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LENGTH MINIMIZING PATHS IN THE HYPERBOLIC PLANE: PROOF VIA PAIRED SUBCALIBRATIONS

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Abstract. Minimization proofs using paired calibrations have in the past been done with vector fields of divergence zero. We generalize this method to find the shortest network connecting four points in the hyperbolic plane.

1. Paired subcalibrations

Jacob Steiner posed the problem of finding the shortest path connecting several points in the plane. Ronald Graham considered the case where the points all lie on the same circle (see [2], [3]). We begin by considering three points p_1, p_2, and p_3 spaced evenly around a circle which form a triangle S in the Poincaré Disk $\mathbb{D}$. Let s_1, s_2, and s_3 be the sides of that triangle and let $Y = \{y_i\}$ be the network connecting $\{p_i\}$ to the origin and separating S into regions C_1, C_2, and C_3 so that y_i originates from p_i and has unit normal $\vec{n}_i$ pointing toward C_i. Then we construct vector fields $\vec{V}_1, \vec{V}_2$, and $\vec{V}_3$, where $\vec{V}_i$ enters through s_i (see Figure 1(a)).

In previous paired calibration proofs, these vector fields were required to have divergence zero; however, we require the following less-restrictive criterion for each i:

$$\text{for all } p \in C_i \quad \operatorname{div} \vec{V}_i(p) \leq \operatorname{div} \vec{V}_j(p),\ j \neq i. \tag{1}$$

Whenever $\{(\vec{V}_i, C_i)\}$ is a system that satisfies (1) it is called a paired subcalibration.

Theorem 1. *Suppose $S = \{s_i\}$ is an n-gon and $Y = \{y_i\}$ is a network separating S into regions $\{C_i\}$ and $\{(\vec{V}_i, C_i)\}$ is a paired subcalibration that satisfy the following:*

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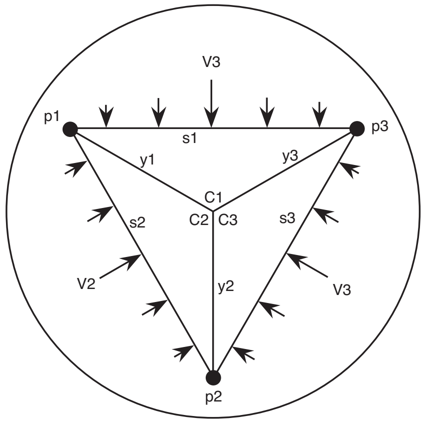
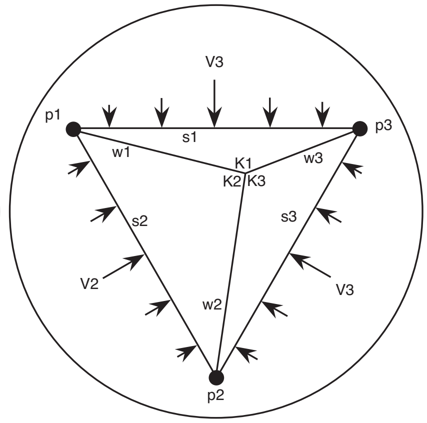


(a) The minimizer



(b) The competitor

FIGURE 1. Three points

(1) *Whenever C_i and C_j $(i \neq j)$ share a common line $y_k \in Y$, $\vec{V}_i - \vec{V}_j$ is a unit normal to y_k.*
(2) *Whenever $W = \{w_i\}$ is a network of the same combinatorial structure as Y, $\Phi_{out}(W) \leq \mathrm{Len}(W)$ with equality when $W = Y$.*

Then Y is the unique minimizer for its combinatorial structure.

Proof. We define the following notation for flux: $\Phi(\vec{V}, y)$ is the flux of the vector field $\vec{V}$ through the oriented curve y (i.e., y has a given unit normal). It is also natural to define Φ_{in} and Φ_{out} as[1]

$$\Phi_{in}(S) = \sum_{i=1}^{n} \Phi(\vec{V}_i, s_i).$$

and

$$\Phi_{out}(Y) = \sum_{i=1}^{n} \Phi_{out}(C_i)$$

where $\Phi_{out}(C_i) = \sum \Phi(V_i, y)$, summing over all $y \in Y$ that separate C_i from C_j $(j \neq i)$.

Then we have

$$\Phi_{out}(Y) - \Phi_{in}(S) = \sum_{i=1}^{n} \iint_{C_i} \operatorname{div} \vec{V}_i dA$$

[1]We will also use Φ_{in} when S is understood by context.

and for any competitor W that divides S into regions $K_1, \ldots, K_n$

$$\Phi_{out}(W) - \Phi_{in}(S) = \sum_{i=1}^{n} \iint_{K_i} \operatorname{div} \vec{V}_i dA \geq \sum_{i=1}^{n} \iint_{C_i} \operatorname{div} \vec{V}_i dA,$$

which gives

$$\Phi_{out}(W) - \Phi_{in}(S) \geq \Phi_{out}(Y) - \Phi_{in}(S).$$

Since $\{\vec{V}_i\}$ are such that $\Phi_{out}(W) \leq \operatorname{Len}(W)$ with equality when $W = Y$, we obtain

$$\operatorname{Len}(Y) = \Phi_{out}(Y) \leq \Phi_{out}(W) \leq \operatorname{Len}(W). \qquad \square$$

Continuing the proof for three points in $\mathbb{D}$, we now define the three vector fields $\vec{V}_1, \vec{V}_2$, and $\vec{V}_3$ to be orthogonal to $\{s_i\}$, respectively, with constant hyperbolic length $1/\sqrt{3}$. So $\vec{V}_2$ and $\vec{V}_3$ are rotations of

$$\vec{V}_1(x, y) = \frac{1 - (x^2 + y^2)}{\sqrt{3}} \langle 0, -1 \rangle, (x, y) \in \mathbb{D}. \tag{2}$$

The general equation for divergence on a space with a metric tensor is given in Eisenhart [4, page 113]. With the metric

$$ds^2 = \frac{1}{(1 - r^2)^2} \left(dx^2 + dy^2 \right)$$

on $\mathbb{D}$, we have the following formula for divergence of a vector field $\vec{V} = \langle f(x, y), g(x, y) \rangle$:

$$\operatorname{div} \vec{V} = f_x(x, y) + g_y(x, y) + \frac{4}{1 - (x^2 + y^2)} \left(x \cdot f(x, y) + y \cdot g(x, y) \right).$$

Now a basic calculation shows that $\{(\vec{V}_i, C_i)\}$ satisfy (1).

Since the difference vectors are of unit length and perpendicular to the respective y_i, by Theorem 1, Y is the minimizer, since there is exactly one combinatorial structure for three points.

2. Lines and equidistant curves

In the proof for four points we will be using vector fields tangent to families of lines and equidistant curves, so we first construct such vector fields and examine their divergences. The formula for a hyperbolic line right of the y-axis, perpendicular to the x-axis is given by

$$f(a, t) = (a - \sqrt{a^2 - 1} \cos t, \sqrt{a^2 - 1} \sin t),$$

where $(a, 0)$ is the center of the Euclidean circle representing the hyperbolic line given by the standard formula

$$a(x, y) = \frac{x^2 + y^2 + 1}{2x}$$

and the tangent vectors are given by

$$F(a,t) = \left\langle \sqrt{a^2-1}\sin t, \sqrt{a^2-1}\cos t \right\rangle$$

or

$$F(x,y) = \left\langle y, \frac{-x^2+y^2+1}{2x} \right\rangle.$$

Therefore, $\mathscr{V}_1$ given by

$$\mathscr{V}_1 = -\frac{x(1-x^2-y^2)}{\sqrt{(x^2+y^2)^2+1-2x^2+2y^2}} \left\langle y, \frac{-x^2+y^2+1}{2x} \right\rangle \tag{3}$$

has constant hyperbolic length and meets the x-axis at 90°. A straightforward calculation shows that this vector field has divergence

$$\operatorname{div} \mathscr{V}_1 = \frac{-4y}{\sqrt{(x^2+y^2)^2+1-2x^2+2y^2}}.$$

We now define the perpendicular vector field

$$\mathscr{V}_4 = \mathscr{V}_1^{\perp} = -\frac{x(1-x^2-y^2)}{\sqrt{(x^2+y^2)^2+1-2x^2+2y^2}} \left\langle \frac{-x^2+y^2+1}{2x}, -y \right\rangle \tag{4}$$

and it can be calculated that this field has divergence 0. We note that these vectors are tangent to the family of curves[2] equidistant from the x-axis.

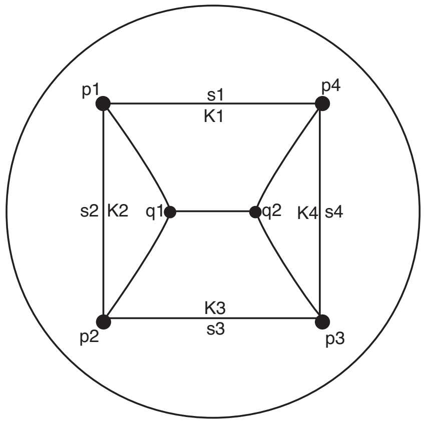
3. Four vertices of a square in $\mathbb{D}$

We have introduced tools useful in proving more complicated problems than are practical with the divergence-zero criterion for paired calibration proofs in the hyperbolic plane. In this section we will explore one interesting result given by these tools.

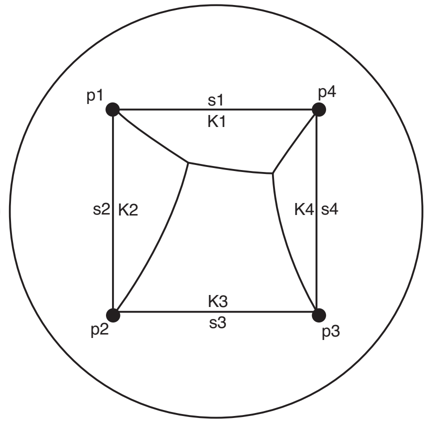
THEOREM 2. *Let p_1, p_2, p_3, and p_4 be four points of a Euclidean square S in the unit disk and centered at the origin. Then the length-minimizing network connecting these four points in $\mathbb{D}$ is unique (up to rotation) and is given by Figure 2(a).*

Proof. We note that for four points there are exactly two combinatorial structures and they are rotations of each other, so we will assume that $\{p_1, p_2\}$ and $\{p_3, p_4\}$ are pairs of siblings and rotate the network if that is not the case. Let $\{s_i\}$ be the sides of S so that s_i intersects p_i and p_{i-1} (where $p_0 = p_4$). Let $Y = \{y_i\}$ be hyperbolic line segments so that for $i = 1, 2, 3, 4$, y_i originates from p_i and so that y_1, y_2, and the x-axis meet at 120°, as do y_3, y_4, and the x-axis, and let y_5 be the portion of the x-axis that lies between the Steiner points q_1 and q_2. Let $\{C_i\}$ be the regions separated by Y so that s_i borders

[2]Note that in the hyperbolic plane a curve C of constant distance from a given line L is not itself a line. For more on equidistant curves, see [5, p 129].



(a) The minimizer



(b) The competitor

FIGURE 2. Four points

C_i. Associate with $y_1, \ldots, y_4$ the unit normal $\vec{n}_i$ that points toward C_i and for y_5, let $\vec{n}_5$ point toward C_1. (See Figure 2(a).)

Our goal is to create vector fields $\mathscr{F}_1$, $\mathscr{F}_2$, $\mathscr{F}_3$, and $\mathscr{F}_4$ so that $\{(\mathscr{F}_i, C_i)\}$, S, and Y together satisfy the hypothesis of Theorem 1. To facilitate this task we begin by constructing vector fields $\mathscr{V}_1$, $\mathscr{V}_2$, $\mathscr{V}_3$, and $\mathscr{V}_4$ around a Y centered at the origin. Then we will translate and reflect the system (see Figure 3).

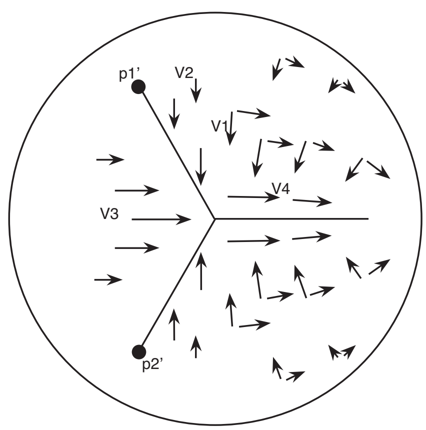


FIGURE 3. Building the vector fields

Let $p_1' = (-t, \sqrt{3}t)$ and $p_2' = (-t, -\sqrt{3}t)$, where t is the unique positive number so that $d_{\mathbb{D}}(p_1', p_2') = d_{\mathbb{D}}(p_1, p_2)$. Let $C_1{}'$ be the first quadrant of the disk, $C_2{}'$ be the region bounded by the y-axis on the right and the line $y = -\sqrt{3}\ x$ on the left, let $C_3{}'$ be the region bounded above by the line $y = -\sqrt{3}x$ and below by the line $y = \sqrt{3}x$. Let $\mathscr{V}_1$ be as given in (3) and defined on the right half-plane, $\mathscr{V}_2 = \frac{1}{2}(1-x^2-y^2)\,\langle 0,-1\rangle$ defined on the left half-plane. Let $\mathscr{V}_3 = \frac{\sqrt{3}}{2}(1-x^2-y^2)\,\langle 1,0\rangle$ be defined on the left half-plane and $\mathscr{V}_4$ be defined as in (4) on the right half-plane. The other vector fields will be reflections of these fields.

The divergence of $\mathscr{V}_1$ and $\mathscr{V}_4$ is given above, and for $\mathscr{V}_2$ and $\mathscr{V}_3$ we have

$$\operatorname{div} \mathscr{V}_2 = -y, \quad \operatorname{div} \mathscr{V}_3 = \sqrt{3}x.$$

Note that $\{(\mathscr{V}_i, C_i)\}$ satisfies (1). (In $C_1{}'$, div$\mathscr{V}_1 \leq 0$ and div$\mathscr{V}_4$=0. In $C_2{}'$, $-y \leq \sqrt{3}x$ and in $C_3{}'$, the opposite is true: $\sqrt{3}x \leq -y$.)

Now we use an isometry φ to translate the entire system to the left so that $p_1' \mapsto p_1$ and $p_2' \mapsto p_2$. For each i we denote $\hat{\mathscr{V}_i} = \varphi(\mathscr{V}_i)$. Let $\mathscr{F}_1$ be the union of $\hat{\mathscr{V}_1}$ and $\hat{\mathscr{V}_2}$ on the left half-plane and as the reflection of that on the right half-plane. Let $\mathscr{F}_2$ be the union of $\hat{\mathscr{V}_3}$ and $\hat{\mathscr{V}_4}$, $\mathscr{F}_3$ be the reflection of $\mathscr{F}_1$ across the x-axis, and $\mathscr{F}_4$ be the reflection of $\mathscr{F}_2$ across the y-axis. Since isometries preserve length, angle, and divergence, we have all the properties we need to prove minimization.

We note that $(\mathscr{F}_1 - \mathscr{F}_2)$, $(\mathscr{F}_2 - \mathscr{F}_3)$, $(\mathscr{F}_3 - \mathscr{F}_4)$, $(\mathscr{F}_1 - \mathscr{F}_4)$, and $(\mathscr{F}_1 - \mathscr{F}_3)$ are unit length and are normal to $y_1, \ldots, y_5$ respectively. Thus, by Theorem 1, Y is the minimizer for its combinatorial structure. Since the only other combinatorial structure is a rotation of that of Y, we say that Y is the unique minimizer up to rotation. □

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