Illinois Journal of Mathematics
Volume 48, Number 3, Fall 2004, Pages 999–1020
S 0019-2082

FREDHOLM PROPERTIES OF EVOLUTION SEMIGROUPS

YURI LATUSHKIN AND YURI TOMILOV

ABSTRACT. We show that the Fredholm spectrum of an evolution semigroup $\{E^t\}_{t\geq 0}$ is equal to its spectrum, and prove that the ranges of the operator $E^t - I$ and the generator $\mathbf{G}$ of the evolution semigroup are closed simultaneously. The evolution semigroup is acting on spaces of functions with values in a Banach space, and is induced by an evolution family that could be the propagator for a well-posed linear differential equation $u'(t) = A(t)u(t)$ with, generally, unbounded operators $A(t)$; in this case $\mathbf{G}$ is the closure of the operator G given by $(Gu)(t) = -u'(t) + A(t)u(t)$.

1. Introduction and main results

An evolution family (propagator) associated with a well posed nonautonomous linear differential equation $u'(t) = A(t)u(t)$ on a Banach space X with (generally, unbounded) operator coefficients generates three important operators acting on spaces of X-valued functions: a differential operator, $\mathbf{G}$, a functional operator, E^t, and a difference operator, D_τ. The objective of the current paper is to study Fredholm and other fine spectral properties of these operators as they are related to the dynamical properties of the evolution family such as its exponential dichotomy. Let $\{U(t,\tau)\}_{t\geq\tau}$, $t,\tau \in \mathbb{R}$, denote a strongly continuous exponentially bounded evolution family on the Banach space X, let $\{E^t\}_{t\geq 0}$ denote the corresponding *evolution semigroup*, defined on the spaces $\mathcal{E}(\mathbb{R}) = L_p(\mathbb{R}; X)$, $1 \leq p < \infty$, or $\mathcal{E}(\mathbb{R}) = C_0(\mathbb{R}; X)$, the space of continuous functions vanishing at $\pm\infty$, by the rule

$$(1.1) \qquad (E^t u)(\tau) = U(\tau, \tau - t)u(\tau - t), \quad \tau \in \mathbb{R}, \quad t \geq 0,$$

Received December 30, 2003; received in final form March 25, 2004.

2000 *Mathematics Subject Classification.* Primary 34D05. Secondary 47D06, 46B20.

This project has been funded in part by the National Academy of Sciences under the Collaboration in Basic Science and Engineering Program / Twinning Program supported by Contract No. INT-0002341 from the NSF. Also, the first author was supported in part by the NSF grant 0338743 and by the CRDF grant UP1-2567-OD-03, and the second author was supported in part by the KBN grant 1 P03A 038 26.

and let $\mathbf{G}$ denote the evolution semigroup generator. If $\{U(t,\tau)\}_{t\geq\tau}$ is the propagator for the nonautonomous differential equation then $\mathbf{G}$ is the closure of the operator $G = -d/dt + A(t)$ when the domain of the latter is the intersection of the domains of the operators of differentiation and multiplication by $A(\cdot)$.

The spectrum, $\sigma(\cdot)$, of the operators E^t and $\mathbf{G}$ and the Fredholm properties of $\mathbf{G}$ and their relations to the asymptotic behavior of the evolution family are fairly well understood; see [9], [13, Ch. VI.9], [29], a newer survey [28], and a recent paper [16] and the bibliographies therein. In particular, as it is well known, unlike many strongly continuous semigroups, the evolution semigroups enjoy the *spectral mapping property* $\sigma(E^t)\backslash\{0\} = \exp t\sigma(\mathbf{G})$, $t \geq 0$; see the above-cited references. In this paper we continue our work in [16], where the Fredholm properties of $\mathbf{G}$ have been related to the exponential dichotomy of the evolution family (see Theorem 2.4 below), and study the Fredholm spectrum, $\sigma_{\text{fred}}(\cdot)$, of the operator E^t.

To put the work in [16] and in the current paper in a broader context, we remark first that for many classes of partial differential equations the operators G and $\mathbf{G}$ coincide; see, e.g., [28]. An understanding of spectral properties of the operator G and the corresponding semigroup is important for several reasons. The study of Fredholm properties of G is crucial, for example, in the stability theory of traveling waves where G appears as a linearization of certain parabolic PDE's [27]. Also, an asymptotically hyperbolic case when the limits $A(\pm\infty) := \lim_{t\to\pm\infty} A(t)$ exist in an appropriate sense, and $\sigma(A_\pm)\cap i\mathbb{R} = \emptyset$, is of special interest in infinite dimensional Morse theory [1], [2], where the operator G appears after linearization of a vector field on a manifold along an orbit connecting two hyperbolic critical points, and where understanding its Fredholm properties is an important issue. Finally, results relating the Fredholm index of G and the spectral flow of the operator path $\{A(t)\}_{t=-\infty}^{\infty}$ provide a set-up for generalizations of the Atiyah-Patody-Singer theory; see [2], [12], [25], [26].

An attempt to obtain a spectral mapping property for the *Fredholm* spectrum of the evolution semigroup has led to our first main result.

THEOREM 1.1. *For the evolution semigroup* (1.1) *on* $\mathcal{E}(\mathbb{R})$ *we have*

$$\sigma_{\text{fred}}(E^t)\backslash\{0\} = \sigma(E^t)\backslash\{0\}, \quad t \geq 0.$$

Thus, a more refined property than Fredholm should be of interest for the evolution semigroups, and in this paper we study conditions when the ranges of $E^t - I$ and $\mathbf{G}$ are closed. For this, we involve a family of difference operators, D_τ, acting on the respective sequence spaces $\mathcal{E}(\mathbb{Z}) = \ell_p(\mathbb{Z}; X)$, $1 \leq p < \infty$, or $\mathcal{E}(\mathbb{Z}) = c_0(\mathbb{Z}; X)$, the space of sequences vanishing at $\pm\infty$, by

$$D_\tau : (x_n)_{n\in\mathbb{Z}} \mapsto (x_n - U(n+\tau, n+\tau-1)x_{n-1})_{n\in\mathbb{Z}}, \quad \tau \in [0,1). \tag{1.2}$$

The interplay between the three operators, the functional operator E^t, the differential operators G or $\mathbf{G}$, and the difference operators D_τ, has been studied by many authors; see, e.g., [3], [4], [6], [9], [14], [16]. In particular, many results relating their spectral properties such as invertibility, correctness (uniform boundedness from below), Fredholm property, etc., are available and can be found in these references. Our second principal result settles the more delicate issue of the closedness of their ranges. For its formulation, recall the definition of the Kato lower bound, $\gamma(T)$, for a closed operator T,

$$\gamma(T) := \inf_{\substack{x\in\operatorname{dom}(T)\\ Tx\neq 0}} \frac{\|Tx\|}{\operatorname{dist}(x, \operatorname{Ker} T)},$$

and note that $\gamma(T) > 0$ if and only if the range $\operatorname{Im} T$ is closed [15, Sec. IV.5.1].

THEOREM 1.2. *For the operators E^1 and $\mathbf{G}$ on $\mathcal{E}(\mathbb{R})$ and the operators D_τ on $\mathcal{E}(\mathbb{Z})$ the following assertions are equivalent:*

(i) $\gamma(E^1 - I) > 0$;
(ii) $\gamma(\mathbf{G}) > 0$;
(iii) $\inf_{\tau\in[0,1)} \gamma(D_\tau) > 0$.

In addition, we obtain results similar to Theorems 1.1 and 1.2 for evolution semigroups acting on spaces of periodic X-valued functions and on spaces of X-valued functions on the half-line; in the latter case we relate Fredholm properties of the corresponding operators to exponential dichotomy of the evolution family on the half-line.

The paper is organized as follows. In Section 2 we introduce notations and recall some known facts about evolution semigroups. The Spectral Mapping Theorem 1.1 for the Fredholm spectrum for evolution semigroups on the line is proved in Section 3, which also contains Theorem 3.1, a rather general result on Fredholm properties of first order autonomous differential operators. Our second main result, Theorem 1.2, is proved in Section 4. In Section 5 we develop a Fredholm theory for evolution semigroups and their generators on spaces of functions on the half-line, and connect it to the exponential dichotomy on the half-line. Finally, in Section 6 we give assertions similar to Theorems 1.1 and 1.2 for evolution semigroups on spaces of periodic functions.

2. Notation and preliminaries

Notation. $\mathbb{R}_+ = [0,\infty)$, $\mathbb{R}_- = (-\infty, 0]$, $\mathbb{Z}_\pm = \mathbb{Z}\cap\mathbb{R}_\pm$; X is a Banach space; X^* is the adjoint space; $\mathcal{L}(X)$ is the set of bounded operators on X; A^*, $\operatorname{dom} A$, $\operatorname{Ker} A$ and $\operatorname{Im} A$ are the adjoint, domain, kernel and range of an operator A; $\sigma(A)$ and $\sigma_{\text{fred}}(A) = \{\lambda\in\mathbb{C} : \lambda - A \quad \text{is not Fredholm}\}$ denote the spectrum and the Fredholm spectrum of A, and $R(\lambda, A)$ is the resolvent of A. The function space $\mathcal{E}(\mathbb{R})$ is one of the spaces $L_p(\mathbb{R}; X), 1 \le p < \infty$, or $C_0(\mathbb{R}; X)$; the sequence space $\mathcal{E}(\mathbb{Z})$ is one of the spaces $\ell_p(\mathbb{Z}; X)$ or $c_0(\mathbb{Z}; X)$.

Similarly, $\mathcal{E}(\mathbb{R}_+)$, resp. $\mathcal{E}_0(\mathbb{R}_+)$, stands for one of the spaces $L_p(\mathbb{R}_+;X)$, $1 \leq p < \infty$, or $C_0(\mathbb{R}_+;X)$, resp. $C_{00}(\mathbb{R}_+;X)$, the space of continuous X-valued functions on $\mathbb{R}_+$ vanishing at zero and at infinity; the sequence space $\mathcal{E}(\mathbb{Z}_+)$ is one of the spaces $\ell_p(\mathbb{Z}_+;X)$ or $c_0(\mathbb{Z}_+;X)$. Finally, $\mathcal{E}([0,2\pi])$ is one of the spaces $L_p([0,2\pi];X)$ or $C_{\mathrm{per}}([0,2\pi];X)$, the space of continuous X-valued 2π-periodic functions on $[0,2\pi]$. We use boldface to denote sequences, e.g., $\mathbf{x} = (x_n)_{n\in\mathbb{Z}}, x_n \in X$. If $A(\cdot)$ is an operator-valued function, then M_A, $(M_A u)(t) = A(t)u(t)$, denotes the operator of multiplication on a function space $\mathcal{E}$ with the maximal domain $\operatorname{dom} M_A = \{u \in \mathcal{E} : u(t) \in \operatorname{dom} A(t)\, \mathbf{a.e.}, A(\cdot)u(\cdot) \in \mathcal{E}\}$.

Evolution semigroups. Let J denote one of the intervals $\mathbb{R}_+$, $\mathbb{R}_-$, or $\mathbb{R}$. A family $\{U(t,\tau)\}_{t\geq\tau}$, $t,\tau \in J$, of bounded linear operators on X is called a *strongly continuous exponentially bounded evolution family* on J if it satisfies:

(1) For each $x \in X$ the map $(t,\tau) \mapsto U(t,\tau)x$ is continuous for all $t \geq \tau$ in J.
(2) $\sup\{\|e^{-\omega(t-\tau)}U(t,\tau)\| : t,\tau \in J, t \geq \tau\} < \infty$ for some $\omega \in \mathbb{R}$.
(3) $U(t,t) = I$, $U(t,\tau) = U(t,s)U(s,\tau)$ for all $t \geq s \geq \tau$ in J.

Throughout, all evolution families are assumed to be strongly continuous and exponentially bounded.

First, consider the evolution semigroup $\{E^t\}_{t\geq 0}$ defined on $\mathcal{E}(\mathbb{R})$ in (1.1). The generator $\mathbf{G}$ of the evolution semigroup can be described as follows (see [20], [9, Proposition 4.32], [9, Lemma 3.16]).

Proposition 2.1. *Let $u, f \in \mathcal{E}(\mathbb{R})$. Then $u \in \operatorname{dom}\mathbf{G}$ and $\mathbf{G}u = f$ if and only if $u \in \mathcal{E}(\mathbb{R}) \cap C_0(\mathbb{R};X)$ and, for all $t \geq \tau$ in $\mathbb{R}$,*

$$u(t) = U(t,\tau)u(\tau) - \int_\tau^t U(t,s)f(s)ds. \tag{2.1}$$

If $\{U(t,\tau)\}_{t\geq\tau}$ solves the abstract Cauchy problem

$$\dot{x}(t) = A(t)x(t), \quad x(\tau) = x_\tau, \quad x_\tau \in \operatorname{dom} A(\tau), \quad t \geq \tau,$$

in the sense of [9, Definition 3.2], then by [9, Theorem 3.12] and [28], $\mathbf{G}$ is a closed extension of the operator G, $(Gu)(t) = -u'(t) + A(t)u(t)$, which is defined in $\mathcal{E}(\mathbb{R})$ with the domain $\operatorname{dom}(d/dt) \cap \operatorname{dom}(M_A)$. If, for example, $A(\cdot) \in C_b(\mathbb{R};\mathcal{L}(X))$, then $\mathbf{G} = G$. For more general situations when $\mathbf{G} = G$ see [28] and the references therein, and, in addition, recent work in [24], [25] that includes, e.g., the case of Fredholm elliptic differential operators on compact manifolds.

Next, consider an evolution semigroup, $\{E_+^t\}_{t\geq 0}$, on $\mathcal{E}_0(\mathbb{R}_+)$ defined[1] as

$$(E_+^t u)(\tau) = \begin{cases} U(\tau, \tau - t)u(\tau - t), & \tau \geq t, \\ 0, & 0 \leq \tau \leq t, \end{cases} \tag{2.2}$$

and let $\mathbf{G}_0^+$ denote its generator. Extending functions from $\mathcal{E}_0(\mathbb{R}_+)$ by zero on $\mathbb{R}_-$, we may identify $\mathcal{E}_0(\mathbb{R}_+)$ with a subspace of $\mathcal{E}(\mathbb{R})$. The semigroup (1.1) leaves this subspace invariant, and E_+^t is the restriction of E^t on this subspace. Arguing as in [13, p. 60], we conclude that $\mathbf{G}_0^+$ is the restriction of $\mathbf{G}$ on this subspace. Similarly to Proposition 2.1, $\mathbf{G}_0^+$ can be described as follows (see [20, Lemma 1.1] and [21], and note that Lemma 3.16 in [9] also holds for the half-line case).

PROPOSITION 2.2. *Let $u, f \in \mathcal{E}_0(\mathbb{R}_+)$. Then $u \in \operatorname{dom} \mathbf{G}_0^+$ and $\mathbf{G}_0^+ u = f$ if and only if $u \in \mathcal{E}_0(\mathbb{R}_+) \cap C_{00}(\mathbb{R}_+; X)$ and, for all $t \geq 0$,*

$$u(t) = -\int_0^t U(t, s) f(s) ds. \tag{2.3}$$

Note that (2.3) implies (2.1) for all $t \geq \tau$ in $\mathbb{R}_+$. If $\{U(t, \tau)\}_{t\geq\tau\geq 0}$ is the propagator of a differential equation $u' = A(t)u(t)$ on $\mathbb{R}_+$, then (2.3) corresponds to the inhomogeneous equation $u' = A(t)u(t) + f(t)$ with the boundary condition $u(0) = 0$. We will also consider the following operator, $\mathbf{G}^+$, on $\mathcal{E}(\mathbb{R}_+)$ (see [4], [20], [21]).

DEFINITION 2.3. Let $u, f \in \mathcal{E}(\mathbb{R}_+)$. Then $u \in \operatorname{dom} \mathbf{G}^+$ and $\mathbf{G}^+ u = f$ if and only if $u \in \mathcal{E}(\mathbb{R}_+) \cap C_0(\mathbb{R}_+; X)$ and (2.1) holds for all $t \geq \tau$ in $\mathbb{R}_+$.

By [20, Lemma 1.1] and [21, Lemma 1.1], the operator $\mathbf{G}^+$ is well-defined and closed on $\mathcal{E}(\mathbb{R}_+)$; also, $\operatorname{dom} \mathbf{G}_0^+ = \{u \in \operatorname{dom} \mathbf{G}^+ : u(0) = 0\}$ and $\mathbf{G}_0^+ u = \mathbf{G}^+ u$ for $u \in \operatorname{dom} \mathbf{G}_0^+$. In addition, $\mathbf{G}^+$ on $\mathcal{E}(\mathbb{R}_+) = C_0(\mathbb{R}_+; X)$ is related to the generator of the following evolution semigroup; see [20, Lemma 1.1(b)]:

$$(\tilde{E}_+^t u)(\tau) = \begin{cases} U(\tau, \tau - t)u(\tau - t), & \tau \geq t, \\ U(\tau, 0)u(0), & 0 \leq \tau \leq t. \end{cases}$$

Finally, note that

$$\operatorname{Ker} \mathbf{G}^+ = \{u \in \mathcal{E}(\mathbb{R}_+) : u(t) = U(t, \tau)u(\tau) \text{ for all } t \geq \tau \geq 0\}, \tag{2.4}$$

$$\operatorname{Ker} \mathbf{G}_0^+ = \{0\}, \tag{2.5}$$

and define the subspace $X_0 \subseteq X$ of stable initial data by

$$X_0 = \{x = u(0) : u \in \operatorname{Ker} \mathbf{G}^+\}. \tag{2.6}$$

[1] Recall that if $\mathcal{E}_0(\mathbb{R}_+) = C_{00}(\mathbb{R}_+; X)$ then $u(0) = u(+\infty) = 0$.

Dichotomy. Recall that the evolution family $\{U(t,\tau)\}_{t\geq\tau}$ is said to have an *exponential dichotomy* $\{P_t\}_{t\in J}$ on J with dichotomy constants $M\geq 1$ and $\alpha>0$ (see [11], [13], [14], [18]) if P_t, $t\in J$, are bounded projections on X, and for all $t\geq\tau$ in J the following assertions hold:

(i) $U(t,\tau)P_\tau = P_t U(t,\tau)$.

(ii) The restriction $U(t,\tau)|_{\operatorname{Ker} P_\tau}$ of the operator $U(t,\tau)$ is an invertible operator from $\operatorname{Ker} P_\tau$ to $\operatorname{Ker} P_t$.

(iii) The following stable and unstable dichotomy estimates hold:

$$\|U(t,\tau)|_{\operatorname{Im} P_\tau}\| \leq Me^{-\alpha(t-\tau)} \text{ and } \|(U(t,\tau)|_{\operatorname{Ker} P_\tau})^{-1}\| \leq Me^{-\alpha(t-\tau)}. \tag{2.7}$$

Also, recall that a pair of subspaces (W,V) in X is called a *Fredholm pair* provided $\alpha(W,V):=\dim(W\cap V)<\infty$, the subspace $W+V$ is closed, and $\beta(W,V):=\operatorname{codim}(W+V)<\infty$; the *Fredholm index* of the pair is defined as $\operatorname{ind}(W,V)=\alpha(W,V)-\beta(W,V)$; see, e.g. [15, Sec. IV.4.1].

The Fredholm property of the operator $\mathbf{G}$ is related to the exponential dichotomies of $\{U(t,\tau)\}_{t\geq\tau}$ on $\mathbb{R}_+$ and $\mathbb{R}_-$. The following particular case of a result from [16] is a generalization of the celebrated Dichotomy Theorem; cf. [5], [6], [22], [23], [27], [30].

Theorem 2.4 ([16, Theorem 1.2]). *If the Banach space X is reflexive and the family $\{U(t,\tau)\}_{t,\tau\in\mathbb{R}}$ consists of invertible operators, then the operator $\mathbf{G}$ is Fredholm on $\mathcal{E}(\mathbb{R})$ if and only if the following conditions hold:*

(i) *The evolution family $\{U(t,\tau)\}_{t\geq\tau}$, $t,\tau\in\mathbb{R}$, admits exponential dichotomies $\{P_t^-\}_{t\leq 0}$ and $\{P_t^+\}_{t\geq 0}$ on $\mathbb{R}_-$ and $\mathbb{R}_+$, respectively.*

(ii) *The pair of subspaces $(\operatorname{Ker} P_0^-, \operatorname{Im} P_0^+)$ is Fredholm in X.*

Also, $\dim\operatorname{Ker}\mathbf{G}=\alpha(\operatorname{Ker} P_0^-,\operatorname{Im} P_0^+)$, $\operatorname{codim}\operatorname{Im}\mathbf{G}=\beta(\operatorname{Ker} P_0^-,\operatorname{Im} P_0^+)$, *and* $\operatorname{ind}\mathbf{G}=\operatorname{ind}(\operatorname{Ker} P_0^-,\operatorname{Im} P_0^+)$.

Theorem 2.5 ([16, Theorem 1.4]). *The range $\operatorname{Im}\mathbf{G}$ of the operator $\mathbf{G}$ is closed in $\mathcal{E}(\mathbb{R})$ if and only if the range $\operatorname{Im} D_0$ of the operator D_0 defined in* (1.2) *is closed in $\mathcal{E}(\mathbb{Z})$. Also,* $\dim\operatorname{Ker}\mathbf{G}=\dim\operatorname{Ker} D_0$ *and* $\operatorname{codim}\operatorname{Im}\mathbf{G}=\operatorname{codim}\operatorname{Im} D_0$. *In particular, $\mathbf{G}$ is Fredholm if and only if D_0 is Fredholm, and* $\operatorname{ind}\mathbf{G}=\operatorname{ind} D_0$.

Let $W_\tau(t,s)=U(t+\tau,s+\tau)$, $t\geq s$, $\tau\in[0,1)$, and define on $\mathcal{E}(\mathbb{R})$ the shift group, $\{S(t)\}_{t\in\mathbb{R}}$, by

$$(S(t)u)(s)=u(s-t),\quad s,t\in\mathbb{R}. \tag{2.8}$$

If $E^t_{W_\tau}$ and E^t_U denote the evolution semigroups on $\mathcal{E}(\mathbb{R})$ induced by the evolution families $\{W_\tau(t,s)\}$ and $\{U(t,s)\}$, then $S(\tau)E^t_{W_\tau}S(-\tau)=E^t_U$ and $S(\tau)\mathbf{G}_{W_\tau}S(-\tau)=\mathbf{G}_U$. Thus, Theorem 2.5 holds if the operator D_0 is replaced by any operator $D_\tau, \tau\in(0,1)$.

The spectral properties of the operators $\mathbf{G}^+$ and $\mathbf{G}_0^+$ are related to the dichotomy of $\{U(t,\tau)\}_{t\geq\tau\geq 0}$ on $\mathbb{R}_+$. The following result from [20] and [21] has a long prehistory going back to classical characterizations of the dichotomy in terms of the operator $G^+ = -d/dt + M_A$ for $A(\cdot) \in C_b(\mathbb{R}_+; \mathcal{L}(X))$; see [11] and the references therein.

THEOREM 2.6 ([20, Theorem 4.3], [21, Theorem 3.1]). *The evolution family $\{U(t,s)\}_{t\geq s\geq 0}$ has an exponential dichotomy $\{P_t\}_{t\geq 0}$ on $\mathbb{R}_+$ if and only if the operator $\mathbf{G}^+$ is surjective on $\mathcal{E}(\mathbb{R}_+)$ and the subspace X_0 defined in* (2.6) *is complemented in X.*

Sun-duals. Given a strongly continuous semigroup $\{e^{tA}\}_{t\geq 0}$ on X, the adjoint semigroup $\{(e^{tA})^*\}_{t\geq 0}$ on the Banach space X^* is, in general, not a strongly continuous semigroup. The subspace

$$X^{\odot} := \{x^* \in X^* \colon \|(e^{tA})^* x - x^*\| \to 0 \text{ as } t \downarrow 0\}$$

is a closed linear subspace of X^* and $(e^{tA})^*(X^{\odot}) \subseteq X^{\odot}$ for all $t \geq 0$. The restrictions $e^{tA^{\odot}}$ of $(e^{tA})^*$ to $X^{\odot}$ define a strongly continuous semigroup in $X^{\odot}$; moreover, $X^{\odot}$ is equal to the norm closure of $\operatorname{dom}(A^*)$ in X^*, so that $X^{\odot} = \overline{R(\lambda, A)^*(X^*)}$ for all $\lambda \in \mathbb{C} \setminus \sigma(A)$.

REMARK 2.7. The definition of $X^{\odot}$ implies that $\operatorname{Ker}(I - e^{tA})^* \subset X^{\odot}$ and $\operatorname{Ker}(I - e^{tA})^* = \operatorname{Ker}(I - e^{tA^{\odot}})$ for every $t \geq 0$, and that $\operatorname{Ker}(A^* - \mu) \subset X^{\odot}$ and $\operatorname{Ker}(A^* - \mu) = \operatorname{Ker}(A^{\odot} - \mu)$ for any $\mu \in \mathbb{C}$; see, e.g. [10, Ch. II.2].

3. Fredholm property implies invertibility

Proof of Theorem 1.1. It suffices to prove that the operator $E^1 - I$ is invertible provided it is Fredholm. First, we claim that $\operatorname{Ker} \mathbf{G} \neq \{0\}$ implies $\dim \operatorname{Ker}(E^1 - I) = \infty$. To prove the claim, for each $k \in \mathbb{Z}$ define a bounded operator $M = M_k$ on $\mathcal{E}(\mathbb{R})$ by $(Mu)(\tau) = e^{2\pi i k \tau} u(\tau)$, $\tau \in \mathbb{R}$. Then $ME^t = e^{2\pi i k t} E^t M$ for all $t \geq 0$ and $M(\mathbf{G} - 2\pi i k) = \mathbf{G} M$. Therefore,

$$\operatorname{Ker}(\mathbf{G} - 2\pi i k) = M^{-1} \operatorname{Ker} \mathbf{G}. \tag{3.1}$$

Fix a nonzero $u \in \operatorname{Ker} \mathbf{G}$ and let $u_k = M_k^{-1} u$, $k \in \mathbb{Z}$. Then $u_k \in \operatorname{Ker}(\mathbf{G} - 2\pi i k)$ is a nonzero eigenfunction for $\mathbf{G}$ that corresponds to its eigenvalue $2\pi i k$. The functions in the family $\{u_k : k \in \mathbb{Z}\}$ are linearly independent since nonzero eigenfunctions corresponding to different eigenvalues of a linear operator are linearly independent. Since

$$\operatorname{Ker}(E^1 - I) = \overline{\operatorname{lin}}\{\operatorname{Ker}(\mathbf{G} - 2\pi i k) : k \in \mathbb{Z}\} \tag{3.2}$$

by [13, p. 278], we have $u_k \in \operatorname{Ker}(E^1 - I)$ and thus $\dim \operatorname{Ker}(E^1 - I) = \infty$, proving the claim. Next, if $\operatorname{Ker}(E^1 - I) \neq \{0\}$, then $\operatorname{Ker} \mathbf{G} \neq 0$ by (3.2) and (3.1). Thus $\operatorname{Ker}(E^1 - I) \neq \{0\}$ implies $\dim \operatorname{Ker}(E^1 - I) = \infty$ by the claim above. Finally, let $\{(E^t)^{\odot}\}_{t\geq 0}$ be the sun-dual semigroup for $\{E^t\}_{t\geq 0}$. Since

$(M^{-1})^*(E^1)^*M^* = e^{-2\pi ik}(E^1)^*$, we note that $M^*((\mathcal{E}(\mathbb{R}))^{\odot}) \subset (\mathcal{E}(\mathbb{R}))^{\odot}$, and moreover $M^*(\mathbf{G}^{\odot} - 2\pi ik)(M^{-1})^* = \mathbf{G}^{\odot}$. Using Remark 2.7 for $A = \mathbf{G}$, and arguing as above, we infer that the assumption $\operatorname{Ker}(E^1 - I)^* \neq \{0\}$ leads to $\dim \operatorname{Ker}(E^1 - I)^* = \infty$. □

We conclude this section with a general result related to Theorem 1.1 regarding constant coefficient first order differential operators on $\mathcal{E}(\mathbb{R})$. Let A be a closed linear operator on X with a dense domain. Consider the operator of differentiation $Du = -u'$ on $\mathcal{E}(\mathbb{R})$ with the maximal domain. Let $\mathcal{D} = \operatorname{dom} D \cap \operatorname{dom} M_A$, consider the sum $D + M_A$ with $\operatorname{dom}(D + M_A) = \mathcal{D}$ and consider an operator, $\mathcal{G}_A$, which is a closed extension of $D + M_A$ such that $\mathcal{D}$ is a core for $\mathcal{G}_A$ (i.e., $\mathcal{G}_A$ is the closure of $\mathcal{G}_A\Big|_{\mathcal{D}}$). Remark that $\operatorname{dom} D$, $\operatorname{dom} M_A$ and $\mathcal{D}$ are invariant for the isometric shift group $\{S(t)\}_{t\in\mathbb{R}}$ defined in (2.8). An example of the operator $\mathcal{G}_A$ is furnished by $\mathcal{G}_A = \mathbf{G}$ with $\mathbf{G}$ induced by $U(t,\tau) = e^{(t-\tau)A}$, where A is the generator of a strongly continuous semigroup.

THEOREM 3.1. *The operator $\mathcal{G}_A$ is Fredholm on $\mathcal{E}(\mathbb{R})$ if and only $\mathcal{G}_A$ is invertible on $\mathcal{E}(\mathbb{R})$.*

Proof. First, we claim that the domain and the range of $\mathcal{G}_A$ are shift-invariant. Indeed, suppose $u \in \operatorname{dom} \mathcal{G}_A$. Since $\mathcal{D}$ is a core for $\mathcal{G}_A$, there exists $\{u_n : n \geq 0\} \subset \mathcal{D}$ such that $\|u_n - u\|_{\mathcal{E}} \to 0$ and $\|\mathcal{G}_A u_n - \mathcal{G}_A u\|_{\mathcal{E}} \to 0$ as $n \to \infty$. Since the core $\mathcal{D}$ is shift-invariant, for every $t \in \mathbb{R}$ we have $\|S(t)u_n - S(t)u\|_{\mathcal{E}} \to 0$ and $\|\mathcal{G}_A S(t)u_n - S(t)\mathcal{G}_A u\|_{\mathcal{E}} \to 0$ as $n \to \infty$. Since the operator $\mathcal{G}_A$ is closed, it follows that $S(t)u \in \operatorname{dom} \mathcal{G}_A$ and $S(t)\mathcal{G}_A u = \mathcal{G}_A S(t)u$, proving the claim. Moreover,

$$S(t)\,(\operatorname{Im} \mathcal{G}_A) = \operatorname{Im} \mathcal{G}_A, \quad t \in \mathbb{R}. \tag{3.3}$$

Next, suppose that $\mathcal{G}_A$ is Fredholm on $\mathcal{E}(\mathbb{R})$, and assume that $\dim \operatorname{Ker} \mathcal{G}_A > 0$. For any $u \in \operatorname{Ker} \mathcal{G}_A$ we have $S(t)u \in \operatorname{Ker} \mathcal{G}_A$, $t \in \mathbb{R}$. Hence the operator group $\{S(t)\}_{t\in\mathbb{R}}$ given by (2.8) is well-defined on the Banach space $\mathcal{X} := (\operatorname{Ker} \mathcal{G}_A, \|\cdot\|_{\mathcal{E}})$ and is isometric there. Since $\operatorname{Ker} \mathcal{G}_A$ is finite-dimensional, $S(t) = e^{tB}, t \in \mathbb{R}$, for some $B \in \mathcal{L}(\mathcal{X})$. Since $\{S(t)\}_{t\in\mathbb{R}}$ is isometric, $\sigma(B)$ belongs to $i\mathbb{R}$ and consists of eigenvalues of B. So, there exists a $\xi \in \mathbb{R}$ and a nonzero $u_0 \in \operatorname{Ker} \mathcal{G}_A$ such that $S(t)u_0 = e^{it\xi}u_0$. Hence, for every $t \in \mathbb{R}$ we have $u_0(s+t) = e^{it\xi}u_0(s)$ for a.e. $s \in \mathbb{R}$. In particular, $u_0(s + 2\pi/\xi) = u_0(s)$ for a.e. $s \in \mathbb{R}$. But then u_0 does not belong to $\mathcal{E}(\mathbb{R})$, a contradiction. Thus, $\operatorname{Ker} \mathcal{G}_A = \{0\}$. Finally, consider the quotient space $Y := \mathcal{E}(\mathbb{R})/\operatorname{Im} \mathcal{G}_A$ and assume that $\dim Y > 0$. Since $\operatorname{Im} \mathcal{G}_A$ is $S(t)$-invariant, the quotient group $\{\hat{S}(t)\}_{t\in\mathbb{R}}$ is well-defined on Y, and, if $f \in \hat{f}$, the equivalence class in Y, then

$$\|\hat{S}(t)\hat{f}\|_Y = \inf_{g\in\operatorname{Im}\mathcal{G}_A} \|S(t)f + g\|_{\mathcal{E}} = \inf_{g\in\operatorname{Im}\mathcal{G}_A} \|f + g\|_{\mathcal{E}} = \|\hat{f}\|_Y,$$

so that $\{\hat{S}(t)\}_{t\in\mathbb{R}}$ is isometric on the finite dimensional space Y. Since $\dim Y < \infty$, there exists a finite dimensional subspace N of $\mathcal{E}(\mathbb{R})$ isomorphic to Y such that $\operatorname{Im}\mathcal{G}_A \oplus N = \mathcal{E}(\mathbb{R})$. Using the isomorphic image of $\{\hat{S}(t)\}_{t\in\mathbb{R}}$ on N, as above, we infer that there exists a nonzero $f \in N$ such that $f(s+2\pi/\xi) = f(s)$ for some $\xi \in \mathbb{R}$ and a.e. s. This leads to a contradiction again. Thus, $Y = \{0\}$, and $\mathcal{G}_A$ must be invertible. □

A particular case of Theorem 3.1 for concrete classes of pseudo-differential operators has been proved by M. Shubin in [31, Theorem 11.1].

4. Ranges of the generators of evolution semigroups

In this section we prove our main result, Theorem 1.2, saying that the ranges of the operators D_τ, $\mathbf{G}$ and $E^1 - I$ are closed simultaneously. We mention related work in [3]–[5] dealing with correctness (uniform boundedness from below) and invertibility of these operators. Note that the implication (ii)⇒(i) in Theorem 1.2, asserting that the range of the operator $F(\mathbf{G}) = e^{\mathbf{G}} - I$ is closed provided the range of $\mathbf{G}$ is closed, is a consequence of the very special structure of the evolution semigroup. Generally, the assertion "range of T is closed implies range of $F(T)$ is closed" fails even for bounded operators T and the function $F(T) = T^n$, $n \in \mathbb{N}$. Indeed, by a result in [8, p. 124], for every sequence $\{n_k\} \subset \mathbb{N} \setminus \{1\}$ there is a $T \in \mathcal{L}(X)$ with closed range so that the ranges of T^{n_k} are not closed. On the other hand, if $T \in \mathcal{L}(X)$ and $e^T - I$ has closed range, then T has closed range; see [19].

LEMMA 4.1. *For the operators D_τ, $\tau \in [0,1)$, defined in* (1.2),

(i) *If $(z_n)_{n\in\mathbb{Z}} \in \operatorname{Ker} D_\tau$ then $(U(n, n+\tau-1)z_{n-1})_{n\in\mathbb{Z}} \in \operatorname{Ker} D_0$.*
(ii) *If $(z_n)_{n\in\mathbb{Z}} \in \operatorname{Ker} D_0$ then $(U(\tau+n, n)z_n)_{n\in\mathbb{Z}} \in \operatorname{Ker} D_\tau$.*

Proof. (i) For any $\tau \in [0,1)$,

$$\operatorname{Ker} D_\tau = \{(x_n)_{n\in\mathbb{Z}} : x_n = U(n+\tau, m+\tau)x_m \text{ for all } n \ge m \text{ in } \mathbb{Z}\}. \tag{4.1}$$

If $(z_n)_{n\in\mathbb{Z}} \in \operatorname{Ker} D_\tau$ then $z_n = U(n+\tau, n+\tau-1)z_{n-1} = U(n+\tau, n)U(n, n+\tau-1)z_{n-1}$. If $x_n = U(n, n+\tau-1)z_{n-1}$, then

$$\begin{aligned} U(n, n-1)x_{n-1} &= U(n, n-1)U(n-1, n+\tau-2)z_{n-2} \\ &= U(n, n+\tau-2)z_{n-2} = U(n, n+\tau-1)U(n+\tau-1, n+\tau-2)z_{n-2} \\ &= U(n, n+\tau-1)z_{n-1} = x_n,\ n \in \mathbb{Z}. \end{aligned}$$

(ii) If $(z_n)_{n\in\mathbb{Z}} \in \operatorname{Ker} D_0$ then $z_n = U(n, n-1)z_{n-1}$ and

$$\begin{aligned} D_\tau(U(\tau+n, n)z_n)_{n\in\mathbb{Z}} &= \big(U(\tau+n, n)z_n - U(\tau+n, n-1)z_{n-1}\big)_{n\in\mathbb{Z}} \\ &= \big(U(\tau+n, n)z_n - U(\tau+n, n)U(n, n-1)z_{n-1}\big)_{n\in\mathbb{Z}} = 0. \end{aligned}$$ □

Fix a smooth function $\alpha : [0,1] \to [0,1]$ such that $\alpha(\tau) = 0$ for $\tau \in \left[0, \frac{1}{3}\right]$ and $\alpha(\tau) = 1$ for $\tau \in \left[\frac{2}{3}, 1\right]$; cf. [9, p. 39]. For a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{Z}}$ define a function, $B\mathbf{x}$, so that if $t \in [n, n+1]$, $n \in \mathbb{Z}$, then

$$(B\mathbf{x})(t) = U(t,n)[\alpha(t-n)x_n + (1-\alpha(t-n))U(n,n-1)x_{n-1}]. \tag{4.2}$$

Recall that the evolution family $\{U(t,\tau)\}_{t\geq\tau}$ is exponentially bounded; thus $C := \sup\{\|U(t,\tau)\| : 0 \leq t-\tau \leq 1\} < \infty$. We will use the following estimates valid for any $t \in [n, n+1]$, $n \in \mathbb{Z}$, and $x \in X$:

$$\begin{aligned} \|U(n+1,n)x\| = \|U(n+1,t)U(t,n)x\| \leq C\|U(t,n)x\|, \\ \|U(t,n)x\| \leq C\|x\|. \end{aligned} \tag{4.3}$$

Lemma 4.2.

(i) $B : \mathcal{E}(\mathbb{Z}) \to \mathcal{E}(\mathbb{R})$ *is a bounded linear operator;*
(ii) $B : \operatorname{Ker} D_0 \to \operatorname{Ker} \mathbf{G}$ *is an isomorphism;*
(iii) $(E^1 - I)B = -BD_0$.

Proof. Since $(B\mathbf{x})(n) = U(n,n-1)x_{n-1}$, $n \in \mathbb{Z}$, the function $B\mathbf{x}$ is continuous. Assertion (i) follows from (4.3). Assertion (iii) follows from a direct calculation. To prove (ii), recall from Proposition 2.1 that $u \in \operatorname{Ker}\mathbf{G}$ if and only if $u \in L_p(\mathbb{R};X) \cap C_0(\mathbb{R};X)$ (resp., $u \in C_0(\mathbb{R};X)$), and $u(t) = U(t,\tau)u(\tau)$ for all $t \geq \tau$ in $\mathbb{R}$. Also, $\mathbf{x} \in \operatorname{Ker} D_0$ if and only if $x_n = U(n,n-1)x_{n-1}$, $n \in \mathbb{Z}$. If $\mathbf{x} \in \operatorname{Ker} D_0$ then $(B\mathbf{x})(t) = U(t,n)x_n$ for $t \in [n,n+1]$, $n \in \mathbb{Z}$, and thus $B\mathbf{x} \in \operatorname{Ker}\mathbf{G}$ (see also (iii) in the lemma). If $\mathbf{x} \in \operatorname{Ker} D_0$ and $B\mathbf{x} = 0$ then $\mathbf{x} = 0$. Thus, $B : \operatorname{Ker} D_0 \to \operatorname{Ker}\mathbf{G}$ is an injection. To see that it is a surjection, take $u \in \operatorname{Ker}\mathbf{G}$ and let $\mathbf{x} = (u(n))_{n\in\mathbb{Z}}$. Then $(B\mathbf{x})(t) = U(t,n)u(n) = u(t)$. It remains to check that $\mathbf{x} \in \ell_p(\mathbb{Z};X)$. This follows from (4.3):

$$\begin{aligned} \|\mathbf{x}\|_{\ell_p}^p = \|(u(n))_{n\in\mathbb{Z}}\|_{\ell_p}^p &= \sum_{n\in\mathbb{Z}} \int_n^{n+1} \|U(n+1,t)U(t,n)u(n)\|^p dt \\ &\leq C^p \sum_{n\in\mathbb{Z}} \int_n^{n+1} \|U(t,n)u(n)\|^p dt \\ &= C^p \sum_{n\in\mathbb{Z}} \int_n^{n+1} \|u(t)\|^p dt = C^p \|u\|_{L_p}^p. \end{aligned}$$

□

Proof of Theorem 1.2. We give the proof for L_p-spaces; the case of $C_0(\mathbb{R};X)$ is similar.

(i)⇒(ii) Using Lemma 4.2(iii), and setting $\gamma = \gamma(E^1 - I)$, we have

$$\begin{aligned} \|D_0\mathbf{x}\|_{\ell_p} &\geq \|B\|^{-1}\|BD_0\mathbf{x}\|_{L_p} = \|B\|^{-1}\|(E^1 - I)B\mathbf{x}\|_{L_p} \\ &\geq \gamma\|B\|^{-1} \operatorname{dist}(B\mathbf{x}, \operatorname{Ker}(E^1 - I)) \geq \frac{\gamma}{2\|B\|}\|B\mathbf{x} - u\|_{L_p} \end{aligned} \tag{4.4}$$

for some $u \in \operatorname{Ker}(E^1 - I)$. Due to (3.2), we may assume that $u = \sum_{|k|\le K} u_k$, where $u_k \in \operatorname{Ker}(\mathbf{G} - 2\pi i k)$. Using (3.1) and Lemma 4.2(ii), find $\mathbf{z}^{(k)} \in \operatorname{Ker} D_0$ such that $u_k(t) = e^{-2\pi i k t}(B\mathbf{z}^{(k)})(t)$, $t \in \mathbb{R}$, $|k| \le K$. Recall that $\mathbf{z}^{(k)} = (z_n^{(k)})_{n\in\mathbb{Z}} \in \operatorname{Ker} D_0$ implies $z_{n+1}^{(k)} = U(n+1,n)z_n^{(k)}$ and $(B\mathbf{z}^{(k)})(t) = U(t,n)z_n^{(k)}$ for $t \in [n+1,n]$ and $n \in \mathbb{Z}$. Thus, with C from (4.3) and using (4.2), we continue estimate (4.4) as follows:

$$
\begin{aligned}
C^p \|D_0\mathbf{x}\|_{\ell_p}^p &\ge \left(\frac{\gamma C}{2\|B\|}\right)^p \|B\mathbf{x} - u\|_{L_p}^p \\
&= \left(\frac{\gamma C}{2\|B\|}\right)^p \sum_{n\in\mathbb{Z}} \int_n^{n+1} \Big\| U(t,n)\Big[\alpha(t-n)x_n \\
&\qquad + (1-\alpha(t-n))U(n,n-1)x_{n-1} - \sum_{|k|\le K} e^{-2\pi i k t} z_n^{(k)}\Big]\Big\|^p dt \\
&\ge \left(\frac{\gamma}{2\|B\|}\right)^p \sum_{n\in\mathbb{Z}} \int_n^{n+1} \Big\| U(n+1,n)\Big[\alpha(t-n)x_n \\
&\qquad + (1-\alpha(t-n))U(n,n-1)x_{n-1} - \sum_{|k|\le K} e^{-2\pi i k t} z_n^{(k)}\Big]\Big\|^p dt.
\end{aligned}
$$

Since $\alpha(t) = 1$ for $t \in \left[\frac{2}{3}, 1\right]$ and $U(n+1,n)z_n^{(k)} = z_{n+1}^{(k)}$, we infer:

$$
\begin{aligned}
C\|D_0\mathbf{x}\|_{\ell_p} &\ge \frac{\gamma}{2\|B\|} \left(\int_{2/3}^1 \sum_{n\in\mathbb{Z}} \Big\| U(n+1,n)x_n - \sum_{|k|\le K} e^{-2\pi i k t} z_{n+1}^{(k)} \Big\|^p dt \right)^{1/p} \\
&= \frac{\gamma}{2\|B\|} \left(\int_{2/3}^1 \sum_{n\in\mathbb{Z}} \Big\| x_{n+1} - (x_{n+1} - U(n+1,n)x_n) \right. \\
&\qquad \left. - \sum_{|k|\le K} e^{-2\pi i k t} z_{n+1}^{(k)} \Big\|^p dt \right)^{1/p} \\
&= \frac{\gamma}{2\|B\|} \left(\int_{2/3}^1 \Big\| \mathbf{x} - D_0\mathbf{x} - \sum_{|k|\le K} e^{-2\pi i k t} \mathbf{z}^{(k)} \Big\|_{\ell_p}^p dt \right)^{1/p}.
\end{aligned}
$$

Since $\sum_k e^{-2\pi i k t}\mathbf{z}^{(k)} \in \operatorname{Ker} D_0$ for each $t \in \left[\frac{2}{3}, 1\right]$, we may continue as follows:

$$
\begin{aligned}
C\|D_0\mathbf{x}\|_{\ell_p} &\geq \frac{\gamma}{2\|B\|}\left(\int_{2/3}^{1}\operatorname{dist}(\mathbf{x}-D_0\mathbf{x},\operatorname{Ker}D_0)^p\,dt\right)^{1/p}\\
&= \frac{\gamma}{2(3)^{1/p}\|B\|}\inf_{\mathbf{y}\in\operatorname{Ker}D_0}\|\mathbf{x}-D_0\mathbf{x}-\mathbf{y}\|_{\ell_p}\\
&\geq \frac{\gamma}{2(3)^{1/p}\|B\|}\operatorname{dist}(\mathbf{x},\operatorname{Ker}D_0)-\frac{\gamma}{2(3)^{1/p}\|B\|}\|D_0\mathbf{x}\|_{\ell_p}.
\end{aligned}
$$

Thus, $\gamma(D_0)>0$. By Theorem 2.5, assertion (ii) in Theorem 1.2 is proved.

(ii)$\Rightarrow$(iii) By Theorem 2.5, $\gamma(\mathbf{G})>0$ implies $\gamma(D_0)>0$. Using (4.3), for each $\tau\in[0,1)$ we infer:

$$
\begin{aligned}
&C\|D_\tau\mathbf{x}\|_{\ell_p} = C\left(\sum_{n\in\mathbb{Z}}\|x_n-U(n+\tau,n+\tau-1)x_{n-1}\|^p\right)^{1/p}\\
&\geq \left(\sum_{n\in\mathbb{Z}}\|U(n+1,n+\tau)x_n-U(n+1,n+\tau)U(n+\tau,n+\tau-1)x_{n-1}\|^p\right)^{1/p}\\
&= \|D_0(U(n,n+\tau-1)x_{n-1})_{n\in\mathbb{Z}}\|_{\ell_p}\\
&\geq \gamma(D_0)\operatorname{dist}((U(n,n+\tau-1)x_{n-1})_{n\in\mathbb{Z}},\operatorname{Ker}D_0)\\
&\geq (\gamma(D_0)/2)\|(U(n,n+\tau-1)x_{n-1})_{n\in\mathbb{Z}}-\mathbf{z}\|_{\ell_p}
\end{aligned}
$$

for some $\mathbf{z}=(z_n)_{n\in\mathbb{Z}}\in\operatorname{Ker}D_0$. Using (4.3) again,

$$
\begin{aligned}
&\|U(n+\tau,n+\tau-1)x_{n-1}-U(n+\tau,n)z_n\|\\
&\quad=\|U(n+\tau,n)[U(n,n+\tau-1)x_{n-1}-z_n]\|\\
&\quad\leq C\|U(n,n+\tau-1)x_{n-1}-z_n\|.
\end{aligned}
$$

Since $(U(n+\tau,n)z_n)_{n\in\mathbb{Z}}\in\operatorname{Ker}D_\tau$ for $\mathbf{z}\in\operatorname{Ker}D_0$ by Lemma 4.1(ii),

$$
\begin{aligned}
C\|D_\tau\mathbf{x}\|_{\ell_p} &\geq \frac{\gamma(D_0)}{2C}\left(\sum_{n\in\mathbb{Z}}\|U(n+\tau,n+\tau-1)x_{n-1}-U(n+\tau,n)z_n\|^p\right)^{1/p}\\
&= \frac{\gamma(D_0)}{2C}\|\mathbf{x}-D_\tau\mathbf{x}-(U(n+\tau,n)z_n)_{n\in\mathbb{Z}}\|_{\ell_p}\\
&\geq \frac{\gamma(D_0)}{2C}\operatorname{dist}(\mathbf{x},\operatorname{Ker}D_\tau)-\frac{\gamma(D_0)}{2C}\|D_\tau\mathbf{x}\|_{\ell_p}.
\end{aligned}
$$

Assertion (iii) in Theorem 1.2 is proved.

(iii)$\Rightarrow$(i) Set $\gamma=\inf_{\tau\in[0,1)}\gamma(D_\tau)>0$, consider a continuous compactly supported function $u:\mathbb{R}\to X$, and note that the set of such functions is dense in $L_p(\mathbb{R};X)$. We have:

$$\|(E^1-I)u\|^p_{L_p} = \sum_{n\in\mathbb{Z}} \int_n^{n+1} \|u(t)-U(t,t-1)u(t-1)\|^p\,dt \tag{4.5}$$
$$= \int_0^1 \|D_\tau\left((u(\tau+n))_{n\in\mathbb{Z}}\right)\|^p_{\ell_p}\,d\tau.$$

Using Lemma 4.1(i) and (4.3), for any $\mathbf{y}\in\ell_p$ and $\tau\in[0,1)$ we infer:

$$\begin{aligned}\operatorname{dist}(\mathbf{y},\operatorname{Ker}D_\tau) &= \inf_{\mathbf{x}\in\operatorname{Ker}D_\tau}\|\mathbf{y}-\mathbf{x}\|_{\ell_p}\\ &\ge C^{-1}\inf_{\mathbf{x}\in\operatorname{Ker}D_\tau}\|(U(n,n+\tau-1)y_{n-1})_{n\in\mathbb{Z}}\\ &\qquad -(U(n,n+\tau-1)x_{n-1})_{n\in\mathbb{Z}}\|_{\ell_p}\\ &\ge C^{-1}\operatorname{dist}\left((U(n,n+\tau-1)y_{n-1})_{n\in\mathbb{Z}},\operatorname{Ker}D_0\right).\end{aligned}\tag{4.6}$$

Using (4.6) for $\mathbf{y}=(u(\tau+n))_{n\in\mathbb{Z}}$, we have:

$$\begin{aligned}\|D_\tau\left((u(\tau+n))_{n\in\mathbb{Z}}\right)\|_{\ell_p} &\ge \gamma\operatorname{dist}\left((u(\tau+n))_{n\in\mathbb{Z}},\operatorname{Ker}D_\tau\right)\\ &\ge \gamma C^{-1}\operatorname{dist}\left((U(n,n+\tau-1)u(\tau+n-1))_{n\in\mathbb{Z}},\operatorname{Ker}D_0\right).\end{aligned}\tag{4.7}$$

Next, we claim that for each $\tau\in[0,1)$ one can choose $\mathbf{x}=(x_n)_{n\in\mathbb{Z}}\in\operatorname{Ker}D_0$, $\mathbf{x}=\mathbf{x}(\tau)$, so that the function $\mathbf{x}:[0,1)\to\ell_p$ is continuous, and that the following inequality holds:

$$\begin{aligned}&\operatorname{dist}\left((U(n,n+\tau-1)u(\tau+n-1))_{n\in\mathbb{Z}},\operatorname{Ker}D_0\right)\\ &\qquad\ge \frac{1}{2}\|(U(n,n+\tau-1)u(\tau+n-1))_{n\in\mathbb{Z}}-\mathbf{x}\|_{\ell_p}.\end{aligned}\tag{4.8}$$

To prove the claim, consider a continuous function $\mathbf{u}:[0,1]\to\ell_p$ defined by the rule $\mathbf{u}(\tau):=(U(n,n+\tau-1)u(\tau+n-1))_{n\in\mathbb{Z}}$. By the choice of u, the values $\mathbf{u}(\tau)$, $\tau\in[0,1]$, are sequences with finite support and therefore do not belong to $\operatorname{Ker}D_0$. Let $\epsilon:=\inf_{\tau\in[0,1]}\operatorname{dist}(\mathbf{u}(\tau),\operatorname{Ker}D_0)>0$ and choose an $\eta>0$ so that $\|\mathbf{u}(t)-\mathbf{u}(t')\|_{\ell_p}<\epsilon/10$ provided $|t-t'|<\eta$. Let $\{\tau_0=0,\dots,\tau_n=1\}$ be a partition of $[0,1]$ with the size η. Choose $\mathbf{y}_i\in\operatorname{Ker}D_0$ so that $\|\mathbf{u}(\tau_i)-\mathbf{y}_i\|<(11/10)\operatorname{dist}(\mathbf{u}(\tau_i),\operatorname{Ker}D_0)$, and define a piecewise constant function $\mathbf{x}_0$ by $\mathbf{x}_0(\tau)=\mathbf{y}_i$ for $\tau\in[\tau_i,\tau_{i+1})$. Then, for $\tau\in[\tau_i,\tau_{i+1})$, $i=0,\dots,n-1$, we have:

$$\begin{aligned}\|\mathbf{u}(\tau)-\mathbf{x}_0(\tau)\| &\le \|\mathbf{u}(\tau)-\mathbf{u}(\tau_i)\|+(11/10)\operatorname{dist}(\mathbf{u}(\tau_i),\operatorname{Ker}D_0)\\ &\le \epsilon/10+(11/10)(\epsilon/10+\operatorname{dist}(\mathbf{u}(\tau),\operatorname{Ker}D_0))\\ &\le (3/2)\operatorname{dist}(\mathbf{u}(\tau),\operatorname{Ker}D_0).\end{aligned}\tag{4.9}$$

Extending $\mathbf{u}$ and $\mathbf{x}_0$ periodically to $[1,2)$, for a $\delta\in(0,1)$ consider a continuous on $[0,1]$ function $\mathbf{x}$ defined by $\mathbf{x}(\tau)=(1/\delta)\int_\tau^{\tau+\delta}\mathbf{x}_0(s)\,ds$. Using (4.9), for

$\tau \in [0,1)$ we infer:

$$\begin{aligned}\|\mathbf{u}(\tau)-\mathbf{x}(\tau)\| &\le (1/\delta)\int_\tau^{\tau+\delta}\|\mathbf{u}(\tau)-\mathbf{x}_0(s)\|\,ds\\ &\le (1/\delta)\int_\tau^{\tau+\delta}\|\mathbf{u}(s)-\mathbf{x}_0(s)\|\,ds+(1/\delta)\int_\tau^{\tau+\delta}\|\mathbf{u}(\tau)-\mathbf{u}(s)\|\,ds\\ &\le (1/\delta)\int_\tau^{\tau+\delta}(3/2)\operatorname{dist}(\mathbf{u}(s),\operatorname{Ker}D_0)\,ds+\sup_{s\in[\tau,\tau+\delta]}\|\mathbf{u}(\tau)-\mathbf{u}(s)\|.\end{aligned}$$

Choosing δ so that the first term is less than $(3/2)\operatorname{dist}(\mathbf{u}(\tau),\operatorname{Ker}D_0)+(\epsilon/4)$ and the second term is less than $\epsilon/4$, we conclude that $\|\mathbf{u}(\tau)-\mathbf{x}(\tau)\| \le 2\operatorname{dist}(\mathbf{u}(\tau),\operatorname{Ker}D_0)$, $\tau\in[0,1)$, and the claim is proved.

Next, using (4.7) and (4.8), taking into account (4.3) again, and letting $w(\tau+n)=U(\tau+n,n)x_n(\tau)$ for $n\in\mathbb{Z}$ and $\tau\in[0,1)$, we have:

(4.10)

$$\begin{aligned}&C\|D_\tau(u(\tau+n))_{n\in\mathbb{Z}}\|_{\ell_p}\\ &\qquad\ge(\gamma/2)\|\left(U(n,n+\tau-1)u(n+\tau-1)-x_n\right)_{n\in\mathbb{Z}}\|_{\ell_p}\\ &\qquad\ge\gamma(2C)^{-1}\|\left(U(\tau+n,n)[U(n,n+\tau-1)u(n+\tau-1)-x_n]\right)_{n\in\mathbb{Z}}\|_{\ell_p}\\ &\qquad=\gamma(2C)^{-1}\|\left(U(\tau+n,n+\tau-1)u(\tau+n-1)-U(\tau+n,n)x_n\right)_{n\in\mathbb{Z}}\|_{\ell_p}\\ &\qquad=\gamma(2C)^{-1}\|\left((E^1u)(\tau+n)-w(\tau+n)\right)_{n\in\mathbb{Z}}\|_{\ell_p}.\end{aligned}$$

Since $\mathbf{x}\in\operatorname{Ker}D_0$, for any $n\in\mathbb{Z}$ and $\tau\in[0,1)$ we obtain:

$$\begin{aligned}(E^1w)(\tau+n)&=U(\tau+n,\tau+n-1)w(\tau+n-1)=U(\tau+n,n-1)x_{n-1}\\ &=U(\tau+n,n)U(n,n-1)x_{n-1}=U(\tau+n,n)x_n=w(\tau+n).\end{aligned}$$

Thus $w\in\operatorname{Ker}(E^1-I)$. Using (4.5) and (4.10), we thus infer:

$$\begin{aligned}C^2\|(E^1-I)u\|_{L_p}&\ge(\gamma/2)\left(\int_0^1\|\left((E^1u)(\tau+n)-w(\tau+n)\right)_{n\in\mathbb{Z}}\|_{\ell_p}^p\,d\tau\right)^{1/p}\\ &=(\gamma/2)\,\|E^1u-w\|_{L_p}\ge(\gamma/2)\left(\|u-w\|_{L_p}-\|(E^1-I)u\|_{L_p}\right).\end{aligned}$$

Since $w\in\operatorname{Ker}(E^1-I)$, assertion (i) in Theorem 1.2 follows. □

5. Fredholm operators on the half-line

In this section we study Fredholm properties of the generator $\mathbf{G}_0^+$ of the evolution semigroup (2.2) on $\mathcal{E}_0(\mathbb{R}_+)$ and the operator $\mathbf{G}^+$ on $\mathcal{E}(\mathbb{R}_+)$, described in Definition 2.3. On the sequence space $\mathcal{E}(\mathbb{Z}_+)$ we introduce the following difference operator $D^+:\mathbf{x}=(x_n)_{n\ge0}\mapsto((D^+\mathbf{x})_n)_{n\ge0}$, where

$$(D^+\mathbf{x})_n=\begin{cases}x_0, & n=0,\\ x_n-U(n,n-1)x_{n-1}, & n\ge1,\end{cases}\tag{5.1}$$

and remark that $\operatorname{Ker} D^+ = \{0\}$. First, we will provide an analog of Theorem 2.5 for the operators $\mathbf{G}_0^+$ and D^+. To start, fix a continuous 1-periodic function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\alpha(0) = \alpha(1) = 0$ and $\int_0^1 \alpha(s)ds = 1$, and let $\mathbf{x} = (x_n)_{n\in\mathbb{Z}_+} \in \mathcal{E}(\mathbb{Z}_+)$. Define bounded linear operators $R^0 : \mathcal{E}(\mathbb{R}_+) \to \mathcal{E}(\mathbb{Z}_+)$ and $S : \mathcal{E}(\mathbb{Z}_+) \to \mathcal{E}(\mathbb{R}_+)$ as follows:

$$(R^0 f)_n = \begin{cases} 0, & n = 0, \\ -\int_{n-1}^n U(n,s)f(s)\,ds, & n \geq 1; \end{cases} \tag{5.2}$$
$$(S\mathbf{x})(t) = \alpha(t)U(t,n)x_n, \quad t \in [n, n+1], \quad n \geq 0.$$

Lemma 5.1.

(i) *If* $\mathbf{y} = D^+\mathbf{x}$ *for some* $\mathbf{x} \in \mathcal{E}(\mathbb{Z}_+)$ *then* $\mathbf{G}_0^+ u = S\mathbf{y}$ *for some* $u \in \operatorname{dom} \mathbf{G}_0^+$*;*
(ii) *if* $S\mathbf{y} = \mathbf{G}_0^+ u$ *for some* $u \in \operatorname{dom} \mathbf{G}_0^+$ *then* $\mathbf{y} = D^+\mathbf{x}$ *for an* $\mathbf{x} \in \mathcal{E}(\mathbb{Z}_+)$*;*
(iii) *if* $f = \mathbf{G}_0^+ u$ *for some* $u \in \operatorname{dom} \mathbf{G}_0^+$ *then* $R^0 f = D^+(u(n))_{n\in\mathbb{Z}^+}$*;*
(iv) *if* $R^0 f = D^+\mathbf{x}$ *for an* $\mathbf{x} \in \mathcal{E}(\mathbb{Z}_+)$ *then* $f = \mathbf{G}_0^+ u$ *for some* $u \in \operatorname{dom} \mathbf{G}_0^+$.

Proof. We give the proof only for L_p-spaces. The argument for the space $C_{00}(\mathbb{R}_+; X)$ is similar.

(i) Define $u(t) = U(t,n)(y_n - x_n) - \int_n^t U(t,s)S\mathbf{y}(s)\,ds$ for $t \in [n, n+1], n \geq 0$. Then $u(0) = 0$. A calculation similar to [9, p. 117] shows that $u \in L_p(\mathbb{R}_+; X) \cap C_{00}(\mathbb{R}_+; X)$ and that u satisfies (2.3) with $f = S\mathbf{y}$. Thus $\mathbf{G}_0^+ u = S\mathbf{y}$.

(ii) For $u \in L_p(\mathbb{R}_+; X) \cap C_{00}(\mathbb{R}_+; X)$ satisfying (2.3) with $f = S\mathbf{y}$ equation (2.1) holds for all $t \geq \tau$ in $\mathbb{R}_+$. In particular, for $t = n+1$ and $\tau = n$,

$$\begin{aligned} u(n+1) &= U(n+1,n)u(n) - \int_n^{n+1} U(n+1,s)\alpha(s)U(s,n)y_n\,ds \\ &= U(n+1,n)u(n) - U(n+1,n)y_n, \quad n \geq 0. \end{aligned}$$

Thus, $\mathbf{y} = D^+(y_n - u(n))_{n\in\mathbb{Z}_+}$. Moreover, $\mathbf{u} = (u(n))_{n\in\mathbb{Z}_+} \in \ell_p(\mathbb{Z}_+; X)$. Indeed, (2.1) implies that

$$\|u(n)\| \leq C(\|u(t)\| + \|y_{n-1}\|), \quad t \in [n-1, n], n \geq 1, \tag{5.3}$$

with $C > 0$ from (4.3). Then, using the inequality $(a+b)^p \leq 2^{p-1}(a^p + b^p)$ and integrating (5.3) along $[n-1, n]$, we have $\|\mathbf{u}\|_{\ell_p}^p \leq 2^{p-1}C^p(\|u\|_{L_p}^p + \|\mathbf{y}\|_{\ell_p}^p)$.

(iii) Since u and f satisfy (2.1) for all $t \geq \tau$ in $\mathbb{R}_+$, letting $t = n$ and $\tau = n-1$, we have that $-\int_{n-1}^n U(n,s)f(s)\,ds = u(n) - U(n,n-1)u(n-1)$, $n \geq 1$. As above, $\mathbf{u} = (u(n))_{n\in\mathbb{Z}_+} \in \ell_p(\mathbb{Z}_+; X)$. By (5.2), $(R^0 f)_0 = u(0) = 0$.

(iv) For $\mathbf{x} = (x_n)_{n\in\mathbb{Z}_+}$ such that $R^0 f = D^+\mathbf{x}$ define

$$u(t) = U(t,n)x_n - \int_n^t U(t,s)f(s)ds, \quad t \in [n, n+1], \quad n \in \mathbb{Z}_+.$$

Note that $u(0) = x_0 = (R^0 f)_0 = 0$. A calculation similar to [9, p. 117] again shows that $u \in L_p(\mathbb{R}_+; X) \cap C_{00}(\mathbb{R}_+; X)$, and that u and f satisfy (2.1) for all $t \geq \tau$ in $\mathbb{R}_+$. Thus, $\mathbf{G}_0^+ u = f$. □

THEOREM 5.2. *The range* $\operatorname{Im} \mathbf{G}_0^+$ *is closed in* $\mathcal{E}_0(\mathbb{R}_+)$ *if and only if* $\operatorname{Im} D^+$ *is closed in* $\mathcal{E}(\mathbb{Z}_+)$. *Also,* $\operatorname{codim} \operatorname{Im} \mathbf{G}_0^+ = \operatorname{codim} \operatorname{Im} D^+$. *In particular, the operator* $\mathbf{G}_0^+$ *is Fredholm if and only if* D^+ *is Fredholm, and* $\operatorname{ind} \mathbf{G}_0^+ = \operatorname{ind} D^+$.

The proof of Theorem 5.2 is identical to the proof of [16, Theorem 1.4] with [16, Lemma 6.1] replaced by Lemma 5.1, and is therefore omitted. Recall that $\operatorname{Ker} \mathbf{G}_0^+ = \{0\}$; see (2.5). The main result of this section is given next.

THEOREM 5.3. *Let* X *be a reflexive Banach space and assume the family* $\{U(t,\tau)\}_{t,\tau \in \mathbb{R}}$ *consists of invertible operators. Then the following statements are equivalent.*

(i) *The operator* $\mathbf{G}_0^+$ *is Fredholm on* $\mathcal{E}_0(\mathbb{R}_+)$.

(ii) *The evolution family* $\{U(t,s)\}_{t \geq s \geq 0}$ *admits an exponential dichotomy* $\{P_t\}_{t \geq 0}$ *on* $\mathbb{R}_+$ *and* $\operatorname{codim} \operatorname{Im} P_0 < \infty$.

Also, $\operatorname{ind} \mathbf{G}_0^+ = -\operatorname{codim} \operatorname{Im} P_0$.

Proof. Extend the evolution family $\{U(t,\tau)\}_{t \geq \tau \geq 0}$ from $\mathbb{R}_+$ to an evolution family $\{V(t,\tau\}_{t \geq \tau}$ on $\mathbb{R}$ as follows:

$$V(t,\tau) = \begin{cases} U(t,\tau) & \text{for } t \geq \tau \geq 0, \\ U(t,0)e^{\tau} & \text{for } t \geq 0 \geq \tau, \\ e^{-(t-\tau)} & \text{for } 0 \geq t \geq \tau. \end{cases} \tag{5.4}$$

On $\mathcal{E}(\mathbb{R})$ consider the generator $\mathbf{G}_V$ of the evolution semigroup associated with $\{V(t,\tau)\}_{t \geq \tau}$; cf. Proposition 2.1. Let D_V denote the corresponding difference operator $D_V\left((x_n)_{n \in \mathbb{Z}}\right) = (x_n - V(n,n-1)x_{n-1})_{n \in \mathbb{Z}}$ on $\mathcal{E}(\mathbb{Z})$. In the direct sum decomposition $\mathcal{E}(\mathbb{Z}) = \mathcal{E}(\mathbb{Z} \cap (-\infty,-1]) \oplus \mathcal{E}(\mathbb{Z}_+)$ the operator D_V allows the following matrix representation:

$$D_V = \begin{bmatrix} D_V^- & 0 \\ D_V^{\pm} & D_V^+ \end{bmatrix} = \begin{bmatrix} D_V^- & 0 \\ D_V^{\pm} & I \end{bmatrix} \cdot \begin{bmatrix} I & 0 \\ 0 & D_V^+ \end{bmatrix}. \tag{5.5}$$

Here $D_V^- = D_V|_{\mathcal{E}(\mathbb{Z} \cap (-\infty,-1])}$, $D_V^+ : (x_n)_{n \geq 0} \mapsto (x_0, x_1 - V(1,0)x_0, \ldots)$, and $D_V^{\pm} : (x_n)_{n \leq -1} \mapsto (-V(0,-1)x_{-1}, 0, \ldots)$. Note that $D_V^- = I - e^{-1}S$, where $S : (x_n)_{n \leq -1} \mapsto (x_{n-1})_{n \leq -1}$ is the backward shift, and the operator D_V^- is invertible because $\|S\| = 1$. Since the first factor in the product (5.5) is an invertible operator, we infer that D_V is Fredholm if and only if D_V^+ is Fredholm, and that $\operatorname{ind} D_V = \operatorname{ind} D_V^+$. Thus, by Theorem 5.2, D_V is Fredholm if and only if $\mathbf{G}_0^+$ is Fredholm, and $\operatorname{ind} D_V = \operatorname{ind} \mathbf{G}_0^+$. Next, we claim that D_V is Fredholm if and only if $\mathbf{G}_V$ is Fredholm and that $\operatorname{ind} D_V = \operatorname{ind} \mathbf{G}_V$. Indeed, by Theorem 2.5 applied to the evolution family $\{V(t,\tau)\}_{t \geq \tau}$, $t, \tau \in \mathbb{R}$, we know that $\operatorname{Im}\mathbf{G}_V$ and $\operatorname{Im} D_V$ are closed at the same time with $\operatorname{codim}\operatorname{Im}\mathbf{G}_V = \operatorname{codim}\operatorname{Im} D_V$. By Lemma 4.2(ii), we have $\dim\operatorname{Ker}\mathbf{G}_V = \dim\operatorname{Ker} D_V$ (in fact, $\operatorname{Ker} D_V \subset \operatorname{Ker} D_V^+ = \{0\}$), and the claim is proved. Thus, $\mathbf{G}_0^+$ is Fredholm if and only if $\mathbf{G}_V$ is Fredholm and $\operatorname{ind}\mathbf{G}_0^+ = \operatorname{ind}\mathbf{G}_V$.

By Theorem 2.4, $\mathbf{G}_V$ is Fredholm if and only if the evolution families $\{U(t,\tau)\}_{t\geq\tau\geq 0}$ and $\{U(t,\tau)\}_{0\geq t\geq\tau}$ admit exponential dichotomies $\{P_t^+\}_{t\geq 0}$ and $\{P_t^-\}_{t\leq 0}$ on $\mathbb{R}_+$ and $\mathbb{R}_-$, and the pair of subspaces $(\operatorname{Ker} P_0^-, \operatorname{Im} P_0^+)$ is Fredholm (note that the dichotomy subspaces Ker P_0^- and $\operatorname{Im} P_0^+$ are uniquely defined; cf. [11, Remark IV.3.4]). But, using formula (5.4), one has $P_t^- = I$, $t \in \mathbb{R}_-$, and $\operatorname{Ker} P_0^- = \{0\}$. So, the pair $(\operatorname{Ker} P_0^-, \operatorname{Im} P_0^+)$ is Fredholm if and only if $\operatorname{codim}\operatorname{Im} P_0^+ < \infty$. Moreover, by Theorem 2.4, $\operatorname{ind}\mathbf{G}_0^+ = \operatorname{ind}\mathbf{G}_V = \operatorname{ind}(\operatorname{Ker} P_0^-, \operatorname{Im} P_0^+) = -\operatorname{codim}\operatorname{Im} P_0$. □

In particular, if $\dim X < \infty$ then the operator $\mathbf{G}_0^+$ is Fredholm if and only if $\{U(t,\tau)\}_{t\geq\tau\geq 0}$ admits an exponential dichotomy on $\mathbb{R}_+$. Turning to the study of Fredholm properties of $\mathbf{G}^+$, we will assume in the remaining part of this section that the evolution family $\{U(t,\tau)\}_{t\geq\tau\geq 0}$ consists of *invertible* operators. We note the following fact; cf. [6, Lemma 5.2].

LEMMA 5.4. *The range* $\operatorname{Im}\mathbf{G}^+$ *is dense in* $\mathcal{E}(\mathbb{R}_+)$.

Proof. For any function $f \in \mathcal{E}(\mathbb{R}_+)$ with compact support the function $u(t) := \int_t^\infty U(t,s)f(s)\,ds$, $t \geq 0$, has compact support and satisfies (2.1) for all $t \geq \tau$ in $\mathbb{R}_+$. So, $u \in \operatorname{dom}\mathbf{G}^+$ and $\mathbf{G}^+u = f$. The lemma follows from the density of such f in $\mathcal{E}(\mathbb{R}_+)$. □

In our next result we recast Theorem 2.6 in the current context.

THEOREM 5.5. *If X is a Banach space and the family $\{U(t,\tau)\}_{t,\tau\in\mathbb{R}}$ consists of invertible operators then the following statements are equivalent.*

(i) *The operator $\mathbf{G}^+$ is Fredholm on $\mathcal{E}(\mathbb{R}_+)$.*
(ii) *The family $\{U(t,s)\}_{t\geq s\geq 0}$ admits an exponential dichotomy $\{P_t\}_{t\geq 0}$ on $\mathbb{R}_+$ and* $\dim\operatorname{Im} P_0 < \infty$.
(iii) *The operator $\mathbf{G}^+$ is surjective on $\mathcal{E}(\mathbb{R}_+)$ and* $\dim X_0 < \infty$ *for the subspace X_0 defined in* (2.6).

Also, $\operatorname{ind}\mathbf{G}^+ = \dim X_0$.

Proof. If $\{U(t,\tau)\}_{t\geq\tau}$ admits an exponential dichotomy on $\mathbb{R}_+$, then $\operatorname{Im} P_0 = X_0$. Indeed, if $u \in \operatorname{Ker}\mathbf{G}$ then, from (2.4) and the dichotomy estimates (2.7),

$$\|u(t)\| = \|U(t,0)u(0)\| \geq M^{-1}e^{\alpha t}\|(I-P_0)u(0)\| - Me^{-\alpha t}\|P_0u(0)\|, \quad t \in \mathbb{R}_+.$$

Since $u \in \mathcal{E}(\mathbb{R})$, we have $(I-P_0)u(0) = 0$, i.e., $X_0 \subset \operatorname{Im} P_0$. The inverse inclusion follows from (2.4) and (2.6).

Since any finite-dimensional subspace is complemented in X, the equivalence (ii)$\Leftrightarrow$(iii) follows from Theorem 2.6.

(i)$\Rightarrow$(ii) If $\mathbf{G}^+$ is Fredholm then $\operatorname{Im}\mathbf{G}^+$ is closed, and thus $\operatorname{Im}\mathbf{G}^+ = \mathcal{E}(\mathbb{R}_+)$ by Lemma 5.4. Since $\dim\operatorname{Ker}\mathbf{G}^+ < \infty$, we also have $\dim X_0 < \infty$ due to (2.4), and X_0 is complemented in X. Since (iii)$\Rightarrow$(ii), $\{U(t,\tau)\}_{t\geq\tau\geq 0}$ admits an exponential dichotomy $\{P_t\}_{t\geq 0}$ on $\mathbb{R}_+$ with $\operatorname{Im}P_0 = X_0$.

(ii)$\Rightarrow$(i) By the implication (ii)$\Rightarrow$(iii), $\mathbf{G}^+$ is surjective. By (2.4), $u \in \operatorname{Ker}\mathbf{G}^+$ if and only if $u(0) \in \operatorname{Im}P_0$ and $u(t) = U(t,0)u(0)$, $t \geq 0$. Thus, the map $u(0) \mapsto u(\cdot)$, $u(t) = U(t,0)u(0)$, is a bijection from $\operatorname{Im}P_0$ on $\operatorname{Ker}\mathbf{G}^+$, and thus $\dim\operatorname{Ker}\mathbf{G}^+ = \dim X_0 = \dim\operatorname{Im}P_0 < \infty$. □

To conclude this section, we remark that Theorem 1.1 (with a similar proof) holds on $\mathcal{E}_0(\mathbb{R}_+)$ with E^t replaced by E^t_+.

6. Evolution semigroups on spaces of periodic functions

Let $\{e^{At}\}_{t\geq 0}$ be a strongly continuous semigroup on the Banach space X. Define an evolution semigroup, $\{e^{t\mathbf{G}_p}\}_{t\geq 0}$, on the space $\mathcal{E}([0,2\pi])$ of 2π-periodic functions by the formula

$$(e^{t\mathbf{G}_p}u)(\tau) = e^{At}u([\tau - t](\operatorname{mod}2\pi)), \quad \tau \in [0,2\pi], t \geq 0. \tag{6.1}$$

Its generator $\mathbf{G}_p$ is the closure of the operator $G_p u = -u' + M_A u$ defined on $\operatorname{dom}(d/dt) \cap \operatorname{dom}(M_A)$; cf. [9, p. 38].

Lemma 6.1. *Let T be a bounded linear operator on a Banach space X. For the multiplication operator M_T on $\mathcal{E}([0,2\pi])$ the following assertions hold.*

(i) $\dim\operatorname{Ker}M_T < \infty$ *if and only if* $\operatorname{Ker}M_T = \{0\}$;

(ii) $\operatorname{codim}\operatorname{Im}M_T < \infty$ *if and only if* $\operatorname{Im}M_A = \mathcal{E}([0,2\pi])$;

(iii) $\operatorname{Im}T$ *is closed in* X *if and only if* $\operatorname{Im}M_T$ *is closed in* $\mathcal{E}([0,2\pi])$.

Proof. (i) For $k \in \mathbb{N}$ consider functions $\varphi_k : [0,2\pi] \to [0,1]$ defined so that $\varphi_k((2k)^{-1}) = 1$, $\varphi_k(t) = 0$ for $t \in [0,2\pi] \setminus [(2k-1)^{-1},(2k+1)^{-1}]$, and φ_k is linear on $[(2k-1)^{-1},(2k)^{-1})$ and $((2k)^{-1},(2k+1)^{-1}]$. Suppose that there is a nonzero $x \in \operatorname{Ker}T$ and let $n = \dim\operatorname{Ker}M_T$. The functions in the family

$$S_x = \{\varphi_k(\cdot)x\colon 1 \leq k \leq n+1\} \subset \mathcal{E}([0,2\pi])$$

are linearly independent. Indeed, if $\sum_{k=1}^{n+1}\lambda_k\varphi_k(\cdot) = 0$ for $\lambda_k \in \mathbb{C}$, then, applying the functionals $F_k = \langle\cdot,\varphi_k(\cdot)x^*\rangle \in (\mathcal{E}([0,2\pi]))^*$ with $\langle x,x^*\rangle = 1$, we obtain $\lambda_k = 0$, $1 \leq k \leq n+1$. Thus, $\dim\operatorname{Ker}M_T \geq n+1$, a contradiction.

(ii) Suppose that there is an $x \in X \setminus \operatorname{Im}T$ and let $n = \dim(X/\operatorname{Im}M_T)$. Consider the family S_x constructed above. If $\varphi_k(\cdot)x - \varphi_m(\cdot)x \in \operatorname{Im}M_T$ for $k \neq m$ then $x \in \operatorname{Im}T$, which contradicts $x \in X\setminus\operatorname{Im}T$. Therefore, $\varphi_k(\cdot)x$ belong to different quotient classes of $X/\operatorname{Im}M_T$. Moreover, if $\sum_{k=1}^{n+1}\lambda_k\varphi_k(\cdot) \in \operatorname{Im}M_T$ then for each k we have $\lambda_k\varphi_k(t) \in \operatorname{Im} M_T$ for a.e. $t \in [(2k-1)^{-1}, (2k+1)^{-1}]$. This can hold only if $\lambda_k = 0$. Thus, the quotient classes containing $\varphi_k(\cdot)x, 1 \le k \le n+1$, are linearly independent, and $\dim(X/\operatorname{Im} M_T) \ge n+1$, a contradiction.

(iii) This part of the lemma is proved[2] in [7]. □

The following well-known spectral mapping theorem relates $\sigma(e^{2\pi A})$ on X and the spectra of $\mathbf{G}_p$ and $e^{2\pi\mathbf{G}_p}$ on $\mathcal{E}([0, 2\pi])$; see [9, Theorem 2.30].

Theorem 6.2. *The following statements are equivalent.*

(i) $e^{2\pi A} - I$ *is invertible in* X*;*
(ii) $\mathbf{G}_p$ *is invertible in* $\mathcal{E}([0, 2\pi])$*;*
(iii) $e^{2\pi\mathbf{G}_p} - I$ *is invertible in* $\mathcal{E}([0, 2\pi])$*.*

The next proposition shows, once again, that an analog of Theorem 6.2 for the Fredholm spectra can hold only trivially; cf. Theorem 1.1.

Proposition 6.3. *For each* $\lambda \in \mathbb{C}\backslash\{0\}$, *the operator* $e^{2\pi\mathbf{G}_p} - \lambda$ *is Fredholm in* $\mathcal{E}([0, 2\pi])$ *if and only if it is invertible.*

Indeed, this holds by Lemma 6.1 because

$$(e^{2\pi\mathbf{G}_p}u)(\tau) = e^{2\pi A}u(\tau), \quad u \in \mathcal{E}([0, 2\pi]), \quad \tau \in [0, 2\pi]. \tag{6.2}$$

We will need a description of $\operatorname{dom}\mathbf{G}_p$; cf. Proposition 2.1 and [17].

Lemma 6.4. *A function* u *belongs to* dom $\mathbf{G}_p$ *on* $\mathcal{E}([0, 2\pi])$ *if and only if* $u \in C_{\mathrm{per}}([0, 2\pi]; X)$ *and there exists an* $f \in \mathcal{E}([0, 2\pi])$ *such that*

$$u(t) = e^{tA}u(0) + \int_0^t e^{(t-s)A}f(s)ds, \ t \in [0, 2\pi]. \tag{6.3}$$

Proof. Define a closed operator, $\mathbf{G}_{p,1}$, as $\mathbf{G}_{p,1}u = f$ for u and f satisfying (6.3). The set $\mathcal{P}$ of trigonometric polynomials on $[0, 2\pi]$ with values in $\operatorname{dom} A$ is a core for $\mathbf{G}_{p,1}$. Since $\mathbf{G}_{p,1}u = \mathbf{G}_p u$ for every $u \in \mathcal{P}$, we infer that $\mathbf{G}_p = \mathbf{G}_{p,1}$. □

The next result is an analog of Theorem 1.2 for the space $\mathcal{E}([0, 2\pi])$.

Theorem 6.5. *The following statements are equivalent.*

(i) $\operatorname{Im}(e^{2\pi A} - I)$ *is closed in* X*;*
(ii) $\operatorname{Im}\mathbf{G}_p$ *is closed in* $\mathcal{E}([0, 2\pi])$*;*
(iii) $\operatorname{Im}(e^{2\pi\mathbf{G}_p} - I)$ *is closed in* $\mathcal{E}([0, 2\pi])$*.*

[2] We thank L. Burlando for making her preprint [7] available.

Proof. (i)$\Leftrightarrow$(iii) This follows directly from (6.2) and Lemma 6.1(iii).

(i)$\Rightarrow$(ii) Consider a sequence $\{f_k = \mathbf{G}_p u_k : k \in \mathbb{N}\} \subset \operatorname{Im} \mathbf{G}_p$ such that $f_k \to f$ in $\mathcal{E}([0,2\pi])$ as $k \to \infty$. By Lemma 6.4,

$$(I - e^{2\pi A})u_k(0) = \int_0^{2\pi} e^{(2\pi - s)A} f_k(s)\, ds \in \operatorname{Im}(I - e^{2\pi A}).$$

Since $f_k \to f$, we obtain $(I - e^{2\pi A})u_k(0) \to \int_0^{2\pi} e^{(2\pi-s)A} f(s)\, ds$ in X as $k \to \infty$. Since $\operatorname{Im}(e^{2\pi A} - I)$ is closed, there exists an $x \in X$ such that $\int_0^{2\pi} e^{(2\pi-s)A} f(s)\, ds = (I - e^{2\pi A})x$. Define $u \in C_{\text{per}}([0,2\pi]; X)$ by

$$u(t) = e^{tA}x + \int_0^t e^{(t-s)A} f(s)\, ds \quad \text{for} \quad t \in [0, 2\pi].$$

By Lemma 6.4, $f = \mathbf{G}_p u$ and thus $\operatorname{Im} \mathbf{G}_p$ is closed.

(ii)$\Rightarrow$(i) Suppose that $\gamma(e^{2\pi A} - I) = 0$ and choose a sequence $\{x_n \colon n \in \mathbb{N}\} \subset X$ such that $\|x_n\| = 1$, $n \in \mathbb{N}$, and also assertions (a) $\|(e^{2\pi A} - I)x_n\| \le n^{-1}$ and (b) $q := \inf_{n\in\mathbb{N}} \inf_{y \in \operatorname{Ker}(e^{2\pi A} - I)} \|x_n - y\| > 0$ hold. As in [9, p. 39], let $\alpha : [0,2\pi] \to [0,1]$ be a smooth function such that $\alpha(\tau) = 0$ provided $\tau \in [0, 2\pi/3]$ and $\alpha(\tau) = 1$ provided $\tau \in [4\pi/3, 2\pi]$. Define a sequence of functions $\{g_n : n \ge 0\}$ in $\mathcal{E}([0,2\pi])$ by the formula

$$g_n(\tau) = (1 - \alpha(\tau))e^{(2\pi+\tau)A}x_n + \alpha(\tau)e^{\tau A}x_n, \quad \tau \in [0, 2\pi].$$

We claim that $\inf_{n\in\mathbb{N}} \operatorname{dist}(g_n, \operatorname{Ker} \mathbf{G}_p) > 0$ and $\|\mathbf{G}_p g_n\|_{\mathcal{E}} \to 0$ as $n \to \infty$. This implies $\gamma(\mathbf{G}_p) = 0$, a contradiction with (ii).

To prove the claim, we note, first, that $u \in \operatorname{Ker} \mathbf{G}_p$ if and only if $u(\tau) = e^{A\tau}u(0)$, $\tau \in [0, 2\pi]$, and $e^{2\pi A}u(0) = u(0)$. Hence,

$$\operatorname{dist}(g_n, \operatorname{Ker} \mathbf{G}_p) = \inf_{u \in \operatorname{Ker} \mathbf{G}_p} \|g_n - u\| = \inf_{y \in \operatorname{Ker}(e^{2\pi A} - I)} \|g_n - e^{(\cdot)A}y\|. \tag{6.4}$$

Set $a := \max\{|\alpha'(\tau)| \colon \tau \in [0,2\pi]\}$ and $b := \max\{\|e^{\tau A}\| \colon \tau \in [0, 2\pi]\}$.

If $\mathcal{E}([0,2\pi]) = C_{\text{per}}([0,2\pi]; X)$ then, as in [9, p. 39], $\{g_n \colon n \ge 0\} \subset \operatorname{dom} \mathbf{G}_p$ and $\|\mathbf{G}_p g_n\| \le ab/n$. Moreover, if $y \in \operatorname{Ker}(e^{2\pi A} - I)$ then

$$\begin{aligned} \|g_n - e^{(\cdot)A}y\|_{C_{\text{per}}} &\ge \|g_n(0) - y\| = \|e^{2\pi A}x_n - y\| \\ &\ge \|x_n - y\| - \|e^{2\pi A}x_n - x_n\| \ge q/2 > 0 \end{aligned}$$

for sufficiently large n since $e^{2\pi A}x_n - x_n \to 0$ as $n \to \infty$. By (6.4), we have $\operatorname{dist}(g_n, \operatorname{Ker} \mathbf{G}_p) > 0$. If $\mathcal{E}([0,2\pi]) = L_p([0,2\pi]; X)$, $1 \le p < \infty$, then, as in [9,

p. 39], $\{g_n\colon n \geq 0\} \subset \operatorname{dom} \mathbf{G}_p$, and $\|\mathbf{G}_p g_n\| \leq (2\pi)^{1/p} ab/n$. Furthermore,

$$\begin{aligned}
\|g_n - e^{(\cdot)A} y\|_{L_p}^p &\geq \int_0^{2\pi/3} \|e^{\tau A}(e^{2\pi A}x_n - y)\|^p \, d\tau \\
&\geq b^{-1} \int_0^{2\pi/3} \|e^{(2\pi-\tau)A} e^{\tau A}(e^{2\pi A}x_n - y)\|^p \, d\tau \\
&= \frac{2\pi}{3b} \|e^{2\pi A}(e^{2\pi A}x_n - y)\|^p \\
&= \frac{2\pi}{3b} \|e^{2\pi A}(e^{2\pi A}x_n - x_n) + (e^{2\pi A}x_n - x_n) + (x_n - y)\|^p
\end{aligned}$$

for $y \in \operatorname{Ker}(e^{2\pi A} - I)$. Since $e^{2\pi A}x_n - x_n \to 0$ as $n \to \infty$, we have $\|g_n - e^{(\cdot)A}y\|_{L_p} \geq c > 0$ for some $c > 0$ and sufficiently large n, and thus $\operatorname{dist}(g_n, \operatorname{Ker} \mathbf{G}_p) > 0$. □

References

[1] A. Abbondandolo and P. Majer, *Ordinary differential operators in Hilbert spaces and Fredholm pairs*, Math. Z. **243** (2003), 525–562. MR 2004d:58014

[2] ———, *A Morse complex for infinite dimensional manifolds, Part I*, 2003, preprint.

[3] A. G. Baskakov, *Semigroups of difference operators in the spectral analysis of linear differential operators*, Funct. Anal. Appl. **30** (1996), 149–157. MR 98b:47053

[4] ———, *On the well-posedness of linear differential operators*, Sb. Mat. **190** (1999), 323–348. MR 2000i:47074

[5] ———, *On invertibility and the Fredholm property of parabolic differential operators*, Dokl. Akad. Nauk **383** (2002), 583–585. MR 2003d:47054

[6] A. Ben-Artzi and I. Gohberg, *Dichotomy of systems and invertibility of linear ordinary differential operators*, Time-variant systems and interpolation, Oper. Theory Adv. Appl., vol. 56, Birkhäuser, Basel, 1992, pp. 90–119. MR 1 173 919

[7] L. Burlando, *Multiplication operators between vector-valued function spaces which are induced by bounded linear operators between Banach spaces*, preprint, 2003.

[8] S. R. Caradus, *Generalized inverses and operator theory*, Queen's Papers in Pure and Applied Mathematics, vol. 50, Queen's University, Kingston, Ont., 1978. MR 81m:47003

[9] C. Chicone and Y. Latushkin, *Evolution semigroups in dynamical systems and differential equations*, Mathematical Surveys and Monographs, vol. 70, American Mathematical Society, Providence, RI, 1999. MR 2001e:47068

[10] P. Clément, H. J. A. M. Heijmans, S. Angenent, C. J. van Duijn, and B. de Pagter, *One-parameter semigroups*, CWI Monographs, vol. 5, North-Holland Publishing Co., Amsterdam, 1987. MR 89b:47058

[11] J. Daleckij and M. Krein, *Stability of solutions of differential equations in Banach space*, American Mathematical Society, Providence, R.I., 1974. MR 50 #5126

[12] D. Di Giorgio and A. Lunardi, *On Fredholm properties of $\mathcal{L}u = u' - A(t)u$ for paths of sectorial operators*, preprint, 2003.

[13] K.-J. Engel and R. Nagel, *One-parameter semigroups for linear evolution equations*, Graduate Texts in Mathematics, vol. 194, Springer-Verlag, New York, 2000. MR 2000i:47075

[14] D. Henry, *Geometric theory of semilinear parabolic equations*, Lecture Notes in Mathematics, vol. 840, Springer-Verlag, Berlin, 1981. MR 83j:35084

[15] T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, Berlin, 1976. MR 53 #11389

[16] Y. Latushkin and Y. Tomilov, *Fredholm differential operators with unbounded coefficients*, J. Diff. Eqns. **208** (2005), 388–429.

[17] Y. Latushkin and F. Räbiger, *Fourier multipliers and stability*, Integral Equations Operator Theory, to appear.

[18] B. M. Levitan and V. V. Zhikov, *Almost periodic functions and differential equations*, Cambridge University Press, Cambridge, 1982. MR 84g:34004

[19] B. Nagy, *Spectral mapping theorems for semigroups of operators*, Acta Sci. Math. (Szeged) **38** (1976), 343–351. MR 56 #9320

[20] Nguyen van Minh, F. Räbiger, and R. Schnaubelt, *Exponential stability, exponential expansiveness, and exponential dichotomy of evolution equations on the half-line*, Integral Equations Operator Theory **32** (1998), 332–353. MR 99j:34083

[21] Nguyen van Minh and Nguyen Thieu Huy, *Characterizations of dichotomies of evolution equations on the half-line*, J. Math. Anal. Appl. **261** (2001), 28–44. MR 2003a:34078

[22] K. J. Palmer, *Exponential dichotomies and transversal homoclinic points*, J. Differential Equations **55** (1984), 225–256. MR 86d:58088

[23] ———, *Exponential dichotomies and Fredholm operators*, Proc. Amer. Math. Soc. **104** (1988), 149–156. MR 89k:34052

[24] P. J. Rabier, *Fredholm operators, semigroups and the asymptotic and boundary behavior of solutions of PDEs*, J. Differential Equations **193** (2003), 460–480. MR 2004f:47059

[25] ———, *The Robbin-Salamon index theorem in Banach spaces*, Dynamics of PDEs, to appear.

[26] J. Robbin and D. Salamon, *The spectral flow and the Maslov index*, Bull. London Math. Soc. **27** (1995), 1–33. MR 96d:58021

[27] B. Sandstede, *Stability of travelling waves*, Handbook of dynamical systems, Vol. 2, North-Holland, Amsterdam, 2002, pp. 983–1055. MR 2004e:37121

[28] R. Schnaubelt, *Well-posedness and asymptotic behaviour of non-autonomous linear evolution equations*, Evolution equations, semigroups and functional analysis (Milano, 2000), Progr. Nonlinear Differential Equations Appl., vol. 50, Birkhäuser, Basel, 2002, pp. 311–338. MR 2003i:34140

[29] ———, *Exponential dichotomy of non-autonomous evolution equations*, Habilitation thesis, Tübingen, 1999.

[30] G. R. Sell and Y. You, *Dynamics of evolutionary equations*, Applied Mathematical Sciences, vol. 143, Springer-Verlag, New York, 2002. MR 2003f:37001b

[31] M. A. Shubin, *Almost periodic pseudodifferential operators, and von Neumann algebras*, Trudy Moskov. Mat. Obšč. **35** (1976), 103–164. MR 58 #30521

Yuri Latushkin, Department of Mathematics, University of Missouri, Columbia, MO 65211, USA

E-mail address: `yuri@math.missouri.edu`

Yuri Tomilov, Institute of Mathematics, Polish Academy of Sciencesi, Śniadeckich Str. 8, P.O. Box 21, 00-956 Warsaw 10, Poland, and Faculty of Mathematics and Informatics, Nicholas Copernicus University, Chopin Str. 12/18, 87-100 Torun, Poland

E-mail address: `tomilov@mat.uni.torun.pl`