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NULL SETS FOR THE CAPACITY ASSOCIATED TO RIESZ KERNELS

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Abstract. We prove that the capacity associated to the signed vector-valued Riesz kernel $\frac{x}{|x|^{1+\alpha}}$ in $\mathbb{R}^n$, $0 < \alpha < n$, $\alpha \notin \mathbb{Z}$, vanishes on compact sets with finite α-Hausdorff measure that satisfy an additional density condition.

1. Introduction

The aim of this paper is to continue the study of the capacity γ_α associated to the signed vector-valued Riesz kernel $x/|x|^{1+\alpha}$ in $\mathbb{R}^n$, which was initiated in the articles [P], [MPV] and [Vo]. Given $0 < \alpha < n$ and a compact set $E \subset \mathbb{R}^n$, one sets

$$\gamma_\alpha(E) = \sup |\langle T, 1\rangle|, \tag{1}$$

where the supremum is taken over all real distributions T supported on E such that $T * \frac{x_i}{|x|^{1+\alpha}}$ is a function in $L^\infty(\mathbb{R}^n)$ and $\|T * \frac{x_i}{|x|^{1+\alpha}}\|_\infty \leq 1$, for $1 \leq i \leq n$.

When $n = 2$ and $\alpha = 1$, γ_1 is comparable to analytic capacity by the main result of [T1]. For each $n \geq 2$ the capacity γ_{n-1} is called Lipschitz harmonic capacity and has been considered in [Par], [MP], [V], and more recently in [Vo], where it was shown to be semiadditive.

It is a remarkable fact that the behaviour of γ_α depends on whether α is an integer or not, as was discovered in [P]. For integer values of α it was proved in [MP] that γ_α and the α-dimensional Hausdorff measure $\mathcal{H}^\alpha$ vanish simultaneously for compact subsets of α-dimensional smooth surfaces. It was shown in [P] that if $0 < \alpha < 1$ and $\mathcal{H}^\alpha(E) < \infty$ then, surprisingly, $\gamma_\alpha(E) = 0$. In the same article it was also shown that this result holds for any non-integer value of α between 0 and n provided that the compact set E is assumed to be Ahlfors-David regular of dimension α. Recall that a closed subset E of $\mathbb{R}^n$

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is said to be Ahlfors-David regular of dimension α if it has locally finite and positive α-dimensional Hausdorff measure in a uniform way, i.e.,

$$C^{-1}r^{\alpha} \leq \mathcal{H}^{\alpha}(E \cap B(x,r)) \leq Cr^{\alpha}, \text{ for } x \in E, \ \ r \leq d(E),$$

where $B(x,r)$ is the open ball centered at x of radius r and $d(E)$ is the diameter of E. Notice that if E is a compact Ahlfors-David regular set of dimension α, then $\mathcal{H}^{\alpha}(E) < \infty$.

The difficulty in extending the result just mentioned from the case $\alpha < 1$ to the case of non-integer values $\alpha > 1$ is due to the fact that the Riesz kernels enjoy a special positivity property for $\alpha \leq 1$, which fails for every α in the range $1 < \alpha < n$ (see [P]). This lack of positivity makes the treatment of the case $1 < \alpha < n$ much more difficult (see [Vo]).

In this paper we take one more step towards the understanding of γ_{α} for non-integer indexes $\alpha > 1$. Our main result replaces the Ahlfors-David regularity assumption by a much weaker density condition. It becomes then more and more plausible that one can get $\gamma_{\alpha}(E) = 0$ from $\mathcal{H}^{\alpha}(E) < \infty$ for all compact sets E and every non-integer α between 0 and n.

Theorem 1 (Main Theorem). *Let $0 < \alpha < n$, $\alpha \notin \mathbb{Z}$, and let $E \subset \mathbb{R}^n$ be a compact set with $\mathcal{H}^{\alpha}(E) < \infty$, such that for almost all $x \in E$*

$$0 < \theta_{*}^{\alpha}(x,E) \leq \theta^{*\alpha}(x,E) < \infty.$$

Then $\gamma_{\alpha}(E) = 0$.

Recall that the quantities $\theta_{*}^{\alpha}(x,E)$ and $\theta^{*\alpha}(x,E)$ are the lower and upper densities of E at x, defined by

$$\theta_{*}^{\alpha}(x,E) = \liminf_{r \to 0} \frac{\mathcal{H}^{\alpha}(E \cap B(x,r))}{r^{\alpha}}$$

and

$$\theta^{*\alpha}(x,E) = \limsup_{r \to 0} \frac{\mathcal{H}^{\alpha}(E \cap B(x,r))}{r^{\alpha}}.$$

The proof of the Main Theorem uses an adaptation of a result of Pajot (see [Pa]) on coverings by Ahlfors-David regular sets. This will take us back to the Ahlfors-David regular case. To perform the reduction we also need to study a positive version of γ_{α} , denoted by $\gamma_{\alpha,+}$. For $0 < \alpha < n$, the capacity $\gamma_{\alpha,+}$ is defined in the same way as γ_{α}, except that the supremum in (1) is taken only over positive measures instead of all distributions. We will show that for $0 < \alpha < n$, $\gamma_{\alpha,+}$ is countably semiadditive, and this will play a role in proving the Main Theorem.

We finally mention that the proof of the Main Theorem, as those of the main results in [P], rely on the basic fact that if E is an α-dimensional Ahlfors-David regular compact set, with α non-integer, then the α-Riesz operator is unbounded on $L^2(\mathcal{H}^{\alpha}_{E})$ (see [Vi]). We do not know how to prove this result for general sets with finite $\mathcal{H}^{\alpha}$ measure and non-integer $\alpha > 1$. Such a result would imply that the conclusion of the Main Theorem holds without any density assumptions.

Throughout the paper, the letter C will stand for an absolute constant that may be different at different occurrences. The notation $A \approx B$ means, as usual, that for some constant C one has $C^{-1}B \leq A \leq CB$.

The plan of the paper is the following. Section 2 contains some preliminary definitions and results that will be used throughout the paper. The semiadditivity of the capacity $\gamma_{\alpha,+}$, for $0 < \alpha < n$, is also proved in this section. In Section 3 we prove the Main Theorem.

2. Preliminaries

2.1. L^2-boundedness of Calderón-Zygmund operators. A function $K(x,y)$ defined on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,y) : x = y\}$ is called a Calderón-Zygmund kernel if the following holds:

(1) $|K(x,y)| \leq C|x-y|^{-\alpha}$ for some $0 < \alpha < n$ (α not necessarily an integer) and some positive constant $C < \infty$.

(2) There exists $0 < \epsilon \leq 1$ such that for some constant $0 < C < \infty$

$$|K(x,y) - K(x_0,y)| + |K(y,x) - K(y,x_0)| \leq C\frac{|x-x_0|^{\epsilon}}{|x-y|^{\alpha+\epsilon}},$$

if $|x - x_0| \leq |x-y|/2$.

Let μ be a Radon measure on $\mathbb{R}^n$. Then the Calderón-Zygmund operator T associated to the kernel K and the measure μ is formally defined as

$$Tf(x) = T(f\mu)(x) = \int K(x,y)f(y)d\mu(y).$$

This integral may not converge for many functions f, because for $x = y$ the kernel K may have a singularity. For this reason, we introduce the truncated operators T_ϵ, $\epsilon > 0$, by

$$T_\epsilon f(x) = T_\epsilon(f\mu)(x) = \int_{|x-y|>\epsilon} K(x,y)f(y)d\mu(y).$$

We say that the singular integral operator T is bounded in $L^2(\mu)$ if the operators T_ϵ are bounded in $L^2(\mu)$ uniformly in ϵ.

The maximal operator T^* is defined as

$$T^*f(x) = \sup_{\epsilon>0} |T_\epsilon f(x)|.$$

Let $0 < \alpha < n$ and consider the Calderón-Zygmund operator R_α associated to the antisymmetric vector-valued Riesz kernel $x/|x|^{1+\alpha}$.

For the proof of our Theorem a deep result of Nazarov, Treil and Volberg will be needed (see [NTV3]). This result was originally proved for the Cauchy transform; the modifications needed to use the result for the operators R_α are explained in [P]. In this way one obtains the following $T(b)$-Theorem for the α-Riesz transform R_α:

Theorem 2. *Let μ be a positive measure on $\mathbb{R}^n$ such that $\limsup_{r\to\infty} \mu(B(x,r))/r^\alpha < +\infty$ for μ almost all x, and let b be an $L^\infty(\mu)$ function such that $|\int b d\mu| = \gamma_\alpha$. Assume that $R_\alpha^* \, b \, (x) < +\infty$ for μ almost all x. Then there is a set F with $\mu(F) \geq \gamma_\alpha/4$ such that the α-Riesz transform R_α is bounded in $L^2(\mu_{|F})$.*

2.2. The capacities $\gamma_{\alpha,+}$ and $\gamma_{\alpha,2}$. Recall that the capacity $\gamma_{\alpha,+}$ of a compact set $E \subset \mathbb{R}^n$ is a variant of γ_α, defined by

$$\gamma_{\alpha,+}(E) = \sup\,\{\mu(E)\},$$

where the supremum is taken over those positive Radon measures μ supported on E and such that for all $1 \leq i \leq n$ the i-th α-Riesz potential $\mu * \frac{x_i}{|x|^{1+\alpha}}$ is a function in $L^\infty(\mathbb{R}^n)$ with $\sup_{1\leq i\leq n} \|\mu * \frac{x_i}{|x|^{1+\alpha}}\|_\infty \leq 1$. We clearly have $\gamma_{\alpha,+}(E) \leq \gamma_\alpha(E)$.

We define now an L^2-version of the capacity $\gamma_{\alpha,+}$. For a compact set $E \subset \mathbb{R}^n$ set

$$\gamma_{\alpha,2}(E) = \sup\,\{\mu(E)\},$$

where the supremum is taken over the positive Radon measures μ supported on E with growth $\mu(B(x,r)) \leq r^\alpha$ for $x \in \operatorname{spt}(\mu)$ and $r > 0$, and such that for $1 \leq i \leq n$ the α-Riesz transform R_α^i is bounded on $L^2(\mu)$ with L^2-norm smaller than 1.

We show now that these two capacities are comparable.

Lemma 3. *For $E \subset \mathbb{R}^n$, $\gamma_{\alpha,+}(E) \approx \gamma_{\alpha,2}(E)$.*

For the proof of Lemma 3, we need the following result (see Lemma 4.2 in [MP]) that tells us how to dualize a weak type $(1,1)$-inequality for several linear operators. The result is a modification of Theorem 23 in [Ch] (see also [U]).

Let X be a locally compact Hausdorff space and denote by $\mathcal{M}(X)$ the space of all finite signed Radon measures on X equipped with the total variation norm. For any $T : \mathcal{M}(X) \to \mathcal{C}(X)$ bounded and linear, denote by $T^t : \mathcal{M}(X) \to \mathcal{C}(X)$ its transpose, that is,

$$\int (T\nu_1) d\nu_2 = \int (T^t \nu_2) d\nu_1 \text{ for } \nu_1,\ \nu_2 \in \mathcal{M}(X).$$

Lemma 4 ([MP]). *Let μ be a positive Radon measure on a locally compact Hausdorff space X and let $T_i : \mathcal{M}(X) \to C(X)$, $1 \leq i \leq n$, be bounded linear operators. Suppose that every T_i^t is of weak type $(1,1)$ with respect to μ, that is, there exists a constant $A < \infty$ such that*

$$\mu(\{x : \ |T_i^t \nu(x)| > t\}) \leq A t^{-1} \|\nu\|$$

for $1 \le i \le n$, $t > 0$, *and* $\nu \in \mathcal{M}(X)$. *Then for* $\tau > 0$ *and any Borel set* $E \subset X$ *with* $0 < \mu(E) < \infty$ *there exists* $h : X \to [0,1]$ *in* $L^\infty(\mu)$ *satisfying* $h(x) = 0$ *for* $x \in X \setminus E$,

$$\int_E h d\mu \ge \mu(E)/2 \quad \text{and} \quad \|T_i(hd\mu)\|_\infty \le (n+\tau)A \quad \text{for} \quad 1 \le i \le n.$$

Proof of Lemma 3. We have to prove that for some positive constants a and b

$$a\gamma_{\alpha,+}(E) \le \gamma_{\alpha,2}(E) \le b\gamma_{\alpha,+}(E). \tag{2}$$

For the second inequality in (2), let σ be a positive measure supported on E such that $\sigma(B(x,r)) \le r^\alpha$ for $x \in \operatorname{spt}(\sigma)$ and $r > 0$, R^i_α is bounded on $L^2(\sigma)$ with operator norm smaller than 1, $1 \le i \le n$, and $\sigma(E) \ge \gamma_{\alpha,2}(E)/2$.

From the L^2-boundedness of R^i_α, $1 \le i \le n$, we get that each R^i_α is of weak type $(1,1)$ with respect to the measure σ. This follows from the standard Calderón-Zygmund theory if the measure is doubling, and by an argument given in [NTV2] in the general case.

We would like to dualize this weak type $(1,1)$ inequality applying Lemma 4. Unfortunately, Lemma 4 does not apply to the truncated operators $(R^i_\alpha)_\epsilon$, because they do not map $\mathcal{M}(E)$ to $\mathcal{C}(E)$. This difficulty can be overcome by using the following regularized operators. For $\epsilon > 0$ and $1 \le i \le n$ define

$$R^\psi_{i,\epsilon}\nu(x) = \int \psi\left(\frac{x-y}{\epsilon}\right) \frac{x_i - y_i}{|x-y|^{1+\alpha}} d\nu(y)$$

for Radon measures ν on $\mathbb{R}^n$, and for $f \in L^1(\sigma)$ define

$$R^\psi_{i,\epsilon}(f\sigma)(x) = \int \psi\left(\frac{x-y}{\epsilon}\right) \frac{x_i - y_i}{|x-y|^{1+\alpha}} f(y) d\sigma(y),$$

where $\psi \in \mathcal{C}^\infty(\mathbb{R}^n)$ is some radial function on $\mathbb{R}^n$ with $0 \le \psi \le 1$, $\psi = 0$ on $B(0,1/2)$ and $\psi = 1$ on $\mathbb{R}^n \setminus B(0,1)$.

Set $R_{i,\epsilon} = (R^i_\alpha)_\epsilon$. Notice that for $\epsilon > 0$ and $x \in \mathbb{R}^n$ we have

$$|R^\psi_{i,\epsilon}\nu(x) - R_{i,\epsilon}\nu(x)| \le C\widetilde{M}_\sigma\nu(x), \tag{3}$$

where

$$\widetilde{M}_\sigma\nu(x) = \sup_{r>0} \frac{\nu(B(x,r))}{\sigma(B(x,3r))}$$

is the *modified maximal operator* introduced in [NTV2, pp. 6–7]. Notice that if the measure σ is doubling, then $\widetilde{M}_\sigma\nu \approx M_\sigma\nu$, with constants depending only on those involved in the doubling condition. Here

$$M_\sigma\nu(x) = \sup_{r>0} \frac{\nu(B(x,r))}{\sigma(B(x,r))}$$

is the *centered Hardy-Littlewood maximal operator*.

By Lemma 3.1 in [NTV2] the operator $\widetilde{M}_\sigma \nu$ satisfies a weak (1,1)-inequality with respect to σ,

$$\sigma(\{x \in E : \ \widetilde{M}_\sigma \nu(x) > t\}) \leq Ct^{-1}\|\sigma\| \ , \text{for } \nu \in \mathcal{M}(E). \tag{4}$$

It follows from (4) and (3) that if $R_{i,\epsilon}$ satisfies a weak type $(1,1)$-inequality, so does $R^{\psi}_{i,\epsilon}$ and vice versa. The advantage of the operators $R^{\psi}_{i,\epsilon}$ is that they do map $\mathcal{M}(E)$ to $\mathcal{C}(E)$, so we may apply Lemma 4 to them instead of $R_{i,\epsilon}$. Observe that $(R^{\psi}_{i,\epsilon})^t = -R^{\psi}_{i,\epsilon}$. Thus for any compact set K in E with $0 < \sigma(K) < \infty$ we can find for each $\epsilon > 0$ a function h_ϵ supported on K and satisfying

$$0 \leq h_\epsilon(x) \leq 1 \text{ for all } x, \tag{5}$$

$$\int_K h_\epsilon d\sigma \geq \sigma(K)/2$$

and

$$\|R^{\psi}_{i,\epsilon}(h_\epsilon \sigma)\|_{L^\infty(K)} \leq 2nA. \tag{6}$$

In view of (3), (5), (6) and the growth condition $\sigma(B(x,r)) \leq C_0 r^\alpha$ for $x \in \operatorname{spt}(\sigma)$ and $r > 0$, we have $\|R_{i,\epsilon}(h_\epsilon\sigma)\|_{L^\infty(K)} \leq C$. But we also want $R_{i,\epsilon}(h_\epsilon\sigma)$ to be bounded outside of K.

We claim now that for all $\eta > \epsilon$ we have $\|R_{i,\eta}(h_\epsilon\sigma)\|_{L^\infty(K)} \leq C$. To see this, let first $\epsilon \leq \eta \leq 2\epsilon$. Then using (3), (5), (6) and the growth condition for σ, we have

$$\begin{aligned}\|R_{i,\eta}(h_\epsilon\sigma)\|_{L^\infty(K)} \leq \|R_{i,\eta}(h_\epsilon\sigma) - R_{i,\epsilon}(h_\epsilon\sigma)\|_{L^\infty(K)} \\ + \|R_{i,\epsilon}(h_\epsilon\sigma)\|_{L^\infty(K)} \leq C.\end{aligned}$$

If $\eta > 2\epsilon$, then $R_{i,\eta} = (R^{\psi}_{i,\epsilon})_\eta$. Using (5) and (6), Cotlar's inequality (see Theorem 7.1 in [NTV2]) implies that the maximal operator $(R^{\psi}_{i,\epsilon})^*(h_\epsilon\sigma)$ is uniformly bounded on K. Hence for all $\eta > 2\epsilon$,

$$\|R_{i,\eta}(h_\epsilon\sigma)\|_{L^\infty(K)} = \|(R^{\psi}_{i,\epsilon})_\eta(h_\epsilon\sigma)\|_{L^\infty(K)} \leq \|(R^{\psi}_{i,\epsilon})^*(h_\epsilon\sigma)\|_{L^\infty(K)} \leq C.$$

Thus the operators $R_{i,\eta}(h_\epsilon\sigma)$ are uniformly bounded in ϵ and η.

Let $\{\epsilon_j\}_j$ be an arbitrary sequence tending monotonically to 0 and let h be a weak-star limit of some subsequence of $\{h_{\epsilon_j}\}$ in $L^\infty(K)$; by passing to some subsequence we may assume that $h_{\epsilon_j} \to h$ in the weak-star topology. Then h is supported on K, $0 \leq h \leq 1$, $\int h d\sigma \geq C\sigma(K)$ and $\|R_{i,\eta}(h\sigma)\|_{L^\infty(K)} \leq C$ uniformly in η.

If we can prove that $\|R_{i,\epsilon}(h\sigma)\|_{L^\infty(K^c)} \leq C$, then we are done with the lower inequality in (2) because $\mu = h\sigma$ is an admissible measure for $\gamma_{\alpha,+}$ and so we have

$$\gamma_{\alpha,+}(E) \geq \int_E h d\sigma \geq C\sigma(E) \geq C\gamma_{\alpha,2}(E).$$

Consider any $x \in \mathbb{R}^n \setminus K$, set $d = \operatorname{dist}(x, K)$ and choose $y \in K$ so that $d = |x - y|$. Fix $\epsilon > 0$ and distinguish the following three cases:

(1) If $\epsilon \geq 4d$, then

$$|R_{i,\epsilon}(h\sigma)(x)| \leq |R_{i,\epsilon}(h\sigma)(x) - R_{i,\epsilon}(h\sigma)(y)| + \|R_{i,\epsilon}(h\sigma)\|_{L^\infty(K)}$$

and

$$\begin{aligned}
&|R_{i,\epsilon}(h\sigma)(x) - R_{i,\epsilon}(h\sigma)(y)| \\
&\quad \leq \left| \int_{\{w:\ |w-x|>\epsilon,\ |w-y|>\epsilon\}} h(w) \left(\frac{x_i - w_i}{|x-w|^{1+\alpha}} - \frac{y_i - w_i}{|y-w|^{1+\alpha}} \right) d\sigma(w) \right| \\
&\qquad + \left| \int_{\{w:\ |w-y|\leq\epsilon,\ |w-x|>\epsilon\}} h(w) \frac{x_i - w_i}{|x-w|^{1+\alpha}} d\sigma(w) \right| \\
&\qquad + \left| \int_{\{w:\ |w-x|\leq\epsilon,\ |w-y|>\epsilon\}} h(w) \frac{y_i - w_i}{|y-w|^{1+\alpha}} d\sigma(w) \right| \\
&\quad = A + B + C.
\end{aligned}$$

To deal with A, note that $|y - w| > \epsilon \geq 4d = 4|x - y| \geq 2|x - y|$. Hence using the standard estimates for the Calderón-Zygmund kernels, $0 \leq h \leq 1$ and the α-growth of σ we get

$$\begin{aligned}
A &\leq C \sum_{j=0}^{\infty} \int_{\{w:\ 2^j\epsilon\leq|y-w|\leq 2^{j+1}\epsilon\}} \frac{|x-y|}{|y-w|^{1+\alpha}} |h(w)| d\sigma(w) \\
&\leq Cd \sum_{j=0}^{\infty} \frac{1}{(2^j\epsilon)^{1+\alpha}} \int_{\{|y-w|\leq 2^{j+1}\epsilon\}} |h(w)| d\sigma(w) \\
&\leq C \frac{d}{\epsilon} \sup_{r>0} \frac{1}{r^\alpha} \int_{|y-w|<r} |h(w)| d\sigma(w) \sum_{j=1}^{\infty} 2^{-j} \leq C,
\end{aligned}$$

where the last inequality comes from the α-growth of the measure σ and the boundedness of h.

For the term B we have

$$B \leq \frac{1}{\epsilon^\alpha} \int_{|w-y|\leq\epsilon} |h(w)| d\sigma(w) \leq C.$$

Term C is treated in the same way as B, but with the roles of x and y interchanged.

(2) If $d/2 \leq \epsilon < 4d$, then

$$\begin{aligned}
|R_{i,\epsilon}(h\sigma)(x)| &\leq |R_{i,4d}(h\sigma)(x)| \ + \ |R_{i,\epsilon}(h\sigma)(x) - R_{i,4d}(h\sigma)(x)| \\
&\leq C + C \sup_{r>0} \frac{1}{r^\alpha} \int_{B(y,r)} |h(w)| d\sigma(w) \leq C,
\end{aligned}$$

by using the previous case to bound $|R_{i,4d}(h\sigma)(x)|$ and the α-growth condition on σ and $0 \le h \le 1$ to bound the difference $|R_{i,\epsilon}(h\sigma)(x) - R_{i,4d}(h\sigma)(x)|$.

(3) If $\epsilon < d/2$, then $R_{i,\epsilon}(h\sigma)(x) = R_{i,d/2}(h\sigma)(x)$, which leads us to the second case.

For the first inequality in (2), let σ be a positive measure supported on E such that $\sigma(E) \ge \frac{\gamma_{\alpha,+}(E)}{2}$ and $\|\sigma * \frac{x_i}{|x|^{1+\alpha}}\|_\infty \le 1$, $1 \le i \le n$.

To see that σ is admissible for $\gamma_{\alpha,2}$, we check first that it satisfies the growth condition $\sigma(B(x,r)) \le Cr^\alpha$. Take an infinitely differentiable function φ, supported on $B(x,2r)$ such that $\varphi = 1$ on $B(x,r)$, and $\|\partial^s\varphi\|_\infty \le C_s r^{-|s|}$, $|s| \ge 0$. Here $s = (s_1, \ldots, s_n)$, with $0 \le s_i \in \mathbb{Z}$, $|s| = s_1 + s_2 + \cdots + s_n$ and $\partial^s = (\partial/\partial x_i)^{s_1} \ldots (\partial/\partial x_n)^{s_n}$. Assume first that n is odd and of the form $n = 2k+1$. Then, by Lemma 11 in [P],

$$
\begin{aligned}
\sigma(B(x,r)) \le \int \varphi d\sigma &= c_{n,\alpha} \int \left(\sum_{i=1}^{n} \Delta^k \partial_i \varphi * \frac{1}{|x|^{n-\alpha}} * \frac{x_i}{|x|^{1+\alpha}} \right)(y) d\sigma(y) \\
&= -c_{n,\alpha} \sum_{i=1}^{n} \int \left(\sigma * \frac{x_i}{|x|^{1+\alpha}} \right)(y) \left(\Delta^k \partial_i \varphi * \frac{1}{|x|^{n-\alpha}} \right)(y) dy \\
&\le C \left\{ \sum_{i=1}^{n} \int_{B(x,3r)} \left| \left(\Delta^k \partial_i \varphi * \frac{1}{|x|^{n-\alpha}} \right)(y) \right| dy \right. \\
&\qquad \left. + \int_{\mathbb{R}^n \setminus B(x,3r)} \left| \left(\Delta^k \partial_i \varphi * \frac{1}{|x|^{n-\alpha}} \right)(y) \right| dy \right\}.
\end{aligned}
$$

Arguing as in Lemma 12 in [P] we get that the last two integrals can be estimated by Cr^α.

When n is even we use the corresponding representation formula in Lemma 11 of [P].

We are left now with the L^2-boundedness of the α-Riesz transform R^i_α for $i = 1, \cdots, n$. By assumption $\|\sigma * \frac{x_i}{|x|^{1+\alpha}}\|_\infty \le 1$ for $1 \le i \le n$. In particular, this implies that we can apply the $T(1)$ Theorem (Theorem 2 with $b \equiv 1$) and so we get the L^2-boundedness of R^i_α for $1 \le i \le n$. This means that σ is admissible for $\gamma_{\alpha,2}$. Thus

$$\gamma_{\alpha,2}(E) \ge C\sigma(E) \ge C\gamma_{\alpha,+}(E),$$

which finishes the proof of the lemma. □

From this lemma we can deduce the semiadditivity of the capacity $\gamma_{\alpha,+}$. In fact, $\gamma_{\alpha,+}$ is countably semiadditive.

Corollary 5. *Let $E \subset \mathbb{R}^n$ be compact. Let E_i, $i \geq 1$, be Borel sets such that $E = \bigcup_{i=1}^{\infty} E_i$. Then*

$$\gamma_{\alpha,+}(E) \leq C \sum_{i=1}^{\infty} \gamma_{\alpha,+}(E_i),$$

where C is some absolute constant.

Proof. Let μ be an admissible measure for $\gamma_{\alpha,2}(E)$. Then using Lemma 3 and the fact that the measures $\mu_{|E_i}$ are admissible for the capacity $\gamma_{\alpha,2}(E_i)$, we obtain

$$\begin{aligned}\gamma_{\alpha,+}(E) \approx \gamma_{\alpha,2}(E) \approx \mu(E) = \mu\left(\bigcup_i E_i\right) \leq \sum_i \mu(E_i) \\ \leq C \sum_i \gamma_{\alpha,2}(E_i) \approx \sum_i \gamma_{\alpha,+}(E_i).\end{aligned}$$

□

3. Proof of the Main Theorem

We need the following result, which is inspired by a theorem of H. Pajot (see Proposition 4.4 in [Pa]). Pajot's result says that under a certain density condition every compact set of $\mathbb{R}^n$ with finite $\mathcal{H}^\alpha$-measure can be covered by a countable union of α-dimensional Ahlfors-David regular sets. Pajot proved the result for sets in $\mathbb{R}^n$ of integer dimension α, but with some minor changes in the proof the same result holds also for sets in $\mathbb{R}^n$ of non-integer dimension α with $0 < \alpha < n$. That is, we have:

Theorem 6. *Let $E \subset \mathbb{R}^n$ be a compact set with $\mathcal{H}^\alpha(E) < \infty$ such that for almost all $x \in E$*

$$0 < \theta_*^\alpha(x, E) \leq \theta^{*\alpha}(x, E) < \infty.$$

Then

$$E \subset \bigcup_{i=0}^{\infty} E_i,$$

where $\mathcal{H}^\alpha(E_0) = 0$ and for all $i \in \mathbb{N}$, E_i are compact Ahlfors-David regular sets of dimension α.

Proof of the Main Theorem. Suppose $\gamma_\alpha(E) > 0$. Applying Lemma 8 in [P] we find a measure of the form $\nu = b\mathcal{H}^\alpha$, with $b \in L^\infty(\mathcal{H}^\alpha, E)$ such that the signed α-Riesz potential $R_\alpha(\nu) = \nu * \frac{x}{|x|^{1+\alpha}}$ is in $L^\infty(\mathbb{R}^n)$ and $\int_E b \, d\mathcal{H}^\alpha = \gamma_\alpha(E)$. We can apply now Theorem 2 to get a set $F \subset E$ of positive $\mathcal{H}^\alpha$-measure such that the operator R_α is bounded on $L^2(\mathcal{H}^\alpha, F)$. This implies that $\gamma_{\alpha,2}(E) > 0$. By Lemma 3, $\gamma_{\alpha,+}(E) > 0$. From Theorem 6 one can deduce that

$$E \subset \bigcup_{i=0}^{\infty} E_i,$$

where $\mathcal{H}^\alpha(E_0) = 0$ and for $i \geq 1$ the sets E_i are α-dimensional compact Ahlfors-David regular sets.

Since sets with zero $\mathcal{H}^\alpha$ measure have zero γ_α capacity (see Lemma 12 in [P]), we have $\gamma_{\alpha,+}(E_0) = 0$.

The semiadditivity of the capacity $\gamma_{\alpha,+}$, stated in Corollary 5 implies then that

$$0 < \gamma_{\alpha,+}(E) \leq C \sum_{i=1}^{\infty} \gamma_{\alpha,+}(E_i).$$

Therefore, for some $k \neq 0$, $\gamma_{\alpha,+}(E_k) > 0$. For this set E_k we then have

$$0 < \gamma_{\alpha,+}(E_k) \leq \gamma_\alpha(E_k).$$

Applying now Theorem 2 in [P] to the Ahlfors-David regular set E_k, we get that α must be an integer. □

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