SPIKED TRAVELING WAVES AND ILL-POSEDNESS FOR THE CAMASSA-HOLM EQUATION ON THE CIRCLE

PETER BYERS

Abstract. We will show that the Camassa-Holm equation possesses periodic traveling wave solutions with spikes, i.e., peaks where the first derivative is unbounded. Moreover, we will show that such a solution can be chosen to be ρ-periodic for arbitrarily small $\rho > 0$.

This family of solutions (parametrized by ρ) has the important property that, for $q \in [1, 3)$, $\|u_0'\|_{L^q(\mathbb{T})}$ is uniformly bounded above and below, where u_0 is the initial data. Using this property with $q = 2$ we are able to prove that the corresponding Cauchy problem is not locally well-posed in the Sobolev space $H^1(\mathbb{T})$. Similarly, we will show ill-posedness in the corresponding L^q Sobolev space, $W^{1,q}(\mathbb{T})$, for any $q \in [1, 3)$.

1. Introduction

In this paper, we consider the following partial differential equation:

$$\partial_t u - \partial_t \partial_x^2 u + \frac{3}{2}\partial_x \left(u^2\right) - \frac{1}{2}\partial_x^3 \left(u^2\right) + \frac{1}{2}\partial_x \left((\partial_x u)^2\right) = 0, \tag{1.1}$$

which is formally equivalent to

$$\partial_t u - \partial_x^2 \partial_t u + 3u\partial_x u - 2\partial_x u \partial_x^2 u - u\partial_x^3 u = 0. \tag{1.2}$$

This equation, which now is generally called the Camassa-Holm equation (CH), was derived in different ways by and Fokas and Fuchssteiner [4] and by Camassa and Holm [1].

We will also study the corresponding periodic initial value problem, namely

$$\begin{cases} \partial_t u - \partial_t \partial_x^2 u + \frac{3}{2}\partial_x \left(u^2\right) - \frac{1}{2}\partial_x^3 \left(u^2\right) + \frac{1}{2}\partial_x \left((\partial_x u)^2\right) = 0, \\ u(x, 0) = u_0(x), \end{cases} \tag{1.3}$$

where $t \in \mathbb{R}$ and $x \in \mathbb{T}$.

We will say that an initial value problem is *locally well-posed* in a Banach space E if for every $r > 0$ there exists $T > 0$ such that

Received January 6, 2004; received in final form June 24, 2004.

2000 *Mathematics Subject Classification.* Primary 35G25, 35Q58. Secondary 35Q53, 46E35, 46F10.

Partially supported by the University of Notre Dame.

(i) for each $u_0 = u_0(x) \in B(0,r) \doteq \{\varphi \in E \mid \|\varphi\|_E \leq r\}$ there exists a unique solution $u = u(x,t) \in C([-T,T] : E)$ of the initial value problem;
(ii) the map from $B(0,r)$ into $C([-T,T] : E)$ given by $u_0 \mapsto u$ is uniformly continuous.

The definition implies, in particular, that for each $t \in [-T,T]$, $u_0 \mapsto u(t)$ is a uniformly continuous map from $B(0,r)$ into E, where $u(t)$ is understood to mean $u(-,t)$.

Remarks. (1) One can also think of $u_0 \mapsto u$ as a map from E into $\bigcup_{T>0} C([-T,T] : E)$.
(2) Another common definition of well-posedness requires only continuous dependence on the initial data, not uniformly continuous dependence.

Constantin and Escher proved in [2] that (1.3) is locally well-posed in the Sobolev space H^3. More recently, it has been shown (see [7] and [13]) that (1.3) is locally well-posed in the Sobolev space H^s for every $s > 3/2$. (Throughout this paper H^s will be understood to mean $H^s(\mathbb{T})$.)

It is not known whether (1.3) is well-posed in H^s for any $s \leq 3/2$, but some existence and uniqueness results have been proven for the space H^1. Constantin and Escher showed in [2] that a unique global weak solution exists for initial data $u_0 \in H^1$ under the additional assumption that $u_0 - \partial_x^2 u_0$ is a positive Radon measure. Their method involved approximating u_0 by a sequence $u_0^n \in C^\infty$ and taking a limit of the corresponding solutions u^n of (1.3).

More recently, Xin and Zhang showed in [15] that a global weak solution (not necessarily unique) exists for *any* initial data $u_0 \in H^1$ by using solutions of a certain viscous problem, which converge to a solution of (1.3).

Our main result in this paper is that the periodic CH equation is not well-posed in H^1. More specifically:

Theorem 1.1. *There exists a ball $B(0,r) \subset H^1(\mathbb{T})$ such that for every $T > 0$ the map from $B(0,r)$ to $C([-T,T] : H^1(\mathbb{T}))$ given by $u_0 \mapsto u$, where u solves* (1.3)*, is not uniformly continuous.*

The proof of Theorem 1.1 will rely heavily on traveling wave solutions of the Camassa-Holm equation, i.e., solutions of the form

$$u(x,t) = f(x - \sigma t), \tag{1.4}$$

where σ is a constant. Substituting (1.4) into (1.1), we see that the profile f must be a solution of

$$-\sigma f' + \sigma f''' + \frac{3}{2}\frac{d}{dx}\left(f^2\right) - \frac{1}{2}\frac{d^3}{dx^3}\left(f^2\right) + \frac{1}{2}\frac{d}{dx}\left((f')^2\right) = 0; \tag{1.5}$$

hence

$$2\sigma f'' - 2\sigma f + 3f^2 - \frac{d^2}{dx^2}\left(f^2\right) + (f')^2 = M \tag{1.6}$$

for a constant M.

In Section 2 we will construct solutions to (1.6) with certain special properties; these will then be used in Section 3 to prove Theorem 1.1.

2. Traveling waves with unbounded slope

We'll begin with a brief overview of known types of traveling wave solutions to the Camassa-Holm equation. The simplest is the non-periodic function $u = \sigma e^{-|x-\sigma t|}$ for any $\sigma \in \mathbb{R}$.

For the periodic case, a greater variety of traveling wave solutions exist. For example, [3] showed that the CH equation has C^∞ periodic traveling waves. Another type of solution can be defined explicitly as follows: Pick any $r > 0$ and let

$$f_r(x) = \frac{e^x + e^{-x}}{e^r + e^{-r}}$$

for all $x \in [-r, r]$, and $f_r(x + 2kr) = f_r(x)$ for every $k \in \mathbb{Z}$. Then for every $\sigma \in \mathbb{R}$, $u(x,t) = \sigma f_r(x - \sigma t)$ is a solution to (1.1) with period $2r$. (In particular, if $r = 1/(2n)$ for some positive integer n, then $f_r : \mathbb{T} \to \mathbb{R}$.)

This function is similar to the non-periodic wave $u = \sigma e^{-|x-\sigma t|}$, in that both have peaks with a "corner", i.e., points where $\partial_x u$ is discontinuous but bounded (and in fact $(\partial_x u)^2$ is continuous, with removable singularities).

Our main focus in this section is to construct traveling wave solutions with "spikes"—that is to say peaks where the derivative goes to infinity. For simplicity, we will only consider traveling waves with unit speed, i.e., solutions of (1.6) with $\sigma = 1$. Other traveling waves can then be obtained by the well-known scaling property of the CH equation: if $u = u(x,t)$ is a solution of (1.1), then $u_\lambda(x,t) \doteq \lambda u(x, \lambda t)$ is a solution as well for any $\lambda \in \mathbb{R}$.

Let us start by unpacking equation (1.6) a little (cf. [3]). For a given function f, let $S \subset \mathbb{R}$ be the largest open set on which f is C^∞ (i.e., the complement of the singular support of f). Then on S, (1.6) is equivalent to

$$2f'' - 2f + 3f^2 - 2ff'' - (f')^2 = M. \tag{2.1}$$

If we multiply by f', (2.1) is equivalent on the set $\tilde{S} \doteq \{x \in S \mid f'(x) \neq 0\}$ to

$$\begin{aligned} Mf' &= 2f'f'' - 2ff' + 3f^2 f' - 2ff'f'' - (f')^3 \\ &= \frac{d}{dx}\left[(f')^2 - f^2 + f^3 - f(f')^2\right] \end{aligned}$$

which is to say

$$\begin{aligned} (f-1)(f')^2 &= f^3 - f^2 - Mf + L \qquad (2.2)\\ &= (f-a)(f-b)(f-c) \qquad (2.3) \end{aligned}$$

for constants L, a, b, and c. Then the coefficient of f^2 in (2.2) implies that $a+b+c=1$. Conversely if $a+b+c=1$ then (2.3) holds for some M and L.

We can now prove the following result:

PROPOSITION 2.1. *For any $\rho > 0$ sufficiently small, there is a periodic solution f of (1.5) with period ρ, such that $f \leq 1$ and $\sup|f-1| \to 0$ as $\rho \to 0$. Furthermore,*

$$C_1\rho \leq \|f'\|^q_{L^q([0,\rho])} \leq C_2\rho \tag{2.4}$$

for any $q \in [1,3)$, where C_1, C_2 are positive constants independent of ρ; and

$$f' < 0 \text{ on } (0,\rho/2) \quad \text{and} \quad f' > 0 \text{ on } (\rho/2,\rho). \tag{2.5}$$

Proof. For convenience let us first write $f = 1-g$, $a = 1+A$, $b = 1-\epsilon$, and $c = \epsilon - 1 - A$, so (2.3) becomes

$$g(g')^2 = (g+A)(\epsilon - g)(A+2-\epsilon-g). \tag{2.6}$$

Fix any $A > 0$ and $\epsilon_0 \in (0, 1+A/2)$. Then for any $\epsilon \in (0,\epsilon_0)$ we have $\epsilon < A+2-\epsilon$, and hence the right hand side of (2.6) is positive for all $g \in (0,\epsilon)$.

Now if, for some $x_0 \in \mathbb{R}$, we prescribe $g(x_0) = y_0 \in (0,\epsilon)$ and take $g' > 0$, then (2.6) will have an increasing local solution, with

$$g' = \sqrt{(g+A)(\epsilon-g)(A+2-\epsilon-g)/g} \tag{2.7}$$

Clearly there exists $\tilde{x}_0 < x_0$ such that $g(\tilde{x}_0) = 0$ (since $g' > 0$ and $g' \to \infty$ as $g \to 0$) and without loss of generality we can take $\tilde{x}_0 = 0$, i.e., $g(0) = 0$.

If we write

$$\frac{dx}{dg} = \sqrt{\frac{g}{(g+A)(\epsilon-g)(A+2-\epsilon-g)}},$$

we see that the inverse of g is given explicitly by

$$\begin{aligned} g^{-1}(y) &= \int_0^y \sqrt{\frac{t}{(t+A)(\epsilon-t)(A+2-\epsilon-t)}}dt \\ &= \int_0^y \frac{ß(t)\sqrt{t}}{\sqrt{\epsilon-t}}dt, \end{aligned}$$

where

$$ß(t) \doteq \frac{1}{\sqrt{(t+A)(A+2-\epsilon-t)}}.$$

Note that $0 < c_1 \leq ß(t) \leq c_2$ for all $t \in [0, \epsilon]$, with $c_1 \doteq 1/\sqrt{(A+\epsilon_0)(A+2)}$ and $c_2 \doteq 1/\sqrt{A(A+2-2\epsilon_0)}$. Now for every $y \in (0, \epsilon)$,

$$\begin{aligned} g^{-1}(y) &\leq c_2\sqrt{\epsilon}\int_0^y (\epsilon - t)^{-1/2} dt \\ &= 2c_2\epsilon - 2c_2\sqrt{\epsilon}\sqrt{\epsilon - y} \\ &\leq 2c_2\epsilon. \end{aligned}$$

In other words, there is some positive real number, let us say $\rho/2 \leq 2c_2\epsilon$, such that $g(\rho/2) = \epsilon$.

On the other hand,

$$\begin{aligned} g^{-1}(y) &\geq \frac{c_1}{\sqrt{\epsilon}}\int_0^y t^{1/2} dt \\ &= \frac{2c_1}{3\sqrt{\epsilon}} t^{3/2}\Big|_0^y \\ &= \frac{2c_1}{3\sqrt{\epsilon}} y^{3/2}, \end{aligned} \tag{2.8}$$

so that $\rho = 2g^{-1}(\epsilon) \geq \frac{4c_1}{3}\epsilon$. Moreover, if we set

$$x_1 \doteq g^{-1}(\epsilon/2) \geq \frac{2c_1}{3\sqrt{\epsilon}}(\epsilon/2)^{3/2} = \left(\frac{c_1}{3\sqrt{2}}\right)\epsilon,$$

then for all $x \in [0, x_1]$, $g(x) \leq \epsilon/2$ implies $\epsilon - g(x) \geq g(x)$, and hence

$$\begin{aligned} g'(x) &\geq \frac{\sqrt{\epsilon - g}}{c_2\sqrt{g}} \\ &\geq 1/c_2. \end{aligned}$$

Therefore, since $f'(x) = -g'(x)$,

$$\begin{aligned} \|f'\|^q_{L^q([0,\rho/2])} &\geq \|g'\|^q_{L^q([0,x_1])} \\ &\geq c_2^{-q} x_1 \\ &\geq c_3\rho, \end{aligned} \tag{2.9}$$

where we used $x_1 \geq \left(\frac{c_1}{3\sqrt{2}}\right)\epsilon$ and $\epsilon \geq \frac{\rho}{4c_2}$.

On the other hand, for $x \in [x_1, \rho/2]$,

$$g'(x) \leq \frac{\sqrt{\epsilon - g}}{c_1\sqrt{g}} \leq \frac{1}{c_1},$$

and for $y \in (0, \epsilon/2)$ we can calculate

$$\begin{aligned} g^{-1}(y) &\leq c_2 \sqrt{\frac{2}{\epsilon}} \int_0^y t^{1/2} dt \\ &= \frac{2c_2\sqrt{2}}{3\sqrt{\epsilon}} y^{3/2}, \end{aligned}$$

which implies

$$g(x) \geq \left(\frac{3}{2c_2\sqrt{2}} \sqrt{\epsilon} x \right)^{2/3} = c_4 \left(\epsilon x^2 \right)^{1/3} \tag{2.10}$$

for all $x \in [0, x_1]$, and

$$\begin{aligned} g'(x) &\leq \frac{\sqrt{\epsilon - g}}{c_1\sqrt{g}} \\ &\leq \frac{\sqrt{\epsilon}}{c_1 \sqrt{c_4 \left(\epsilon x^2 \right)^{1/3}}} \\ &= c_5 \left(\frac{\epsilon}{x} \right)^{1/3}. \end{aligned}$$

So we have shown

$$g'(x) \leq \max \left(\frac{1}{c_1}, c_5 \left(\frac{\epsilon}{x} \right)^{1/3} \right)$$

for all $x \in (0, \rho/2]$, and therefore

$$\begin{aligned} \|f'\|^q_{L^q([0,\rho/2])} &= \|g'\|^q_{L^q([0,\rho/2])} \\ &\leq c_1^{-q}\rho + c_5^q \int_0^{\rho/2} \epsilon^{q/3} x^{-q/3} dx \\ &= c_1^{-q}\rho + \frac{c_5^q}{1 - q/3} \epsilon^{q/3} (\rho/2)^{1-q/3} \\ &\leq c_6 \rho, \end{aligned} \tag{2.11}$$

where we used the fact that $q < 3$ and $\epsilon \leq \frac{3}{4c_1}\rho$.

Until now we have only been dealing with the function $g : [0, \rho/2] \to \mathbb{R}$. Now let us extend this function to $g : [-\rho/2, \rho/2] \to \mathbb{R}$ by defining $g(-x) = g(x)$. Furthermore, we can then extend g to a periodic function with period ρ, by assigning $g(x + np) = g(x)$ for every integer $n \neq 0$ and for all $x \in [-\rho/2, \rho/2]$. (Note that g is C^1 at $x = \rho/2$, with $g'(\rho/2) = 0$.)

The period of g is

$$\rho = 2 \int_0^{\epsilon} \sqrt{\frac{t}{(t + A)(\epsilon - t)(A + 2 - \epsilon - t)}} dt, \tag{2.12}$$

which depends continuously on ϵ, and we know that $\frac{4c_1}{3}\epsilon \le \rho \le 4c_2\epsilon$. Hence for any ρ sufficiently small, there will always be a corresponding value of $\epsilon \in (0, \epsilon_0)$.

The estimate (2.4) follows immediately from (2.9) and (2.11). Also, it is obvious from the construction that $f' < 0$ on $(0, \rho/2)$, $f' > 0$ on $(\rho/2, \rho)$, and $\sup|f-1| = \sup|g| = \epsilon \to 0$ as $\rho \to 0$.

It still remains to show that $f = 1 - g$ satisfies (1.6). We will start by showing that

$$2f'' - 2f + 3f^2 - \frac{d^2}{dx^2}\left(f^2\right) + (f')^2$$

is a measurable function. This is equivalent to showing that the expression

$$\begin{aligned} 2f'' - \frac{d^2}{dx^2}(f^2) &= \frac{d}{dx}(2f' - 2ff') \\ &= \frac{d}{dx}(2gg') \end{aligned}$$

is a measurable function (since clearly f and f' are measurable functions, and hence $-2f + 3f^2 + (f')^2$ is as well).

We know that on the interval $(0, \rho/2)$, gg' is continuous and satisfies

$$gg' = \sqrt{g(g+A)(\epsilon - g)(A + 2 - \epsilon - g)}. \tag{2.13}$$

So as $x \to 0^+$, $g(x) \to 0$, and hence $g(x)g'(x) \to 0$ by (2.13). Similarly, $g(x)g'(x) \to 0$ as $x \to 0^-$. So $g(x)g'(x)$ is continuous at $x = 0$, and by periodicity it is continuous everywhere. But we also know that gg' is smooth except on a discrete set, so $\frac{d}{dx}(2gg')$ is a measurable function, as desired.

Hence the left hand side of (1.6) is a measurable function. We already know that on the set $\tilde{S} = \mathbb{R} - \{n\rho/2 \mid n \in \mathbb{Z}\}$, f is smooth, satisfies (2.3), and has $f' \neq 0$. Therefore f satisfies (1.6) on this set, and in fact everywhere, (in the distribution sense). □

Remark 2.2. In this paper we are mainly interested in ρ small; a word could be said, however, about ρ large. It is fairly easy to show from (2.12) that ρ becomes arbitrarily large if we take ϵ sufficiently close to $1+A/2$. (Note that this necessarily means taking ϵ_0 close to $1 + A/2$ as well, which affects the constants $c_1, c_2, \ldots$)

Therefore, the stipulation "for any $\rho > 0$ sufficiently small" in Proposition 2.1 could be changed to "for any $\rho \in (0, \rho_0)$", where ρ_0 is any positive constant we choose (and where it is understood that the constants C_1 and C_2 in (2.4) depend on ρ_0).

It is also worth noting that if we instead take $\epsilon = 1 + A/2$, which is to say that if we take $b = c = -A/2$ in (2.3), then the integral in (2.12) becomes infinite; that is, we get a non-periodic function $f : \mathbb{R} \to \mathbb{R}$. Note, however, that $\lim_{x\to\pm\infty} f(x) = -A/2$. So if we want $\lim_{x\to\pm\infty} f(x) = 0$, we must take

$A = 0$ rather than $A > 0$. In this case f is none other than the function $f(x) = e^{-|x+c|}$ (which has corner-type peaks, rather than spikes).

3. Proof of Theorem 1.1

Proposition 2.1 and Remark 2.2 tell us, in particular, that for every positive integer k there is a periodic solution $f_k : \mathbb{T} \to \mathbb{R}$ of (1.5) with period $\rho_k = 1/k$. Then $0 < f_k \leq 1$, so $\|f_k\|^q_{L^q(\mathbb{T})} \leq 1$, and (2.4) implies

$$C_1 \leq \|f'_k\|^q_{L^q(\mathbb{T})} \leq C_2 \tag{3.1}$$

for any $q \in [1, 3)$.

Now let us recall the definition of the L^p-Sobolev spaces on the torus:

$$W^{m,p}(\mathbb{T}) = \{h \in L^p(\mathbb{T}) : h^{(k)} \in L^p(\mathbb{T}), k \leq m\},$$

for any nonnegative integer m and any $p \in [0, \infty)$, which is a Banach space with respect to the norm

$$\|h\|^p_{W^{m,p}} \doteq \sum_{k=1}^{m} \|h^{(k)}\|^p_{L^p}.$$

In particular, $W^{m,2} = H^m$, with equivalent norms. (Of course, $W^{m,p}$ can be defined for arbitrary $m \geq 0$ and $p \in [0, \infty]$, but we will not use the general definition since it will not be needed in this paper.)

So (3.1) implies that $f_k \in W^{1,q}(\mathbb{T})$ and

$$\|f_k\|_{W^{1,q}(\mathbb{T})} = \left(\|f_k\|^q_{L^q(\mathbb{T})} + \|f'_k\|^q_{L^q(\mathbb{T})}\right)^{1/q} \leq (1 + C_2)^{1/q} \doteq C_3.$$

Using these functions f_k we will now prove Theorem 1.1. In fact, we will show that (1.3) is not well-posed in the space $W^{1,q}(\mathbb{T})$ for any $q \in [1, 3)$:

Theorem 3.1. *Let $q \in [1, 3)$ and $T > 0$. Then there exist sequences $u^k, v^k \in C[\mathbb{R} : W^{1,q}(\mathbb{T})]$ which solve (1.1), such that*

$$\|u^k_0\|_{W^{1,q}} \leq C \tag{3.2}$$

for all k, where $C > 0$ is a constant independent of T, and

$$\|u^k_0 - v^k_0\|_{W^{1,q}} \to 0 \text{ as } k \to \infty, \tag{3.3}$$

but

$$\|u^k(T) - v^k(T)\|_{W^{1,q}} \not\to 0. \tag{3.4}$$

Proof. By the preceding discussion, we can take

$$u^k(x, t) = f_k(x - t)$$

for all positive integers k. Also set $\sigma_k = 1 + \frac{1}{2kT}$ and

$$v^k(x, t) = \sigma_k f_k(x - \sigma_k t).$$

Then we have

$$\|u_0^k\|_{W^{1,q}} = \|f_k\|_{W^{1,q}} \leq C_3.$$

We also have

$$\|u_0^k - v_0^k\|_{W^{1,q}} = (\sigma_k - 1)\|f_k\|_{W^{1,q}} \leq \frac{C_3}{2kT},$$

which goes to zero as $k \to \infty$. Now at time $t = T$, we see from (2.5) that the derivatives $\partial_x u^k(x, T) = f_k'(x - T)$ and $\partial_x v^k(x, T) = \sigma_k f_k'(x - T - 1/(2k))$ will never have the same sign. Therefore

$$|\partial_x u^k(x, T) - \partial_x v^k(x, T)| \geq |\partial_x u^k(x, T)|$$

and hence

$$\begin{aligned}
\|u^k(T) - v^k(T)\|_{W^{1,q}} &\geq \|\partial_x u^k(T) - \partial_x v^k(T)\|_{L^q} \\
&\geq \|\partial_x u^k(T)\|_{L^q} \\
&= \|f_k'\|_{L^q} \\
&\geq C_1^{1/q}.
\end{aligned}$$

□

Remark. In the definition of local well-posedness, the existence time T is allowed to depend on the size of the initial data. It is important, therefore, that the uniform bound (3.2) was independent of T.

Acknowledgements. The author would like to thank Alex Himonas, who suggested the ill-posedness question, and Gerard Misiołek, for helpful conversations.

References

[1] R. Camassa and D. D. Holm, *An integrable shallow water equation with peaked solitons*, Phys. Rev. Lett. **71** (1993), 1661–1664. MR 94f:35121

[2] A. Constantin and J. Escher, *Well-posedness, global existence, and blowup phenomena for a periodic quasi-linear hyperbolic equation*, Comm. Pure Appl. Math. **51** (1998), 475–504. MR 98k:35165

[3] A. Constantin and H. P. McKean, *A shallow water equation on the circle*, Comm. Pure Appl. Math. **52** (1999), 949–982. MR 2000m:37146

[4] A. S. Fokas and B. Fuchssteiner, *Symplectic structures, their Bäcklund transformations and hereditary symmetries*, Phys. D **4** (1981/82), 47–66. MR 84j:58046

[5] A. A. Himonas and G. Misiołek, *The Cauchy problem for an integrable shallow-water equation*, Differential Integral Equations **14** (2001), 821–831. MR 2002c:35228

[6] ———, *Analyticity of the Cauchy problem for an integrable evolution equation*, Math. Ann. **327** (2003), 575–584. MR 2 021 030

[7] D. Holm, S. Kouranbaeva, J. Marsden, T. Ratiu, and S. Shkoller, *A nonlinear analysis of the averaged Euler equations*, preprint, 1998.

[8] D. D. Holm, J. E. Marsden, and T. S. Ratiu, *The Euler-Poincaré equations and semidirect products with applications to continuum theories*, Adv. Math. **137** (1998), 1–81. MR 99e:58070

[9] C. E. Kenig, G. Ponce, and L. Vega, *On the ill-posedness of some canonical dispersive equations*, Duke Math. J. **106** (2001), 617–633. MR 2002c:35265

[10] Y. A. Li and P. J. Olver, *Well-posedness and blow-up solutions for an integrable nonlinearly dispersive model wave equation*, J. Differential Equations **162** (2000), 27–63. MR 2002a:35185

[11] H. P. McKean, *Breakdown of a shallow water equation*, Asian J. Math. **2** (1998), 867–874. MR 2000k:35262

[12] G. Misiołek, *A shallow water equation as a geodesic flow on the Bott-Virasoro group*, J. Geom. Phys. **24** (1998), 203–208. MR 99d:58018

[13] ———, *Classical solutions of the periodic Camassa-Holm equation*, Geom. Funct. Anal. **12** (2002), 1080–1104. MR 2003k:37125

[14] G. Rodríguez-Blanco, *On the Cauchy problem for the Camassa-Holm equation*, Nonlinear Anal. **46** (2001), 309–327. MR 2002i:35172

[15] Z. Xin and P. Zhang, *On the weak solutions to a shallow water equation*, Comm. Pure Appl. Math. **53** (2000), 1411–1433. MR 2001m:35278

Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556, USA

E-mail address: pbyers@nd.edu