Illinois Journal of Mathematics
Volume 51, Number 3, Fall 2007, Pages 767–803
S 0019-2082

ASYMPTOTIC ℓ_p HEREDITARILY INDECOMPOSABLE BANACH SPACES

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Abstract. For every $1 < p < \infty$ we construct an asymptotic ℓ_p Banach space which is hereditarily indecomposable and such that its dual is asymptotic ℓ_q hereditarily indecomposable, where q is the conjugate of p. We prove that c_0 is finitely representable in these spaces and that every bounded linear operator on these spaces is a strictly singular perturbation of a multiple of the identity.

1. Introduction

In recent years the study of the geometry of hereditarily indecomposable (HI) Banach spaces has revealed new structural phenomena in Banach space theory. A Banach space X is called HI if no infinite-dimensional closed subspace of X can be written as the direct sum of two infinite-dimensional closed subspaces. The study of the geometry of HI Banach spaces has been initiated after the solution of the unconditional basic sequence problem by W.T. Gowers and B. Maurey [15], and the dichotomy theorem due to W.T. Gowers [14]. Concerning the geometry of HI spaces, S.A. Argyros and the first named author [2] provided examples of asymptotic ℓ_1 HI spaces, and V. Ferenczi [11] using complex interpolation arguments gave examples of uniformly convex HI Banach spaces. Concerning the structure of the dual as well as the quotients of an HI space, V. Ferenczi [12] proved that the dual and the quotients of the Gowers-Maurey space are HI spaces. The Argyros-Felouzis dichotomy [5] shows that in general this is not the case, since from their results it follows that the classical spaces are quotients of HI spaces. Moreover, S.A. Argyros and A. Tolias [8] proved that every separable Banach space Z not containing ℓ_1 is a quotient of an HI space X and Z^* is complemented in X^*. We refer the reader to the handbook article of B. Maurey [17] and the lecture notes by S.A. Argyros [7] for a comprehensive study of HI spaces.

Received September 2, 2005; received in final form February 23, 2006.
2000 *Mathematics Subject Classification.* 46B20, 46B03.
Research partially supported by the EPEAEK program Pythagoras.

In this paper we provide examples of asymptotic ℓ_p HI Banach spaces, for $1 < p < \infty$. We recall the definition of an asymptotic ℓ_p Banach space.

Definition 1.1. Let $1 \leq p \leq \infty$. A Banach space X with a normalized basis $(e_n)_n$ is said to be an asymptotic ℓ_p space if there exists a constant C such that, for every $n \in \mathbb{N}$, any sequence of normalized vectors $(x_i)_{i=1}^n$ with $n \leq \operatorname{supp} x_1 < \cdots < \operatorname{supp} x_n$ is C–equivalent to the unit vector basis of ℓ_p^n.

Examples of asymptotic ℓ_1 HI Banach spaces were given in [2]. In the present paper we construct, for every $1 < p < \infty$, a real asymptotic ℓ_p HI space $X_{(p)}$. We also prove that the spaces $X_{(p)}$ have the following properties:

(1) For every infinite-dimensional closed subspace Y of $X_{(p)}$, every bounded linear operator $T : Y \to X_{(p)}$ is of the form $T = \lambda I_Y + S$, where λ is a scalar, I_Y is the inclusion operator from Y to $X_{(p)}$ and S is a strictly singular operator. Real asymptotic ℓ_1 spaces with the same property were constructed in [8]. Recall also that, as proved by V. Ferenczi in [10] (see also [15]), this property characterizes complex HI Banach spaces.

(2) c_0 is finitely representable in every subspace of $X_{(p)}$. It was proved in [3] that the asymptotic ℓ_1 HI Banach spaces constructed in [2] also have this property. It follows that the spaces $X_{(p)}$ do not contain uniformly convex subspaces. We do not know whether similar constructions can yield examples of uniformly convex HI spaces. This would be interesting in particular in the case $p = 2$, since it is related to the question whether there exists a weak Hilbert HI Banach space.

(3) The dual $X_{(p)}^*$ of the space $X_{(p)}$ is also an HI space. Since the dual of an asymptotic ℓ_p space is asymptotic ℓ_q, where $\frac{1}{p} + \frac{1}{q} = 1$, we get, for every $1 < q < \infty$, an asymptotic ℓ_q HI space $X_{(q)}$, and a dually defined asymptotic ℓ_q HI space $X_{(p)}^*$, which are proved to be totally incomparable. We note that asymptotic ℓ_1 HI Banach spaces with HI duals were constructed in [8].

Our construction is based on a p-convexified mixed Tsirelson space. Namely, we use as frame the mixed Tsirelson spaces $X = T[(\mathcal{S}_{n_j}, \theta_j)_j]$ introduced in [2], where $(n_j)_j$ is an increasing sequence of positive integers, $(\theta_j)_j$ is a sequence of positive reals decreasing to zero and, for $n \in \mathbb{N}$, $\mathcal{S}_n$ is the Schreier family of order n. Let us recall the definition.

Definition 1.2 ([1]). We set

$$\mathcal{S}_0 = \{\{n\} : n \in \mathbb{N}\} \cup \{\emptyset\}, \quad S_1 = \{F \subset \mathbb{N} : \#F \leq \min F\} \cup \{\emptyset\}$$

and for $n = 1, 2, \ldots,$

$$\mathcal{S}_{n+1} = \left\{ \bigcup_{i=1}^k F_i : F_i \in \mathcal{S}_n \text{ for all } i \leq k, F_1 < \cdots < F_k \right.$$
$$\left. \text{and } (\min F_i)_{i=1}^k \in \mathcal{S}_1 \right\} \cup \{\emptyset\}.$$

We next give a brief description of the space $X_{(p)}$. Denote by q the conjugate of p. We first consider the unconditional counterpart $X_{u,p}$ of the space $X_{(p)}$. This is an asymptotic ℓ_p Banach space with an unconditional basis. The ball of $X^*_{u,p}$ is closed as a subset of ℓ_∞ under the $(\mathcal{S}_{n_j}, \theta_j)$ q-convex operation for every $j \in \mathbb{N}$. This means that if $f_1 < \cdots < f_d$ is a $\mathcal{S}_{n_j}$-admissible family in $B_{X^*_{u,p}}$ (Definition 2.1), then $\theta_j \sum_{i=1}^d \beta_i f_i \in B_{X^*_{u,p}}$ for every choice of coefficients $(\beta_i)_{i=1}^d \in B_{\ell_q}$. Moreover, $B_{X^*_{u,p}}$ is minimal with this property. In fact, $X_{u,p}$ is the p-convexification of the asymptotic ℓ_1 space $T[(\mathcal{S}_{n_j}, \theta_j)_j]$ constructed in [2].

We now turn to the space $X_{(p)}$. $X_{(p)}$ has a norming set $D \subset B_{X^*_{(p)}}$, which is closed under the $(\mathcal{S}_{n_{2j}}, \theta_{2j})$ q-convex operation for every $j \in \mathbb{N}$. This implies in particular that, if $(x_i)_{i=1}^d$ is a block sequence in $X_{(p)}$ with $d \leq \min \operatorname{supp} x_1$, then

$$\left\| \sum_{i=1}^d x_i \right\| \geq \theta_2 \left(\sum_{i=1}^d \|x_i\|^p \right)^{1/p} .$$

Using an appropriate coding function σ, i.e., an injective map from the countable set of finite block sequences of vectors with rational coordinates to the natural numbers, we define for every $j \in \mathbb{N}$ some special $\mathcal{S}_{n_{2j+1}}$-admissible sequences of vectors, which we call $(\sigma, 2j+1)$-sequences. If $(f_1, \ldots, f_{2d})$ is a $(\sigma, 2j+1)$-sequence, then every functional of the form

$$\theta_{2j+1} E \sum_{i=1}^d \gamma_i (f_{2i-1} + f_{2i}),$$

where $(2^{1/q}\gamma_i)_{i=1}^d \in B_{\ell_q}$ and E is an interval of $\mathbb{N}$, is called a $(\mathcal{S}_{n_{2j+1}}, \theta_{2j+1})$-special functional.

The norming set D is rationally convex and minimal with the property of being closed under the $(\mathcal{S}_{n_{2j}}, \theta_{2j})$ q-convex operation for every $j \in \mathbb{N}$ as well as under the formation of $(\mathcal{S}_{n_{2j+1}}, \theta_{2j+1})$ special functionals for every $j \in \mathbb{N}$.

The use of some type of special functionals defined by means of a coding function is common in every construction of an HI space so far, starting from [15]. Let us add a few comments on our choice of special functionals:

The fact that the coefficients $(\gamma_i)_{i=1}^d$ are chosen freely yields that the space satisfies an upper ℓ_p estimate (Proposition 2.9). On the other hand, the restriction that the two elements of each pair (f_{2i-1}, f_{2i}) have the same coefficient allows us to prove the following:

For every $j \in \mathbb{N}$ and every pair of block subspaces Y and Z of $X_{(p)}$, there exist a sequence of vectors $(x_k)_{k=1}^{2d}$ in $B_{X_{(p)}}$ with $x_{2i-1} \in Y$ and $x_{2i} \in Z$, for every $i = 1, \ldots, d$, and a p-convex combination $x = \sum_{i=1}^d a_i (x_{2i-1} + x_{2i})$, such that:

(i) x is well normed by a $(\mathcal{S}_{n_{2j+1}}, \theta_{2j+1})$-special functional of the form $\theta_{2j+1} \sum_{i=1}^{d} \gamma_i (f_{2i-1} + f_{2i})$, where, for each $k = 1, \ldots, 2d$, $f_k(x_k) = 1$.

(ii) The norm of $\tilde{x} = \sum_{i=1}^{d} a_i (x_{2i-1} - x_{2i})$ is less than $C\theta_{2j+1}\|x\|$ (where C is a constant).

This yields that the space $X_{(p)}$ is hereditarily indecomposable using the following standard characterization of HI spaces. A Banach space X is hereditarily indecomposable if and only if, for every $\varepsilon > 0$ and all infinite-dimensional subspaces Y, Z of X there exist $y \in Y$ and $z \in Z$ with $\|y - z\| \leq \varepsilon \|y + z\|$.

Our terminology and notation is standard and can be found in [16].

Acknowledgment. We would like to thank S. Argyros for valuable discussions.

2. The space $X_{(p)}$ and its norming set

In this section we give the definition of the space $X_{(p)}$, by constructing its norming set. We give some properties of the norming set of this space and we prove that $X_{(p)}$ is an asymptotic ℓ_p space (Proposition 2.9). We show that the dual space $X^*_{(p)}$ is an asymptotic ℓ_q-space, where q is the conjugate of p, and moreover that the asymptotic ℓ_q spaces $X^*_{(p)}$ and $X_{(q)}$ are totally incomparable.

Let $(e_i)_{i=1}^{\infty}$ be the standard basis of the linear space c_{00} of finitely supported sequences. For $x = \sum_{i=1}^{\infty} a_i e_i \in c_{00}$, the *support* of x is the set $\operatorname{supp} x = \{i \in \mathbb{N} : a_i \neq 0\}$. The *range* of x, written $\operatorname{ran} x$, is the smallest interval of $\mathbb{N}$ containing the support of x. For finite subsets E, F of $\mathbb{N}$, $E < F$ means $\max E < \min F$ or either E or F is empty. For $n \in \mathbb{N}$, $E \subset \mathbb{N}$, $n < E$ (resp. $E < n$) means $n < \min E$ (resp. $\max E < n$). For x, y in c_{00}, $x < y$ means $\operatorname{supp} x < \operatorname{supp} y$. For $n \in \mathbb{N}$, $x \in c_{00}$, we write $n < x$ (resp. $x < n$) if $n < \operatorname{supp} x$ (resp. $\operatorname{supp} x < n$). We say that the sets $E_i \subset \mathbb{N}$, $i = 1, \ldots, n$, are *successive* if $E_1 < E_2 < \cdots < E_n$. Similarly, the vectors x_i, $i = 1, \ldots n$, are *successive* if $x_1 < x_2 < \cdots < x_n$. For $x = \sum_{i=1}^{\infty} a_i e_i$ and E a subset of $\mathbb{N}$, we denote by Ex the vector $Ex = \sum_{i \in E} a_i e_i$.

Let $\mathcal{M}$ be a family of finite subsets of $\mathbb{N}$. We say that $\mathcal{M}$ is *compact* if it is closed in the topology of pointwise convergence in $2^{\mathbb{N}}$. $\mathcal{M}$ is *hereditary* if whenever $B \subset A$ and $A \in \mathcal{M}$, then $B \in \mathcal{M}$. $\mathcal{M}$ is *spreading* if whenever $A = \{m_1, \ldots, m_k\} \in \mathcal{M}$ and $B = \{n_1, \ldots, n_k\}$ is such that, for all $i = 1, \ldots k$, $m_i \leq n_i$, then $B \in \mathcal{M}$.

It is easy to see that, for every $n \in \mathbb{N}$, the Schreier family $\mathcal{S}_n$ (Definition 1.2) is compact, hereditary and spreading.

Definition 2.1. Let $\mathcal{M}$ be a family of finite subsets of $\mathbb{N}$.

(a) A finite sequence $(E_i)_{i=1}^{d}$ of subsets of $\mathbb{N}$ is $\mathcal{M}$-*admissible* if there exists $(k_i)_{i=1}^{d} \in \mathcal{M}$ such that $k_1 \leq E_1 < k_2 \leq E_2 < \cdots < k_d \leq E_d$.

If $\mathcal{M}$ is spreading, this is equivalent to $E_1 < E_2 < \cdots < E_d$ and $\{\min E_1, \ldots, \min E_d\} \in \mathcal{M}$.

(b) A finite sequence $(x_i)_{i=1}^n$ of vectors in c_{00} is $\mathcal{M}$-*admissible* if the sequence $(\operatorname{supp} x_i)_{i=1}^n$ is $\mathcal{M}$-admissible.

Fix $p \in [1, \infty)$. In the sequel, whenever we write m_j or n_j we shall refer to a pair of fixed recursively defined sequences of natural numbers satisfying the following properties: $m_1 = 4$, $n_1 = 1$ and, for $j \in \mathbb{N}$,

$$m_{j+1} \text{ is a power of 2 and } m_{j+1} \geq m_j^5, \tag{1}$$

and

$$(\log_2(m_j)+1)(\gamma(j)(n_{j-1}+1)+1) < n_j, \text{ where } \gamma(j) = [3p \log_{m_1}(m_j)]+2. \tag{2}$$

Let

$$\mathcal{W} = \Big\{(f_1, \ldots, f_d) : f_i \in c_{00}, f_i \neq 0, \ f_i(n) \in \mathbb{Q} \quad \text{for all } i \leq d, n \in \mathbb{N}, \ \text{ and } \ f_1 < \cdots < f_d\Big\}.$$

We consider a partition of $\mathbb{N}$ into two infinite sets N_1 and N_2. We also consider a coding function $\sigma : \mathcal{W} \to \{2j : j \in N_2\}$ with the following properties.

(i) σ is one-to-one.

(ii) $\sigma(f_1, \ldots, f_{i-1}) < \sigma(f_1, \ldots, f_{i-1}, f_i)$.

(iii) $m_{\sigma(f_1,\ldots,f_i)} > \max\left\{\frac{1}{|f_j(e_k)|} : k \in \operatorname{supp} f_j, \quad j = 1, \ldots, i\right\} \cdot \max \operatorname{supp} f_i$ for every $(f_1, \ldots, f_d) \in \mathcal{W}$ and $i = 1, \ldots, d$.

Definition 2.2. Let $(K_{2i})_{i\in\mathbb{N}}$ be a sequence of subsets of c_{00} consisting of vectors with rational coordinates and let $j \in \mathbb{N}$. A sequence $(f_1, \ldots, f_d) \in \mathcal{W}$ is called a (σ, j)-sequence (with respect to the family $(K_{2i})_{i\in\mathbb{N}}$) if:

(i) $f_1 \in K_{2j_1}$ for some $j_1 \in N_1$ with $2j_1 \geq j + 5$, and $f_i \in K_{\sigma(f_1,\ldots,f_{i-1})}$ for $i = 2, \ldots, d$.

(ii) The set $(f_1, \ldots, f_d)$ is S_{n_j}-admissible.

We now construct a sequence $(K^n)_n$ of subsets of c_{00} as follows:

We set $K^0 = \{\pm e_n : n \in \mathbb{N}\}$ and $D^0 = \operatorname{conv}_{\mathbb{Q}} K^0$, where $\operatorname{conv}_{\mathbb{Q}} M$ denotes the rational convex hull of M. Assume that for some $n \geq 0$ we have defined the sets K_j^n, $j \in \mathbb{N}$, and we have set $K^n = \bigcup_j K_j^n$ and $D^n = \operatorname{conv}_{\mathbb{Q}} K^n$. Then, for $j \in \mathbb{N}$, we set

$$K_{2j}^{n+1} = K_{2j}^n \cup \Big\{\frac{1}{m_{2j}} \sum_{i=1}^d \beta_i f_i : \ d \in \mathbb{N}, \ f_i \in D^n, \beta_i \in \mathbb{Q} \text{ for } \ i = 1, 2, \ldots, d, \quad (\beta_i)_{i\leq d} \in B_{\ell_q} \setminus \{0\} \ \text{ and } (f_1, f_2, \ldots, f_d) \text{ is } \mathcal{S}_{n_{2j}}\text{-admissible}\Big\}$$

and

$$K^{n+1}_{2j+1} = K^n_{2j+1} \cup \Big\{ \frac{1}{m_{2j+1}} E \sum_{i=1}^{d} \gamma_i (f_{2i-1} + f_{2i}) : E \text{ is an interval of } \mathbb{N},\ d \in \mathbb{N}, \gamma_i \in \mathbb{Q} \text{ for } i = 1, \ldots, d, (2^{1/q}\gamma_i)_{i \le d} \in B_{\ell_q} \setminus \{0\} \text{ and } (f_1, \ldots, f_{2d}) \text{ is a } (\sigma, 2j+1)-\text{sequence with respect to the family } (K^n_{2i})_{i\in\mathbb{N}} \Big\}.$$

We set $K^{n+1} = \bigcup_j K^{n+1}_j$ and $D^{n+1} = \mathrm{conv}_{\mathbb{Q}} K^{n+1}$.

For every $i \in \mathbb{N}$, we set $L_i = \bigcup_{n\in\mathbb{N}} K^n_i$. For $j \in \mathbb{N}$, the functionals contained in the set $L_{2j+1} = \bigcup_{n\in\mathbb{N}} K^n_{2j+1}$ are called $2j+1$-special functionals.

Finally, we set $K = \bigcup_n K^n$ and $D = \bigcup_n D^n$. The space $X_{(p)}$ is defined as the completion of c_{00} under the norm $\|x\| = \|x\|_D = \sup\{\langle f, x\rangle : f \in D\}$.

Remarks 2.3. (i) It is easily verified that the sets K and D are symmetric and closed under the projection of their elements on intervals. It follows that the sequence $(e_n)_{n\in\mathbb{N}}$ is a bimonotone basis of $X_{(p)}$.

(ii) A simple inductive argument shows that $K^n \subset B_{\ell_q}$ for every $n \in \mathbb{N}$ and therefore $D \subset B_{\ell_q}$. It follows that, for every $f \in D$ and $\sum_{i\in A} a_i e_i \in c_{00}$, we have $|f(\sum_{i\in A} a_i e_i)| \le \|(a_i)_{i\in A}\|_{\ell_p}$.

(iii) By the definition of the norm and since D is rationally convex it follows that $B_{X^*_{(p)}} = \overline{D}^{w^*} = \overline{D}^p$, where $\overline{D}^p$ is the pointwise closure of D.

(iv) The set D is closed under the $(\mathcal{S}_{n_{2j}}, 1/m_{2j})$-operation for every $j \in \mathbb{N}$, i.e., if $f_1 < \cdots < f_d$ is an $\mathcal{S}_{n_{2j}}$-admissible sequence of elements of D and $(\beta_i)_{i=1}^d$ is a rational q-convex combination, then $1/m_{2j} \sum_{i=1}^d \beta_i f_i \in K \subset D$. If some β_i are not rational, then it follows by (iii) that $1/m_{2j} \sum_{i=1}^d \beta_i f_i \in B_{X^*_{(p)}}$.

Definition 2.4 (The tree T_f corresponding to $f \in D$). Let $f \in D$. By a *tree* analysis of f (or a *tree corresponding to* f) we mean a finite family $T_f = (f_t)_{t\in\mathcal{T}}$ indexed by a finite tree $\mathcal{T}$ with a unique root $0 \in \mathcal{T}$ such that the following conditions are satisfied:

1. $f_0 = f$ and $f_t \in D$ for all $t \in \mathcal{T}$.
2. If $t \in \mathcal{T}$, then t is maximal if and only if $f_t \in K^0$.
3. For every $t \in \mathcal{T}$ which is not maximal we denote by S_t the set of immediate successors of t in $\mathcal{T}$. Then exactly one of the following three statements holds:
   (a) $S_t = \{s_1, \ldots, s_d\}$, where $f_{s_1} < \cdots < f_{s_d}$ and there exist $j \in \mathbb{N}$ and $(\beta_i)_{i=1}^d \in B_{\ell_q}$ such that the family $(f_{s_1}, \ldots, f_{s_d})$ is $\mathcal{S}_{n_{2j}}$-admissible and

$$f_t = \frac{1}{m_{2j}} \sum_{i=1}^{d} \beta_i f_{s_i}.$$

(b) There exist $j \in \mathbb{N}$, a $(\sigma, 2j+1)$-sequence with respect to the family $(L_{2i})_{i\in\mathbb{N}}$, $(g_1, \ldots, g_{2d})$, coefficients $(\gamma_i)_{i=1}^d \in \frac{1}{2^{1/q}} B_{\ell_q}$ and an interval E such that

$$f_t = \frac{1}{m_{2j+1}} E \sum_{i=1}^{d} \gamma_i (g_{2i-1} + g_{2i});$$

in this case $\{f_s : s \in S_t\} = \{Eg_i : i \leq 2d,\ Eg_i \neq 0\}$.

(c) $S_t = \{s_1, \ldots, s_d\}$ and there exists a family of positive rationals $(r_i)_{i=1}^d$ with $\sum_{i=1}^d r_i = 1$ such that $f_t = \sum_{i=1}^d r_i f_{s_i}$. Moreover, for every $i \leq d$, $f_{s_i} \in K$ and $\operatorname{ran} f_{s_i} \subset \operatorname{ran} f_t$.

An easy inductive argument shows that every $f \in D$ admits a tree analysis.

Notation 2.5. For every $f \in K \setminus K^0$, if f is of the form $f = \frac{1}{m_j} \sum_{l=1}^d \beta_l f_l$, where $(f_l)_{l \leq d} \subset D$ is $\mathcal{S}_{n_j}$-admissible and $(\beta_l)_{l\leq d} \in B_{\ell_q}$, then we say that the *weight* of f is m_j and denote it by $w(f)$. Note that the weight of a functional is not necessarily uniquely determined. However, when we refer to a $2j+1$-special functional $f = \frac{1}{m_{2j+1}} E \sum_{l=1}^d \gamma_l (f_{2l-1} + f_{2l})$, then by $w(f_i)$ we shall always mean $w(f_i) = m_{\sigma(f_1, \ldots, f_{i-1})}$ for $i \geq 2$ and $w(f_1)$ will satisfy $w(f_1) \geq m_{2j+4}$.

Lemma 2.6.

(a) *Let* $f = \frac{1}{m_{2j}} \sum_{i=1}^k \beta_i f_i \in K$. *Then, for every subset* A *of* $\{1, \ldots, k\}$, *we have*

$$\left\| \frac{1}{m_{2j}} \sum_{i\in A} \beta_i f_i \right\| \leq \|(\beta_i)_{i\in A}\|_{\ell_q}.$$

(b) *Let* $f = \frac{1}{m_{2j+1}} E \sum_{i=1}^d \gamma_i (f_{2i-1} + f_{2i}) \in L_{2j+1}$ *be a special functional. Then, for every interval* $I \subseteq \{1, \ldots, d\}$, *we have*

$$\left\| \frac{1}{m_{2j+1}} E \sum_{i\in I} \gamma_i (f_{2i-1} + f_{2i}) \right\| \leq 2^{1/q} \|(\gamma_i)_{i\in I}\|_{\ell_q}.$$

Proof. The proofs of (a) and (b) are similar, so we shall prove only (b). We set

$$\delta_i = \frac{\gamma_i}{2^{1/q} \|(\gamma_i)_{i\in I}\|_{\ell_q}}, \text{ for } i \in I,$$

and $\delta_i = 0$ otherwise. Then we have $(2^{1/q}\delta_i)_{i\in I} \in S_{\ell_q}$ and hence

$$\frac{1}{m_{2j+1}} E \sum_{i\in I} \frac{\gamma_i}{2^{1/q} \|(\gamma_i)_{i\in I}\|_{\ell_q}} (f_{2i-1} + f_{2i}) = \frac{1}{m_{2j+1}} E \sum_{i=1}^{d} \delta_i (f_{2i-1} + f_{2i})$$

belongs to $B_{X^*_{(p)}}$. Therefore,

$$\left\| \frac{1}{m_{2j+1}} E \sum_{i \in I} \gamma_i (f_{2i-1} + f_{2i}) \right\| \leq 2^{1/q} \left\| (\gamma_i)_{i \in I} \right\|_{\ell_q} . \qquad \square$$

REMARK 2.7. In the sequel, for a special functional

$$f = \frac{1}{m_{2j+1}} E \sum_{i=1}^{d} \gamma_i (f_{2i-1} + f_{2i})$$

we shall also use the notation

$$f = \frac{1}{m_{2j+1}} E \sum_{i=1}^{2d} \beta_i f_i,$$

where $\beta_{2i-1} = \beta_{2i} = \gamma_i$ for $i = 1, \ldots, d$. In this case, for every interval I, denoting by I' be the smallest interval of positive integers of the form $[2n-1, 2m]$ which contains I, it follows from the previous lemma that

$$\left\| \frac{1}{m_{2j+1}} E \sum_{i \in I} \beta_i f_i \right\| \leq \left\| (\beta_i)_{i \in I'} \right\|_{\ell_q} .$$

NOTATION 2.8. (a) Let $(a_k)_{k \in \mathbb{N}}$ be a non-zero element of ℓ_p. By the *conjugate sequence* of $(a_k)_{k \in \mathbb{N}}$ we shall mean the unique sequence $(\beta_k)_{k \in \mathbb{N}} \in S_{\ell_q}$ which satisfies $\sum_k a_k \beta_k = (\sum_k |a_k|^p)^{1/p}$.

(b) We introduce the following general terminology, which will be used repeatedly in the sequel. Let $(X, (e_n)_n)$ be a Banach space with a shrinking basis, let $(z_k)_{k=1}^m$ be a block sequence in X and $(g_l)_{l=1}^d$ a block sequence of $(e_n^*)_n$ in X^*. For $k = 1, \ldots, m$ we say that z_k *is split* by the sequence $(g_l)_{l=1}^d$ if the set

$$A_k = \{l = 1, \ldots, d : \operatorname{ran} z_k \cap \operatorname{ran} g_l \neq \emptyset\}$$

has at least two elements. If on the contrary the set A_k is a singleton and $A_k = \{l\}$, then we say that z_k *is covered* by g_l.

PROPOSITION 2.9. *For every block sequence $(x_r)_{r \in F}$ in $X_{(p)}$ we have*

$$\left\| \sum_{r \in F} x_r \right\| \leq 4 \left(\sum_{r \in F} \|x_r\|^p \right)^{1/p} . \tag{2.1}$$

Moreover, if the sequence $(x_r)_{r \in F}$ is $\mathcal{S}_{n_{2j}}$ admissible for some $j \in \mathbb{N}$, then

$$\frac{1}{m_{2j}} \left(\sum_{r \in F} \|x_r\|^p \right)^{1/p} \leq \left\| \sum_{r \in F} x_r \right\| . \tag{2.2}$$

In particular, $X_{(p)}$ is an asymptotic ℓ_p space.

Proof. Let $(x_r)_{r\in F}$ be a block sequence. To prove (2.1) we show by induction on n that, for every $f\in K^n$,

$$\left|f\left(\sum_{r\in F}x_r\right)\right|\leq 4\left(\sum_{r\in A}\|x_r\|^p\right)^{1/p}, \tag{2.3}$$

where $A=\{r:\operatorname{supp}f\cap\operatorname{ran}x_r\neq\emptyset\}$. It is clear that if (2.3) holds for every $f\in K^n$, then the same holds for every $f\in D^n$. (2.3) is trivially true for every $f\in K^0$. Assume that it holds true for every $f\in D^n$ and let $f=\frac{1}{m_j}\sum_{l=1}^{d}\beta_l f_l\in K^{n+1}$, $j\in\mathbb{N}$. We set

$$F_2=\left\{r\in A: x_r \text{ is split by the family } (f_l)_{l=1}^d\right\} \text{ and } F_1=A\setminus F_2\,.$$

For every $l\leq d$ we set

$$G_l=\{r\in F_1:\operatorname{supp}f_l\cap\operatorname{ran}x_r\neq\emptyset\}$$

and

$$I=\{l\leq d: G_l\neq\emptyset\}\,.$$

Then, since for every $r\in F_1$, $\operatorname{ran}x_r\cap\operatorname{supp}f_l\neq\emptyset$ for at most one l, from the inductive hypothesis and the fact that $4\frac{1}{m_j}\leq 1$ for every j, we get

$$\begin{aligned}\left|\frac{1}{m_j}\sum_{l=1}^{d}\beta_l f_l\left(\sum_{r\in F_1}x_r\right)\right| &= \frac{1}{m_j}\left|\sum_{l\in I}\beta_l f_l\left(\sum_{r\in G_l}x_r\right)\right| \\ &\leq \frac{1}{m_j}\sum_{l\in I}|\beta_l|\cdot 4\left(\sum_{r\in G_l}\|x_r\|^p\right)^{1/p} \\ &\leq \left(\sum_{r\in F_1}\|x_r\|^p\right)^{1/p}.\end{aligned} \tag{2.4}$$

For every $r\in F_2$ we set

$$B_r=\{l\leq d:\operatorname{supp}f_l\cap\operatorname{ran}x_r\neq\emptyset\}\ .$$

Notice that $\#(B_{r_1}\cap B_{r_2})\leq 1$ for $r_1\neq r_2\in F_2$. From Remark 2.7 we get

$$\frac{1}{m_j}\left|\sum_{l\in B_r}\beta_l f_l(x_r)\right|\leq\left(\sum_{l\in B_r'}|\beta_l|^q\right)^{1/q}\|x_r\|\ ,$$

where B_r' is the smallest interval of the form $[2n-1,2m]$ which contains B_r. It is easy to see that for every $l\in\mathbb{N}$ there exist at most three $r\in F_2$ such that $l \in B'_r$, and hence by Hölder's inequality,

$$\left|\sum_{r\in F_2}\frac{1}{m_j}\sum_{l\in B_r}\beta_l f_l(x_r)\right| \leq \sum_{r\in F_2}\left(\sum_{l\in B'_r}|\beta_l|^q\right)^{1/q}\|x_r\| \leq 3\left(\sum_{r\in F_2}\|x_r\|^p\right)^{1/p}. \tag{2.5}$$

From (2.4) and (2.5) it follows that

$$\left|f\left(\sum_{r\in A}x_r\right)\right| \leq \frac{1}{m_j}\left|\sum_{l=1}^{d}\beta_l f_l\left(\sum_{r\in F_1}x_r\right)\right| + \frac{1}{m_j}\left|\sum_{l=1}^{d}\beta_l f_l\left(\sum_{r\in F_2}x_r\right)\right| \leq 4\left(\sum_{r\in F}\|x_r\|^p\right)^{1/p}.$$

For the lower estimate, let $\varepsilon > 0$, let $(\beta_r)_{r\in F}$ be the conjugate sequence of $(\|x_r\|)_{r\in F}$, and let $f_r \in D$ be such that $f_r(x_r) \geq (1-\varepsilon)\|x_r\|$ and $\operatorname{ran} f_r \subset \operatorname{ran} x_r$. It follows that the sequence $(f_r)_{r\in F}$ is $\mathcal{S}_{n_{2j}}$-admissible and hence $f = \frac{1}{m_{2j}}\sum_{r\in F}\beta_r f_r \in B_{X^*_{(p)}}$. Therefore,

$$\left\|\sum_{r\in F}x_r\right\| \geq f\left(\sum_{r\in F}a_r x_r\right) = \frac{1}{m_{2j}}\sum_{r\in F}\beta_r f_r(x_r) \geq (1-\varepsilon)\left(\sum_{r\in F}\|x_r\|^p\right)^{1/p},$$

and since ε was arbitrarily chosen, (2.2) holds. □

For $p = 1$ it was proved in [2] that the space $X_{(1)}$ is reflexive. For $1 < p < \infty$, every asymptotic ℓ_p space is reflexive and its dual is asymptotic ℓ_q, where $\frac{1}{p} + \frac{1}{q} = 1$ (see [18]). In particular, for our spaces we have:

Corollary 2.10. *Let $1 < p < \infty$. Then $X_{(p)}$ is a reflexive Banach space and its dual space $X^*_{(p)}$ is asymptotic ℓ_q, where q is the conjugate of p. In particular, for every $\mathcal{S}_1$-admissible block sequence $(f_i)_{i\in F}$ in $X^*_{(p)}$,*

$$\frac{1}{4}\left(\sum_{i\in F}\|f_i\|^q\right)^{1/q} \leq \left\|\sum_{i\in F}f_i\right\| \leq m_2\left(\sum_{i\in F}\|f_i\|^q\right)^{1/q}.$$

Corollary 2.11. *For $1 < p < \infty$ and q the conjugate of p, the spaces $X^*_{(p)}$ and $X_{(q)}$ are totally incomparable.*

Proof. This is a consequence of the fact that $X_{(q)}$ satisfies an upper ℓ_q estimate but does not contain ℓ_q. Assume on the contrary that there exist subspaces Z of $X^*_{(p)}$ and Y of $X_{(q)}$ which are isomorphic. By standard arguments, there exist normalized block sequences $(f_n)_n \subset X^*_{(p)}$, $(y_n)_n \subset X_{(q)}$

which are C-equivalent for some constant C. By Proposition 2.9, for every $k \in \mathbb{N}$, we have

$$\left\| \sum_{n=1}^{k} b_n y_n \right\| \leq 4 \left(\sum_{n=1}^{k} |b_n|^q \right)^{1/q}, \text{ for every } (b_n) \subset \mathbb{R}.$$

By the same proposition and standard duality arguments it follows that the space $X^*_{(p)}$ satisfies a lower ℓ_q estimate so that for every $k \in \mathbb{N}$,

$$\frac{1}{4} \left(\sum_{n=1}^{k} |b_n|^q \right)^{1/q} \leq \left\| \sum_{n=1}^{k} b_n f_n \right\|, \text{ for every } (b_n) \subset \mathbb{R}.$$

Therefore, for every $(b_n) \subset \mathbb{R}$ and $k \in \mathbb{N}$,

$$\frac{1}{4} \left(\sum_{n=1}^{k} |b_n|^q \right)^{1/q} \leq \left\| \sum_{n=1}^{k} b_n f_n \right\| \leq C \left\| \sum_{n=1}^{k} b_n y_n \right\| \leq 4C \left(\sum_{n=1}^{k} |b_n|^q \right)^{1/q},$$

which yields that ℓ_q is isomorphic to a subspace of $X_{(q)}$. Since, as proved in Theorem 5.8, $X_{(q)}$ is an HI space, we have arrived at a contradiction. $\square$

3. The auxiliary space $T_{(p)} = T_{(p)}[(\mathcal{S}_{n_j+1}, \theta/m_j)_j]$

As in a series of papers which deal with the construction of HI spaces using the mixed Tsirelson spaces as frame ([2], [6], [8], [9]), here we also use an auxiliary space in order to obtain upper estimates for the norm of special vectors. Our auxiliary space is the space $T_{(p)} = T_{(p)}[(\mathcal{S}_{n_j+1}, \frac{\theta}{m_j})_j]$, where $\theta = 4^{1/q}$.

We fix $p \geq 1$ and define the spaces $T_{(p)} = T_{(p)}[(\mathcal{S}_{n_j+1}, \frac{\theta}{m_j})_j]$ and, if $p > 1$, $Z = T[(\mathcal{S}_{n_j+1}, (\frac{\theta}{m_j})^p)_j]$, via their norming sets $B_{(p)}$ and W, respectively.

Let q be the conjugate exponent of p. We set $B_{(p)} = \bigcup_{n=0}^{\infty} B^n$ and $W = \bigcup_{n=0}^{\infty} W^n$, where the sequences $(B^n)_{n=0}^{\infty}$, $(W^n)_{n=0}^{\infty}$ are recursively defined as follows:

$$B^0 = W^0 = \{\pm e_l : \ l \in \mathbb{N}\},$$

and for $n = 0, 1, \ldots$,

$$B^{n+1} = \Big\{ \frac{\theta}{m_j} \sum_{i=1}^{d} \beta_i f_i : \ j \in \mathbb{N},\ d \in \mathbb{N},\ (\beta_i)_{i=1}^{d} \in B_{\ell_q},\ f_i \in B^n \text{ for } i \leq d$$
$$\text{and the sequence } (f_1, f_2, \ldots, f_d) \text{ is } \mathcal{S}_{n_j+1} \text{ admissible} \Big\}$$

and

$$W^{n+1}=\Bigg\{\left(\frac{\theta}{m_j}\right)^p\sum_{i=1}^{d}g_i:\ j\in\mathbb{N},\ d\in\mathbb{N},\ g_i\in W^n\ \text{ for } i=1,2,\dots,d$$
$$\text{and the sequence } (g_1,g_2,\dots,g_d) \text{ is } \mathcal{S}_{n_j+1} \text{ admissible}\Bigg\}.$$

The space $T_{(p)}$ is defined to be the completion of c_{00} endowed with the norm

$$\|x\|_{(p)}=\sup\left\{f(x):f\in B_{(p)}\right\}$$

and the space Z is the completion of c_{00} endowed with the norm

$$\|x\|_{Z}=\sup\left\{g(x):g\in W\right\}.$$

The weight of a functional $f=\frac{\theta}{m_j}\sum_{i=1}^{d}\beta_i f_i\in B_{(p)}$ is defined as $w(f)=\frac{m_j}{\theta}$ and, similarly, for $g=(\frac{\theta}{m_j})^p\sum_{i=1}^{d}g_i\in W$, we define $w(g)=(\frac{m_j}{\theta})^p$.

The next lemma states in particular that $T_{(p)}$ is the p-convexification of Z.

Lemma 3.1. *For every $x=\sum_k a_k e_k\in c_{00}$ with $a_k\geq 0$ and every $f\in B_{(p)}$ (respectively, W) there exists $g\in W$ (respectively $B_{(p)}$) such that $w(f)^p=w(g)$ and*

$$\left|f\left(\sum_k a_k e_k\right)\right|\leq\left|g\left(\sum_k a_k^p e_k\right)\right|^{1/p},\tag{3.1}$$

respectively

$$\left|g\left(\sum_k a_k^p e_k\right)\right|^{1/p}\leq\left|f\left(\sum_k a_k e_k\right)\right|.$$

As a consequence,

$$\left\|\sum_k a_k e_k\right\|_{(p)}=\left\|\sum_k a_k^p e_k\right\|_{Z}^{1/p}.\tag{3.2}$$

Proof. We shall prove by induction on n that for every $f\in B^n$ there exists a functional $g\in W^n$ such that $\operatorname{supp} f=\operatorname{supp} g$, $w(f)^p=w(g)$ and

$$\left|f\left(\sum_k a_k e_k\right)\right|\leq\left|g\left(\sum_k a_k^p e_k\right)\right|^{1/p}\leq\left\|\sum_k a_k^p e_k\right\|_{Z}^{1/p}.$$

For $f,g\in B^0=W^0$ the conclusion is trivially true. Assume that it holds true for every $f\in B^n$ for some $n\geq 0$. Let $f=\frac{\theta}{m_j}\sum_{i\in A}\beta_i f_i\in B^{n+1}$, where $(f_i)_{i\in A}$ is $\mathcal{S}_{n_j+1}$-admissible, $(\beta_i)_{i\in A}\in B_{\ell_q}$. For $i\in A$, set $F_i=\operatorname{supp} x\cap\operatorname{supp} f_i$

and choose $g_i \in W^n$ to satisfy the inductive hypothesis for f_i and $x_i = \sum_{k\in F_i} a_k e_k$. Then we get, for appropriate $\varepsilon_i = \pm 1$, $i \in A$,

$$\begin{aligned}
\left| f\left(\sum_k a_k e_k\right)\right| &= \frac{\theta}{m_j}\left|\sum_{i\in A}\beta_i f_i\left(\sum_{k\in F_i} a_k e_k\right)\right| \\
&\le \frac{\theta}{m_j}\sum_{i\in A}|\beta_i|\ |g_i\left(\sum_{k\in F_i} a_k^p e_k\right)|^{1/p} \\
&\le \frac{\theta}{m_j}\left(\sum_{i\in A}\varepsilon_i g_i\left(\sum_{k\in F_i} a_k^p e_k\right)\right)^{1/p} \\
&= \left(\left(\frac{\theta}{m_j}\right)^p\sum_{i\in A}\varepsilon_i g_i\left(\sum_{k\in F_i} a_k^p e_k\right)\right)^{1/p} \\
&= \left| g\left(\sum_k a_k^p e_k\right)\right|^{1/p},
\end{aligned}$$

where $g = (\frac{\theta}{m_j})^p \sum_{i\in A}\varepsilon_i g_i \in W^{n+1}$ since, for $\varepsilon_i = \pm 1$, $\varepsilon_i g_i \in W^n$ and $(g_i)_{i\in A}$ is $\mathcal{S}_{n_j+1}$-admissible.

The proof of the symmetric statement is similar. □

Let us recall that we have set $\gamma(j) = [3p\log_2(m_j)] + 2$ and by the choice of the sequence $(n_j)_j$ we have $\gamma(j)(n_{j-1}+1) < n_j$.

DEFINITION 3.2 (Basic p-special combinations). Let $\varepsilon > 0$ and $j \in \mathbb{N}$, $j \ge 2$. A basic combination $\sum_{k\in F} a_k e_k$ is said to be an (ε, j) *basic p-convex special combination* ((ε, j) basic p-convex s.c.), if it satisfies:

(1) $F \in \mathcal{S}_{n_j}$.
(2) a_k is a positive real for every $k \in F$ and $\sum_{k\in F} a_k^p = 1$.
(3) For every $P \in \mathcal{S}_{\gamma(j)(n_{j-1}+1)}$ we have that $\left(\sum_{k\in P} a_k^p\right)^{1/p} < \varepsilon$.

REMARK 3.3. If $\varepsilon > 0$, $j \in \mathbb{N}$ with $j > 1$ and $L \in [\mathbb{N}]$ with $\min L \ge \frac{3}{\varepsilon^p}$, then for each $n \in \mathbb{N}$ with $\gamma(j)(n_{j-1}+1) < n \le n_j$ there exists an (ε, j) basic p-convex s.c. $\sum_{k\in F} a_k e_k$ with $F \subset L$ and, moreover, $F \in \mathcal{S}_n$.

Indeed, it was shown in Proposition 2.3 of [8] that there exists an (ε^p, j) basic 1-convex special combination $\sum_{k\in F} b_k e_k$, where F is the maximal $\mathcal{S}_n$ initial segment of L. Setting $a_k = b_k^{1/p}$ we see that $\sum_{k\in F} a_k e_k$ is an (ε, j)-basic p-convex s.c.

The next proposition gives estimates for the action of functionals on p-convex special combinations of the basis.

Proposition 3.4 (Estimates on the basis of $T_{(p)}$). *Let $\varepsilon > 0$ and $j \in \mathbb{N}$ with $j > 1$ such that $\varepsilon \leq \frac{\theta}{m_j}$, and let $x = \sum_{k\in F} a_k e_k$ be an (ε, j) basic p-convex s.c. Then:*

(i) *For every $f \in B_{(p)}$*

$$|f(x)| \leq \begin{cases} \varepsilon & \text{if } f = \pm e_n^*, \\ \dfrac{\theta}{m_i} & \text{if } w(f) = m_i/\theta,\ i \geq j, \\ \dfrac{2\theta^2}{m_i m_j} & \text{if } w(f) = m_i/\theta,\ i < j, \end{cases}$$

and thus $|f(x)| \leq \frac{\theta}{m_j}$. It follows in particular that $\|x\| = \frac{\theta}{m_j}$.

(ii) *If $\varepsilon \leq \frac{1}{m_j^3}$ and the functional $f \in B_{(p)}$ has a tree analysis $T_f = (f_\alpha)_{\alpha\in\mathcal{T}}$ with $w(f_\alpha) \neq m_j/\theta$ for all $\alpha \in \mathcal{T}$, then $|f(x)| \leq \frac{2\theta}{m_j^3}$.*

Proof. Let $\sum_{k\in F} a_k e_k$ be a p-convex special combination and let $f \in B_{(p)}$. Then we have that $\sum_{k\in F} a_k^p e_k$ is an (ε^p, j) 1-special combination and from Lemma 3.1 there exists $g \in W$ with $w(f) = w(g)^{1/p}$ such that

$$\left| f\left(\sum_{k\in F} a_k e_k\right)\right| \leq \left| g\left(\sum_{k\in F} a_k^p e_k\right)\right|^{1/p}.$$

So it is enough to prove the result for the 1-special combination $\sum_{k\in F} a_k^p e_k$ in the space Z. The proof of this is given in Proposition 3.19 of [8] and we omit it. ☐

4. Special vectors and the basic inequality

In this section we shall define the p-convex special combinations and the rapidly increasing sequences (R.I.S.), and we shall prove that we can compare the action of a functional on an R.I.S. to the action of a functional of the auxiliary space on a p-convex special combination of its basis (basic inequality, Proposition 4.9). The basic inequality will be an essential tool in the proof that the spaces and their duals are HI as well as in the proof that c_0 is finitely disjointly representable in these spaces.

Definition 4.1 (p-convex special combinations). Let $\varepsilon > 0$ and $j \in \mathbb{N}$ with $j > 1$, and let $(x_k)_{k\in\mathbb{N}}$ be a block sequence in $X_{(p)}$.

(a) A p-convex combination $x = \sum_{k\in F} a_k x_k$ of the sequence $(x_k)_{k\in\mathbb{N}}$ is called an (ε, j) *p-convex special combination* ((ε, j) p-convex s.c.) of $(x_k)_{k\in\mathbb{N}}$ if, for $r_k = \min\operatorname{supp} x_k$, $k \in \mathbb{N}$, $\sum_{k\in F} a_k e_{r_k}$ is an (ε, j) basic p-convex special combination.

(b) If, in addition, the sequence $(x_k)_{k\in\mathbb{N}}$ satisfies $\|x_k\| \leq C$ for all k, and there exists a functional $f \in D$ with $f(x) = 1$ and $\operatorname{ran} f \subset \operatorname{ran} x$, then we say that $\sum_{k\in F} a_k x_k$ is a C- *seminormalized* (ε, j) *p-convex special combination* of $(x_k)_{k\in\mathbb{N}}$.

Remark 4.2. Let $(x_k)_{k\in\mathbb{N}}$ be a block sequence in $X_{(p)}$, and let $\varepsilon > 0$ and $j \in \mathbb{N}$ with $j > 1$. Then, for every $n \in \mathbb{N}$ such that $\gamma(j)(n_{j-1}+1) < n \leq n_j$ there exists an (ε, j) p-convex s.c. $\sum_{k\in F} a_k x_k$ with the additional property $\{\min\operatorname{supp} x_k :\ k \in F\} \in \mathcal{S}_n$.

Indeed, if $r_k = \min\operatorname{supp} x_k$ for $k = 1, 2, \ldots$, then by Remark 3.3, there exists an (ε, j) basic p-convex s.c. $\sum_{k\in F} a_k e_{r_k}$ with $\{r_k :\ k \in F\} \in \mathcal{S}_n$. Thus $\sum_{k\in F} a_k x_k$ is the desired (ε, j) p-convex s.c.

The next proposition ensures that for every $\lambda > 2$ there exists a λ-seminormalized $(\varepsilon, 2j)$-p-convex s.c. in every block subspace.

Proposition 4.3. *Let $(x_k)_{k\in\mathbb{N}}$ be a normalized block sequence in $X_{(p)}$, and let $\varepsilon > 0$ and $j \in \mathbb{N}$ with $j > 1$. Then, there exist a block sequence $(y_k)_{k\in\mathbb{N}}$ of $(x_k)_{k\in\mathbb{N}}$ with $\|y_k\| \leq 1$ for all k and an $(\varepsilon, 2j)$ p-convex s.c. $\sum_{k\in F} a_k y_k$ with the following properties:*

(1) $\|\sum_{k\in F} a_k y_k\| \geq \frac{1}{2}$.
(2) *The family $\{y_k :\ k \in F\}$ is $\mathcal{S}_{\gamma(2j)(n_{2j-1}+1)+1}$ admissible.*

It follows that for every $\lambda > 2$ there exists a λ-seminormalized $(\varepsilon, 2j)$-p-convex s.c.

The proof appears in [2] and [8] for $p = 1$, so we omit it. The next proposition is the dual of the above proposition. It will be an essential tool in the proof that $X^*_{(p)}$ is HI as well as in the result concerning the structure of the space of operators on $X_{(p)}$.

Proposition 4.4. *Let $(x^*_k)_{k\in\mathbb{N}}$ be a normalized block sequence in $X^*_{(p)}$ such that $x^*_k \in D$ for all k, and let $\varepsilon > 0$ and $j \in \mathbb{N}$ with $j > 1$. Then, there exists a finite block sequence $(y^*_k)_{k\in F}$ of $(x^*_k)_{k\in\mathbb{N}}$ such that:*

(1) *There exists $(\beta_k)_{k\in F} \in B_{\ell_q}$ such that $\sum_{k\in F} \beta_k y^*_k$ is an $(\varepsilon, 2j)$ q-convex special combination, i.e., if $r_k = \operatorname{minsupp} y^*_k$, then $\sum_{k\in F} \beta_k e^*_{r_k}$ is an $(\varepsilon, 2j)$ basic q-convex special combination.*
(2) *The set $\{\operatorname{minsupp} y^*_k : k \in F\}$ is a maximal $\mathcal{S}_{\gamma(2j)(n_{2j-1}+1)+1}$ set.*
(3) *We have $\|y^*_k\| > 1/2$ and $\|\sum_{k\in F} \beta_k y^*_k\| \leq 1$.*

Proof. Assume the contrary. We set $r = \gamma(2j)(n_{2j-1}+1)+1$ and $m = \log_2(m_{2j}) + 1$ and observe that $m > 2$ and $mr \leq n_{2j}$. We proceed by constructing, for every $l = 0, \ldots, m$, a block sequence $(f^l_k)_{k\in\mathbb{N}}$ of $(x^*_k)_k$ satisfying the following conditions:

(i) For all $l \geq 1$ and $k \in \mathbb{N}$, $f_k^l = \sum_{i \in G_k^l} \beta_i f_i^{l-1}$ is an $(\varepsilon, 2j)$- q-convex s.c. of $(f_i^{l-1})_{i \in G_k^l}$.

(ii) The set $\{\operatorname{minsupp} f_i^{l-1} : i \in G_k^l\}$ is a maximal $\mathcal{S}_r$ set.

(iii) $2^{l-1} < \|f_k^l\| \leq m_{2j}$ for $1 \leq l \leq m$ and all k.

After the completion of the inductive construction, (iii) yields that $2^{m-1} < m_{2j}$ which contradicts the choice of m.

The construction is done by induction on l. We set $f_k^0 = x_k^*$ for all k. Suppose that the sequences $(f_k^s)_{k \in \mathbb{N}}$, $s \leq l-1$, have been chosen to satisfy the inductive assumptions. From Remark 4.2 we may choose a block sequence $(f_k^l)_k$ such that, for each k, $f_k^l = \sum_{i \in G_k^l} \beta_i f_i^{l-1}$ is an $(\varepsilon, 2j)$ q-convex s.c. of $(f_i^{l-1})_{i \in \mathbb{N}}$ and $\{\min \operatorname{supp} f_i^{l-1} : i \in G_k^l\}$ is a maximal $\mathcal{S}_r$ set.

By our inductive assumptions (i), (ii) and the definition of f_k^l we get that each f_k^l may be written as $f_k^l = \sum_{i \in F_k^l} \gamma_i x_i^*$, where the set $\{\operatorname{minsupp} x_i^* : i \in F_k^l\}$ is $\mathcal{S}_{lr}$ maximal and $(\gamma_i)_{i \in F_k^l} \in B_{\ell_q}$. Since $lr \leq n_{2j}$ we get that $\{\operatorname{minsupp} x_i^* : i \in F_k^l\}$ is $\mathcal{S}_{n_{2j}}$ admissible and hence

$$\|f_k^l\| = \left\| \sum_{i \in F_k^l} \gamma_i x_i^* \right\| \leq m_{2j}.$$

If $\|f_k^l\| \leq 2^{l-1}$ for some k, then we get

$$\left\| \sum_{i \in G_k^l} \beta_i \left(\frac{1}{2^{l-1}} f_i^{l-1} \right) \right\| \leq 1.$$

By our inductive condition (iii) we also have $\|\frac{1}{2^{l-1}} f_i^{l-1}\| > \frac{1}{2}$ for all i. This leads to a contradiction since we have assumed that the proposition is not true. Therefore, $\|f_k^l\| > 2^{l-1}$ for each k. This completes the inductive construction and the proof of the proposition. □

Proposition 4.5. *Let $C > 0$. Let $(x_k)_{k \in \mathbb{N}}$ be a block sequence in $X_{(p)}$ with $\|x_k\| \leq C$ for each k, let $1 < j \in \mathbb{N}$, and $0 < \varepsilon < 1/3m_j$ and let $x = \sum_{k \in F} a_k x_k$ be an (ε, j) p-convex s.c. Then for every $i < j$ and*

$$f = \frac{1}{m_i} \sum_{r=1}^{d} \beta_r f_r \in K$$

we have

$$\left| f \left(\sum_{k \in F} a_k x_k \right) \right| \leq \frac{5C}{m_i} \left(\sum_{r \in I} |\beta_r|^q \right)^{1/q},$$

where I is the smallest interval of the form $[2n-1, 2m]$ which contains $\{r \leq d : \operatorname{ran} f_r \cap \operatorname{ran} x \neq \emptyset\}$.

Proof. Fix $i < j$, let $f_1 < f_2 < \cdots < f_d$ be an $\mathcal{S}_{n_i}$ admissible sequence in D and let $(\beta_r)_{r=1}^d \in B_{\ell_q}$ be such that

$$f = \frac{1}{m_i}(\beta_1 f_1 + \beta_2 f_2 + \cdots + \beta_d f_d) \in K.$$

Let I be the smallest interval of the form $[2n-1, 2m]$ which contains $\{r \leq d : \operatorname{ran} f_r \cap \operatorname{ran} x \neq \emptyset\}$.

We set

$$F_2 = \{k \in F : x_k \text{ is split by the family } (f_r)_{r\in I}\} \text{ and } F_1 = F \setminus F_2 .$$

Using the fact that $X_{(p)}$ satisfies an upper ℓ_p-estimate (Proposition 2.9) and setting

$$D_r = \{k \in F_1 : \operatorname{ran} f_r \cap \operatorname{ran} x_k \neq \emptyset\}$$

for $r \in I$, we get

$$\begin{aligned}(4.1)\qquad \left|\left(\sum_{r=1}^d \beta_r f_r\right)\left(\sum_{k\in F_1} a_k x_k\right)\right| &= \left|\sum_{r\in I} \beta_r f_r \left(\sum_{k\in D_r} a_k x_k\right)\right| \\ &\leq \sum_{r\in I} |\beta_r| \cdot 4 \left(\sum_{k\in D_r} \|a_k x_k\|^p\right)^{1/p} \\ &\leq 4C\left(\sum_{r\in I} |\beta_r|^q\right)^{1/q},\end{aligned}$$

where the last inequality follows from Hölder's inequality. To estimate the action of f on $\sum_{k\in F_2} a_k x_k$, we set $l_k = \min\operatorname{supp} x_k$ for each k and observe that $\min\operatorname{supp} x_1 \geq 2$. For each $k \in F_2$ we choose $r_k \in \{1, 2, \ldots, d\}$ such that

$$\min\operatorname{supp} x_k \leq \min\operatorname{supp} f_{r_k} \leq \max\operatorname{supp} x_k.$$

Since $\{\min\operatorname{supp} f_r : r = 1, 2, \ldots, d\} \in \mathcal{S}_{n_i}$, it follows that $\{\min\operatorname{supp} f_{r_k} : k \in F_2\} \in \mathcal{S}_{n_i}$, and thus $\{l_k : k \in F_2 \setminus \{\min F_2\}\} \in \mathcal{S}_{n_i}$. So the set $\{l_k : k \in F_2\}$ is $\mathcal{S}_{n_i+1}$ admissible. Since $n_i + 1 \leq \gamma(j)(n_{j-1}+1)$, we get that $\{l_k : k \in F_2\} \in \mathcal{S}_{\gamma(j)(n_{j-1}+1)}$. The fact that $\sum_{k\in F} a_k x_k$ is an (ε, j) p-convex s.c. yields that $\sum_{k\in F_2} a_k^p < \varepsilon^p$.

For every $k \in F_2$, let

$$A_k = \{r \in I : \operatorname{ran} f_r \cap \operatorname{ran} x_k \neq \emptyset\}.$$

It is clear that A_k is an interval. Hence, letting A_k' be the smallest interval of the form $[2n-1, 2m]$ which contains A_k, it follows from Remark 2.7 that

$$(4.2)\qquad \left|\frac{1}{m_i}\sum_{r\in A_k} \beta_r f_r(x_k)\right| \leq \left(\sum_{r\in A_k'} |\beta_r|^q\right)^{1/q} \|x_k\| \leq C\left(\sum_{r\in A_k'} |\beta_r|^q\right)^{1/q}.$$

Since every r belongs to at most three A'_k's, by (4.2) and Hölder's inequality we get

$$\left|\left(\sum_{r=1}^{d}\beta_r f_r\right)\left(\sum_{k\in F_2} a_k x_k\right)\right| \le m_i C \sum_{k\in F_2} a_k \left(\sum_{r\in A'_k}|\beta_r|^q\right)^{1/q} \tag{4.3}$$

$$\le C m_i \left(\sum_{k\in F_2} a_k^p\right)^{1/p} 3 \left(\sum_{r\in I}|\beta_r|^q\right)^{1/q}$$

$$= (3Cm_i\cdot\varepsilon)\left(\sum_{r\in I}|\beta_r|^q\right)^{1/q}$$

$$< C\left(\sum_{r\in I}|\beta_r|^q\right)^{1/q}.$$

From (4.1) and (4.3) we conclude that

$$\left|\left(\sum_{r=1}^{d}\beta_r f_r\right)\left(\sum_{k\in F} a_k x_k\right)\right| < 5C\left(\sum_{r\in I}|\beta_r|^q\right)^{1/q}.$$

This completes the proof. □

Definition 4.6 (Rapidly increasing sequences). Let $C>0$ and $(j_k)_{k\in\mathbb{N}}$ be an increasing sequence of positive integers. A block sequence $(x_k)_{k\in\mathbb{N}}$ in $X_{(p)}$ is a $(C,(j_k)_{k\in\mathbb{N}})$-rapidly increasing sequence (R.I.S), if the following are satisfied for every $k\in\mathbb{N}$:

(a) $\|x_k\|\le C$.
(b) $\frac{1}{m_{j_{k+1}}}(\max\operatorname{supp} x_k) < \frac{1}{m_{j_k}}$.
(c) For every $i<j_k$ and every $f\in K$ with $w(f)=m_i$, we have $|f(x_k)|\le \frac{C}{m_i}$.

Remark 4.7. Let $(x_k)_{k\in\mathbb{N}}$ be a $(C,(j_k)_{k\in\mathbb{N}})$-rapidly increasing sequence and $k\in\mathbb{N}$. Note that if $i<j_k$ and

$$f=\frac{1}{m_i}\sum_r \beta_r f_r \in K,$$

then by Lemma 2.6 we get that

$$|f(x_k)|\le \frac{C}{m_i}\left(\sum_{r\in I}|\beta_r|^q\right)^{1/q},$$

where I is the smallest interval of the form $[2n-1,2m]$ which contains $\{r : \operatorname{supp} f_r\cap\operatorname{ran} x_k\neq\emptyset\}$.

PROPOSITION 4.8 (Existence of an R.I.S.). *Let $C > 0$ and $(z_l)_l$ be a block sequence in $X_{(p)}$ with $\|z_l\| \leq C$ for every l. Suppose that $(j_k)_{k\in\mathbb{N}}$ is a strictly increasing sequence of integers and $(x_k)_{k\in\mathbb{N}}$ is a block sequence such that each x_k is a $(1/3m_{j_k}, j_k)$- p-convex s.c. of the sequence $(z_l)_l$. Then there exists a subsequence $(x_{k_n})_n$ of $(x_k)_k$ which is a $(5C, (j_{k_n})_{n\in\mathbb{N}})$ R.I.S.*

Proof. Choose a subsequence $(x_{k_n})_n$ of $(x_k)_k$ such that $m_{j_{k_n}} \max\operatorname{supp}(x_{k_n}) < m_{j_{k_{n+1}}}$ for all n. From Proposition 4.5 we get that the sequence $(x_{k_n})_n$ satisfies condition (c) of the definition of the R.I.S with constant $5C$. Therefore this subsequence is a $(5C, (j_{k_n})_n)$ R.I.S. □

PROPOSITION 4.9 (The basic inequality). *Recall that we have set $\theta = 4^{1/q}$ and that $B_{(p)}$ is the norming set of the auxiliary space $T_{(p)}$. Let $(x_k)_{k\in\mathbb{N}}$ be a $(C, (j_k)_{k\in\mathbb{N}})$ R.I.S. in $X_{(p)}$. Then:*

(a) *For every sequence $(\lambda_k)_k$ of scalars and every $f \in K$, we can find functionals g_1 and g_2 in c_{00}, such that either $g_1 = h_1$ or $g_1 = e_r^* + h_1$ with $r \notin \operatorname{supp} h_1$, where $h_1 \in \operatorname{conv}_{\mathbb{Q}}\{h \in B_{(p)} : w(h) = w(f)/\theta\}$, while $\|g_2\|_{\ell_q} \leq \frac{2}{m_{j_1}}$, such that*

$$\left|f\left(\sum \lambda_k x_k\right)\right| \leq C(g_1 + g_2)\left(\sum |\lambda_k| e_{r_k}\right), \tag{4.4}$$

where $r_k = \min\operatorname{supp} x_k$ for each $k = 1, 2, \ldots$.

(b) *If in addition we assume that there exists $j_0 < j_1 - 1$ such that, for every $\phi \in K$ with $w(\phi) = m_{j_0}$ and every interval E of natural numbers, we have*

$$\left|\phi\left(\sum_{k\in E} \lambda_k x_k\right)\right| \leq C\left(\max_{k\in E} |\lambda_k| + \frac{1}{m_{j_0+1}}\left(\sum_{k\in E} |\lambda_k|^p\right)^{1/p}\right), \tag{4.5}$$

then h_1 may be selected to have a tree analysis $(h_t)_{t\in\mathcal{T}}$ with $w(h_t) \neq \frac{m_{j_0}}{\theta}$ for all $t \in \mathcal{T}$ but this time the functional g_2 satisfies $\|g_2\|_{\ell_q} \leq \frac{2}{m_{j_0+1}}$.

The proof of the basic inequality is an adaptation of the proof of Proposition 4.3 in [8] and we omit it.

PROPOSITION 4.10. *Let $C > 1$, let $(x_k)_{k\in\mathbb{N}}$ be a $(C, (j_k)_{k\in\mathbb{N}})$ rapidly increasing sequence in $X_{(p)}$, $j \in \mathbb{N}$ with $1 < j < j_1$ and $\varepsilon > 0$ with $\varepsilon \leq \frac{1}{m_j^3}$.*

(1) *Let $\sum_{k\in F} a_k x_k$ be an (ε, j) p-convex s.c. of the sequence $(x_k)_{k\in\mathbb{N}}$. Then:*

(a) *For every $f \in K$ with $w(f) < m_j$ we have*

$$\left|f\left(\sum_{k\in F} a_k x_k\right)\right| \leq \frac{33C}{w(f) m_j}.$$

(b) *For every $f \in K$ with $m_j \leq w(f)$ we have*

$$\left| f\left(\sum_{k\in F} a_k x_k \right) \right| \leq C\left(\frac{4}{w(f)} + \frac{2}{m_j^3} \right).$$

In particular,

$$\left\| \sum_{k\in F} a_k x_k \right\| \leq \frac{5C}{m_j}. \tag{4.6}$$

(2) *If the sequence $(c_k)_{k\in F}$ is such that $\sum_{k\in F} |c_k| x_k$ is an (ε, j) p-convex s.c. and assumption (b) of the basic inequality is fulfilled for $j_0 = j$ and for the linear combination $\sum_{k\in F} c_k x_k$ of $(x_k)_{k\in\mathbb{N}}$, then*

$$\left\| \sum_{k\in F} c_k x_k \right\| \leq \frac{10C}{m_j^3}.$$

Proof. For the proof of (1a) and (1b), let $f \in K$. It follows from the basic inequality that there exist $h_1 \in \operatorname{conv}_{\mathbb{Q}}\left\{ h \in B_{(p)} : w(h) = w(f)/\theta \right\}$, $r \in \mathbb{N}$, with $r \notin \operatorname{supp} h_1$ and $g_2 \in c_{00}$ with $\|g_2\|_{\ell_q} \leq \frac{2}{m_{j_1}}$ such that

$$\begin{aligned} \left| f\left(\sum_{k\in F} a_k x_k \right) \right| &\leq C\left((h_1 + e_r^*) + g_2 \right)\left(\sum_{k\in F} a_k e_{r_k} \right) \\ &\leq C\left(h_1\left(\sum_{k\in F} a_k e_{r_k} \right) + \max_{k\in F} a_k + \|g_2\|_{\ell_q} \|(a_k)_k\|_{\ell_p} \right) \\ &\leq C\left(h_1\left(\sum_{k\in F} a_k e_{r_k} \right) + \varepsilon + \frac{2}{m_{j_1}} \right), \end{aligned} \tag{4.7}$$

where $r_k = \min \operatorname{supp} x_k$ for each k. From Proposition 3.4 we get

$$\left| f\left(\sum_{k\in F} a_k x_k \right) \right| \leq \begin{cases} C\left(\frac{\theta}{w(f)} + \frac{2}{m_j^3} \right) \leq \dfrac{4C}{w(f)} + \dfrac{2C}{m_j^3} & \text{if } w(f) \geq m_j, \\ C\left(\frac{2\theta^2}{w(f)m_j} + \frac{2}{m_j^3} \right) \leq \dfrac{33C}{w(f)m_j} & \text{if } w(f) < m_j, \end{cases} \tag{4.8}$$

since $\theta = 4^{1/q}$. From (4.8) we get (1a) and (1b) and moreover $\| \sum_{k\in F} a_k x_k \| \leq \frac{5C}{m_j}$.

We now pass to the proof of (2). Let $f \in K$. Assume first that $w(f) \neq m_j$. Since the linear combination $\sum_{k\in F} c_k x_k$ satisfies assumption (b) of the basic inequality for $j_0 = j$, we may choose h_1 and g_2 to satisfy

$$\left| f\left(\sum_{k\in F} c_k x_k \right) \right| \leq C\left(h_1\left(\sum_{k\in F} |c_k| e_{r_k} \right) + \max_{k\in F} |c_k| + g_2\left(\sum_{k\in F} |c_k| e_{r_k} \right) \right),$$

assuming in addition that h_1 has a tree $(f_t)_{t\in\mathcal{T}}$ with $w(f_t)\neq\frac{m_j}{\theta}$ for all $t\in\mathcal{T}$.

Using Lemma 3.4(ii), we conclude that

$$\left|f\left(\sum_{k\in F}c_kx_k\right)\right|\leq C\left(\frac{2\theta}{m_j^3}+\varepsilon+\|g_2\|_{\ell_q}\right)\leq\frac{10C}{m_j^3}.$$

The case $w(f)=m_j$ follows immediately, since our assumption that part (b) of the basic inequality is satisfied implies that

$$\left|f\left(\sum_{k\in F}c_kx_k\right)\right|\leq C\left(\max_k|c_k|+\frac{1}{m_{j+1}}\left(\sum_{k\in F}|c_k|^p\right)^{1/p}\right)$$
$$\leq C\left(\frac{1}{m_j^3}+\frac{1}{m_{j+1}}\right)\leq\frac{2C}{m_j^3}.$$

This completes the proof of (2). □

5. The space $X_{(p)}$ is hereditarily indecomposable

Definition 5.1 (Exact pair). Let $C>33$ and $j\in\mathbb{N}$. A pair (x,ϕ) with $x\in X_{(p)}$ and $\phi\in K$ is said to be a $(C,2j)$ exact pair if the following conditions are satisfied:

(i) There exist a $(\frac{C}{33},(j_k)_k)$−R.I.S. $(y_k)_k$ with $2j<j_1$ and $\|y_k\|>1$, for every $k\in\mathbb{N}$, and a $(\frac{1}{m_{2j}^3},2j)$ p-convex s.c. $\sum_{k\in F}a_ky_k$ such that $x=m_{2j}\sum_{k\in F}a_ky_k$.

(ii) $w(\phi)=m_{2j}$.

(iii) $\phi(x)=1$ and $\operatorname{ran}x=\operatorname{ran}\phi$.

Remark 5.2. Let $C>33$, $j\in\mathbb{N}$, and let (x,ϕ) be a $(C,2j)$ exact pair, and let $\psi\in K$ with $w(\psi)=m_i\neq m_{2j}$. Then, by Proposition 4.10, we have:

(a) $1<\|x\|<C$.

(b) If $i<2j$, then $|\psi(x)|\leq\frac{C}{m_i}$.

(c) If $i>2j$, then $|\psi(x)|\leq\frac{C}{m_{2j}^2}$.

Proposition 5.3. *Let $j=2,3,\ldots$ and $C>330$. Then every block subspace of $X_{(p)}$ contains a $(C,2j)$-exact pair.*

Proof. Let $(x_k)_{k\in\mathbb{N}}$ be a block sequence in $X_{(p)}$ and $\lambda=\frac{C}{165}>2$. It follows from Proposition 4.3 that there exists a block sequence $(z_k)_{k\in\mathbb{N}}$ of $(x_k)_{k\in\mathbb{N}}$ such that, for every k, z_k is a λ-seminormalized $(\frac{1}{m_{2k}^2},2k)$ p-convex s.c. (see Definition 4.1 (b)). In particular, for every k, $\|z_k\|\leq\lambda$ and there exists $z_k^*\in D$ with $z_k^*(z_k)=1$ and $\operatorname{ran}z_k^*\subset\operatorname{ran}z_k$. Using Proposition 4.8, we may pass to a subsequence $(y_k)_k$ of $(z_k)_k$ which is a $(5\lambda,(2j_k)_{k\in\mathbb{N}})$ R.I.S. for an appropriate sequence $(j_k)_k$.

Now let $y = \sum_{k\in F} a_k y_k$ be a $(\frac{1}{m_{2j}^3}, 2j)$ p-convex s.c. of $(y_k)_k$. For every $k \in F$, let $y_k^* \in D$ with $y_k^*(y_k) = 1$ and $\operatorname{ran} y_k^* \subset \operatorname{ran} y_k$ and let $(\beta_k)_{k\in F}$ be the conjugate sequence of $(a_k)_{k\in F}$. We may assume that β_k is rational for every $k \in F$. Then $\phi = \frac{1}{m_{2j}} \sum_{k\in F} \beta_k y_k^*$ belongs to K and $\phi(y) = \frac{1}{m_{2j}}$. After a small perturbation we may also assume that $\operatorname{ran}\phi = \operatorname{ran} y$. So setting $x = m_{2j} y$, we get that (x, ϕ) is a $(C, 2j)$ exact pair. □

LEMMA 5.4. *Let $C > 33$ and let i, r and $(j_k)_{k=1}^r$ be positive integers with $i + 3 < 2j_1 < 2j_2 < \cdots < 2j_r$. Let $(x_k)_{k=1}^r \subset X_{(p)}$, $(\phi_k)_{k=1}^r \subset X_{(p)}^*$ be block sequences such that, for every $k \le r$, (x_k, ϕ_k) is a $(C, 2j_k)$ exact pair. Suppose that, for some $d \in \mathbb{N}$, $(f_l)_{l=1}^d$ is a $\mathcal{S}_{n_i}$-admissible sequence of functionals in K such that $i + 3 < w(f_1) < \cdots < w(f_d)$ and $w(f_l) \neq m_{2j_k}$ for every $l \le d$ and $k \le r$. Then, for every choice of coefficients $(c_k)_{k=1}^r$, and $(\beta_l)_{l=1}^d$ with $(\beta_l)_{l=1}^d \in B_{\ell_q}$, we have*

$$\left| \left(\sum_{l=1}^d \beta_l f_l \right) \left(\sum_{k=1}^r c_k x_k \right) \right| \le \frac{C}{m_{i+2}} \left(\sum_{k=1}^r |c_k|^p \right)^{1/p}. \tag{5.1}$$

Proof. We start with the following remark which is needed for the proof of Lemma 6.2: The result of the lemma remains true if the assumption that each (x_k, ϕ_k) is a $(C, 2j_k)$ exact pair is relaxed as follows: For every $k = 1, \ldots, r$, x_k satisfies condition (i) of Definition 5.1 for $j = j_k$. In other words, the functionals ϕ_k, $k = 1, \ldots, r$, do not play any role in the proof.

Set

$$J = \left\{ k : \operatorname{ran}(x_k) \cap \left(\bigcup_{l=1}^d \operatorname{ran}(f_l) \right) \neq \emptyset \right\}.$$

We partition the set J into two sets F and G as follows:

$$G = \{ k \in J : x_k \text{ is split by the sequence } (f_l)_{l=1}^d \} \text{ and } F = J \setminus G.$$

We first consider $\sum_{k\in F} c_k x_k$. For each $l = 1, \ldots, d$, let

$$F_l = \{ k \in F : x_k \text{ is covered by } f_l \}.$$

Suppose that

$$f_l = \frac{1}{w(f_l)} \sum_{i\in A_l} \gamma_i h_i,$$

where $w(f_l) = m_{r_l}$, $(h_i)_{i\in A_l}$ is a n_{r_l}-admissible sequence of functionals in D and $(\gamma_i)_{i\in A_l} \in B_{\ell_q}$. Partition the set F_l as follows:

$$F_l^1 = \{ k \in F_l : x_k \text{ is covered by some } h_i,\ i \in A_l \},$$
$$F_l^2 = \{ k \in F_l : x_k \text{ is split by the sequence } (h_i)_{i\in A_l} \text{ and } w(f_l) < m_{2j_k} \},$$
$$F_l^3 = \{ k \in F_l : x_k \text{ is split by the sequence } (h_i)_{i\in A_l} \text{ and } m_{2j_k} < w(f_l) \}.$$

We first turn to the set $\bigcup_{l=1}^d F_l^1$. If we set, for fixed l and $i \in A_l$,

$$B_i = \left\{k \in F_l^1 : x_k \text{ is covered by } h_i\right\},$$

then using Proposition 2.9 we get

$$\begin{aligned}\left|f_l\left(\sum_{k\in F_l^1} c_k x_k\right)\right| &\leq \frac{1}{w(f_l)}\sum_{i\in A_l}|\gamma_i|\left|h_i\left(\sum_{k\in B_i} c_k x_k\right)\right| \\ &\leq \frac{4C}{w(f_l)}\sum_{i\in A_l}|\gamma_i|\left(\sum_{k\in B_i}|c_k|^p\right)^{1/p} \\ &\leq \frac{4C}{w(f_l)}\left(\sum_{k\in F_l^1}|c_k|^p\right)^{1/p},\end{aligned}$$

where the last inequality follows by Hölder's inequality. It follows that

$$\begin{aligned}\left|\sum_{l=1}^d \beta_l f_l\left(\sum_{k\in F_l^1} c_k x_k\right)\right| &\leq 4C\sum_{l=1}^d|\beta_l|\frac{1}{w(f_l)}\left(\sum_{k\in F_l^1}|c_k|^p\right)^{1/p} \\ &\leq \frac{4C}{w(f_1)}\left(\sum_{k\in\bigcup_l F_l^1}|c_k|^p\right)^{1/p}.\end{aligned} \tag{5.2}$$

We now turn to $\bigcup_{l=1}^d F_l^2$. For fixed $l = 1,\dots,d$ and each $k \in F_l^2$, we set

$$R_k = \{i \in A_l : \operatorname{ran} h_i \cap \operatorname{ran} x_k \neq \emptyset\}$$

and we let $\tilde{R}_k$ be the smallest interval of the form $[2n-1, 2m]$ which contains R_k. Note that each $i \in A_l$ belongs to at most three $\tilde{R}_k$'s. Using this, for $k \in F_l^2$, $w(f_l) < m_{2j_k}$, as well as Remark 5.2(b) and Lemma 2.6, we get

$$\begin{aligned}\left|f_l\left(\sum_{k\in F_l^2} c_k x_k\right)\right| &\leq \sum_{k\in F_l^2}|c_k|\left|\frac{1}{w(f_l)}\left(\sum_{i\in R_k}\gamma_i h_i\right)(x_k)\right| \\ &\leq \frac{C}{w(f_l)}\sum_{k\in F_l^2}|c_k|\left(\sum_{i\in\tilde{R}_k}|\gamma_i|^q\right)^{1/q} \\ &\leq \frac{3^{1/q}C}{w(f_l)}\left(\sum_{k\in F_l^2}|c_k|^p\right)^{1/p}.\end{aligned}$$

It follows that

$$\left|\sum_{l=1}^{d} \beta_l f_l \left(\sum_{k\in F_l^2} c_k x_k\right)\right| \le \sum_{l=1}^{d} |\beta_l| \frac{3^{1/q} C}{w(f_l)} \left(\sum_{k\in F_l^2} |c_k|^p\right)^{1/p} \le \frac{3^{1/q} C}{w(f_1)} \left(\sum_{k\in \bigcup_l F_l^2} |c_k|^p\right)^{1/p}. \tag{5.3}$$

Next we turn to $\bigcup_{l=1}^{d} F_l^3$. If $k \in F_l^3$ for some $l = 1, \ldots, d$, then by Remark 5.2(c) it follows that $|f_l(x_k)| \le \frac{C}{m_{2j_k}^2}$. Hence,

$$\begin{aligned}\left|\sum_{l=1}^{d} \beta_l f_l \left(\sum_{k\in F_l^3} c_k x_k\right)\right| &\le \sum_{l=1}^{d} |\beta_l| \sum_{k\in F_l^3} |c_k| \frac{C}{m_{2j_k}^2} \\ &\le C \sum_{l=1}^{d} |\beta_l| \left(\sum_{k\in F_l^3} \frac{1}{m_{2j_k}^{2q}}\right)^{1/q} \left(\sum_{k\in F_l^3} |c_k|^p\right)^{1/p} \\ &\le \frac{C}{m_{2j_1}} \sum_{l=1}^{d} |\beta_l| \left(\sum_{k\in F_l^3} |c_k|^p\right)^{1/p} \\ &\le \frac{C}{m_{2j_1}} \left(\sum_{k\in \bigcup_l F_l^3} |c_k|^p\right)^{1/p}.\end{aligned} \tag{5.4}$$

Finally, we turn to the set

$$G = \left\{k \in J : x_k \text{ is split by } (f_l)_{l=1}^{d}\right\}.$$

For every $k \in G$, let

$$L_k = \{l = 1, \ldots, d : \operatorname{ran} f_l \cap \operatorname{ran} x_k \ne \emptyset\}.$$

We partition L_k into

$$M_k = \{l \in L_k : w(f_l) < m_{2j_k}\} \text{ and } N_k = L_k \setminus M_k.$$

From Remark 5.2(b), we get $|f_l(x_k)| \le \frac{C}{w(f_l)}$ for every $l \in M_k$. So, for fixed $k \in G$,

$$\left|\sum_{l\in M_k}\beta_l f_l(x_k)\right| \le C\sum_{l\in M_k}|\beta_l|\frac{1}{w(f_l)}$$
$$\le C\left(\sum_{l\in M_k}|\beta_l|^q\right)^{1/q}\left(\sum_{l\in M_k}\frac{1}{w(f_l)^p}\right)^{1/p}$$
$$\le \frac{2C}{w(f_{l_k^1})}\left(\sum_{l\in M_k}|\beta_l|^q\right)^{1/q}\le\frac{2C}{w(f_{l_k^1})},$$

where $l_k^1=\min M_k$. It follows that

$$\left|\sum_{k\in G}c_k\left(\sum_{l\in M_k}\beta_l f_l\right)(x_k)\right|\le 2C\left(\sum_{k\in G}|c_k|^p\right)^{1/p}\left(\sum_{k\in G}\frac{1}{w(f_{l_1^k})^q}\right)^{1/q}.$$

Note that $l_k^1\neq l_r^1$, for $k\neq r$ in G. Hence

$$\left(\sum_{k\in G}\frac{1}{w(f_{l_1^k})^q}\right)^{1/q}\le\sum_{j>i+3}\frac{1}{m_j}<\frac{1}{m_{i+3}}.$$

So,

$$\left|\sum_{k\in G}c_k\left(\sum_{l\in M_k}\beta_l f_l\right)(x_k)\right|\le\frac{2C}{m_{i+3}}\left(\sum_{k\in G}|c_k|^p\right)^{1/p}. \tag{5.5}$$

Now we consider the set N_k for fixed $k\in G$. Let $x_k=m_{2j_k}\sum_{t\in I_k}a_t y_t$, where $\sum_{t\in I_k}a_t y_t$ is a $(\frac{1}{m_{2j_k}^3},2j_k)$ p-convex s.c. of a $(\frac{C}{33},(i_t)_t)$ R.I.S. $(y_t)_t$ as in Definition 5.1. We set

$$T=\{t\in I_k : y_t \text{ is split by the family } (f_l)_{l\in N_k}\}$$

and, for $l\in N_k$, we set

$$J_l=\{t\in I_k : f_l \text{ covers } y_t\}.$$

Since the family $(f_l)_{l=1}^d$ is $\mathcal{S}_{n_i}$-admissible, it follows that the family $(y_t)_{t\in T}$ is $\mathcal{S}_{n_i+1}$-admissible, and since $i+3<2j_k$, we get

$$\left(\sum_{t\in T}a_t^p\right)^{1/p}<\frac{1}{m_{2j_k}^3}.$$

Note also that the functional $\frac{1}{m_i}\sum_{l=1}^d\beta_l f_l$ may not belong to $B_{X_{(p)}^*}$ (if i is odd), but $\frac{1}{m_{i+1}}\sum_{l=1}^d\beta_l f_l$ does belong to $B_{X_{(p)}^*}$. Using also Proposition 2.9

and the fact that $\|y_t\| < C$, it follows that

$$\left|\left(\sum_{l\in N_k}\beta_l f_l\right)\left(\sum_{t\in T} m_{2j_k} a_t y_t\right)\right| \leq 4Cm_{i+1}m_{2j_k}\left(\sum_{t\in T}|a_t|^p\right)^{1/p} \leq \frac{4Cm_{i+1}m_{2j_k}}{m_{2j_k}^3} < \frac{C}{m_{2j_k}}. \tag{5.6}$$

It remains to estimate $\sum_{l\in N_k}\beta_l f_l(\sum_{t\in J_l} a_t y_t)$. By the basic inequality,

$$\left|\sum_{l\in N_k}\beta_l f_l\left(\sum_{t\in J_l} a_t y_t\right)\right| \leq C\sum_{l\in N_k}|\beta_l|\left(h_l^1 + e_{r_l}^* + g_l^2\right)\left(\sum_{t\in J_l} a_t e_{s_t}\right), \tag{5.7}$$

where $s_t = \min\operatorname{supp} y_t$, and for every l, $\operatorname{supp}(h_l^1 + e_{r_l}^* + g_l^2) \subset \{e_{s_t} : t\in J_l\}$, $h_l^1 \in \operatorname{conv}_{\mathbb{Q}}\left\{h\in B_{(p)} : w(h) = w(f_l)/\theta\right\}$, $r_l\in\mathbb{N}$ with $r_l\notin\operatorname{supp} h_l^1$ and $\|g_l^2\|_{\ell_q} \leq \frac{2}{m_{2j_k+1}} < \frac{1}{m_{2j_k}^3}$. Since the sets J_l, $l\in N_k$, are pairwise disjoint, we get

$$\sum_{l\in N_k}|\beta_l| g_l^2\left(\sum_{t\in J_l} a_t e_{s_t}\right) \leq \frac{1}{m_{2j_k}^3}\left(\sum_{t\in\cup J_l}|a_t|^p\right)^{1/p} \leq \frac{1}{m_{2j_k}^3}. \tag{5.8}$$

Notice also that the set $\{e_{r_l}^* : l\in N_k\}$ is at most $\mathcal{S}_{n_i+1}$-admissible, so, since $\sum_{t\in I_k} a_t e_{s_t}$ is a $(\frac{1}{m_{2j_k}^3}, 2j_k)$ p-convex s.c., it follows that

$$\sum_{l\in N_k}|\beta_l| e_{r_l}^*\left(\sum_{t\in J_l} a_t e_{s_t}\right) \leq \frac{1}{m_{2j_k}^3}. \tag{5.9}$$

Moreover,

$$\sum_{l\in N_k}|\beta_l| h_l^1\left(\sum_{t\in J_l} a_t e_{s_t}\right) \leq \sum_{l\in N_k}|\beta_l|\frac{\theta}{w(f_l)} \leq \theta\sum_{l\in N_k}\frac{1}{w(f_l)} \leq \theta\sum_{j>2j_k}\frac{1}{m_j} \leq \frac{4}{m_{2j_k}^2}. \tag{5.10}$$

Therefore, by (5.7), (5.8), (5.9) and (5.10), we get

$$\left|\left(\sum_{l\in N_k}\beta_l f_l\right)\left(m_{2j_k}\sum_{t\in\cup J_l} a_t y_t\right)\right| \leq Cm_{2j_k}\left(\frac{2}{m_{2j_k}^3} + \frac{4}{m_{2j_k}^2}\right) \leq \frac{5C}{m_{2j_k}}.$$

Combining this with (5.6), we get

$$\left|\left(\sum_{l\in N_k}\beta_l f_l\right)(x_k)\right| \leq \frac{6C}{m_{2j_k}}.$$

So,

$$\left|\sum_{k\in G} c_k \left(\sum_{l\in N_k} \beta_l f_l\right)(x_k)\right| \leq 6C \left(\sum_{k\in G} \frac{1}{m_{2j_k}^q}\right)^{1/q} \left(\sum_{k\in G} |c_k|^p\right)^{1/p} \leq \frac{6C}{m_{i+3}} \left(\sum_{k\in G} |c_k|^p\right)^{1/p}. \tag{5.11}$$

Combining this with (5.5), we conclude that

$$\left|\left(\sum_{l=1}^{d} \beta_l f_l\right)\left(\sum_{k\in G} c_k x_k\right)\right| \leq \frac{8C}{m_{i+3}} \left(\sum_{k\in G} |c_k|^p\right)^{1/p}. \tag{5.12}$$

Set now

$$\varepsilon = \frac{4}{w(f_1)} + \frac{3}{w(f_1)} + \frac{1}{m_{2j_1}} + \frac{8}{m_{i+3}} < \frac{9}{m_{i+3}} < \frac{1}{m_{i+2}}.$$

It follows from (5.2), (5.3), (5.4) and (5.12), that

$$\left|\left(\sum_{l=1}^{d} \beta_l f_l\right)\left(\sum_{k=1}^{r} c_k x_k\right)\right| \leq \varepsilon C \left(\sum_{k=1}^{r} |c_k|^p\right)^{1/p}.$$

This completes the proof of the lemma. □

Definition 5.5 (Dependent sequence). Let $j \in \mathbb{N}$, $d \in \mathbb{N}$ and $C > 33$. A double sequence $(x_k, x_k^*)_{k=1}^{2d}$ with $x_k \in X_{(p)}$ and $x_k^* \in K$ is said to be a $(C, 2j+1)$ dependent sequence if there exists a sequence $(2j_k)_{k=1}^{2d}$ of even integers such that the following conditions are fulfilled:

(i) $(x_k^*)_{k=1}^{2d}$ is a $(\sigma, 2j+1)$- sequence with $w(x_k^*) = m_{2j_k}$ for all $k \leq 2d$.
(ii) Each (x_k, x_k^*) is a $(C, 2j_k)$ exact pair.
(iii) There exists a sequence $(a_i)_{i=1}^{d}$ of positive numbers such that $\sum_{i=1}^{d} a_i(x_{2i-1} + x_{2i})$ is a $(\frac{1}{m_{2j+2}}, 2j+1)$ p-convex s.c. of $(x_k)_{k=1}^{2d}$.

Proposition 5.6. *Let $j \in \mathbb{N}$ and $C > 330$. Then, for every pair of block subspaces Y_1, Y_2 of $X_{(p)}$, there exists a $(C, 2j+1)$ dependent sequence $(x_k, x_k^*)_{k=1}^{2d}$ with $x_{2i-1} \in Y_1$ and $x_{2i} \in Y_2$ for all $i = 1, \dots, d$.*

Proof. By an inductive application of Proposition 5.3, we can choose a double sequence $(x_k, x_k^*)_{k=1}^{\infty}$ satisfying the following: $(x_k)_{k=1}^{\infty}$ is a block sequence with $x_{2i-1} \in Y_1$ and $x_{2i} \in Y_2$ for all $i = 1, \dots$ and each pair (x_k, x_k^*) is a $(C, 2j_k)$ exact pair, where $2j_1 \geq 4j + 6$ and, for all $k \geq 2$, $2j_k = \sigma(x_1^*, \dots, x_{k-1}^*)$.

By Remark 4.2 there exist $d \in \mathbb{N}$ and a $(\frac{1}{2m_{2j+2}}, 2j+1)$ p-convex s.c. $\sum_{k=1}^{2d} b_k x_k$. For $i = 1, \dots, d$, we set $a_i = (\frac{b_{2i-1}^p + b_{2i}^p}{2})^{1/p}$.

It follows that $(x_k^*)_{k=1}^{2d}$ is a $(\sigma, 2j+1)$-sequence and $\sum_{i=1}^{d} a_i(x_{2i-1}+x_{2i})$ is a $(\frac{1}{m_{2j+2}}, 2j+1)$ p-convex s.c. of $(x_k)_{k=1}^{2d}$. Hence the sequence $(x_k, x_k^*)_{k=1}^{d}$ is the desired dependent sequence. □

Proposition 5.7. *Let $j, d \in \mathbb{N}$ and $C > 33$. Let $(x_k, x_k^*)_{k=1}^{2d}$ be a $(C, 2j+1)$ dependent sequence and suppose that the positive coefficients $(a_i)_{i=1}^{d}$ are such that $\sum_{i=1}^{d} a_i(x_{2i-1}+x_{2i})$ is a $(\frac{1}{m_{2j+2}}, 2j+1)$ p-convex s.c. Then*

$$\left\| \sum_{i=1}^{d} a_i(x_{2i-1} - x_{2i}) \right\| \leq \frac{10C}{m_{2j+1}^3}.$$

Proof. To simplify notation we set $c_{2i-1} = a_i$ and $c_{2i} = -a_i$, for $i = 1, \ldots, d$, so that

$$\sum_{i=1}^{d} a_i(x_{2i-1} - x_{2i}) = \sum_{k=1}^{2d} c_k x_k.$$

Since $(x_k^*)_{k=1}^{2d}$ is a $(\sigma, 2j+1)$-sequence (Definition 2.2), it follows that

$$m_{2j_{k+1}} = m_{\sigma(x_1^*, \ldots, x_k^*)} > m_{2j_k} \cdot \max\operatorname{supp} x_k^* = m_{2j_k} \cdot \max\operatorname{supp} x_k\,.$$

From the definition of exact pairs (Definition 5.1) and Remark 5.2 it follows that

$$\|x_k\| \leq C \text{ and } |f(x_k)| \leq \frac{C}{w(f)} \text{ for every } f \in K \text{ with } w(f) < m_{2j_k}\,.$$

Hence $(x_k)_{k=1}^{2d}$ is a $(C, (2j_k)_{k\in\mathbb{N}})$ R.I.S. (Definition 4.6).

The conclusion will follow from Proposition 4.10(2) provided that we show that assumption (b) of the basic inequality (Proposition 4.9) is satisfied by $j_0 = 2j+1$ and by the linear combination $\sum_{k=1}^{2d} c_k x_k$. We need to show that, for every interval E and for every $\phi \in K$ with $w(\phi) = m_{2j+1}$, we have

$$\left| \phi\left(\sum_{k\in E} c_k x_k \right) \right| \leq C \left(\max_{k\in E} |c_k| + \frac{1}{m_{2j+2}} \left(\sum_{k\in E} |c_k|^p \right)^{1/p} \right).$$

Let $\phi \in K$ with $w(\phi) = m_{2j+1}$. Then ϕ has the form

$$\phi = \frac{1}{m_{2j+1}} F\Big(\gamma_1(x_1^* + x_2^*) + \gamma_2(x_3^* + x_4^*) + \cdots + \gamma_r(x_{2r-1}^* + x_{2r}^*) + \gamma_{r+1}(f_{2r+1} + f_{2r+2}) + \cdots + \gamma_s(f_{2s-1} + f_{2s}) \Big),$$

where $(2^{1/q}\gamma_i)_{i=1}^{s} \in B_{\ell_q}$, $x_1^*, x_2^*, \ldots, x_{2r}^*, f_{2r+1}, f_{2r}, \ldots, f_{2s}$ is a $(\sigma, 2j+1)$-sequence, F is an interval of the natural numbers and either $r = d$ or $0 \leq r \leq d-1$ and one of the following three cases holds:

(a) $r \geq 1$, $w(f_{2r+1}) = w(x_{2r+1}^*)$ and $f_{2r+1} \neq x_{2r+1}^*$.
(b) $r \geq 0$, $f_{2r+1} = x_{2r+1}^*$, $w(f_{2r+2}) = w(x_{2r+2}^*)$ and $f_{2r+2} \neq x_{2r+2}^*$.

(c) $r = 0$ and $f_1 \neq x_1^*$. In this case, it follows from the definition of $(\sigma, 2j+1)$ sequences that $w(f_1) \geq m_{2j+6}$.

Let $E \subset \{1, \dots, 2d\}$ be an interval of integers. Without loss of generality we may assume that $r \geq 1$, $F = [m, \infty)$, where $m \in \operatorname{ran} x_{2t-1}^*$ for some $t \leq r$ and that $[2t-1, 2r] \subset E$. The other cases are treated similarly. We have the following estimates:

(i) For $k < 2t-1$, $\phi(x_k) = 0$.

(ii) $|\phi(x_{2t-1})| = \frac{1}{m_{2j+1}} |\gamma_t| |F x_{2t-1}^*(x_{2t-1})| \leq \frac{C|\gamma_t|}{m_{2j+1}}$.

(iii) $|\phi(x_{2t})| = \frac{1}{m_{2j+1}} |\gamma_t| x_{2t}^*(x_{2t}) = \frac{|\gamma_t|}{m_{2j+1}}$.

(iv) For $i = t+1, \dots, r$,

$$\begin{aligned}\phi(c_{2i-1}x_{2i-1} + c_{2i}x_{2i}) &= a_i \phi(x_{2i-1} - x_{2i}) \\ &= \frac{1}{m_{2j+1}} a_i \gamma_i \left(x_{2i-1}^*(x_{2i-1}) - x_{2i}^*(x_{2i})\right) = 0.\end{aligned}$$

(v) $|\phi(x_{2r+1})| \leq \frac{C}{m_{2j+1}}$ and $|\phi(x_{2r+2})| \leq \frac{C}{m_{2j+1}}$ by Remark 5.2 on exact pairs.

(vi) Finally,

$$\phi\left(\sum_{k=2r+3}^{2d} c_k x_k\right) = \frac{1}{m_{2j+1}} \left(\sum_{i=r+1}^{s} \gamma_i (f_{2i-1} + f_{2i})\right) \left(\sum_{k=2r+3}^{2d} c_k x_k\right),$$

where, for $2r+3 \leq k \leq 2d$ and $2r+1 \leq l \leq 2s$, we have $w(f_l) \neq w(x_k^*)$ by the injectivity of the function σ. It follows that the sequences $(x_k)_{k=2r+3}^{2d}$ and $(f_l)_{l=2r+1}^{2s}$ satisfy the assumptions of Lemma 5.4 with $i = 2j+1$. So, for the given interval E, we get

$$\begin{aligned}\left|\phi\left(\sum_{\substack{k\in E\\ k\geq 2r+3}} c_k x_k\right)\right| &= \frac{1}{m_{2j+1}} \left|\left(\sum_{i=r+1}^{s} \gamma_i (f_{2i-1} + f_{2i})\right)\left(\sum_{\substack{k\in E\\ k\geq 2r+3}} c_k x_k\right)\right| \\ &\leq \frac{C}{m_{2j+1} m_{2j+3}} \left(\sum_{k\in E} |c_k|^p\right)^{1/p}.\end{aligned}$$

Combining (i)–(vi) we conclude that

$$\begin{aligned}\left|\phi\left(\sum_{k\in E} c_k x_k\right)\right| &\leq \frac{4C}{m_{2j+1}} \max_{k\in E} |c_k| + \frac{C}{m_{2j+3}} \left(\sum_{k\in E} |c_k|^p\right)^{1/p} \\ &\leq C\left(\max_{k\in E} |c_k| + \frac{1}{m_{2j+3}} \left(\sum_{k\in E} |c_k|^p\right)^{1/p}\right).\end{aligned}$$

This shows that assumption (b) of the basic inequality is satisfied by the sequences $(x_k)_{k=1}^{2d}$ and $(c_k)_{k=1}^{2d}$ and completes the proof of the proposition. □

From the above proposition and Proposition 5.6 we get the following theorem.

THEOREM 5.8. *The space $X_{(p)}$ is hereditarily indecomposable.*

Proof. Let Y, Z be infinite dimensional block subspaces of $X_{(p)}$. Choose $C > 330$. By Proposition 5.6 for $j \in \mathbb{N}$ there exists a $(C, 2j+1)$ dependent sequence $(x_k, x_k^*)_{k=1}^{2d}$ with $x_{2i-1} \in Y$ and $x_{2i} \in Z$ for every $i = 1, \ldots, d$. Let $\sum_{i=1}^{d} a_i(x_{2i-1} + x_{2i})$ be a $(\frac{1}{m_{2j+2}}, 2j)$ p-convex s.c of $(x_k)_{k=1}^{2d}$. Then $y = \sum_{i=1}^{d} a_i y_{2i-1} \in Y$ and $z = \sum_{i=1}^{d} a_i x_{2i} \in Z$. Let $(2^{1/q}\beta_i)_{i=1}^{d}$ be the conjugate sequence of $(2^{1/p}a_i)_{i=1}^{d}$. Then $\frac{1}{m_{2j+1}} \sum_{1=1}^{d} \beta_i(x_{2i-1}^* + x_{2i}^*) \in B_{X_{(p)}^*}$, so

$$\|y + z\| \geq \frac{1}{m_{2j+1}} \sum_{i=1}^{d} \beta_i(x_{2i-1}^* + x_{2i}^*) \left(\sum_{i=1}^{d} a_i(x_{2i-1} + x_{2i}) \right) = \frac{1}{m_{2j+1}}.$$

On the other hand, it follows from Proposition 5.7 that

$$\|y - z\| = \left\| \sum_{i=1}^{d} a_i(x_{2i-1} - x_{2i}) \right\| \leq \frac{10C}{m_{2j+1}^3}.$$

Since $j \in \mathbb{N}$ was arbitrary, we conclude that $X_{(p)}$ is hereditarily indecomposable. ☐

6. The dual space $X_{(p)}^*$ and the space of operators $\mathcal{L}(X_{(p)})$

In this section we shall present the following results concerning the structure of $X_{(p)}$, $X_{(p)}^*$ and the spaces of operators $\mathcal{L}(Y, X_{(p)})$, where Y is a closed subspace of $X_{(p)}$:

(1) For every closed subspace Y of $X_{(p)}$, every bounded linear operator $T : Y \mapsto X_{(p)}$ is of the form $T = \lambda I_Y + S$, where $\lambda \in \mathbb{R}$, I_Y is the inclusion operator from Y to $X_{(p)}$ and S is a strictly singular operator.
(2) c_0 is finitely representable in every infinite-dimensional subspace of $X_{(p)}$. Therefore $X_{(p)}$ does not contain any uniformly convex subspace.
(3) The dual $X_{(p)}^*$ of $X_{(p)}$ is an HI space.

An essential tool for the proofs of (1) and (3) is the fact that the set D is rationally convex, so that $B_{X_{(p)}^*} = \overline{D}^p$. All results of the previous sections could have been obtained without this condition. However, the rational convexity of D seems essential in the proofs of the results about the dual of $X_{(p)}$ and the structure of the spaces of operators $\mathcal{L}(Y, X_{(p)})$.

The proofs of these results follow the same lines as the proofs of the corresponding results in [3] and [8]. Here we only outline the steps needed in order to reduce the present cases to the already known cases. For complete proofs we refer the reader to the corresponding papers.

We proceed to the first result which concerns the spaces of operators $\mathcal{L}(Y, X_{(p)})$.

THEOREM 6.1. *For every closed subspace Y of $X_{(p)}$, every bounded linear operator $T : Y \mapsto X_{(p)}$ is of the form $T = \lambda I_Y + S$, where $\lambda \in \mathbb{R}$, $I_Y : Y \mapsto X_{(p)}$ is the inclusion operator and S is a strictly singular operator.*

The main step of the proof is contained in the following lemma.

LEMMA 6.2. *Let $j_k \nearrow \infty$ and $(x_k)_k$ be a block sequence in $X_{(p)}$ such that each x_k is a 4-seminormalized $(\frac{1}{m^2_{2j_k}}, 2j_k)$ p-convex s.c. Suppose that Y is a closed subspace of $X_{(p)}$ which contains the sequence $(x_k)_k$ and $T : Y \mapsto X_{(p)}$ is a bounded linear operator. Then*

$$\lim_k \operatorname{dist}(Tx_k, \mathbb{R}x_k) = 0\,.$$

The proof of Lemma 6.2 is based on Proposition 5.3, Lemma 5.4 and part (b) of the basic inequality, following the steps given in Chapter 9 of [8]. Lemma 6.2 combined with Propositions 4.3 and 5.7 yield a proof of Theorem 6.1 following the ideas used for the analogous results in [15] . We omit the proofs.

Let us note that Theorem 6.1 implies that the space $X_{(p)}$ is HI. However, for its proof we need all the machinery that was used in the proof of Theorem 5.8.

In the next result we show that $X_{(p)}$ is not uniformly convex by proving that c_0 is finitely disjointly representable in every block subspace of $X_{(p)}$. More precisely, the following is true.

THEOREM 6.3. *For every $\varepsilon > 0$, every infinite-dimensional block subspace Y of $X_{(p)}$ contains, for every $n \in \mathbb{N}$, a sequence of disjointly supported vectors $(y_r)_{r=1}^n$ which is $(1+\varepsilon)$-equivalent to the canonical basis of ℓ_∞^n.*

This theorem is similar to Theorem 1.6 of [3] and we shall only sketch its proof.

Sketch of the proof of Theorem 6.3. We set $\theta = 4^{1/q}$ as in Section 3. Fix $n \in \mathbb{N}$ and a block subspace Y of $X_{(p)}$. By standard arguments it suffices to find a sequence $(x_r)_{r=1}^n$ of disjointly supported vectors in Y which is C-equivalent to the canonical basis of ℓ_∞^n, where $C = 145 \cdot 15 \cdot 75$.

Choose integers s_0, i_0 and $j_k, k = 1, \ldots, n$, such that

$$60n < \Big(\frac{m_1}{\theta}\Big)^{s_0 p}, \quad m_{2i_0}^p > 420n \quad \text{and} \quad s_0 i_0 < j_1 < j_2 < \cdots < j_n.$$

Choose also sequences $(t_i)_{i\in\mathbb{N}}$ and $(z_i)_{i\in\mathbb{N}}$ such that $(t_i)_{i\in\mathbb{N}}$ is an increasing sequence in $\mathbb{N}$ with

$$m_{2t_1} > 4nm_{2j_n} \tag{6.1}$$

and $(z_i)_{i\in\mathbb{N}}$ is a $(15,(2t_i)_{i\in\mathbb{N}})$ R.I.S. in Y, while each z_i is a 3-seminormalized $(\frac{1}{m_{2t_i}^2}, 2t_i)$ p-convex special combination. Such a choice is possible according to Propositions 4.3 and 4.8.

Let $l_i = \operatorname{minsupp} z_i$, $i \in \mathbb{N}$, and let $L = \{l_i : i \in \mathbb{N}\}$. As in the proof of Theorem 1.6 of [3] we can choose a finite tree $\mathcal{T}$ of height n, and two corresponding trees $(l_\beta)_{\beta\in\mathcal{T}}$ of integers in L and $(a_\beta)_{\beta\in\mathcal{T}}$ of real numbers, with the following properties:

(i) For $1 \le r < n$ and $\alpha \in \mathcal{T}$ with length $|\alpha| = r$ we denote by S_α the set of immediate successors of α in $\mathcal{T}$, $S_\alpha = \{\beta \in \mathcal{T} : |\beta| = r+1, \alpha \prec \beta\}$ ($\prec$ is the natural partial order of $\mathcal{T}$). Then, if $r < n$ and $\alpha, \alpha' \in \mathcal{T}$ with $|\alpha| = |\alpha'| = r$ and $l_\alpha < l_{\alpha'}$, we have: For every $\beta \in S_\alpha$, $l_\alpha < l_\beta < l_{\alpha'}$.

(ii) For each $r = 1, \dots, n$, we set

$$v_r = \sum_{\beta\in\mathcal{T}, |\beta|=r} \left(\prod_{\gamma\preceq\beta} a_\gamma \right) e_{l_\beta}.$$

Then, v_r is a $(\frac{1}{m_{2j_r}^3}, 2j_r)$-basic p-convex s.c.

Let $(z_\beta)_{\beta\in\mathcal{T}} \subseteq \{z_i : i \in \mathbb{N}\}$ be the corresponding tree of vectors, that is, $z_\beta = z_{i_\beta}$, where $l_\beta = \operatorname{minsupp} z_{i_\beta}$, for every $\beta \in \mathcal{T}$.

For each $r = 1, \dots, n$, we set

$$y_r = \sum_{\beta\in\mathcal{T}, |\beta|=r} \left(\prod_{\gamma\preceq\beta} a_\gamma \right) z_\beta.$$

We have the following estimates:

(i) By Proposition 3.4,

$$\|v_r\|_{T_{(p)}} = \frac{\theta}{m_{2j_r}}.$$

(ii) By Proposition 4.10 and the fact that, for each i, $1 \le \|z_i\| \le 3$, we get

$$\frac{1}{m_{2j_r}} \le \|y_r\|_{X_{(p)}} \le \frac{75}{m_{2j_r}}.$$

We set

$$x_r = \frac{y_r}{\|y_r\|}, \quad r = 1, \dots, n,$$

and we show that the sequence $(x_r)_{r=1}^\infty$ is C-equivalent to the canonical basis of ℓ_∞^n.

Let $(\lambda_r)_{r=1}^n$ be any reals. We note first that

$$\left\| \sum_{r=1}^n \lambda_r x_r \right\| \ge \frac{1}{75} \max_{1\le r\le n} |\lambda_r|.$$

To see this, fix $r_0 \leq n$. For every $\beta \in \mathcal{T}$ with $|\beta| = r_0$, since z_β is a 3-seminormalized $(\frac{1}{m_{2t_i}^3}, 2t_i)$ p-convex s.c. (see Definition 4.1), there exists $f_\beta \in D$ with $f_\beta(z_\beta) = 1$ and $\operatorname{ran} f_\beta \subseteq \operatorname{ran} z_\beta$.

Let $\mu_\beta = \prod_{\gamma \preceq \beta} a_\gamma$ and take $(\rho_\beta)_{\beta\in\mathcal{T},|\beta|=r_0}$ to be the conjugate sequence of $(\mu_\beta)_{\beta\in\mathcal{T},|\beta|=r_0}$. Then, the functional

$$\varphi = \operatorname{sign}(\lambda_{r_0}) \frac{1}{m_{2j_{r_0}}} \sum_{\beta\in\mathcal{T},|\beta|=r_0} \rho_\beta f_\beta$$

belongs to $B_{X^*_{(p)}}$ and

$$\begin{aligned} \left\| \sum_{r=1}^n \lambda_r x_r \right\| &\geq \varphi\left(\sum_{r=1}^n \lambda_r x_r \right) = \varphi(\lambda_{r_0} x_{r_0}) \\ &= \frac{|\lambda_{r_0}|}{m_{2j_{r_0}} \|y_{r_0}\|} \sum_{\beta\in\mathcal{T},|\beta|=r_0} \rho_\beta \mu_\beta f_\beta(z_\beta) \\ &= \frac{|\lambda_{r_0}|}{m_{2j_{r_0}} \|y_{r_0}\|} \geq \frac{1}{75} |\lambda_{r_0}|. \end{aligned}$$

For the upper estimate, we use the basic inequality (Proposition 4.9). For $r = 1, \ldots, n$, we set $u_r = \frac{1}{\|y_r\|} v_r$ and we note that

$$\frac{\theta}{75} \leq \|u_r\|_{T_{(p)}} \leq \theta\,.$$

Let $f \in K$. Then, there exist functionals g_1, g_2 such that

$$\left| f\left(\sum_{r=1}^n \lambda_r x_r \right) \right| \leq 15(g_1 + g_2)\left(\sum_{r=1}^n |\lambda_r| u_r \right),$$

where $\|g_2\|_{\ell_q} \leq \frac{2}{m_{2t_1}}$ and g_1 either belongs to $B_{(p)} \subseteq B_{T_{(p)}}$ or $g_1 = e_l^* + h_1$ for some $l \in \mathbb{N}$ and $h_1 \in B_{(p)}$.

It follows that

$$\frac{1}{15} \left\| \sum_{r=1}^n \lambda_r x_r \right\|_{X_{(p)}} \leq \frac{2}{m_{2t_1}} \left\| \sum_{r=1}^n |\lambda_r| u_r \right\|_{\ell_p} + \left\| \sum_{r=1}^n |\lambda_r| u_r \right\|_\infty + \left\| \sum_{r=1}^n |\lambda_r| u_r \right\|_{T_{(p)}}.$$

By our choice of t_1 (equation (6.1)), we have $\frac{2}{m_{2t_1}} < \frac{1}{2nm_{2j_n}}$ and since, for every $r = 1, \ldots, n$, $\|u_r\|_{\ell_p} \leq m_{2j_r} \leq m_{2j_n}$, we get

$$\begin{aligned} \frac{2}{m_{2t_1}} \left\| \sum_{r=1}^n |\lambda_r| u_r \right\|_{\ell_p} &= \frac{2}{m_{2t_1}} \left(\sum_{r=1}^n |\lambda_r|^p \|u_r\|_{\ell_p}^p \right)^{1/p} \\ &\leq \frac{m_{2j_n}}{2nm_{2j_n}} \left(\sum_{r=1}^n |\lambda_r|^p \right)^{1/p} \leq \frac{1}{2} \max_{1\leq r\leq n} |\lambda_r|. \end{aligned}$$

Also, clearly,

$$\left\|\sum_{r=1}^{n}|\lambda_r|u_r\right\|_\infty \leq \frac{1}{2}\max_{1\leq r\leq n}|\lambda_r|.$$

It remains to estimate $\|\sum_{r=1}^{n}|\lambda_r|u_r\|_{T_{(p)}}$. We set, for $r=1,\ldots,n$,

$$w_r = \frac{1}{\|y_r\|^p}\sum_{\beta\in\mathcal{T},|\beta|=r}\left(\prod_{\gamma\leq\beta}a_\gamma\right)^p e_{l_\beta}.$$

Then, by Lemma 3.1,

$$\left\|\sum_{r=1}^{n}|\lambda_r|u_r\right\|_{T_{(p)}} = \left\|\sum_{r=1}^{n}|\lambda_r|^p w_r\right\|_Z^{1/p}.$$

But it follows from the proof of Theorem 1.6 of [3] that the sequence $(w_r)_{r=1}^n$ in Z is $36\theta^p$-equivalent to the canonical basis of ℓ_∞^n. In particular,

$$\left\|\sum_{r=1}^{n}|\lambda_r|^p w_r\right\|_Z \leq 36\theta^p\max_{1\leq r\leq n}|\lambda_r|^p.$$

It follows that

$$\left\|\sum_{r=1}^{n}|\lambda_r|u_r\right\|_{T_{(p)}} \leq 36\theta\max_{1\leq r\leq n}|\lambda_r| \leq 144\max_{1\leq r\leq n}|\lambda_r|.$$

We conclude that

$$\frac{1}{75}\max_{1\leq r\leq n}|\lambda_r| \leq \left\|\sum_{r=1}^{n}\lambda_r x_r\right\| \leq 145\cdot 15\max_{1\leq r\leq n}|\lambda_r|.$$

So, setting $C = 145\cdot 15\cdot 75$, $(x_r)_{r=1}^\infty$ is C-equivalent to the canonical basis of ℓ_∞^n. □

Remark 6.4. Following [3], [4], modified versions of the spaces $X_{(p)}$ can be defined. It follows readily from the definitions that c_0 is not disjointly finitely representable in the modified spaces. However it is not clear to the authors whether modified versions of the $X_{(p)}$ spaces can provide uniformly convex HI Banach spaces.

For the dual space $X_{(p)}^*$ of $X_{(p)}$ we have the following result.

Theorem 6.5. *The dual space $X_{(p)}^*$ of $X_{(p)}$ is an asymptotic ℓ_q hereditarily indecomposable Banach space.*

We saw in Section 2 (Corollary 2.10) that $X_{(p)}^*$ is asymptotic ℓ_q. The proof that $X_{(p)}^*$ is HI follows the arguments of Chapter 8 of [8], and it is based on the following two lemmas.

LEMMA 6.6. *Let $f \in B_{X^*_{(p)}} \cap c_{00}$. Then, for every $\varepsilon > 0$, there exists $x^* \in D$ such that $\|f - x^*\| < \varepsilon$ and $\operatorname{ran} f = \operatorname{ran} x^*$.*

Proof. Let $f \in B_{X^*_{(p)}} \cap c_{00}$ and $F = \operatorname{ran} f$. Since $B_{X^*_{(p)}} = \overline{D}^p$ there exists $x^* \in D$ such that $|f(e_k) - x^*(e_k)| < \frac{\varepsilon}{\#F}$ for every $k \in F$. Since the set D is closed under interval projections we may assume that $\operatorname{ran} x^* = F$. Then x^* is the desired functional. □

LEMMA 6.7. *Let $(z^*_k)_k$ be a normalized block sequence in $X^*_{(p)}$, $\varepsilon > 0$ and $j \in \mathbb{N}$. Then there exist a functional $x^* \in D$ and a 4-seminormalized $(\frac{1}{m^2_{2j}}, 2j)$ p-convex s.c. $x \in X_{(p)}$ such that, setting $Z = \langle (z^*_k)_k \rangle$, we have:*

(i) $\operatorname{dist}(x^*, Z) < \varepsilon$.

(ii) $x^*(x) = 1$ *and* $\operatorname{ran} x^* = \operatorname{ran} x$.

For the proof we refer the reader to Lemma 8.2 of [8]. We note however that the main step in our case is Proposition 4.4 of the present paper. We proceed to the proof of Theorem 6.5.

Sketch of the proof of Theorem 6.5. (See also Theorem 8.3 of [8].) Let Y, Z be a pair of block subspaces of $X^*_{(p)}$, $0 < \delta < 1/10$ and $C > 330$. Let $j \in \mathbb{N}$ be such that $m^2_{2j+1} > \frac{21C}{\delta}$.

By Lemma 6.7 and an argument similar to that of Theorem 8.3 [8], we can choose a $(C, 2j+1)$-dependent sequence $(x_k, x^*_k)_{k=1}^{2n}$ such that $\operatorname{dist}(x^*_{2i-1}, Y) < 1/2^{2i-1}$ and $\operatorname{dist}(x^*_{2i}, Z) < 1/2^{2i}$ for every $i = 1, \ldots, n$.

Let $x = \sum_{i=1}^n a_i(x_{2i-1} + x_{2i})$ be a $(1/m_{2j+2}, 2j+1)$ p-convex s.c. of $(x_k)_{k=1}^{2n}$.

We also consider the vector $x' = \sum_{i=1}^n a_i(x_{2i-1} - x_{2i})$. From Proposition 5.7 we have $\|x'\| \le \frac{10C}{m^3_{2j+1}}$. Let $(2^{1/q}\gamma_i)_{i=1}^n$ be the conjugate sequence of $(2^{1/p}a_i)_{i=1}^n$ and define the functionals

$$f_Y = \frac{1}{m_{2j+1}} \sum_{i=1}^n \gamma_i x^*_{2i-1} \quad \text{and} \quad f_Z = -\frac{1}{m_{2j+1}} \sum_{i=1}^n \gamma_i x^*_{2i}.$$

We have $f_Y - f_Z = \frac{1}{m_{2j+1}} \sum_{i=1}^n \gamma_i(x^*_{2i-1} + x^*_{2i}) \in B_{X^*_{(p)}}$ by the choice of $(\gamma_i)_{i=1}^n$ and $(x^*_k)_{k=1}^{2n}$. Hence $\|f_Y - f_Z\| \le 1$. Also,

$$\begin{aligned} \|f_Y + f_Z\| &= \left\| \frac{1}{m_{2j+1}} \sum_{i=1}^n \gamma_i(x^*_{2i-1} - x^*_{2i}) \right\| \\ &\ge \frac{\frac{1}{m_{2j+1}} \sum_{i=1}^n \gamma_i(x^*_{2i-1} - x^*_{2i}) \left(\sum_{i=1}^n a_i(x_{2i-1} - x_{2i})\right)}{\|\sum_{i=1}^n a_i(x_{2i-1} - x_{2i})\|} \\ &\ge \frac{\frac{1}{m_{2j+1}}}{\frac{10C}{m^3_{2j+1}}} = \frac{m^2_{2j+1}}{10C}. \end{aligned}$$

By the choice of the sequence $(x_k^*)_{k=1}^{2n}$, we get

$$\operatorname{dist}(f_Y, Y) \leq \frac{1}{m_{2j+1}} \sum_{i=1}^{n} \gamma_i \operatorname{dist}(x_{2i-1}^*, Y) < \frac{1}{m_{2j+1}} < \frac{1}{2}$$

and also $\operatorname{dist}(f_Z, Z) < \frac{1}{2}$. Thus we may choose $h_Y \in Y$ and $h_Z \in Z$ such that

$$\|f_Y - h_Y\| < \frac{1}{2} \text{ and } \|f_Z - h_Z\| < \frac{1}{2}. \tag{6.2}$$

From the previous estimates we get that

$$\|h_Y - h_Z\| \leq 2\delta \|h_Y + h_Z\| . \qquad \square$$

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