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RELATIVE FATOU THEOREM FOR α-HARMONIC FUNCTIONS IN LIPSCHITZ DOMAINS

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ABSTRACT. We prove a relative Fatou theorem for α-harmonic functions on bounded Lipschitz domains D that vanish outside D. We also discuss the case when the normalizing function corresponds to the Hausdorff measure on the boundary of D.

1. Introduction

The problem of the boundary behaviour of harmonic functions on bounded regular domains has been studied intensively, both in analytic and in probabilistic contexts (see [2], [3], [15], [27]). These functions are related to the Brownian motion process, and since this process hits the boundary when leaving sufficiently regular domains, it follows that harmonic functions have nontangential limits at the boundary as well as other important properties (see [4], [18], [27], [29]).

When considering α-harmonic functions and the potential theory of Riesz kernels, a probabilistic approach leads to the α-stable rotation invariant Lévy process. This process is a jump process that, usually, leaves the domain without entering its boundary (see [14], [28], [30]). This is the reason why the analytic approach to the boundary behaviour of α-harmonic functions is different from the one used in the classical case. In the α-stable case the natural objects to be considered are ratios of α-harmonic functions vanishing outside the domain. These functions can be analyzed in the context of their boundary behaviour, and they have been investigated in the classical case (see [17], [29]). The α-stable case has been studied in the literature; see [12] for the latest results in smooth domains.

In this paper we present some results in the α-stable case for bounded Lipschitz domains (see below for the details). The main part of the paper follows the general outline presented in [29], but we will need different estimates since

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the α-stable Lévy motion has discontinuous paths when $\alpha < 2$. We will use the estimates for the Martin kernels and the Green functions of Lipschitz domains, smooth domains (see [16], [19], [22]) and cones (see [1], [25]). We will also use a version of the Boundary Harnack Principle, presented in [26].

In Section 2 we present the basic concepts and definitions. Section 3 provides some estimates for the Green function and the Martin kernel in Lipchitz domains. Some of these estimates may be useful in other applications. Section 4 contains the most important results of this paper, including the main theorem (Theorem 4.2), an α-stable version of a relative Fatou theorem for classical harmonic functions (see [29, Theorem 3]). We also give a simple example which shows that the assumptions of this theorem cannot be weakened. Section 5 deals with the case when the normalizing α-harmonic function corresponds to the surface measure σ. The main results of this section, Theorems 5.3 and 5.4, generalize the results of [12] and, together with the counterexample presented, show the difference between Lipschitz domains and smooth domains.

After completing this paper, the authors have been informed by the referee that similar results for κ-fat sets, which are a wider class than Lipschitz sets, have recently been obtained by Panki Kim, with the use of probabilistic methods (see [21]).

2. Preliminaries

We denote by $|\cdot|$ the Euclidean norm of vectors. For a set $B \in \mathbf{R}^d$, $d \geq 2$, we denote its complement by B^c and its characteristic function by $\mathbf{1}_B$. For $x \in \mathbf{R}^d$, $B(x,r)$ denotes an open ball centered at x of radius r. For a Borel set B and $r > 0$ we define $rB = \{rx : x \in B\}$.

Let D denote a bounded open set in $\mathbf{R}^d$. We say that D is a Lipschitz domain if there exist constants R_0 (*localization radius*) and $\lambda > 0$ (*Lipschitz constant*) such that for every $z \in \partial D$ there is a function $F : \mathbf{R}^{d-1} \to \mathbf{R}$ and an orthonormal coordinate system $y = (y_1, \ldots, y_d)$ such that

$$D \cap B(z, R_0) = \{y : y_d > F(y_1, \ldots, y_{d-1})\} \cap B(z, R_0)$$

and the function F is Lipschitz with Lipschitz constant not greater than λ. If, in addition, F is differentiable and ∇F is Lipschitz with Lipschitz constant not greater than λ, then D is called a $C^{1,1}$ domain.

Let (X_t, P^x) be the rotation invariant ('symmetric') α-stable Lévy motion (i.e., homogeneous with independent increments) on $\mathbf{R}^d$ with index $\alpha \in (0,2)$ (see [6]). For a Borel subset B of $\mathbf{R}^d$ let T_B and τ_B denote, respectively, the first entry time and the first exit time of B, i.e., $T_B = \inf\{t \geq 0 : X_t \in B\}$ and $\tau_B = T_{B^c}$.

For $x \in \mathbf{R}^d$ we define the *α-harmonic measure* of D by $\omega_D^x(B) = P^x(X_{\tau_D} \in B)$. If D is Lipschitz then this measure is concentrated on $(\overline{D})^c$ and has a density with respect to the Lebesgue measure, called the *Poisson kernel* (see

[8]). This kernel will be denoted by $P_D(x,y),\ x \in D,\ y \in (\overline{D})^c$. It satisfies the *scaling property*

$$P_D(x,y) = (1/r^d) P_{(1/r)D}(x/r, y/r),\ r > 0. \tag{1}$$

When $D = B(0,r),\ r > 0$, the Poisson kernel is given by the explicit formula

$$P_r(x,y) = C_{d,\alpha} \left(\frac{r^2 - |x|^2}{|y|^2 - r^2} \right)^{\alpha/2} \frac{1}{|x-y|^d}, |x| < r, |y| > r,$$

where $C_{d,\alpha} = \Gamma(d/2)\pi^{-d/2-1}\sin(\pi\alpha/2)$ (see [7], [23]).

For $x \in B$ let $\delta_x(B) = \text{dist}(x, \partial B)$ and $\delta_x = \delta_x(D)$.

In this paper constants are always positive numbers. In equations and inequalities the constants may change under arithmetic transformations, but they will be denoted by the same symbols. A notation of the form $c = c(a,b)$ means that the constant c depends only on a, b.

A nonnegative Borel function h on $\mathbf{R}^d$ is said to be α-harmonic on D if for each bounded open set B with $\overline{B} \subset D$ and for all $x \in B$ we have

$$h(x) = E^x h(X_{\tau_B}) < \infty. \tag{2}$$

If $h \equiv 0$ on D^c then h is called *singular α-harmonic* on D. If B can be replaced by D in (2) then h is called *regular α-harmonic* on D. In particular, for B fixed the harmonic measure $\omega_D^x(B)$ is regular α-harmonic on D as a function of x (see [8]).

From now on we will assume that $r > 0,\ x \in D,\ y \in (\partial D)^c,\ z, Q \in \partial D$. $x_0 \in D$ will be a fixed reference point. For $r \le R_0/32$ and $Q \in \partial D$ we denote by $A_{Q,r}$ a point for which $B(A_{Q,r}, \kappa r) \subset B(Q,r)$ for a certain absolute constant $\kappa = \kappa(D) = 1/(2\sqrt{1+\lambda^2})$. The set of such points is nonempty and $A_{Q,r}$ is not unique. For $r > R_0/32$ we set $A_{Q,r} = x_1$, where $x_1 \in D$ is another fixed point such that $|x_0 - x_1| = R_0/4$. See [19] for details.

The following theorems (*Harnack Principles*) constitute some of the basic tools in our paper.

Theorem 2.1 (Harnack Principle). *Let B be an open set. Assume that for some positive integer k and all $x, y \in B$ we have $|x-y| < 2^k \min(\delta_x(B), \delta_y(B))$. Let u be α-harmonic in $B(x, \delta_x(B)) \cup B(y, \delta_y(B))$. Then there exists a constant $C = C(d,\alpha)$ such that*

$$C^{-1} 2^{-k(d+\alpha)} u(y) \le u(x) \le C 2^{k(d+\alpha)} u(y).$$

Theorem 2.2 (Boundary Harnack Principle). *Let D be an open set, $z \in \partial D, r, \rho \in (0,1)$, and $B(A, \rho r)$ a ball in $D \cap B(z,r)$. Then there exists a constant $C = C(d,\alpha) > 1$ such that for any two functions u, v that are positive regular α-harmonic in $D \cap B(z, 2r)$ and vanish in $D^c \cap B(z,r)$ we have*

$$C^{-1} \rho^{d+\alpha} \frac{v(x)}{v(A)} \le \frac{u(x)}{u(A)} \le C \rho^{d+\alpha} \frac{v(x)}{v(A)},\ x \in D \cap B(z, r/2).$$

Theorem 2.1 is a version of [8, Lemma 2] and Theorem 2.2 can be found in [26].

For all nonnegative Borel measurable functions f we define the *Riesz potential* of f by

$$Uf(x) = E^x \int_0^\infty f(X_t)dt = \int A_{d,\alpha}|x-y|^{\alpha-d} f(y)dy,$$

where $A_{d,\alpha} = 2^{-\alpha}\pi^{-d/2}\Gamma((d-\alpha)/2)/\Gamma(\alpha/2)$ (see [6]).

For an open set B we define the *Green potential* of f by

$$G_B f(x) = E^x \int_0^{\tau_B} f(X_t)dt = \int G_B(x,y)f(y)dy,$$

where $G_B(x,y)$ is the *Green function* of B defined by

$$G_B(x,y) = A_{d,\alpha}(|x-y|^{\alpha-d} - E^x|x - X_{\tau_B}|^{\alpha-d}),\ x,y \in B, x \neq y, \tag{3}$$

$G_B(x,x) = \infty,\ x \in B$, and $G_B(x,y) = 0$ otherwise. This function is symmetric (i.e., $G_B(x,y) = G_B(y,x)$), positive in $\mathrm{int}(B)$, and if $B_1 \subset B_2$ then $G_{B_1} \le G_{B_2}$. Furthermore, G_B satisfies the *scaling property*

$$G_B(x,y) = (1/r^{d-\alpha})G_{(1/r)B}(x/r, y/r),\ r > 0. \tag{4}$$

For other properties of the Green function see [9] and [16].

Every nonnegative function that is singular α-harmonic on D has a unique representation (called *Martin representation*)

$$f(x) = \int_{\partial D} M(x,z)\mu(dz), \tag{5}$$

where μ is a finite Borel measure on ∂D (see [9]). The kernel function $M(x,z)$, called *Martin kernel*, may be defined by

$$M(x,z) = \lim_{D\ni y\to z} \frac{G_D(x,y)}{G_D(x_0,y)},\ x \in D, z \in \partial D. \tag{6}$$

The existence of this limit follows from the Boundary Harnack Principle (see [9]). We will use the following estimates for M (see [19, Theorem 3]):

$$c\,\frac{\phi(x)}{|x-z|^{d-\alpha}\phi^2(A_{z,|x-z|})} \le M(x,z) \le C\frac{\phi(x)}{|x-z|^{d-\alpha}\phi^2(A_{z,|x-z|})}, \tag{7}$$

where c, C depend on d, α, λ and

$$\phi(x) = \min(G_D(x,x_0), C_{d,\alpha}(R_0/4)^{\alpha-d}). \tag{8}$$

Note that for x sufficiently close to ∂D we simply have $\phi(x) = G_D(x,x_0)$.

3. Estimates for Green functions and Martin kernels

First we define an *unbounded circular cone* with vertex at $0 = (0, 0, \dots, 0)$ and symmetric with respect to the d-th axis as a set V of the form

$$V = \{x : \eta \cdot |(x_1, x_2, \dots, x_{d-1})| < x_d\},$$

where $\eta \in (-\infty, \infty)$. The *aperture* of V is the angle $\gamma = \arccos(\eta/\sqrt{1+\eta^2}) \in (0, \pi)$. More generally, an unbounded cone with vertex at Q is a set V' isomorphic to the cone V defined above and such that $r(V' - Q) = V' - Q$ for any $r > 0$. For a cone V with vertex at Q we denote by $\mathbf{1}$ a point on the axis of V such that $|\mathbf{1} - Q| = 1$.

Let V be a cone with vertex at 0 and aperture $\gamma \in (0, \pi)$. Assume that $\mathbf{1} = (0, 0, \dots, 0, 1)$. By [1, Theorem 3.2] there exists a so-called *Martin kernel with pole at infinity.* This is a unique nonnegative function M_V on $\mathbf{R}^d$ such that $M_V(\mathbf{1}) = 1$, $M_V \equiv 0$ on V^c, and M_V is regular α-harmonic on every open bounded subset of V. Moreover, M_V is locally bounded on $\mathbf{R}^d$ and homogeneous of degree $\beta \in [0,\ \alpha)$, that is,

$$M_V(x) = |x|^\beta M_V(x/|x|),\ x \in V. \tag{9}$$

Furthermore, $\beta = \beta(V, \alpha)$ is a strictly decreasing function of γ (see [1, Lemma 3.3]). We will call β the *characteristics* of V. If $\gamma = \pi/2$ then V is the half-space $\{(x_1, x_2, \dots, x_d) : x_d > 0\}$. In this case $M_V(x) = x_d^{\alpha/2}, x \in V$, and this gives $\beta = \alpha/2$ (see [1]).

From [25, Lemma 3.3] we have

$$c|x - Q|^{\beta - \alpha/2}\delta_x^{\alpha/2}(V) \le M_V(x) \le C|x - Q|^{\beta - \alpha/2}\delta_x^{\alpha/2}(V), \tag{10}$$

where c, C depend on d, α and the aperture of V.

For $r > 0$ and any cone V with vertex at Q we define a *bounded cone* V_r by $V_r = V \cap B(Q, r)$.

By the properties of Lipschitz domains (see [9]) we know that there exists $R_0 > 0$ such that for every $z \in \partial D$ and every $r \le R_0$ there exist cones Γ, Γ' with vertices at z and such that

$$\Gamma_r \subset D \cap B(z, r), \Gamma'_r \subset D^c \cap B(z, r).$$

We call Γ_r an *inner cone* and Γ'_r an *exterior cone.*

If Γ_r is an exterior cone with its vertex at $z \in \partial D$ then $\widetilde{\Gamma}_r = \overline{\Gamma}_r^c$ is also a cone with its vertex at z. We see that

$$D \cap B(z, r) \subset \widetilde{\Gamma}_r.$$

We will call $\widetilde{\Gamma}_r$ *a covering cone.*

In our subsequent analysis we will often refer to inner cones and covering cones with vertices located at $z \in \partial D$. We denote the characteristics of these cones by β and $\widetilde{\beta}$, respectively. Furthermore, we note that by the Lipschitz property there exists a number γ_0 such that for *all* $z \in \partial D$ there exists an inner cone Γ_{R_0} of aperture γ_0 and vertex at z and a covering cone $\widetilde{\Gamma}_{R_0}$ of aperture $\pi - \gamma_0$ and vertex at z. Simple calculations yield

$$\gamma_0 = \arccos(\lambda/\sqrt{1+\lambda^2}).$$

We will denote the characteristics of these cones by β_0 and $\widetilde{\beta}_0$, respectively.

Because of [1, Lemma 3.3], for any $z \in \partial D$ it suffices to consider those inner bounded cones Γ_r and covering bounded cones $\widetilde{\Gamma}_r$ for which $0 < \widetilde{\beta}_0 \le \widetilde{\beta} \le \beta \le \beta_0 < \alpha$. Note that $\beta, \widetilde{\beta}$ may depend on z, r, while $\beta_0, \widetilde{\beta}_0$ do not depend on z. Furthermore, for any $z \in \partial D$ and $r \le R_0/32$ we may assume that $B(A_{z,r}, \kappa r) \subset \Gamma_r \subset \widetilde{\Gamma}_r$.

We now present some estimates for Martin kernels. We start with the following lemma.

LEMMA 3.1. *Let $r \le R_0$. For $Q \in \partial D$ consider two cones $\Gamma, \widetilde{\Gamma}$ with vertices at Q and characteristics $\beta, \widetilde{\beta}$, respectively, such that Γ_r is an inner bounded cone and $\widetilde{\Gamma}_r$ is a covering bounded cone. Then there exist constants $c = c(Q, \beta, \widetilde{\beta})$ and $C = C(Q, \beta, \widetilde{\beta})$ such that:*

(i) *If x is in $\Gamma_{r/4}$ then*

$$\phi(x) \ge c \frac{|x-Q|^{\beta-\alpha/2} \delta_x^{\alpha/2}(\Gamma)}{r^\beta} \phi(A_{Q,r}).$$

(ii) *If x is in $D \cap B(Q, r/4)$ then*

$$\phi(x) \le C \frac{|x-Q|^{\widetilde{\beta}-\alpha/2} \delta_x^{\alpha/2}(\widetilde{\Gamma})}{r^{\widetilde{\beta}}} \phi(A_{Q,r}) \le C \left(\frac{|x-Q|}{r}\right)^{\widetilde{\beta}} \phi(A_{Q,r}).$$

Proof. We may assume that $x_0 \notin \widetilde{\Gamma}_{2r}$. Set $B_r = \Gamma \cap (B(Q,2r) \setminus B(Q,r))$ and define

$$f_D(x) = P^x(X_{\tau_{D \cap B(Q,r)}} \in B_r),$$
$$f_\Gamma(x) = P^x(X_{\tau_{\Gamma \cap B(Q,r)}} \in B_r),$$
$$f_{\widetilde{\Gamma}}(x) = P^x(X_{\tau_{\widetilde{\Gamma} \cap B(Q,r)}} \in B_r).$$

Obviously, $f_\Gamma \le f_D \le f_{\widetilde{\Gamma}}$ on $\mathbf{R}^d$. Next, the scaling property of the harmonic measures implies that

$$f_{\widetilde{\Gamma}}(A_{Q,r}) = P^{A_{Q,r}}(X_{\tau_{\widetilde{\Gamma} \cap B(Q,r)}} \in B_r) = P^{A_{Q,1}}(X_{\tau_{\widetilde{\Gamma} \cap B(Q,1)}} \in B_1) > 0,$$

and, similarly,

$$f_\Gamma(A_{Q,r}) = P^{A_{Q,1}}(X_{\tau_{\Gamma \cap B(Q,1)}} \in B_1) > 0.$$

Hence $c f_{\widetilde{\Gamma}}(A_{Q,r}) \le f_D(A_{Q,r}) \le C f_\Gamma(A_{Q,r})$, and consequently

$$c \frac{f_\Gamma(x)}{f_\Gamma(A_{Q,r})} \le \frac{f_D(x)}{f_D(A_{Q,r})} \le C \frac{f_{\widetilde{\Gamma}}(x)}{f_{\widetilde{\Gamma}}(A_{Q,r})}, \tag{11}$$

where c, C depend on the apertures of $\Gamma, \widetilde{\Gamma}$, and therefore on $\beta, \widetilde{\beta}$.

The construction of $A_{Q,r}$ implies that

$$\kappa r \leq c|A_{Q,r} - Q| \leq \delta_{A_{Q,r}}(\Gamma) \leq C|A_{Q,r} - Q| \leq r$$

and

$$\kappa r \leq c|A_{Q,r} - Q| \leq \delta_{A_{Q,r}}(\widetilde{\Gamma}) \leq C|A_{Q,r} - Q| \leq r.$$

Therefore, by (10) we get $cr^{\beta} \leq M_{\Gamma}(A_{Q,r}) \leq Cr^{\beta}$ and $cr^{\beta} \leq M_{\widetilde{\Gamma}}(A_{Q,r}) \leq Cr^{\widetilde{\beta}}$. Note that the functions $f_D(\cdot)/f_D(A_{Q,r})$, $f_{\widetilde{\Gamma}}(\cdot)/f_{\widetilde{\Gamma}}(A_{Q,r})$, and $f_{\Gamma}(\cdot)/f_{\Gamma}(A_{Q,r})$ are regular α-harmonic on $D \cap B(Q,r)$, $\widetilde{\Gamma} \cap B(Q,r)$, and $\Gamma \cap B(Q,r)$, respectively, vanish on $D^c \cap B(Q,r)$, $\widetilde{\Gamma}^c \cap B(Q,r)$, and $\Gamma^c \cap B(Q,r)$, respectively, and are equal to 1 at $A_{Q,r}$. Therefore, if $|x - Q| \leq r/4$, using the Boundary Harnack Principle, (8) and (10), we obtain

$$c\frac{f_D(x)}{f_D(A_{Q,r})} \leq \frac{\phi(x)}{\phi(A_{Q,r})} = \frac{G_D(x, x_0)}{G_D(A_{Q,r}, x_0)} \leq C\frac{f_D(x)}{f_D(A_{Q,r})},\ x \in D.$$

In a similar way we get

$$\frac{f_{\Gamma}(x)}{f_{\Gamma}(A_{Q,r})} \geq c\frac{M_{\Gamma}(x)}{M_{\Gamma}(A_{Q,r})} \geq c\frac{|x - Q|^{\beta - \alpha/2}\delta_x^{\alpha/2}(\Gamma)}{r^{\beta}},\ x \in \Gamma_{r/4}$$

and

$$\begin{aligned}\frac{f_{\widetilde{\Gamma}}(x)}{f_{\widetilde{\Gamma}}(A_{Q,r})} &\leq C\frac{M_{\widetilde{\Gamma}}(x)}{M_{\widetilde{\Gamma}}(A_{Q,r})} \leq C\frac{|x - Q|^{\widetilde{\beta} - \alpha/2}\delta_x^{\alpha/2}(\widetilde{\Gamma})}{r^{\widetilde{\beta}}} \\ &\leq C\left(\frac{|x - Q|}{r}\right)^{\widetilde{\beta}},\ x \in D,\end{aligned}$$

which, combined with (11), completes the proof. □

LEMMA 3.2. *Let $r \leq R_0$. For $Q \in \partial D$ consider two cones $\Gamma, \widetilde{\Gamma}$ with vertices at Q and characteristics $\beta, \widetilde{\beta}$, respectively, such that Γ_r is an inner bounded cone and $\widetilde{\Gamma}_r$ is a covering bounded cone. Moreover, let $x \in D, z \in \partial D$ and $|x - Q| \leq |x - z|$. Then there exist constants $c = c(r, D, \beta, \widetilde{\beta})$ and $C = C(r, D, \beta, \widetilde{\beta})$ such that*

$$c\left(\frac{|x - Q|}{|x - z|}\right)^{d - \alpha + 2\beta} \leq \frac{M(x,z)}{M(x,Q)} \leq C\left(\frac{|x - Q|}{|x - z|}\right)^{d - \alpha + 2\widetilde{\beta}}.$$

Proof. By (7) we have

$$c\left(\frac{|x - Q|}{|x - z|}\right)^{d - \alpha} \frac{\phi^2(A_{Q,|x-Q|})}{\phi^2(A_{z,|x-z|})} \leq \frac{M(x,z)}{M(x,Q)} \leq C\left(\frac{|x - Q|}{|x - z|}\right)^{d - \alpha} \frac{\phi^2(A_{Q,|x-Q|})}{\phi^2(A_{z,|x-z|})},$$

so we need to show that

$$c\left(\frac{|x-Q|}{|x-z|}\right)^{\beta} \le \frac{\phi(A_{Q,|x-Q|})}{\phi(A_{z,|x-z|})} \le C\left(\frac{|x-Q|}{|x-z|}\right)^{\widetilde{\beta}}. \tag{12}$$

First assume that $|x-Q| \le |x-z| \le r/4$. Recall that we can choose $A_{Q,|x-Q|}$ so that $B(A_{Q,|x-Q|}, \kappa|x-Q|) \subset \Gamma_r$. Then, using Lemma 3.1 with $r := 4|x-z|$, we obtain

$$\begin{aligned} c\frac{|A_{Q,|x-Q|}-Q|^{\beta-\alpha/2}\delta^{\alpha/2}_{A_{Q,|x-Q|}}(\Gamma)}{(4|x-z|)^{\beta}}\phi(A_{Q,|x-z|}) &\le \phi(A_{Q,|x-Q|}) \\ &\le C\left(\frac{|A_{Q,|x-Q|}-Q|}{(4|x-z|)}\right)^{\widetilde{\beta}}\phi(A_{Q,|x-z|}). \end{aligned} \tag{13}$$

From the definition of $A_{Q,|x-Q|}$ we have

$$\kappa|x-Q| \le \delta_{A_{Q,|x-Q|}}(\Gamma) \le \delta_{A_{Q,|x-Q|}} \le |A_{Q,|x-Q|}-Q| \le |x-Q|$$

and

$$\delta_{A_{z,|x-z|}} \ge \kappa|x-z|, \quad \delta_{A_{Q,|x-z|}} \ge \kappa|x-z|.$$

Combining this with (13) we obtain

$$c\left(\frac{|x-Q|}{|x-z|}\right)^{\beta}\phi(A_{Q,|x-z|}) \le \phi(A_{Q,|x-Q|}) \le C\left(\frac{|x-Q|}{|x-z|}\right)^{\widetilde{\beta}}\phi(A_{Q,|x-z|}). \tag{14}$$

If $|x-Q| \le |x-z|$ then $|z-Q| \le |x-z| + |x-Q| \le 2|x-z|$. Therefore, we get

$$\begin{aligned} |A_{z,|x-z|} - A_{Q,|x-z|}| &\le |A_{z,|x-z|} - z| + |z-Q| + |A_{Q,|x-z|} - Q| \\ &\le 4|x-z| \le 4/\kappa(\delta_{A_{Q,|x-z|}} \wedge \delta_{A_{z,|x-z|}}). \end{aligned} \tag{15}$$

Hence from the Harnack Principle we obtain

$$c\phi(A_{z,|x-z|}) \le \phi(A_{Q,|x-z|}) \le C\phi(A_{z,|x-z|}),$$

and together with (14) we get (12).

Now let $|x-Q| \le r/4 \le |x-z|$. As in the previous case we get, again using Lemma 3.1,

$$c\left(\frac{|x-Q|}{r}\right)^{\beta}\phi(A_{Q,r}) \le \phi(A_{Q,|x-Q|}) \le C\left(\frac{|x-Q|}{r}\right)^{\widetilde{\beta}}\phi(A_{Q,r}). \tag{16}$$

Since $\delta_{A_{z,|x-z|}} \ge \kappa|x-z| \ge \kappa r/4$ we see that

$$0 < \inf\{\phi(x) : \delta_x \ge \kappa r/4\} \le \phi(A_{z,|x-z|}) \le C_{d,\alpha}(R_0/4)^{\alpha-d},$$

and the same estimates hold for $\phi(A_{Q,r})$. This implies that $c\phi(A_{z,|x-z|}) \le \phi(A_{Q,r}) \le C\phi(A_{z,|x-z|})$, where c, C depend on r and D. Moreover,

$|x-z|r/\operatorname{diam}(D) \le r \le 4|x-z|$. Combining this with (16) we again obtain (12).

The case $r/4 \le |x-Q| \le |x-z|$ is similar to the previous one, but simpler as it does not require Lemma 3.1. The proof is complete. □

Recall that $\beta_0, \widetilde{\beta}_0$ are the characteristics of the inner cones and the covering cones that are suitable for all $z \in \partial D$. Therefore, from Lemma 3.2 we obtain the following corollary:

Corollary 3.3. *Let $z, z' \in \partial D$. If $|x-z'| \le |x-z|$ then there exist constants $c = c(R_0, D, \beta_0, \widetilde{\beta}_0)$ and $C = C(R_0, D, \beta_0, \widetilde{\beta}_0)$ such that*

$$c\left(\frac{|x-z'|}{|x-z|}\right)^{d-\alpha+2\beta_0} \le \frac{M(x,z)}{M(x,z')} \le C\left(\frac{|x-z'|}{|x-z|}\right)^{d-\alpha+2\widetilde{\beta}_0}.$$

Note that if D is a $C^{1,1}$ domain, then from [16] we get

$$c\delta_x^{\alpha/2}/|x-z|^d \le M(x,z) \le C\delta_x^{\alpha/2}/|x-z|^d,$$

and Lemma 3.2 gives

$$c\left(\frac{|x-Q|}{|x-z|}\right)^d \le \frac{M(x,z)}{M(x,Q)} \le C\left(\frac{|x-Q|}{|x-z|}\right)^d$$

for all $x \in D$ and $z, Q \in \partial D$. Hence the lemma provides global estimates that are much stronger that those in Corollary 3.3.

Lemma 3.4. *Let $r \le R_0$. For $Q \in \partial D$ consider two cones $\Gamma, \widetilde{\Gamma}$ with vertices at Q and characteristics $\beta, \widetilde{\beta}$, respectively, such that Γ_r is an inner bounded cone and $\widetilde{\Gamma}_r$ is a covering bounded cone. Let $x \in D, z \in \partial D$, and $|z-Q| \le 2|x-z|$. Then there exist constants $c = c(r, D, \beta, \widetilde{\beta})$ and $C = C(r, D, \beta, \widetilde{\beta})$ such that*

$$c\frac{\phi(x)}{|x-z|^{d-\alpha+2\widetilde{\beta}}} \le M(x,z) \le C\frac{\phi(x)}{|x-z|^{d-\alpha+2\beta}}.$$

Furthermore, if $\beta = \beta_0$ and $\widetilde{\beta} = \widetilde{\beta}_0$, then $c = c(R_0, D, \beta_0, \widetilde{\beta}_0)$ and $C = C(R_0, D, \beta_0, \widetilde{\beta}_0)$. In this case the above estimates hold for all $z \in \partial D$, $x \in D$. Finally, by (8), $\phi(x)$ can be replaced by $G_D(x, x_0)$ for x close to ∂D.

Proof. Due to (7) it suffices to show that

$$c|x-z|^\beta \le \phi(A_{z,|x-z|}) \le C|x-z|^{\widetilde{\beta}}. \tag{17}$$

Suppose that $|x-z| \le r/4$. Using Lemma 3.1 and the same arguments as for (14) we obtain

$$c\left(\frac{|x-z|}{r}\right)^\beta \phi(A_{Q,r}) \le \phi(A_{Q,|x-z|}) \le C\left(\frac{|x-z|}{r}\right)^{\widetilde{\beta}} \phi(A_{Q,r}). \tag{18}$$

Since $|z-Q| \le 2|x-z|$, (15) remains true. Hence from the Harnack Principle we obtain $c\phi(A_{z,|x-z|}) \le \phi(A_{Q,|x-z|}) \le C\phi(A_{z,|x-z|})$, and (18) therefore implies (17).

The case $|x-z| \ge r/4$ is analogous to the second case in the proof of Lemma 3.2. The proof is complete. □

4. Main results

In this section we study the behaviour of the ratio of two singular α-harmonic functions. We identify the set of boundary points for which the limit of this ratio exists, either in the usual sense or in the nontangential sense. In Theorem 4.1 we deal with both the ordinary limit and the nontangential limit. In Theorem 4.2, which is the main result of this paper, we investigate nontangential convergence.

We fix two Borel measures μ, ν on $\mathbf{R}^d$, which are finite and concentrated on ∂D, and we define $u(x) = \int_{\partial D} M(x,z)\mu(dz)$, $v(x) = \int_{\partial D} M(x,z)\nu(dz)$. The functions u and v are both (singular) α-harmonic on D (see [9]).

We may represent μ as $d\mu = f d\nu + d\mu_s$, where $f \in L^1(\partial D, \nu)$ and μ_s is singular to ν. Consider all points $Q \in \partial D$ for which

$$\lim_{r\to 0} \frac{\int_{B(Q,r)}(|(f(z)-f(Q)|\nu(dz) + \mu_s(dz))}{\nu(B(Q,r))} = 0. \tag{19}$$

It is well-known that the set of such points Q is of full measure ν.

In this section we prove the following theorems.

Theorem 4.1. *Let $u(x) = \int_{\partial D} M(x,z)\mu(dz)$, $v(x) = \int_{\partial D} M(x,z)\nu(dz)$. Assume that $d\mu = f d\nu$ and that f is continuous at Q. Let $\lim_{x\to Q} v(x)/G_D(x,x_0) = \infty$. Then*

$$\lim_{x\to Q} \frac{u(x)}{v(x)} = f(Q).$$

If we assume that $\lim_{x\to Q} v(x)/G_D(x,x_0) = \infty$ nontangentially, then the limit above must be taken nontangentially.

Remark 1. The condition $\lim_{x\to Q} v(x)/G_D(x,x_0) = \infty$ is essential (see Example 2 below).

Theorem 4.2 (Relative Fatou Theorem). *Let $u(x) = \int_{\partial D} M(x,z)\mu(dz)$, $v(x) = \int_{\partial D} M(x,z)\nu(dz)$. Assume that $d\mu = f d\nu + d\mu_s$, where μ_s is singular to ν. Then for ν-almost every point $Q \in \partial D$ we have*

$$\lim_{x\to Q} \frac{u(x)}{v(x)} = f(Q)$$

as $x \to Q$ nontangentially. More precisely, the convergence holds for every $Q \in \partial D$ such that (19) holds and $\lim_{x\to Q} v(x)/G_D(x,x_0) = \infty$ nontangentially.

To prove these results we need several technical lemmas.

Lemma 4.3. *Let* $Q \in \partial D$ *and* $v(x) = \int_{\partial D} M(x,z)\nu(dz)$. *If* $\lim_{x\to Q} v(x)/G_D(x,x_0) = \infty$ *then for every* $\varepsilon > 0$ *we have*

$$\lim_{x\to Q} \frac{\int_{\partial D\cap\{|z-Q|\geq\varepsilon\}} M(x,z)\mu(dz)}{v(x)} = 0.$$

If we assume that $\lim_{x\to Q} v(x)/G_D(x,x_0) = \infty$ *nontangentially then the limit above must be taken nontangentially.*

Proof. If $|z-Q| \geq \varepsilon$ and $|x-Q| \leq \varepsilon/2$ then $|x-z| \geq \varepsilon/2$. As $G_D(x,x_0) = \phi(x)$ for x close to Q, using Lemma 3.4 we have

$$M(x,z) \leq \frac{CG_D(x,x_0)}{|x-z|^{d-\alpha+2\beta_0}}$$

with $C = C(R_0, D, \beta_0, \widetilde{\beta}_0)$. This implies that

$$\int_{\partial D\cap\{|z-Q|\geq\varepsilon\}} M(x,z)\mu(dz) \leq CG_D(x,x_0)\int_{\partial D\cap\{|z-Q|\geq\varepsilon\}} \frac{\mu(dz)}{|x-z|^{d-\alpha+2\beta_0}}$$

$$\leq \frac{CG_D(x,x_0)\int_{\partial D\cap\{|z-Q|\geq\varepsilon\}}\mu(dz)}{\varepsilon^{d-\alpha+2\beta_0}} \leq C|\mu|G_D(x,x_0).$$

Since $|\mu| < \infty$ we see that

$$\frac{\int_{\partial D\cap\{|z-Q|\geq\varepsilon\}} M(x,z)\mu(dz)}{v(x)} \leq \frac{C|\mu|}{v(x)/G_D(x,x_0)} \to 0$$

as $x \to Q$. The proof is complete. □

Proof of Theorem 4.1. We have

$$\left|\frac{u(x)}{v(x)} - f(Q)\right| \leq \frac{\int_{\partial D}|f(z)-f(Q)|M(x,z)\nu(dz)}{v(x)} = \frac{I_1(x)}{v(x)} + \frac{I_2(x)}{v(x)},$$

where

$$I_1(x) = \int_{\partial D\cap\{|z-Q|\geq\varepsilon\}} |f(z)-f(Q)|M(x,z)\nu(dz),$$
$$I_2(x) = \int_{\partial D\cap\{|z-Q|<\varepsilon\}} |f(z)-f(Q)|M(x,z)\nu(dz).$$

Next, observe that

$$\frac{I_2(x)}{v(x)} \leq \sup\{|f(z)-f(Q)| :\ z\in\partial D, |z-Q|<\varepsilon\}.$$

By the continuity of f, for every $r > 0$ we can choose $\varepsilon > 0$ such that $I_2(x)/v(x) \leq r$. For this ε we have, by Lemma 4.3, $\lim_{x\to Q} I_1(x)/v(x) = 0$. Thus,

$$\limsup_{x\to Q}\left|\frac{u(x)}{v(x)} - f(Q)\right| \leq r$$

for every $r > 0$, which completes the proof. $\square$

The next lemma provides more details on the boundary behaviour of $v(x) = \int_{\partial D} M(x,z)\nu(dz)$ and may be regarded as an α-stable version of [29, Lemma 5.1].

Lemma 4.4. *For ν-almost every point $Q \in \partial D$ we have*

$$\liminf_{x\to Q} v(x) > 0$$

as $x \to Q$ nontangentially.

Proof. Let $|z - Q| < |x - Q|$. Then $|x - z| \leq 2|x - Q|$, so either $|x - z| \leq |x - Q|$ or $(1/2)|x - z| \leq |x - Q| \leq |x - z|$. In each case we can use Corollary 3.3 and we obtain $M(x,z) \geq cM(x,Q)$. This implies that

$$v(x) \geq \int_{\partial D\cap B(Q,|x-Q|)} M(x,z)\nu(dz) \geq cM(x,Q)\nu(B(Q,|x-Q|)).$$

If $x \to Q$ nontangentially then for some C_0 we have

$$\kappa\delta_x \leq \kappa|x - Q| \leq |A_{Q,|x-Q|} - Q| \leq |x - Q| \leq C_0\delta_x.$$

Therefore, by the Harnack Principle (Theorem 2.1) we have

$$cG_D(x,x_0) \leq G_D(A_{Q,|x-Q|},x_0) \leq CG_D(x,x_0),$$

so (7) implies

$$\frac{c}{|x-Q|^{d-\alpha}G_D(A_{Q,|x-Q|},x_0)} \leq M(x,Q) \leq \frac{C}{|x-Q|^{d-\alpha}G_D(A_{Q,|x-Q|},x_0)}.$$

Hence we obtain

$$v(x) \geq \frac{c\nu(B(Q,|x-Q|))}{|x-Q|^{d-\alpha}G_D(A_{Q,|x-Q|},x_0)}. \tag{20}$$

By [8, Lemma 11], there exist constants $c = c(d,\alpha,\lambda)$ and $\rho = \rho(d,\alpha,\lambda) \in (0,1)$ such that for $|x - Q| < R_0$,

$$P^{x_0}(X_{\tau_D} \in B(Q,|x-Q|)) \geq cG_D(A_{Q,\rho|x-Q|/2},x_0)|x-Q|^{d-\alpha}.$$

By the Harnack Principle, for x sufficiently close to Q this implies that

$$P^{x_0}(X_{\tau_D} \in B(Q,|x-Q|)) \geq cG_D(A_{Q,|x-Q|},x_0)|x-Q|^{d-\alpha}.$$

Combining this with (20) we obtain

$$v(x) \geq c\frac{\nu(B(Q,|x-Q|))}{P^{x_0}(X_{\tau_D} \in B(Q,|x-Q|))},$$

or, in other words,

$$\frac{\omega_D^{x_0}(B(Q,|x-Q|))}{\nu(B(Q,|x-Q|))} \geq \frac{c}{v(x)}.$$

By [5, Theorem 5] the symmetric derivative

$$\limsup_{x\to Q} \frac{\omega_D^{x_0}(B(Q,|x-Q|))}{\nu(B(Q,|x-Q|))}$$

is finite for ν-almost every point $Q \in \partial D$. This completes the proof. $\square$

REMARK 2. Since $\lim_{x\to Q} G_D(x,x_0) = 0$ (see [22, Theorem 2.24]), Lemma 4.4 implies that if $x \to Q$ nontangentially then $\lim_{x\to Q} v(x)/G_D(x,x_0) = \infty$ for almost every point Q with respect to ν, so Lemma 4.3 holds for such Q. We will use this result below.

LEMMA 4.5 (Nontangential Maximal Estimate). *For any $x \in D$, $Q \in \partial D$ and $t > 0$ such that $|x-Q| \leq t\delta_x$ there exist constants $C = C(t,Q)$, $c = c(t,Q)$ such that*

$$c \inf_{r>0} \frac{\mu(B(Q,r))}{\nu(B(Q,r))} \leq \frac{\int_{\partial D} M(x,z)\mu(dz)}{\int_{\partial D} M(x,z)\nu(dz)} \leq C \sup_{r>0} \frac{\mu(B(Q,r))}{\nu(B(Q,r))}.$$

Proof. For $n \geq 1$ set $B_n = B(Q, 2^n|x-Q|)$ and $A_n = B_n \setminus B_{n-1}, n \geq 2$. Let n_0 be the smallest index for which $2^{n_0}|x-Q| \geq \operatorname{diam}(D)$. Then we have

$$\int_{\partial D} M(x,z)\mu(dz) = \sum_{n=2}^{n_0} \int_{\partial D\cap A_n} M(x,z)\mu(dz) + \int_{\partial D\cap B_1} M(x,z)\mu(dz). \tag{21}$$

If $z \in B_1$ then $|z-Q| < 2|x-Q|$, which implies

$$\frac{1}{t}|x-Q| \leq \delta_x \leq |x-z| \leq |z-Q| + |x-Q| \leq 3|x-Q|.$$

From Corollary 3.3 we get $cM(x,Q) \leq M(x,z) \leq CM(x,Q)$, so

$$\inf_{z\in B_1} M(x,z) \leq a_1 = \sup_{z\in B_1} M(x,z) \leq Cc^{-1} \inf_{z\in B_1} M(x,z).$$

Therefore we obtain

$$ca_1\mu(B_1) \leq \int_{\partial D\cap B_1} M(x,z)\mu(dz) \leq Ca_1\mu(B_1). \tag{22}$$

Now let $z \in A_n$. We have

$$\begin{aligned}(1/4)\cdot 2^n|x-Q| &\leq ((1/2)\cdot 2^n - 1)|x-Q| \leq |z-Q| - |x-Q| \\ &\leq |x-z| \leq |x-Q| + |z-Q| \leq 2\cdot 2^n|x-Q|.\end{aligned}$$

Therefore, if also $z' \in A_n$, then $|x-z'|/8 \leq |x-z| \leq 8|x-z'|$, so, by Corollary 3.3, $cM(x,z') \leq M(x,z) \leq CM(x,z')$ and

$$\inf_{z\in A_n} M(x,z) \leq a_n = \sup_{z\in A_n} M(x,z) \leq Cc^{-1} \inf_{z\in A_n} M(x,z).$$

Thus we obtain

$$ca_n\mu(A_n) \le \int_{\partial D\cap A_n} M(x,z)\mu(dz) \le Ca_n\mu(A_n). \tag{23}$$

Combining (21), (22) and (23) we see that

$$\begin{aligned} c\left(\sum_{n=2}^{n_0} a_n\mu(A_n) + a_1\mu(B_1)\right) &\le \int_{\partial D} M(x,z)\mu(dz) \\ &\le C\left(\sum_{n=2}^{n_0} a_n\mu(A_n) + a_1\mu(B_1)\right). \end{aligned} \tag{24}$$

Now define $b_n = \sup_{k\ge n} a_k,\ n \ge 1$. Obviously, $b_n \ge a_n$. Let $z' \in A_n$ and $z \in A_k,\ k > n$. Then

$$|x - z'| \le |z' - Q| + |x - Q| \le 2 \cdot 2^n|x - Q|$$

and

$$|x - z| \ge |z - Q| - |x - Q| \ge (2^n - 1)|x - Q| \ge (1/2)\cdot 2^n|x - Q|,$$

so $|x - z'| \le 4|x - z|$. From Corollary 3.3 we obtain $M(x,z) \le CM(x,z')$. This implies that $a_k \le Ca_n$ for $k > n$, so $b_n \le Ca_n$, which means that $b_n/C \le a_n \le b_n,\ n \ge 1$. Combining this with (24) we obtain

(25)

$$\begin{aligned} c\left(\sum_{n=2}^{n_0} b_n\mu(A_n) + b_1\mu(B_1)\right) &\le \int_{\partial D} M(x,z)\mu(dz) \\ &\le C\left(\sum_{n=2}^{n_0} b_n\mu(A_n) + b_1\mu(B_1)\right) \\ &= C\left(\sum_{n=2}^{n_0} b_n(\mu(B_n) - \mu(B_{n-1})) + b_1\mu(B_1)\right) \\ &= C\left(\sum_{n=2}^{n_0} (b_{n-1} - b_n)\mu(B_n) + b_{n_0}\mu(B_{n_0})\right), \end{aligned}$$

and the same estimate holds with μ replaced with ν. By definition, $(b_{n-1}-b_n)$ is nonnegative. Thus we obtain

$$\begin{aligned}
u(x) &= \int_{\partial D} M(x,z)\mu(dz) \\
&\le C\left(\sum_{n=2}^{n_0}(b_{n-1}-b_n)\frac{\mu(B_n)}{\nu(B_n)}\nu(B_n) + b_{n_0}\frac{\mu(B_{n_0})}{\nu(B_{n_0})}\nu(B_{n_0})\right) \\
&\le C\sup_{r>0}\frac{\mu(B(Q,r))}{\nu(B(Q,r))}\left(\sum_{n=2}^{n_0}(b_{n-1}-b_n)\nu(B_n) + b_{n_0}\nu(B_{n_0})\right) \\
&\le C\sup_{r>0}\frac{\mu(B(Q,r))}{\nu(B(Q,r))}\int_{\partial D} M(x,z)\nu(dz) = C\sup_{r>0}\frac{\mu(B(Q,r))}{\nu(B(Q,r))}v(x),
\end{aligned}$$

and similarly,

$$u(x) \ge c\inf_{r>0}\frac{\mu(B(Q,r))}{\nu(B(Q,r))}v(x),$$

which had to be shown. □

Proof of Theorem 4.2. Let $\varepsilon > 0$. Define $d\tilde{\mu} = |f(\cdot)-f(Q)|d\nu + d\mu_s$. Then we have

$$\begin{aligned}
\left|\frac{u(x)}{v(x)} - f(Q)\right| &\le \frac{\int_{\partial D} M(x,z)d\tilde{\mu}(dz)}{v(x)} \\
&= \frac{\int_{\partial D\cap\{|z-Q|\ge\varepsilon\}} M(x,z)d\tilde{\mu}(dz)}{v(x)} \\
&\quad + \frac{\int_{\partial D} M(x,z)d\tilde{\mu}|_{B(Q,\varepsilon)}(dz)}{\int_{\partial D} M(x,z)\nu(dz)},
\end{aligned}\tag{26}$$

where $\tilde{\mu}|_{B(Q,\varepsilon)}$ denotes the truncation of $\tilde{\mu}$ to $B(Q,\varepsilon)$. Since $|f(\cdot)-f(Q)| \in L^1(\nu)$, applying Lemma 4.3 and Remark 2 to the measures $\tilde{\mu}$ and ν, we obtain

$$\lim_{x\to Q}\frac{\int_{\partial D\cap\{|z-Q|\ge\varepsilon\}} M(x,z)\tilde{\mu}(dz)}{v(x)} = 0.$$

Hence, applying Lemma 4.5 to the measures $\tilde{\mu}|_{B(Q,\varepsilon)}$ and ν, using (26) we get

$$\begin{aligned}
\limsup_{x\to Q}\left|\frac{u(x)}{v(x)} - f(Q)\right| &\le \limsup_{x\to Q}\frac{\int_{\partial D} M(x,z)\tilde{\mu}|_{B(Q,\varepsilon)}(dz)}{\int_{\partial D} M(x,z)\nu(dz)} \\
&\le C\sup_{r>0}\frac{\int_{\partial D\cap B(Q,r)}\tilde{\mu}|_{B(Q,\varepsilon)}(dz)}{\nu(B(Q,r))} \\
&= C\sup_{r\le\varepsilon}\frac{\int_{\partial D\cap B(Q,r)}(|f(z)-f(Q)|\nu(dz)+\mu_s(dz))}{\nu(B(Q,r))}.
\end{aligned}$$

Letting $\varepsilon \to 0$ and using (19) we obtain the desired result. □

Example 1. Take $D = B(0,1)$ and $x_0 = 0$. In this case

$$M(x,z) = C(1-|x|^2)^{\alpha/2}/|x-z|^d$$

(see [9, Example 1]). Take $z_1, z_2 \in \partial D,\ z_1 \neq z_2$, and let μ, ν be probabilistic measures concentrated at z_1, z_2 respectively. Then

$$u(x)/v(x) = |x-z_2|^d/|x-z_1|^d,$$

so we see that $\lim_{x\to Q} u(x)/v(x) = |Q-z_2|^d/|Q-z_1|^d < \infty$ if $Q \neq z_1$, but $\lim_{x\to z_1} u(x)/v(x) = \infty$. Hence, by Theorem 4.2, the convergence holds for ν-almost every point $Q \in \partial D$ but not everywhere.

5. The case of the surface measure

In this section we consider the case when $\nu = \sigma$, the $(d-1)$-dimensional Hausdorff measure on ∂D. For this measure we denote the function v by N, i.e.,

$$N(x) = \int_{\partial D} M(x,z)\sigma(dz).$$

We consider the most interesting case when μ is absolutely continuous with respect to σ, i.e., $d\mu = f d\sigma, f \in L^1(\partial D, \sigma)$. This case was investigated for $C^{1,1}$ domains in [12] and some sharp results were obtained in this paper. In particular, from [12, Theorem 3.2 and 4.3] we deduce that $\lim_{x\to Q} \delta_x N(x)/G_D(x_0,x) \in (0,\infty)$, so Lemma 4.4 holds for all $Q \in \partial D$. Hence the condition $\lim_{x\to Q} v(x)/G_D(x_0,x) = \infty$ in Theorem 4.1 holds for all $Q \in \partial D$.

On ∂D, σ is a natural substitute of the $(d-1)$-dimensional Lebesgue measure. This is why we are able to describe how Theorem 4.2 depends on the geometry of ∂D. We will also exhibit some phenomena that are different from those in smooth domains.

We use the following property of σ, called *Ahlfors regular condition*: There exist constants $r_0 = r_0(D,d)$, $c = c(D,d)$ and $C = C(D,d)$ such that for every $z \in \partial D$ and $r \leq r_0$,

$$\begin{aligned} cr^{d-1} &\leq c\sigma(\partial D \cap (B(z,r) \setminus B(z,r/2))) \leq \sigma(\partial D \cap B(z,r)) \\ &\leq C\sigma(\partial D \cap (B(z,r) \setminus B(z,r/2))) \leq Cr^{d-1}. \end{aligned}$$

We begin with a technical lemma.

Lemma 5.1. *For r, $t > 0$ define the function $F_t(r)$ by*

$$F_t(r) = \begin{cases} \dfrac{1}{r^{t-d+1}}, & t-d+1 > 0, \\ 1, & t-d+1 < 0, \\ |\ln r| + 1, & t-d+1 = 0. \end{cases}$$

Then there exist constants $c = c(D, d, t)$, $C = C(D, d, t)$ such that for every $x \in D$ we have

$$cF_t(\delta_x) \le \int_{\partial D} \frac{\sigma(dz)}{|x-z|^t} \le CF_t(\delta_x). \tag{27}$$

Furthermore, if $Q \in \partial D$ and $\mathrm{diam}(D)/2 \ge \varepsilon \ge 2|x-Q|$, then

$$cF_t(\varepsilon) \le \int_{\partial D \cap \{|z-Q| \ge \varepsilon\}} \frac{\sigma(dz)}{|x-z|^t} \le CF_t(\varepsilon). \tag{28}$$

Proof. First we prove (28). We set $B_n = B(Q, 2^n\varepsilon)$ and $A_n = B_n \setminus B_{n-1}$. Let n_0 be the largest index for which $r = 2^{n_0}\varepsilon \le r_0$. Then we have

$$\int_{\partial D \cap \{|z-Q| \ge \varepsilon\}} \frac{\sigma(dz)}{|x-z|^t} = \sum_{n=0}^{n_0} \int_{\partial D \cap A_n} \frac{\sigma(dz)}{|x-z|^t} + \int_{\partial D \setminus B(Q,r)} \frac{\sigma(dz)}{|x-z|^t}. \tag{29}$$

We may assume that $\varepsilon < r_0/2$. Then $|x-Q| \le r_0/4$, so

$$\mathrm{diam}(D)/2 \ge |x-z| \ge |z-Q| - |x-Q| \ge r_0/4.$$

Hence we get

$$c \le \int_{\partial D \setminus B(Q,r)} \frac{\sigma(dz)}{|x-z|^t} \le C, \tag{30}$$

where c, C depend on D, d, t.

If $z \in \partial D \cap A_n$ then $(1/2)\cdot 2^n\varepsilon \le |z-Q| \le 2^n\varepsilon$. Since $|x-Q| \le \varepsilon/2$, we get $(1/4)\cdot 2^n\varepsilon \le |x-z| \le 2\cdot 2^n\varepsilon$. Since $c(2^n\varepsilon)^{d-1} \le \sigma(\partial D \cap A_n) \le C(2^n\varepsilon)^{d-1}$, we obtain

$$c\sum_{n=0}^{n_0}(2^n\varepsilon)^{d-1-t} \le \sum_{n=0}^{n_0} \int_{\partial D \cap A_n} \frac{\sigma(dz)}{|x-z|^t} \le C\sum_{n=0}^{n_0}(2^n\varepsilon)^{d-1-t}. \tag{31}$$

If $t - d + 1 > 0$ then

$$\varepsilon^{d-1-t} \le \sum_{n=0}^{n_0}(2^n\varepsilon)^{d-1-t} \le \varepsilon^{d-1-t}\sum_{n=0}^{\infty}(2^n)^{d-1-t} = C\varepsilon^{d-1-t},$$

which, combined with (29) and (30), gives (28).

By the definition of n_0 we have $r_0/2 \le 2^{n_0}\varepsilon \le r_0$. Hence if $t - d + 1 < 0$ then

$$\begin{aligned}(r_0/2)^{d-1-t} \le (2^{n_0}\varepsilon)^{d-1-t} \le \sum_{n=0}^{n_0}(2^n\varepsilon)^{d-1-t} &= \varepsilon^{d-1-t}\frac{2^{(n_0+1)(d-1-t)}-1}{2^{d-1-t}-1}\\ &\le C(2^{n_0}\varepsilon)^{d-1-t} \le Cr_0^{d-1-t},\end{aligned}$$

which again gives (28).

Finally, if $t-d+1 = 0$ then $\sum_{n=0}^{n_0}(2^n\varepsilon)^{d-1-t} = n_0+1$. Since $r_0/2 \le 2^{n_0}\varepsilon \le r_0$ we see that $c(|\ln(1/\varepsilon)|+1) \le n_0 \le C(|\ln(1/\varepsilon)|+1)$. This completes the proof of (28).

We now prove (27). For any $x \in D$ we can find $Q = Q(x) \in \partial D$ such that $\delta_x = |x - Q|$. We have

$$\int_{\partial D} \frac{\sigma(dz)}{|x-z|^t} = \int_{\partial D \cap \{|z-Q| \geq 2\delta_x\}} \frac{\sigma(dz)}{|x-z|^t} + \int_{\partial D \cap \{|z-Q| < 2\delta_x\}} \frac{\sigma(dz)}{|x-z|^t}.$$

Now, using (28) with $\varepsilon = 2\delta_x$, we see that

$$cF_t(\delta_x) \leq \int_{\partial D \cap \{|z-Q| \geq 2\delta_x\}} \frac{\sigma(dz)}{|x-z|^t} \leq CF_t(\delta_x).$$

We may assume that $2\delta_x \leq r_0$, which gives $c(\delta_x)^{d-1} \leq \sigma(\partial D \cap B(Q, 2\delta_x)) \leq C(\delta_x)^{d-1}$. If $|z - Q| < 2\delta_x$ then $\delta_x \leq |x - z| \leq 4\delta_x$ and we obtain

$$c(\delta_x)^{d-1-t} \leq \int_{\partial D \cap \{|z-Q| < 2\delta_x\}} \frac{\sigma(dz)}{|x-z|^t} \leq C(\delta_x)^{d-1-t}.$$

If $t - d + 1 > 0$ then $(\delta_x)^{d-1-t} = F_t(\delta_x)$; otherwise $0 \leq \delta_x \leq (\mathrm{diam}(D))^{d-1-t}$. This completes the proof of the lemma. □

The next lemma describes the set of points Q for which the assumption of Lemma 4.3 holds.

Lemma 5.2. *Assume that for Q and some $r > 0$ there exists a covering bounded cone $\widetilde{\Gamma}_r$ with its vertex at Q and characteristics $\widetilde{\beta} \geq (\alpha - 1)/2$. Let $N(x) = \int_{\partial D} M(x, z)\sigma(dz)$. Then $\lim_{x \to Q} N(x)/G_D(x, x_0) = \infty$.*

Proof. Let $|z - Q| \geq 2|x - Q|$. Then $|z - Q| \leq |x - z| + |x - Q| \leq |x - z| + |z - Q|/2$, so $|z - Q| \leq 2|x - z|$. Applying Lemma 3.4 and using (28) with $\varepsilon = 2|x - Q|$ we get

$$\begin{aligned} N(x) &\geq \int_{\partial D \cap \{|z-Q| \geq 2|x-Q|\}} M(x,z)\sigma(dz) \\ &\geq cG_D(x, x_0) \int_{\partial D \cap \{|z-Q| \geq 2|x-Q|\}} \frac{\sigma(dz)}{|x-z|^{d-\alpha+2\widetilde{\beta}}} \\ &\geq cG_D(x, x_0) F_{d-\alpha+2\widetilde{\beta}}(|x-Q|). \end{aligned}$$

If $\widetilde{\beta} \geq (\alpha - 1)/2$ then $\lim_{x \to Q} F_{d-\alpha+2\widetilde{\beta}}(|x - Q|) = \infty$, which completes the proof. □

As an immediate consequence of Lemma 5.2 and Theorem 4.1 we obtain the following result:

Theorem 5.3. *Let $u(x) = \int_{\partial D} M(x, z)\mu(dz)$, $N(x) = \int_{\partial D} M(x, z)\sigma(dz)$. Assume that $d\mu = f d\sigma$ and that f is continuous at $Q \in \partial D$. Suppose further* *that for some $r > 0$ there exists a covering cone $\widetilde{\Gamma}_r$ with vertex at Q and characteristics $\widetilde{\beta} \geq (\alpha-1)/2$. Then*

$$\lim_{x\to Q} \frac{u(x)}{N(x)} = f(Q).$$

Remark 3. If $\alpha \leq 1$ then, obviously, $\widetilde{\beta} > (\alpha-1)/2$. If $\alpha > 1$ and $d = 2$ we have again $\widetilde{\beta} > (\alpha-1)/2$ (see [20]). In these cases Theorem 5.3 holds for every $Q \in \partial D$.

Remark 4. Assume that there exists a hyperplane tangent to ∂D at Q. Recall that a half-space has characteristics $\beta = \alpha/2$. Furthermore, β is a continuous function of γ (see [25, Theorem 3.2]). It follows that for sufficiently small $r > 0$ a covering bounded cone $\widetilde{\Gamma}_r$ exists for $\widetilde{\beta}$ close to $\alpha/2 > (\alpha-1)/2$. Therefore, Theorem 5.3 holds for such points Q in the case when $\alpha > 1$ and $d \geq 3$. On the other hand, a Lipschitz function is differentiable almost everywhere (by Rademacher's theorem). Hence the set of such points Q is of full measure σ.

Remark 5. If D is a $C^{1,1}$ domain, then a tangent hyperplane exists for every $Q \in \partial D$, so Theorem 5.3 holds regardless of α, which confirms [12, Theorem 4.2].

The condition $\widetilde{\beta} \geq (\alpha-1)/2$ is essential and cannot be omitted in general, as the following example shows.

Example 2. We construct an example of a domain D, a point $Q \in \partial D$, and a function f continuous on ∂D such that $u(x)/N(x)$ does not tend to $f(Q)$ as $x \to Q$ nontangentially. This shows a difference between Lipschitz domains and $C^{1,1}$ domains.

We take an unbounded cone V with its vertex at Q and characteristics β, and let B_V be a $C^{1,1}$ domain such that

$$(B(0,3/2) \setminus B(0,1/16)) \cap V \subset B_V \subset (B(0,2) \setminus B(0,1/32)) \cap V.$$

We notice that the choice of B_V depends only on d and γ. Next we take $D = B_V \cup (V \cap B(0,1/16))$. From [25, Lemma 3.4] we have for all $x \in D$

$$c\delta_x^{\alpha/2}|x-Q|^{\beta-\alpha/2} \leq \phi(x) \leq C\delta_x^{\alpha/2}|x-Q|^{\beta-\alpha/2}, \tag{32}$$

where $c = c(d,\alpha,\beta), C = C(d,\alpha,\beta)$.

If $x \to Q$ nontangentially, then $|x-Q| \leq C_0|x-z|$ for some C_0. Hence, by the definition of $A_{z,|x-z|}$, we have

$$\begin{aligned} \kappa|x-z| \leq \delta_{A_{z,|x-z|}} &\leq |A_{z,|x-z|} - Q| \\ &\leq |A_{z,|x-z|} - z| + |z-x| + |x-Q| \\ &\leq 2|x-z| + |x-Q| \leq (2+C_0)|x-z|. \end{aligned} \tag{33}$$

Therefore, from (32) and (33) we obtain $c|x-z|^\beta \le \phi(A_{z,|x-z|}) \le C|x-z|^\beta$. Combining this with (7), we see that

$$c\int_{\partial D} \frac{\phi(x)\sigma(dz)}{|x-z|^{d-\alpha+2\beta}} \le N(x) \le C\int_{\partial D} \frac{\phi(x)\sigma(dz)}{|x-z|^{d-\alpha+2\beta}}. \tag{34}$$

Let $\alpha > 1$. Assume that D is 'wide'; more precisely, let $\beta < (\alpha-1)/2$. It is possible to find such β for $d \ge 3$ as $\beta \to 0$ if $\gamma \to \pi$ (see [25, Theorem 3.2]). Then from Lemma 5.1 and (34) we obtain $c\phi(x) \le N(x) \le C\phi(x)$, so we see that the assumption of Lemma 4.3 is not satisfied.

Now let $f > 0$ on $\partial D \setminus \{Q\}$ and $f(Q) = 0$. Then we get

$$\begin{aligned} u(x) = \int_{\partial D} M(x,z)f(z)\sigma(dz) &\ge c\int_{\partial D} \frac{f(z)\phi(x)\sigma(dz)}{\phi^2(A_{z,|x-z|})|x-z|^{d-\alpha}} \\ &\ge C\phi(x)\int_{\partial D} f(z)\sigma(dz) \ge C\phi(x), \end{aligned}$$

since $\phi(A_{z,|x-z|})$ and $|x-z|$ are bounded for every $x \in D$ and $z \in \partial D$. Hence we obtain $u(x)/N(x) \ge C > 0$, so $u(x)/N(x)$ does not tend to $f(Q)$.

We conclude this section with a theorem which deals with general measures that are absolutely continuous with respect to σ and which generalizes the results of [12, Theorem 4.2].

THEOREM 5.4. *Let $u(x) = \int_{\partial D} M(x,z)\mu(dz)$, $N(x) = \int_{\partial D} M(x,z)\sigma(dz)$. Assume that $d\mu = f d\sigma$. Then*

$$\lim_{x\to Q} \frac{u(x)}{N(x)} = f(Q)$$

for almost every $Q \in \partial D$ with respect to σ as $x \to Q$ nontangentially. More precisely, if Q is a Lebesgue point of f and for some $r > 0$ there exists a covering cone $\widetilde{\Gamma}_r$ with its vertex at Q and characteristics $\widetilde{\beta} \ge (\alpha-1)/2$ then the convergence holds.

Proof. Q is a Lebesgue point of f if

$$\lim_{\varepsilon\to 0} \frac{\int_{\partial D\cap B(Q,\varepsilon)} |f(z)-f(Q)|\sigma(dz)}{\sigma(B(Q,\varepsilon))} = 0,$$

so the theorem follows directly from Lemma 5.2 and Theorem 4.2. □

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