n. Adding and subtracting (2.18), and (2.18) with n replaced by $-n$, we find that for all $n \in \mathbb{Z}$,

$$\int_0^1 \sin(2\pi n t)\, \sin(\omega t)\, u(1 - t)\, dt = 0 \quad \text{and}$$

$$\int_0^1 \cos(2\pi n t)\, \cos(\omega t)\, u(1 - t)\, dt = 0.$$

The first of these conditions implies that the function $v(t) = \sin(\omega t)\, u(1 - t)$ satisfies $v(1 - t) = v(t)$ for almost all t, and the second condition implies that $w(t) = \cos(\omega t)\, u(1 - t)$ satisfies $w(1 - t) = -w(t)$ for almost all t. That is, for almost all t,

$$\sin\left(\omega(1 - t)\right)\, u(t) = \sin(\omega t)\, u(1 - t) \text{ and}$$

$$\cos(\omega(1 - t))\, u(t) = -\cos(\omega t)\, u(1 - t).$$

Multiplying the first of these equations by $\cos(\omega t)$ and the second by $\sin(\omega t)$ and adding, we find that $u(t) = 0$ almost everywhere. Hence the family $\{u_{\lambda,n} : n \in \mathbb{Z}\}$ is a complete orthonormal basis for L_λ^2.   □

3. Flows, and the space $\mathcal{M}^{\perp}$

We now study in detail the space $\mathcal{M}^{\perp}$ defined in Lemma 2.6; recall the defining relation (2.7).

Given our graph X, we consider the edge set $E = E(X)$ to be the set of ordered pairs xy, where $x, y \in X^0$ and $x \sim y$. We set $\mathsf{r}(xy) = 1/\mathsf{c}(xy)$, the *resistance* of the edge xy. Let $\ell^2(E, \mathsf{r})$ be the Hilbert space of all functions $\Phi : E(X) \to \mathbb{C}$ for which $\langle \Phi, \Phi \rangle < \infty$, where the inner product is

$$\langle \Phi, \Psi \rangle = \frac{1}{2} \sum_{x,y:x\sim y} \Phi(xy)\overline{\Psi(xy)}\,\mathsf{r}(xy)\,.$$

DEFINITION 3.1.   A *flow* on the network (X, c) is a function $\Phi \in \ell^2(E, \mathsf{r})$ such that

(3.2) $\sum_{y:y\sim x} \Phi(xy) = 0 \qquad$ for all $x \in X$.

The flow is called *odd* if $\Phi(xy) = -\Phi(yx)$, and it is called *even* if $\Phi(xy) = \Phi(yx)$.

Our definition requires, in particular, that $\langle \Phi, \Phi \rangle < \infty$. The latter number is often called the *energy*—or, more appropriately, the *power*—of the flow Φ. In the literature, the term *flow* usually applies to what we call an *odd* flow here. In this case, one may imagine each edge $[x, y]$ as a tube with unit length and cross-section $\mathsf{c}(xy)$, the tubes are connected at the vertices of X, and the network of tubes is filled with liquid. Then $\Phi(xy)$ is the amount of liquid per unit time that flows from x to y, whence $-\Phi(xy) = \Phi(yx)$ flows in the reverse direction. The condition (3.2) is Kirchhoff's law: the amount of liquid per time unit that enters at any vertex coincides with the amount that exits. In particular, our flows have no source or sink—they are "passive flows". In the above definition, even flows do not have such a nice physical interpretation. We shall write $\mathcal{J}^e$ and $\mathcal{J}^o$ for the (closed and orthogonal) subspaces of $\ell^2(E, \mathsf{r})$ consisting of all even and odd flows on the network (X, c), respectively.

REMARK 3.3.   A graph is called *bipartite* if we can partition its vertex set X^0 in two classes C_1, C_2 such that every edge has one endpoint in C_1 and the other in C_2. Equivalently, this means that X has no odd cycles (as defined below). On a bipartite network, there is an obvious one-to-one correspondence between odd and even flows:

(3.4) $\Phi \in \mathcal{J}^o \leftrightarrow \widetilde{\Phi} \in \mathcal{J}^e$, where $\widetilde{\Phi}(xy) = (-1)^i \Phi(xy)$, if $x \in C_i$ ($i = 1, 2$) .

Returning to $L^2(X^1, \mathsf{m}^1)$, a function F in that space is called *even* if $F(xy, 1 - t) = F(xy, t)$, and *odd* if $F(xy, 1 - t) = -F(xy, t)$, for all $t \in [0, 1]$ and each $xy \in E(X)$. Each F has an orthogonal decomposition as a sum of its even and odd part.

It is straightforward to verify the following lemma.

LEMMA 3.5.
(a) *If $F \in \mathcal{M}^\perp$ is even (respectively, odd), and $u \in L^2[0,1]$, then*

$$\Phi(xy) = \Phi_{F,u}(xy) = \mathsf{c}(xy) \int_0^1 F(xy,t)u(t)\, dt$$

defines an even (respectively, odd) flow with $\langle \Phi, \Phi \rangle \leq \langle F, F \rangle \langle u, u \rangle$.
(b) *If Φ is an even flow and $u \in L^2[0,1]$ is even (respectively, if Φ is an odd flow and $u \in L^2[0,1]$ is odd), then*

$$F(xy,t) = \Phi(xy)u(t)/\mathsf{c}(xy)$$

defines an even (respectively, odd) function in $\mathcal{M}^\perp$ with $\langle F, F \rangle = \langle u, u \rangle \langle \Phi, \Phi \rangle$.

The simple proof is left to the reader. Regarding (a), note that when one of $F \in \mathcal{M}^\perp$ and $u \in L^2[0,1]$ is even and the other is odd, then $\Phi_{F,u} \equiv 0$. Thus, we may restrict to even u when F is even and to odd u when F is odd. We set $d^e = \dim \mathcal{J}^e$ and $d^o = \dim \mathcal{J}^o$ ($\leq \infty$). In view of Lemma 3.5, the following is now the consequence of basic Fourier analysis.

PROPOSITION 3.6. *Let $\{\Phi_m^e : 0 \leq m < d^e\}$ and $\{\Phi_m^o : 0 \leq m < d^o\}$ be orthonormal bases of the spaces $\mathcal{J}^e$ and $\mathcal{J}^o$, respectively. Then an orthonormal basis of the subspace $\mathcal{M}^\perp$ of $L^2(X^1, \mathsf{m}^1)$ defined by (2.7) is given by the set of all functions*

$$G_{m,0}^e(xy,t) = \frac{\Phi_m^e(xy)}{\mathsf{c}(xy)}, \quad G_{m,n}^e(xy,t) = \frac{\sqrt{2}\,\Phi_m^e(xy)\cos(2\pi nt)}{\mathsf{c}(xy)}, \quad and$$

$$G_{m,n}^o(xy,t) = \frac{\sqrt{2}\,\Phi_m^o(xy)\sin(2\pi nt)}{\mathsf{c}(xy)},$$

where $n \in \mathbb{N} = \{1, 2, \ldots\}$, and $0 \leq m < d^e$ or $0 \leq m < d^o$, respectively.

A *cycle* in X is a sequence $c = [x_0, \ldots, x_n]$ ($n \geq 3$) of vertices such that $x_0, \ldots, x_{n-1}$ are distinct, $x_i x_{i+1} \in E(X)$ for $i = 0, \ldots, n-1$, and $x_n = x_0$. Associated with c, there is the natural flow $\Phi_c \in \mathcal{J}^o$ defined by

(3.7)
$$\Phi_c(x_i x_{i+1}) = 1, \quad \Phi_c(x_{i+1} x_i) = -1 \quad (i = 0, \ldots, n-1),$$
$$\Phi_c(xy) = 0, \quad \text{if } xy \text{ is not an edge on } c.$$

We remark that our cycle $c = [x_0, x_1 \ldots, x_n]$ has an orientation, and that $\Phi_{c^*} = -\Phi_c$ when $c^* = [x_n, x_{n-1} \ldots, x_0]$.

We now want to characterise those networks for which $\mathcal{M}^\perp = \{0\}$. For this purpose we recall the following.

DEFINITION 3.8.   The network (X, c) is called *transient* if $\sum_{n \geq 0} \langle P^n \delta_x, \delta_y \rangle$ $< \infty$ for some (equivalently, for all) $x, y \in X$. Otherwise, the network is called *recurrent*.

For the significance of this probabilistic notion see, e.g., [19] or [17].

PROPOSITION 3.9.   *One has $\mathcal{M}^\perp = \{0\}$ if and only if* (i) X *is a tree and* (ii) *after removal of any edge, at least one of the two connected components is recurrent as a subnetwork.*

Proof. Suppose that X has a cycle c. Then by Lemma 3.5(b) we can use the odd flow $\Phi = \Phi_c$ to construct a non-zero function in $\mathcal{M}^\perp$.

Thus, X has to be a tree if $\mathcal{M} = L^2(X^1, \mathsf{m}^1)$. Now suppose that X is a tree.

It follows from the *flow criterion* for transience of networks that on the tree X there is a non-zero odd flow with finite power if and only if there is an edge that disconnects X into two transient subtrees; see, e.g., [17], Theorems 3.33 and 4.20. Thus, what is left is to show that on a tree, $\mathcal{M}^\perp \neq \{0\}$ if and only if there is a non-zero $\Phi \in \mathcal{J}^o$.

Suppose that Φ is such a flow. Then Lemma 3.5(b) shows how one can construct a non-zero, odd function in $\mathcal{M}^\perp$. Conversely, suppose that $F \in \mathcal{M}^\perp$ is non-zero. If the odd (respectively, even) part F^o (respectively, F^e) of F is non-zero then there must be an odd (respectively, even) function $u \in L^2[0,1]$ such that the odd flow $\Phi_{F^o, u}$ (respectively, even flow $\Phi_{F^e, u}$) defined in Lemma 3.5(a) is non-zero. By Remark 3.3, when $F^e \neq 0$, the even flow $\Phi_{F^e, u}$ can be transformed into a non-zero odd flow, since every tree is bipartite. $\square$

Our final goal in this section is to describe how one finds (orthonormal) bases of $\mathcal{J}^e$ and $\mathcal{J}^o$, when X is a *finite* graph, in which case the flow spaces do not depend on the specific conductances assigned to the edges. A *spanning tree* of the graph X is a subtree Y of X which contains all vertices of X. It defines a subnetwork (Y, c_Y) whose conductance function c_Y is the restriction of c to Y. Recall that $E(X)$ consists of ordered pairs of adjacent vertices, that is, we have associated with each unoriented edge $[x, y]$ two oppositely oriented edges xy and yx. It will be convenient to choose for each unoriented edge of X one of its endpoints as the *initial* and the other as the *terminal* point. We write $\vec{E}(X)$ for the resulting set of oriented edges, so that $E(X)$ is the disjoint union of the sets of ordered pairs $\{xy : xy \in \vec{E}(X)\}$ and $\{xy : yx \in \vec{E}(X)\}$. Also, we set $\vec{E}(Y) = \vec{E}(X) \cap (Y \times Y)$.

Consider an edge $xy \in \vec{E}(X) \setminus \vec{E}(Y)$. Adding this edge to the tree, the new graph has precisely one cycle $c_{xy} = [x_0, \ldots, x_k]$ $(k \geq 3)$ which is oriented such that $x = x_i$ and $y = x_{i+1}$ for some i. We define

(3.10) $\mathsf{Cyc}(X : Y) = \{c_{xy} : xy \in \vec{E}(X) \setminus \vec{E}(Y)\} \,.$

The following is well known.

LEMMA 3.11. *Let Y be a spanning tree of the finite graph X. Then the set of flows $\{\Phi_c : c \in \mathsf{Cyc}(X : Y)\}$ is a basis of $\mathcal{J}^o$. Every odd flow Φ in X has the unique decomposition*

$$\Phi = \sum_{xy \in \vec{E}(X) \setminus \vec{E}(Y)} \Phi(xy) \cdot \Phi_{c_{xy}} \, .$$

Proof. The function

$$\Psi = \Phi - \sum_{xy \in \vec{E}(X) \setminus \vec{E}(Y)} \Phi(xy) \cdot \Phi_{c_{xy}}$$

vanishes on all edges in $E(X) \setminus E(Y)$. Thus, Ψ defines an odd flow in the finite tree Y, whence $\Psi \equiv 0$. Linear independence of the Φ_c, $c \in \mathsf{Cyc}(X : Y)$, is immediate. $\qquad\square$

If X is finite and bipartite, then all cycles are even (have even length), and (3.4) implies that the set of even flows

$$\{\widetilde{\Phi}_c : c \in \mathsf{Cyc}(X : Y)\}$$

is a basis of $\mathcal{J}^e$. In general, the situation is slightly more complicated. We decompose

(3.12) $\qquad\qquad \mathsf{Cyc}(X : Y) = \mathsf{Cyc}^e(X : Y) \cup \mathsf{Cyc}^o(X : Y) \, ,$

where $\mathsf{Cyc}^e(X : Y)$ consists of all even and $\mathsf{Cyc}^o(X : Y)$ consists of all odd cycles in $\mathsf{Cyc}(X : Y)$. If $\mathsf{Cyc}^o(X : Y) \neq \emptyset$, then we can choose an (oriented) edge $x_0 y_0 \in \vec{E}(X) \setminus \vec{E}(Y)$ such that $c_0 = c_{x_0 y_0}$ is an odd cycle. Now let $xy \in \vec{E}(X) \setminus \vec{E}(Y)$ be any other edge such that $c = c_{xy}$ is odd. We can define an associated even flow $\widetilde{\Phi}_{c_0,c}$ with $\widetilde{\Phi}_{c_0,c}(xy) = 1$ by distinguishing the following two cases: (i) if c and c_0 intersect, we can define $\widetilde{\Phi}_{c_0,c}$ by suitably alternating the values 1 and -1 on all edges of $c_0 \cup c$ whose endpoints do not lie on both c and c_0; (ii) If c and c_0 do not intersect, we can define $\widetilde{\Phi}_{c_0,c}$ by suitably alternating the values 1 and -1 on all edges of $c_0 \cup c$, and by suitably alternating the values 2 and -2 on all the edges of the unique path in the tree Y that connects c with c_0. The simple details are best understood by drawing a few figures.

LEMMA 3.13. *Let Y be a spanning tree of the finite graph X. Then the set of flows*

$$\{\widetilde{\Phi}_c : c \in \mathsf{Cyc}^e(X : Y)\} \cup \{\widetilde{\Phi}_{c_0,c} : c \in \mathsf{Cyc}^o(X : Y) \, , \, c \neq c_0\}$$

(where $c_0 \in \mathsf{Cyc}^o(X : Y)$, if the latter set is non-empty) is a basis of $\mathcal{J}^e$. Every even flow Φ in X has the unique decomposition

$$\Phi = \sum_{\substack{xy \in \vec{E}(X) \setminus \vec{E}(Y) \\ c_{xy} \ even}} \Phi(xy) \cdot \widetilde{\Phi}_{c_{xy}} + \sum_{\substack{xy \in \vec{E}(X) \setminus \vec{E}(Y) \\ c_{xy} \ odd, \ xy \neq x_0 y_0}} \Phi(xy) \cdot \widetilde{\Phi}_{c_0, c_{xy}}.$$

Proof. If Ψ is the difference between Φ and the sum on the right hand side, then Ψ is an even flow on the graph obtained from the tree Y by adding the edge $x_0 y_0$ (if X has odd cycles) or just on the tree Y itself (if X is bipartite). If xy is an edge of that graph such that y is the only neighbour of x, (3.2) implies that $\Psi(xy) = 0$, so that Ψ also is an even flow on the graph that remains after deleting x and the edge xy. Thus, after repeatedly "chopping off" finitely many edges where $\Psi = 0$, we are left with an even flow on the odd cycle $c_{x_0 y_0}$, which must vanish on each edge. Thus, $\Psi \equiv 0$. Once more, linear independence of the proposed basis is immediate. $\qquad\square$

If X is finite, the last two lemmas provide a simple algorithm for finding bases of the (finite dimensional) spaces $\mathcal{J}^o$ and $\mathcal{J}^e$, which can be orthonormalized by the Gram-Schmidt method. Then Proposition 3.6 leads to an orthonormal basis of the space $\mathcal{M}^\perp \subset \ker A$.

4. The spectral measure, and proof of the main results

Recall the Spectral Theorem for a normal operator T on a Hilbert space $\mathcal{H}$ (see [13, Chapter 18] or [15, §§ 4.4, 4.5], for example). Let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of bounded linear operators on $\mathcal{H}$. Let $C^*(T)$ denote the closure in $\mathcal{B}(\mathcal{H})$ of the space of polynomials in T and T^*. Then there is an isometric $*$-isomorphism $\Phi : f \mapsto f(T)$ from the C^*-algebra $\mathcal{C}(\mathsf{spec}(T))$ of continuous functions on $\mathsf{spec}(T)$ onto $C^*(T)$. This isomorphism maps the function $f(\lambda) \equiv \lambda^n$ to T^n for $n = 0, 1, \dots$. Now let $\mathcal{F}_\infty(K)$ denote the C^*-algebra of bounded Borel measurable functions on the compact set $K \subset \mathbb{C}$. Then there is a $*$-homomorphism $\Psi : \mathcal{F}_\infty(\mathsf{spec}(T)) \to \mathcal{B}(\mathcal{H})$, also written $f \mapsto f(T)$, which extends Φ, and which is continuous in the following sense: if (f_n) is a uniformly bounded sequence of measurable functions on $\mathsf{spec}(T)$ converging pointwise to a function f on $\mathsf{spec}(T)$, then $\langle f_n(T)x, y \rangle \to \langle f(T)x, y \rangle$ for each $x, y \in \mathcal{H}$.

For each Borel set $B \subset \mathsf{spec}(T)$, denote by $E(B)$ the operator $\Psi(\mathbf{1}_B)$, where $\mathbf{1}_B$ is the indicator function of B. Each $E(B)$ is a self-adjoint projection, and the map $B \mapsto E(B)$ is called the *spectral measure* of T. For each $x, y \in \mathcal{H}$, the map $\mu_{x,y} : B \mapsto \langle E(B)x, y \rangle$ is a regular Borel complex measure, and

$$\langle f(T)x, y \rangle = \int_{\mathsf{spec}(T)} f(\lambda) \, d\mu_{x,y}(\lambda)$$

for each $f \in \mathcal{F}_\infty\,(\mathsf{spec}(T))$. It is convenient to also write $\int_{\mathsf{spec}(T)} f(\lambda)\, d\langle E_\lambda x, y \rangle$ for the integral on the right, interpreting the latter as a Lebesgue-Stieltjes integral with respect to the function $\lambda \mapsto \langle E_\lambda x, y \rangle = \langle E\,((-\infty, \lambda])\, x, y \rangle$.

Now suppose that T is self-adjoint, so that $\mathsf{spec}(T) \subset \mathbb{R}$. If f is a bounded Borel measurable function defined on a Borel set of $\mathbb{R}$ containing $\mathsf{spec}(T)$, then $f(T)$ is by definition $f_{|\,\mathsf{spec}(T)}(T)$. For example, in Lemma 4.8 below, we apply the above spectral theory to the operators A and $J_\lambda = JS + \lambda\,J$, which are self-adjoint, and of norm at most 1, so that their spectra are contained in $[-1, 1]$. So if $f \in \mathcal{F}_\infty([-1, 1])$ we can form the operators $f(A)$ and $f(J_\lambda)$, acting on $L^2(X^1, \mathsf{m}^1)$ and L_λ^2, respectively, whenever $|\lambda| < 1$.

Similarly, it is convenient to define $E(B) = E\,(B \cap \mathsf{spec}(T))$ for any Borel subset B of $\mathbb{R}$. With this notation, if $\lambda \in \mathbb{R}$, then $\lambda \in \mathsf{spec}(T)$ if and only if the operator $E\,((\lambda - \epsilon, \lambda + \epsilon))$ is non-zero for each $\epsilon > 0$. Equivalently, $\lambda \notin \mathsf{spec}(T)$ if and only if $f(A) = 0$, the zero operator, for every continuous function f supported in $(\lambda - \epsilon, \lambda + \epsilon)$. Also, λ is an eigenvalue of T if and only if $E(\{\lambda\}) \neq 0$ (see [13, Lemma 18.5(3) and Proposition 18.14] or [15, Proposition 4.5.10]).

In the sequel, E will always be the spectral measure of the operator P.

The following Perron Frobenius type proposition concerning P can be found in the literature in a few places, mostly under the assumption that $\mathsf{m}^0(\cdot)$ is bounded away from 0 on X^0 (in which case it becomes easier). Since it appears not to be as well known as it should be, we include its proof, whose first part is extrapolated from Kersting [12, Lemma 3.1].

PROPOSITION 4.1.   *The operator P has eigenfunctions in $\ell^2(X^0, \mathsf{m}^0)$ for the eigenvalue 1 if and only if $\mathsf{m}^0(X^0) < \infty$. In this case, the 1-eigenspace $\mathcal{H}_1$ consists of the constant functions on X^0.*

Furthermore, it has eigenfunctions in $\ell^2(X^0, \mathsf{m}^0)$ for the eigenvalue -1 if and only if $\mathsf{m}^0(X^0) < \infty$ and the graph X is bipartite. In this case, the -1-eigenspace $\mathcal{H}_{-1}$ is spanned by the single function $\mathbf{1}_{C_1} - \mathbf{1}_{C_2}$, where C_1 and C_2 are the two bipartite classes.

Proof. If $\mathsf{m}^0(X^0) < \infty$, then the constant functions are in $\ell^2(X^0, \mathsf{m}^0)$ and are eigenfunctions of P for the eigenvalue 1. If $\mathsf{m}^0(X^0) < \infty$ and X is bipartite, then $\mathbf{1}_{C_1} - \mathbf{1}_{C_2}$ is an eigenfunction of P for the eigenvalue -1.

Conversely, suppose that $g \in \ell^2(X^0, \mathsf{m}^0)$ is nonzero, and $Pg = \lambda_0\, g$, where $\lambda_0 \in \{-1, 1\}$. We first show that (X, c) must be recurrent (see Definition 3.8). For $\lambda \in \mathbb{R}$, the operator $E(\{\lambda\})$ is the orthogonal projection of $\ell^2(X^0, \mathsf{m}^0)$ onto the λ-eigenspace of P (non-trivial if and only if $\lambda \in \mathsf{spec}_p(P)$). Then for each fixed $g_1, g_2 \in \ell^2(X^0, \mathsf{m}^0)$,

$$\langle P^n g_1, g_2 \rangle = \int_{[-1,1]} \lambda^n\, d\langle E_\lambda g_1, g_2 \rangle$$
$$= \langle E(\{1\}) g_1, g_2 \rangle + (-1)^n \langle E(\{-1\}) g_1, g_2 \rangle + o(1) \quad \text{as } n \to \infty$$

by the Bounded Convergence Theorem. By hypothesis, $E(\{\lambda_0\})g = g \neq 0$, and there must be an $x \in X$ so that $E(\{\lambda_0\})\delta_x \neq 0$. So

$$\langle P^{2n}\delta_x, \delta_x \rangle = \langle E(\{1\})\delta_x, \delta_x \rangle + \langle E(\{-1\})\delta_x, \delta_x \rangle + o(1)$$
$$= \|E(\{1\})\delta_x\|^2 + \|E(\{-1\})\delta_x\|^2 + o(1)$$

tends to a nonzero limit as $n \to \infty$. Hence $\sum_{k=0}^{\infty}\langle P^k\delta_x, \delta_x \rangle = \infty$, and (X, c) is recurrent.

Now $Pg = \lambda_0\, g$, and so $|g| = |Pg| \leq P|g|$. Let $f = P|g| - |g|$. Since $\|\,|g|\,\|_2 = \|g\|_2$,

$$\left\|\sum_{k=0}^{n-1} P^k f\right\|_2 = \|P^n|g| - |g|\,\|_2 \leq 2\|g\|_2\,,$$

whence we have for each $x \in X^0$

$$\sum_{k=0}^{n-1}\langle P^k\delta_x, \delta_x \rangle f(x) \leq \left\langle \sum_{k=0}^{n-1} P^k f, \delta_x \right\rangle \leq 2\sqrt{\mathsf{m}^0(x)}\,\|g\|_2$$

for each $x \in X^0$ and each integer $n \geq 1$. So if $f(x) > 0$ for some $x \in X^0$, then

$$\sum_{k=0}^{\infty}\langle P^k\delta_x, \delta_x \rangle \leq 2\sqrt{\mathsf{m}^0(x)}\,\|g\|_2 \big/ f(x) < \infty\,,$$

contradicting recurrence. So $f = 0$. Therefore $|g|$ is a nonnegative harmonic function, that is, $P|g| = |g|$, and so is constant by recurrence, see, e.g., [19, Theorem 1.16]. Since the constant is nonzero, and $g \in \ell^2(X^0, \mathsf{m}^0)$, we have $\mathsf{m}^0(X^0) < \infty$. Now fix $x_0 \in X^0$. Multiplying g by a scalar, we may assume that $g(x_0) = 1$. Then $|g(y)| = 1$ for all $y \in X$, and

$$\sum_{y:y\sim x_0} \frac{\mathsf{c}(x_0, y)}{\mathsf{m}^0(x_0)}|g(y)| = 1 = g(x_0) = \lambda_0 \sum_{y:y\sim x_0} \frac{\mathsf{c}(x_0, y)}{\mathsf{m}^0(x_0)} g(y)$$

$$= \left|\sum_{y:y\sim x_0} \frac{\mathsf{c}(x_0, y)}{\mathsf{m}^0(x_0)}\lambda_0\, g(y)\right|.$$

Hence equality holds in the triangle inequality, and therefore $\lambda_0\, g(y) = 1$ for each $y \in X^0$ such that $y \sim x_0$. So if $\lambda_0 = 1$, the connectedness of X implies that $g(y) = 1$ for all $y \in X^0$. If $\lambda_0 = -1$, connectedness of X implies that $g(y) = (-1)^{\text{dist}(x_0, y)}$, and that X^0 is bipartite, with C_1 and C_2 the sets of vertices at even and at odd distance from x_0, respectively. $\square$

It follows from (2.2) that if $g_1, g_2 \in \ell^2(X^0, \mathsf{m}^0)$ are in two mutually orthogonal subspaces of $\ell^2(X^0, \mathsf{m}^0)$ which are also P-invariant, then $\langle F_{g_1, u_1}, F_{g_2, u_2}\rangle = 0$ for any $u_1, u_2 \in L^2[0, 1]$. So if $\mathcal{H}'$ denotes the orthogonal complement

in $\ell^2(X^0, \mathsf{m}^0)$ of the sum of the eigenspaces $\mathcal{H}_1$ and $\mathcal{H}_{-1}$ (which are at most 1-dimensional), then the orthogonal decomposition

$$(4.2) \qquad\qquad \ell^2(X^0, \mathsf{m}^0) = \mathcal{H}_1 + \mathcal{H}_{-1} + \mathcal{H}'$$

gives rise to a corresponding orthogonal decomposition of $\mathcal{M}$:

$$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_{-1} + \mathcal{M}',$$

where $\mathcal{M}'$ is the closure of the linear span of functions $F_{g,u}$, where $g \in \mathcal{H}'$ and $u \in L^2[0,1]$, and $\mathcal{M}_{\pm 1}$ are constructed analogously from $\mathcal{H}_{\pm 1}$.

LEMMA 4.3.   *The subspaces $\mathcal{M}_1$, $\mathcal{M}_{-1}$ and $\mathcal{M}'$ are invariant under A. Let $u \in L^2[0,1]$. For $g \in \mathcal{H}_1$, $AF_{g,u} = F_{g,\langle u, \mathbf{1}\rangle \mathbf{1}}$. For $g \in \mathcal{H}_{-1}$, $AF_{g,u} = 0$.*

Proof. The invariance of the subspaces is immediate from Lemma 2.3. If $g \in \mathcal{H}_1$, then write $u = v + w$ as in (2.10). Then $F_{g,w} = 0$ because g is constant. Thus $F_{g,u} = F_{g,v}$, and $AF_{g,u}$ equals

$$AF_{g,v} = F_{g,JSv} + F_{Pg,Jv} = F_{g,JSv} + F_{g,Jv} = F_{g,JSv+Jv} = F_{g,\langle v,\mathbf{1}\rangle \mathbf{1}} = F_{g,\langle u,\mathbf{1}\rangle \mathbf{1}}$$

because $JSv + Jv = 2Jv$, and applying (2.10) to $2Jv$ in place of u, we see that the "v-component" of $2Jv$ is $Jv + SJv = JSv + SJv = \langle v, \mathbf{1}\rangle \mathbf{1}$ by (2.12).

Similarly, if $g \in \mathcal{H}_{-1}$, then write $u = v + w$ as in (2.10). Then $F_{g,v} = 0$ because $g(y) = -g(x)$ for each edge xy. Thus $F_{g,u} = F_{g,w}$, and $AF_{g,u}$ equals

$$AF_{g,w} = F_{g,JSw} + F_{Pg,Jw} = F_{g,JSw} + F_{-g,Jw} = F_{g,JSw-Jw} = F_{g,-2Jw} = 0$$

because applying (2.10) to $2Jw$ in place of u, we see that the "w-component" of $2Jw$ is $Jw - SJw = -(JSw + SJw) = -\langle w, \mathbf{1}\rangle \mathbf{1} = 0$ by (2.12). $\square$

The following is one of our main tools for linking the spectra of P and A. Recall that E denotes the spectral measure of the operator P.

PROPOSITION 4.4.   *Suppose that $g_1, g_2 \in \ell^2(X^0, \mathsf{m}^0)$ and $u_1, u_2 \in L^2[0,1]$. Then for $n = 0, 1, \ldots$,*

$$(4.5) \qquad \langle A^n F_{g_1,u_1}, F_{g_2,u_2}\rangle = \int_{\mathsf{spec}(P)} \langle J_\lambda^n u_1, u_2\rangle_\lambda \, d\langle E_\lambda g_1, g_2\rangle.$$

Proof. The proof is by induction. By the Spectral Theorem for P,

$$(4.6) \qquad \langle P^k g_1, g_2\rangle = \int_{\mathsf{spec}(P)} \lambda^k \, d\langle E_\lambda g_1, g_2\rangle \quad \text{for } k = 0, 1, \ldots.$$

Case $n = 0$ of (4.5) is immediate from (2.2), together with the cases $k = 0, 1$ of (4.6).

Assume (4.5) holds for n. Then $A^{n+1}F_{g_1,u_1} = A^n(F_{g_1,JSu_1} + F_{Pg_1,Ju_1})$ by Lemma 2.3. This and the induction hypothesis yield

$$\langle A^{n+1}F_{g_1,u_1}, F_{g_2,u_2} \rangle = \langle A^n F_{g_1,JSu_1}, F_{g_2,u_2} \rangle + \langle A^n F_{Pg_1,Ju_1}, F_{g_2,u_2} \rangle$$

$$= \int_{\mathsf{spec}(P)} (\langle J_\lambda^n JSu_1, u_2 \rangle_\lambda + \lambda \langle J_\lambda^n Ju_1, u_2 \rangle_\lambda) \ d\langle E_\lambda g_1, g_2 \rangle$$

$$= \int_{\mathsf{spec}(P)} \langle J_\lambda^{n+1} u_1, u_2 \rangle_\lambda \ d\langle E_\lambda g_1, g_2 \rangle$$

because $\int_{\mathsf{spec}(P)} f(\lambda) \, d\langle E_\lambda Pg_1, g_2 \rangle = \int_{\mathsf{spec}(P)} \lambda f(\lambda) \, d\langle E_\lambda g_1, g_2 \rangle$ for any continuous function $f(\lambda)$ on $[a, b]$, as on sees by first taking $f(\lambda) = \lambda^k$, for $k = 0, 1, \ldots$, and using (4.6). This completes the induction step. $\square$

Let $u_1, u_2 \in L^2[0, 1]$, and let $f : [-1, 1] \to \mathbb{C}$ be bounded and Borel measurable. If $-1 < \lambda < 1$ then the function $\tilde{f} = \tilde{f}_{u_1, u_2}$,

$$(4.7) \qquad\qquad \tilde{f}(\lambda) = \langle f(J_\lambda)u_1, u_2 \rangle_\lambda,$$

is well defined, since J_λ is selfadjoint on the Hilbert space L_λ^2 by Lemma 2.15. When $\lambda = \pm 1$ then we cannot speak of $f(J_\lambda)$, because $\langle \cdot, \cdot \rangle_{\pm 1}$ is degenerate and we have no Hilbert space on which J_λ is selfadjoint.

We set $\mathsf{spec}(P)' = \mathsf{spec}(P) \cap (-1, 1)$. Recall the definition of the subspace $\mathcal{H}'$ of $\ell^2(X^0, \mathsf{m}^0)$ in (4.2).

LEMMA 4.8. *If $u_1, u_2 \in L^2[0, 1]$, and $f : [-1, 1] \to \mathbb{C}$ is continuous, then the function $\tilde{f}$ defined on $(-1, 1)$ by (4.7) is continuous, and*

$$(4.9) \qquad\qquad \langle f(A)F_{g_1,u_1}, F_{g_2,u_2} \rangle = \int_{\mathsf{spec}(P)'} \tilde{f}(\lambda) \, d\langle E_\lambda g_1, g_2 \rangle$$

for all $g_1 \in \mathcal{H}'$ and $g_2 \in \ell^2(X^0, \mathsf{m}^0)$. If f is the indicator function $\mathbf{1}_{\{\mu\}}$ of a singleton, then $\tilde{f}$ is bounded and Borel measurable on $(-1, 1)$, and (4.9) holds.

Proof of Lemma 4.8. First of all, notice that when $g_1 \in \mathcal{H}'$, the integral in Proposition 4.4 can be regarded as over $\mathsf{spec}(P)'$. For $\mathcal{H}_1, \mathcal{H}_{-1}$ and $\mathcal{H}'$ are the images of $E(\{1\})$, $E(\{-1\})$ and $E((-1, 1))$, respectively. Hence $E(B)g_1 = E(B)E((-1, 1))\, g_1 = E(B \cap (-1, 1))\, g_1 = 0$ for any $B \subset \{-1, 1\}$.

Next, for arbitrary $h : (-1, 1) \to \mathbb{C}$ which is bounded and Borel measurable, we have

$$(4.10) \qquad |\tilde{h}(\lambda)| \le \|h(J_\lambda)\|_\lambda \, \|u_1\|_\lambda \, \|u_2\|_\lambda \le 2\|h\|_\infty \, \|u_1\|_2 \, \|u_2\|_2$$

for all $\lambda \in (-1, 1)$, where $\|h(J_\lambda)\|_\lambda$ is the operator norm on L_λ^2.

Now, given f (continuous), choose a sequence (p_n) of polynomials such that $p_n \to f$ uniformly on $[-1, 1]$. Clearly, $\tilde{p}_n(\lambda)$ is a polynomial, and replacing h with $f - p_n$ in (4.10), we see that

$$|\tilde{f}(\lambda) - \tilde{p}_n(\lambda)| \le 2\|f - p_n\|_\infty \, \|u_1\|_2 \, \|u_2\|_2$$

for all $\lambda \in (-1, 1)$. Consequently, $(\tilde{p}_n)$ converges uniformly to $\tilde{f}$, and so the latter is a continuous function on $(-1, 1)$. By (4.5), we have

$$\langle p_n(A) F_{g_1, u_1}, F_{g_2, u_2} \rangle = \int_{\text{spec}(P)'} \tilde{p}_n(\lambda) \, d\langle E_\lambda g_1, g_2 \rangle \,,$$

and letting $n \to \infty$, we see that this also holds for f and $\tilde{f}$ in the place of p_n and $\tilde{p}_n$, respectively.

Now let f be the indicator function $\mathbf{1}_{\{\mu\}}$ of a singleton. Then there is clearly a uniformly bounded sequence (f_n) of continuous functions converging pointwise to f on $[-1, 1]$. Then for each fixed $\lambda \in (-1, 1)$,

$$\tilde{f}_n(\lambda) = \langle f_n(J_\lambda) u_1, u_2 \rangle_\lambda \to \langle f(J_\lambda) u_1, u_2 \rangle_\lambda = \tilde{f}(\lambda)$$

by the Spectral Theorem applied to J_λ on L_λ^2. Hence $\tilde{f}$ is Borel measurable. Moreover, estimating as in (4.10), $|\tilde{f}_n(\lambda)| \leq 2\|f_n\|_\infty \|u_1\|_2 \|u_2\|_2$, and so the functions $\tilde{f}_n$ are uniformly bounded on $(-1, 1)$. Hence (4.9) holds for f by the Bounded Convergence Theorem, and by the Spectral Theorem applied to A, because it holds for each f_n. $\qquad\square$

REMARK 4.11. The assertions in the lemma can be extended in two directions, neither of which we need in the sequel:

(a) In Lemma 4.8, if f is continuous, then a little more work in the proof shows that $\tilde{f}$ has a continuous extension to $[-1, 1]$, and (4.9) is in fact valid for any $g_1 \in \ell^2(X^0, \mathsf{m}^0)$, with $\text{spec}(P)'$ replaced by $\text{spec}(P)$.

(b) Using [13, Lemma 18.1], it is easy to see that the last statement in Lemma 4.8 is valid for any bounded measurable function f.

Proof of Theorem 1.3. We must show that $\text{spec}(A)$ equals $\mathcal{S}$, where

$$\mathcal{S} = \{0\} \cup \{\mu_{\lambda, n} : \lambda \in \text{spec}(P)', \ n \in \mathbb{Z}\} \cup \{1 : 1 \in \text{spec}(P)\} \,.$$

First note that the set $\mathcal{S}$ is closed. For suppose that $\mu \in \mathbb{R}$ is the limit of a sequence of points $\mu_j = \mu_{\lambda_j, n_j}$ in $\mathcal{S} \setminus \{0, 1\}$. If $|n_j| \to \infty$, then $\mu_j \to 0$, and so $\mu = 0 \in \mathcal{S}$. So taking a subsequence, we may assume that there is an $n \in \mathbb{Z}$ so that $n_j = n$ for all j. Since $\text{spec}(P)$ is compact, taking a further subsequence, we may suppose that $\lambda_j \to \lambda \in \text{spec}(P)$. If $\lambda \neq \pm 1$, then $\mu = \lim_{j \to \infty} \mu_j = \mu_{\lambda, n} \in \mathcal{S}$. If $\lambda \in \{-1, 1\}$, then $\mu_j \to 0$ unless $\lambda = 1$ and $n = 0$, in which case $\mu_j \to 1 \in \mathcal{S}$.

To show that $\text{spec}(A) \subset \mathcal{S}$, suppose that $\mu_0 \in [-1, 1] \backslash \mathcal{S}$. For some $\epsilon > 0$, we have $[\mu_0 - \epsilon, \mu_0 + \epsilon] \cap \mathcal{S} = \emptyset$. If $\mu_0 \neq 1$, we may assume that $1 \notin [\mu_0 - \epsilon, \mu_0 + \epsilon]$. Let f be any continuous function on $[-1, 1]$ supported in $[\mu_0 - \epsilon, \mu_0 + \epsilon]$. For any $\lambda \in \text{spec}(P)'$, $f(J_\lambda) = 0$ because $f(\mu) = 0$ for all $\mu \in \text{spec}(J_\lambda)$ ($\subset \mathcal{S}$, by Lemma 2.16). By Lemma 4.8, $\langle f(A) F_{g_1, u_1}, F_{g_2, u_2} \rangle = 0$ for all $g_1 \in \mathcal{H}'$, $g_2 \in \ell^2(X^0, \mathsf{m}^0)$ and $u_1, u_2 \in L^2[0, 1]$. Therefore $f(A) = 0$ on the A-invariant subspace $\mathcal{M}'$ of $L^2(X^1, \mathsf{m}^1)$. By Corollary 2.8 and Lemma 4.3, $A = 0$ on $\mathcal{M}^\perp$ and $\mathcal{M}_{-1}$, and is an orthogonal projection on $\mathcal{M}_1$, and since $f(0) = 0$, we

get $f(A) = 0$ on $\mathcal{M}_{-1} + \mathcal{M}^\perp$. If $\mu_0 \neq 1$, then $f(A) = 0$ on $\mathcal{M}_1$ because then $f(0) = f(1) = 0$, as arranged above. If $\mu_0 = 1$, then $1 \notin \mathsf{spec}(P)$ and $\mathcal{M}_1 = \{0\}$. So in both cases, $f(A) = 0$ on the whole of $L^2(X^1, \mathsf{m}^1)$ for every continuous f supported in $[\mu_0 - \epsilon, \mu_0 + \epsilon]$. Therefore $\mu_0 \notin \mathsf{spec}(A)$, and $\mathsf{spec}(A) \subset \mathcal{S}$.

For the reverse inclusion, suppose that $\mu_0 \in \mathcal{S}$.

Consider first $\mu_0 \neq 0, 1$. Then $\mu_0 = \mu_{\lambda_0, n_0}$ for some $\lambda_0 \in \mathsf{spec}(P)'$ and $n_0 \in \mathbb{Z}$. Suppose that $0 < \epsilon < 1 - |\lambda_0|$. By [15, Proposition 4.4.5], there is a $g \in \ell^2(X^0, \mathsf{m}^0)$ so that $\|Pg - \lambda_0 g\| < \epsilon \|g\|$. Let $u = u_{\lambda_0, n_0}$. Then

$$AF_{g,u} - \mu_0 F_{g,u} = F_{g,JSu} + F_{Pg,Ju} - \mu_0 F_{g,u} = F_{Pg - \lambda_0 g, Ju},$$

and this has norm at most $2\|Pg - \lambda_0 g\| \, \|u\| \leq 2\epsilon \|g\| \, \|u\|$, by (2.2). Again by (2.2),

$$\|F_{g,u}\|^2 \geq (1 - |\lambda_0| - \epsilon)\|g\|^2 \|u\|^2$$

because $|\langle Pg, g \rangle| \leq (|\lambda_0| + \epsilon)\|g\|^2$. Therefore

$$\|AF_{g,u} - \mu_0 F_{g,u}\|^2 \leq \frac{4\epsilon^2}{1 - |\lambda_0| - \epsilon}\|F_{g,u}\|^2,$$

and since $F_{g,u} \neq 0$, it follows that $\mu_0 \in \mathsf{spec}(A)$.

Next, suppose that $\mu_0 = 1 \in \mathcal{S}$. This can only happen when $1 \in \mathsf{spec}(P)$. If 1 is not an isolated point of $\mathsf{spec}(P)$ then there is a sequence (λ_n) in $\mathsf{spec}(P)$ such that $\lambda_n \to 1$ from below. But then we just showed that $\mu_{\lambda_n, 0} \in \mathsf{spec}(A)$, and $\mu_{\lambda_n, 0} \to 1$. Therefore, $1 \in \mathsf{spec}(A)$. If 1 is an isolated point of $\mathsf{spec}(P)$, then it must be an eigenvalue ([13, Proposition 18.14(3)] or [15, Proposition 4.4.5]), and by Proposition 4.1, the constant function $g \equiv 1$ is an associated eigenfunction in $\ell^2(X^0, \mathsf{m}^0)$. But then the constant function $F \equiv 1$ on X^1 is in $L^2(X^1, \mathsf{m}^1)$, whence $1 \in \mathsf{spec}(A)$.

Finally, consider $\mu_0 = 0 \in \mathcal{S}$. If $\mathsf{spec}(P)$ contains some $\lambda \in (-1, 1)$ then $\mathsf{spec}(A) \ni \mu_{\lambda, n} \to 0$ as $|n| \to \infty$, whence $0 \in \mathsf{spec}(A)$. Otherwise, $\mathsf{spec}(P) \subset \{-1, 1\}$, so that by Proposition 4.1, the space $\ell^2(X^0, \mathsf{m}^0) = \mathcal{M}_1 + \mathcal{M}_{-1}$ is at most 2-dimensional, which can happen only when X has exactly two vertices and one edge. But in this case, $\mathsf{spec}(A) = \{0, 1\}$, by Lemma 4.3. $\qquad\square$

Proof of Theorem 1.5. We first show that

$$(4.12) \qquad \mathsf{spec}_p(A) \setminus \{0, 1\} = \left\{ \mu_{\lambda, n} : n \in \mathbb{Z}, \ \lambda \in \mathsf{spec}_p(P) \setminus \{-1, 1\} \right\}.$$

If $\lambda \in \mathsf{spec}_p(P) \setminus \{-1, 1\}$ and $n \in \mathbb{Z}$, let $g \in \ell^2(X^0, \mathsf{m}^0)$ be non-zero, and satisfy $Pg = \lambda g$. Let $u = u_{\lambda, n}$. Then $AF_{g,u} = \mu_{\lambda, n} F_{g,u}$ and $F_{g,u} \neq 0$. So $\mu_{\lambda, n} \in \mathsf{spec}_p(A)$.

Since $\omega = \arccos \lambda \in (0, \pi)$, note that if $n \in \mathbb{Z}$ then,

$$0 < |\mu_{\lambda, n}| = \left| \frac{\sin \omega}{\omega + 2\pi n} \right| \leq \frac{\sin \omega}{\omega} < 1.$$

So $\mu_{\lambda, n} \in \mathsf{spec}_p(A) \setminus \{0, 1\}$.

On the other hand, let $\mu \in \mathsf{spec}_p(A) \setminus \{0,1\}$. Since $\mu \in \mathsf{spec}(A)$, we can write $\mu = \sin\omega/\omega$ for some $\omega \in \mathbb{R}$ with $\lambda = \cos\omega \in \mathsf{spec}(P)$, or equivalently, $\mu = \mu_{\lambda,n}$ for some $n \in \mathbb{Z}$. Since $\mu \neq 0$, a glance at the curve $\omega \mapsto \sin\omega/\omega$ shows that the number of solutions ω to $\sin\omega/\omega = \mu$ is finite. That is, the number of pairs $(\lambda, n) \in \mathsf{spec}(P) \times \mathbb{Z}$ for which $\mu_{\lambda,n} = \mu$ is finite. Thus, the set F_μ of all $\lambda \in \mathsf{spec}(P)$ such that $\mu = \mu_{\lambda,n}$ for some $n \in \mathbb{Z}$ is a finite set. Also, $F_\mu \subset \mathsf{spec}(P)'$, since $\mu \notin \{0,1\}$.

Let $f = \mathbf{1}_{\{\mu\}}$. By assumption, $\mu \in \mathsf{spec}_p(A)$. So $f(A)$, being the orthogonal projection onto the μ-eigenspace of A, is nonzero. But $f(A)F = 0$ for all $F \in \mathcal{M}_1 \cup \mathcal{M}_{-1} \cup \mathcal{M}^\perp$, because $\mu \neq 0, 1$. So $f(A)$ cannot vanish on $\mathcal{M}'$. Thus there exist $g \in \mathcal{H}'$ and $u \in L^2[0,1]$ so that $\langle f(A)F_{g,u}, F_{g,u}\rangle > 0$. Then by (4.9), we must have $\langle f(J_\lambda)u, u\rangle_\lambda > 0$ for some $\lambda \in \mathsf{spec}(P)'$. For any such λ, $f(J_\lambda) \neq 0$, and so $\mu = \mu_{\lambda,n}$ for some n. Therefore $\lambda \in F_\mu$. So the integrand in (4.9) is nonzero only for $\lambda \in F_\mu$, at most, and so the measure $B \mapsto \langle E(B)g, g\rangle$ must assign nonzero measure to F_μ, and therefore to $\{\lambda\}$ for some $\lambda \in F_\mu$. This λ must be in $\mathsf{spec}_p(P)$, and $\mu = \mu_{\lambda,n}$ for some $n \in \mathbb{Z}$. This completes the proof of (4.12).

Now suppose that $1 \in \mathsf{spec}_p(P)$. Then $0,1 \in \mathsf{spec}_p(A)$ because there are eigenfunctions of A in $\mathcal{M}_1$ for both 0 and 1, by Lemma 4.3. If $1 \notin \mathsf{spec}_p(P)$, then $\mathcal{H}' = \ell^2(X^0, \mathsf{m}^0)$ and $\mathsf{spec}(P)' = \mathsf{spec}(P)$. So by Lemma 4.8, applied to $f = \mathbf{1}_{\{0\}}$ and $f = \mathbf{1}_{\{1\}}$, we see that $f(A)F = 0$ for any $F \in \mathcal{M}$, because 0 and 1 are not in the point spectrum of any J_λ, $\lambda \in (-1,1)$. Of course $0 \in \mathsf{spec}_p(A)$ if $\mathcal{M} \subsetneq L^2(X^1, \mathsf{m}^1)$, since A is zero on $\mathcal{M}^\perp$. This, together with Propositions 3.9 and 4.1, proves the last statement of Theorem 1.5. $\qquad\square$

Theorem 1.5 and its proof applies, in particular, to the case when the graph X is finite. In this case, we also obtain an orthonormal basis of eigenfunctions of the operator A by combining Theorem 1.5 with the following. In the next lemma, it is convenient to define L_λ^2 also when $\lambda \in \{-1, 1\}$. We define L_1^2 (respectively L_{-1}^2) to be the set of $u \in L^2[0,1]$ such that $Su = u$ (respectively $Su = -u$). Note that for both these λ's, $\langle u, v\rangle_\lambda = 2\langle u, v\rangle$ for $u, v \in L_\lambda^2$, so that L_λ^2 is a Hilbert space. Recall the definition 2.5 of $\mathcal{M}_0$.

LEMMA 4.13.   *Suppose that X is finite, and let $g_1, \ldots, g_m$ be an orthonormal basis for $\ell^2(X^0, \mathsf{m}^0)$ consisting of eigenfunctions of P, with $Pg_j = \lambda_j\, g_j$ (with $-1 \leq \lambda_j \leq 1$) for each j. Then $\mathcal{M}_0$ is closed, and consists of all functions*

(4.14) $$F = F_{g_1, u_1} + \cdots + F_{g_m, u_m},$$

where $u_j \in L^2[0,1]$ for each j, and where

(4.15) $$Su_j = u_j \;\; \text{if } \lambda_j = 1, \quad \text{and} \quad Su_j = -u_j \;\; \text{if } \lambda_j = -1.$$

Moreover, the map $F \mapsto (u_1, \ldots, u_m)$ is a linear isometry of $\mathcal{M}_0$ onto the orthogonal direct sum of the spaces $L_{\lambda_j}^2$, $j = 1, \ldots, m$.

Proof. Each $g \in \ell^2(X^0, \mathsf{m}^0)$ is a linear combination of the g_j's, and so each $F_{g,u}$ is a sum (4.14) of functions F_{g_j,u_j}. If $\lambda_j = 1$, then g_j is constant on X^0, and so for each edge xy, and any $u \in L^2[0,1]$,

$$F_{g_j,u}(xy,t) = u(1-t)g_j(x) + u(t)g_j(y) = (u(1-t) + u(t))g_j(x).$$

Therefore, if $Su = -u$, we have $F_{g_j,u} \equiv 0$. Applying (2.10) to $u = u_j$, we see that $F_{g_j,u_j} = F_{g_j,v} + F_{g_j,w} = F_{g_j,v}$. Similarly, if $\lambda_j = -1$, then $F_{g_j,u_j} = F_{g_j,w}$, since $g_j(y) = -g_j(x)$ for any edge xy. Therefore each $F_{g,u}$ can be written in the form (4.14), where (4.15) holds.

If F is as in (4.14), then $\langle F_{g_j,u_j}, F_{g_k,u_k} \rangle = 0$ if $j \neq k$, and so

$$\left\| \sum_{j=1}^m F_{g_j,u_j} \right\|^2 = \sum_{j=1}^m \left(\langle g_j, g_j \rangle \langle u_j, u_j \rangle + \lambda_j \langle g_j, g_j \rangle \langle u_j, Su_j \rangle \right) = \sum_{j=1}^m \langle u_j, u_j \rangle_{\lambda_j}.$$

It follows that $\mathcal{M}_0$ is isometric to the direct sum of the Hilbert spaces $L^2_{\lambda_j}$, recalling the special definition of L^2_λ made above when $\lambda = \pm 1$. Therefore $\mathcal{M}_0$ is complete for its inner product, and so closed in $L^2(X^1, \mathsf{m}^1)$. $\square$

COROLLARY 4.16.   *Suppose that X is finite, and let $g_1, \ldots, g_N$ be an orthonormal basis for $\ell^2(X^0, \mathsf{m}^0)$ consisting of eigenfunctions of P, with $Pg_j = \lambda_j g_j$ (where $-1 \leq \lambda_j \leq 1$) for each j. We assume that $\lambda_1 = 1$ and g_1 is constant, and when X is bipartite, that $\lambda_N = -1$ and g_N is constant on each bipartite class. Then*

$$\mathsf{spec}(A) = \mathsf{spec}_p(A) = \{0,1\} \cup \{\mu_{\lambda_j,n} : |\lambda_j| < 1, \ n \in \mathbb{Z}\},$$

with $\mu_{\lambda_j,n}$ as in (2.17).

(i) *An orthonormal basis of the subspace $\mathcal{M}$ of $L^2(X^1, \mathsf{m}^1)$ consisting of eigenvectors of A is obtained as follows. (Note that $\mathsf{m}^1(X^1) = \mathsf{m}^0(X^0)/2$.)*

   (a) *For the eigenvalue $\mu = 1$, the eigenspace is spanned by the function*
   $$F^{(1,0)}(xy,t) = \frac{1}{\sqrt{\mathsf{m}^1(X^1)}}.$$

   (b) *For the eigenvalue $\mu = 0$, the eigenspace is spanned by the functions*
   $$F^{(1,n)}(xy,t) = \frac{\sqrt{2}}{\sqrt{\mathsf{m}^1(X^1)}} \cos(2\pi nt), \quad n \in \mathbb{N},$$
   if X is not bipartite. If X is bipartite, then the eigenspace is spanned by the $F_{1,n}$ and the functions
   $$F^{(-1,n)}(xy,t) = (-1)^{i(x)} \frac{\sqrt{2}}{\sqrt{\mathsf{m}^1(X^1)}} \sin(2\pi nt), \quad n \in \mathbb{N},$$

> *where $i(x) = 1$ or 2 according to whether x lies in the bipartite class C_1 or C_2.*
>
> (c) *For the eigenvalue μ with $0 < |\mu| < 1$, the eigenspace is spanned by the functions*

$$F^{(j,n)}(xy, t) = \frac{\sqrt{2}}{\sin \omega_j} \left(g_j(x) \sin((\omega_j + 2\pi n)(1-t)) + g_j(y) \sin((\omega_j + 2\pi n)t) \right),$$

> *where $(j, n) \in \{1, \dots, N\} \times \mathbb{Z}$ is such that $\mu_{\lambda_j,n} = \mu$ and $\omega_j = \arccos \lambda_j$.*
>
> (ii) *An orthonormal basis of the subspace $\mathcal{M}^\perp \subset \ker A$ is obtained via Proposition 3.6 in combination with Lemmas 3.11 and 3.13.*

5. Final remarks and examples

REMARK 5.1.   One may ask why we call our operator A, defined in (1.2), *the* averaging operator, and not the one which takes the pure m^1-average over balls of radius 1. The latter is given by

$$\widetilde{A}F(xy, t) = \frac{1}{(1-t)\mathsf{m}^0(x) + t\,\mathsf{m}^0(y)}$$
$$\times \left(\sum_{u \sim x} c(xu) \int_0^{1-t} F(xu, s)\, ds + \sum_{v \sim y} c(yv) \int_0^t F(yv, s)\, ds \right).$$

The point is that unlike A, the latter operator does not enjoy a nice and natural compatibility with the transition operator P and the Laplace operator on a network. Note, however, that $A = \widetilde{A}$ when $\mathsf{m}^0(\cdot)$ is constant. This occurs, in particular, when the graph X is locally finite and regular, and $\mathsf{c}(xy) = 1$ for each edge xy, in which case P is called the *simple random walk* (SRW) operator.

We now give a few examples of locally finite, regular graphs with conductances $\mathsf{c}(xy) = 1$ for each edge $[x, y]$, where the spectrum of A can be computed via the (known) spectrum of P.

EXAMPLE 5.2.   Equip the additive group $\mathbb{Z}$ of all integers with the typical graph structure, where the edges are between x and $x + 1$, $x \in \mathbb{Z}$. Then the SRW operator P, associated with conductances $\equiv 1$, is the convolution operator

$$Pf = \varphi * f \quad \text{with} \quad \varphi = \frac{1}{2} \left(\mathbf{1}_{\{1\}} + \mathbf{1}_{\{-1\}} \right).$$

Computing the Fourier transform $\widehat{\varphi}(\omega) = \cos \omega$, $\omega \in [0, 2\pi]$, one finds the very well known fact that

$$\mathsf{spec}(P) = [-1, 1] \quad \text{and} \quad \mathsf{spec}_p(P) = \emptyset.$$

On the other hand, the one-skeleton is the real line $\mathbb{R}$ (with the integer points singled out as vertices), so that A is the convolution operator

$$AF = \varphi * F \quad \text{with} \quad \varphi = \frac{1}{2}\mathbf{1}_{[-1,1]}.$$

Since the Fourier transform of φ is $\widehat{\varphi}(\omega) = (\sin\omega)/\omega$, $\omega \in \mathbb{R}$, one finds that

$$\mathsf{spec}(A) = [(\sin\omega^*)/\omega^*, 1] \quad \text{and} \quad \mathsf{spec}_p(A) = \emptyset,$$

where ω^* is the smallest positive solution of the equation $\tan\omega = \omega$, so that $(\sin\omega^*)/\omega^* = \cos\omega^*$. Numerical computation gives $\omega^* = 4.493409\ldots$ and $\sin(\omega^*)/\omega^* = -0.217233\ldots$.

EXAMPLE 5.3. Equip the additive group $\mathbb{Z}_N = \mathbb{Z}/(N\mathbb{Z}) = \{0,\ldots,N-1\}$ of integers modulo N with the structure of a cycle, where $x \sim x+1 \pmod{N}$ for $x \in \mathbb{Z}_N$. If each edge is assigned conductance 1, then m^0 becomes twice the counting measure. An orthonormal basis of eigenvectors of the SRW operator P acting on $\ell^2(\mathbb{Z}_N, \mathsf{m}^0)$ with associated eigenvalues is given by

$$g_j(x) = (2N)^{-1/2}\exp(2\pi i\, x\, j/N) \text{ and } \lambda_j = \cos(2\pi j/N), \quad j = 0,\ldots,N-1,$$

so that $\lambda_j = \lambda_{N-j}$ $(1 \le j \le N-1)$ has multiplicity 2 unless $j = N/2$ for even N.

The one-skeleton of $\mathbb{Z}_N$ can be identified with the torus, interpreted as the compact additive group $\mathbb{R}/(N\mathbb{R})$ of real numbers modulo N, parametrized by the interval $[0, N)$, or equivalently, $(-N/2, N/2]$, with Lebesgue measure. Thus, as in Example 5.2, the averaging operator A becomes the convolution operator $AF = \varphi * F$ with $\varphi = \frac{1}{2}\mathbf{1}_{[-1,1]}$, but this time modulo N. The Fourier transform (on $\mathbb{Z}$) of φ is $\widehat{\varphi}(n) = (\sin\varpi_n)/\varpi_n$, and

$$\mathsf{spec}(A) = \{(\sin\varpi_n)/\varpi_n : n \in \mathbb{Z}\}, \quad \text{where} \quad \varpi_n = 2\pi n/N.$$

(We set $(\sin 0)/0 = 1$ by continuous extension.) The orthonormal basis of associated eigenfunctions is given by

$$F_n(t) = N^{-1/2}\exp(i\,\varpi_n\, t), \quad t \in [0, N), \quad n \in \mathbb{Z}.$$

Since the eigenvalues corresponding to ϖ_n and ϖ_{-n} coincide, the eigenspace has dimension 2, unless n is a multiple of N, or—when N is even—a multiple of $N/2$.

Relating this with Corollary 4.16, we get the following: for $1 \le j < N/2$, $\mu_{\lambda_j,n} = \mu_{\lambda_{N-j},n} = (\sin\varpi_{j+nN})/\varpi_{j+nN}$, and elementary computations yield for the functions $F^{(j,n)}$ and $F^{(N-j,n)}$ of Corollary 4.16(c) that $F^{(j,n)}(xy,t) = F_{j+nN}(x+t)$ and $F^{(N-j,n)}(xy,t) = F_{-j-nN}(x+t)$, if $x \in \{0,\ldots,N-1\}$, $y = x+1 \pmod{N}$, and $t \in [0,1]$.

The eigenspace associated with the eigenvector 1 consists of course of the constant functions.

Finally, $\ker A$ is spanned by the functions $F_{nN}(t)$, $n \in \mathbb{Z} \setminus \{0\}$, if N is odd, and by the functions $F_{nN/2}(t)$, $n \in \mathbb{Z} \setminus \{0\}$, if N is even. We have

to consider the decomposition $\ker A = (\mathcal{M} \cap \ker A) \oplus \mathcal{M}^\perp$; compare with Corollary 4.16. If N is odd, then $\mathcal{M} \cap \ker A$ is spanned by the functions $F^{(1,n)}(xy,t) = \sqrt{2}\,\Re\left(F_{nN}(x+t)\right)$, and—since $\mathcal{J}^e = \{0\}$ and $\dim \mathcal{J}^o = 1$— the space $\mathcal{M}^\perp$ is spanned by the functions $G^o_{0,n}(xy,t) = \sqrt{2}\,\Im\left(F_{nN}(x+t)\right)$, $n \in \mathbb{N}$, where $\Re(\cdot)$ and $\Im(\cdot)$ denote real and imaginary part.

When N is even, the situation is slightly more complicated. In this case, $\mathcal{J}^e$ is one-dimensional and spanned by the even flow with value $(-1)^x$ on the edge $[x,y]$ with $y = x+1$, $x = 0, \ldots, N-1$. The functions $F^{(-1,n)}$, $n \in \mathbb{N}$, of Corollary 4.16(b) are non-differentiable at the vertices, and the functions $G^e_{0,n}$, $n \in \mathbb{N}_0$, of Proposition 3.6 are even discontinuous, whence they have to be expressed as Fourier series in terms of the functions $F_{nN/2}$, $n \in \mathbb{Z}$.

EXAMPLE 5.4.   Let $\mathbb{T}$ be the homogeneous tree with degree $q+1$, where $q \geq 1$. It is well known that the spectrum of the SRW operator on $\mathbb{T}$ is

$$\mathsf{spec}(P) = [-\rho, \rho]\,, \quad \rho = \frac{2\sqrt{q}}{q+1}\,, \quad \text{and} \quad \mathsf{spec}_p(P) = \emptyset\,;$$

see [14, §7.B] and the references given there. Removal of any edge splits $\mathbb{T}$ into two transient pieces. Therefore, in view of our theorems, $\mathsf{spec}_p(A) = \{0\}$,

$$\mathsf{spec}(A) = \left\{ \frac{\sin \omega}{\omega} : \omega > 0, |\cos \omega| \leq \rho \right\} \cup \{0\}\,, \quad \text{and}$$

$$\rho(A) = \frac{q-1}{q+1} \Big/ \arccos \frac{2\sqrt{q}}{q+1}.$$

The last formula for the spectral radius of A was first found by Saloff-Coste and Woess [16] by a completely different method, and indeed, the latter was the starting point for the present investigation.

A closer look at $\mathsf{spec}(A)$ may be of interest. Let $\omega_{\pm\rho} = \arccos(\pm\rho)$, where $\rho = \rho(P)$, so that $0 < \omega_\rho < \pi/2$ and $\omega_{-\rho} = \pi - \omega_\rho$. For $n \geq 0$, let m_n and M_n be the minimum and maximum, respectively, of all numbers $|\sin\omega|/\omega$, where $n\pi + \omega_\rho \leq \omega \leq n\pi + \omega_{-\rho}$. In particular, $M_0 = \rho(A)$. Also, $M_n > M_{n+1}$ and $m_n > m_{n+1}$. Then

$$\mathsf{spec}(A) = \{0\} \cup \bigcup_{n \geq 0} [m_{2n}, M_{2n}] \cup \bigcup_{n \geq 0} [-M_{2n+1}, -m_{2n+1}]\,.$$

A closer analysis of these intervals shows that there is an n_0 such that $[m_n, M_n]$ and $[m_{n+2}, M_{n+2}]$ overlap for all $n \geq n_0$, whence $\mathsf{spec}(A)$ is a union of finitely many intervals, one of which contains $[-M_{n_0}, M_{n_0+1}]$ if n_0 is odd, and $[-M_{n_0+1}, M_{n_0}]$ if n_0 is even.

When $q = 2$, one can verify that $n_0 = 0$, whence $\mathsf{spec}(A) = [-M_1, M_0]$, but as q increases, $\mathsf{spec}(A)$ decreases and becomes a finite disjoint union of more and more intervals. Indeed, in the limit, as $q \to \infty$, $\mathsf{spec}(A)$ tends to the set $\{0\} \cup \{(-1)^n/(\frac{2n+1}{2}\pi) : n \geq 0\}$.

Finally, we determine $-M_1 = \min \mathsf{spec}(A)$. As in Example 5.2, let $\omega^* \in (\pi, \frac{3}{2}\pi)$ be the smallest positive solution of $\tan \omega = \omega$. Then

$$
-M_1 = \begin{cases} \dfrac{\sin \omega^*}{\omega^*} = \cos \omega^*, & \text{if } \pi + \omega_\rho < \omega^*, \\[2ex] \dfrac{\sin(\pi + \omega_\rho)}{\pi + \omega_\rho} = -\dfrac{q-1}{q+1} \Big/ \left(\pi + \arccos \dfrac{2\sqrt{q}}{q+1} \right), & \text{otherwise.} \end{cases}
$$

The first of these two cases holds precisely when $\tan(\pi + \omega_\rho) < \pi + \omega_\rho$, or equivalently, when $\frac{q-1}{2\sqrt{q}} < \pi + \arccos \frac{2\sqrt{q}}{q+1}$, that is, for $q \le 82$ by numerical computation.

It seems unlikely that these results could have been found without using the relation between the spectra of P and A.

EXAMPLE 5.5.   Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be two homogeneous trees with degrees $q+1$ and $r+1$, respectively. In each of the trees, we choose a boundary point (end) and the associated Busemann (horocycle) function $h : \mathbb{T}_i \to \mathbb{Z}$. The *Diestel Leader graph* $\mathsf{DL}(q,r)$ is the horocyclic product of the two trees, i.e., the subgraph of their direct product,

$$
\mathsf{DL}(q,r) = \{ x_1 x_2 \in \mathbb{T}_1 \times \mathbb{T}_2 : h(x_1) + h(x_2) = 0 \}.
$$

See, e.g., [1] for a detailed description and further references. In particular, in [1] it is shown that for the SRW operator P on $\mathsf{DL} = \mathsf{DL}(q,r)$, the spectrum $\mathsf{spec}(P) = [-\rho(P), \rho(P)]$ is pure point, i.e., it is the closure of the point spectrum, and there is an orthonormal basis of $\ell^2(\mathsf{DL})$ consisting of finitely supported eigenfunctions of P. One has

$$
\mathsf{spec}_p(P) = \left\{ \rho(P) \cos \frac{m}{n}\pi : n \ge 2, \ 1 \le m \le n-1 \right\}, \quad \rho(P) = \frac{2\sqrt{qr}}{q+r}.
$$

By our theorems, we can compute the spectrum of A, which is also pure point and contains 0, since DL is not a tree.

In the specific case when $r = q$, $\mathsf{DL}(q,r)$ is a Cayley graph of the *lamplighter group* (wreath product) $\mathbb{Z}_q \wr \mathbb{Z}$; see again [1] for details. In that case, the spectrum of P had been determined previously in [8], $\rho(P) = 1$, and the point spectrum of A has the following particularly nice form:

$$
\mathsf{spec}_p(A) = \{0\} \cup \left\{ \frac{\sin \frac{m}{n}\pi}{(\frac{m}{n} + 2k)\pi} : n \ge 2, \ 1 \le m \le n-1, \ k \in \mathbb{Z} \right\} \qquad (q = r).
$$

In [1], an orthonormal basis of $\ell^2(\mathsf{DL}^0)$ consisting of eigenvectors of P is computed. One can of course adapt Corollary 4.16 in order to transfer the latter into an orthonormal basis of the subspace $\mathcal{M}$ of $L^2(\mathsf{DL}^1)$ consisting of eigenvectors of A.

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