Illinois Journal of Mathematics
Volume 48, Number 3, Fall 2004, Pages 945–952
S 0019-2082

MODULES OF G-DIMENSION ZERO OVER LOCAL RINGS OF DEPTH TWO

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ABSTRACT. Let R be a commutative noetherian local ring. Denote by $\operatorname{mod} R$ the category of finitely generated R-modules, and by $\mathcal{G}(R)$ the full subcategory of $\operatorname{mod} R$ consisting of all R-modules of G-dimension zero. Suppose that R is henselian and non-Gorenstein, and that there is a non-free R-module in $\mathcal{G}(R)$. Then it is known that $\mathcal{G}(R)$ is not contravariantly finite in $\operatorname{mod} R$ if R has depth at most one. In this paper, we prove that the same statement holds if R has depth two.

1. Introduction

Throughout the present paper, we assume that all rings are commutative noetherian rings and all modules are finitely generated modules.

Auslander [1] has introduced a homological invariant for modules, which is called Gorenstein dimension, or G-dimension for short. This invariant has a lot of properties similar to those of projective dimension. For example, it is well-known that the finiteness of projective dimension characterizes the regular property of the base ring: any module over a regular local ring has finite projective dimension, and a local ring whose residue class field has finite projective dimension is regular. The finiteness of G-dimension characterizes the Gorenstein property of the base ring.

Over a Gorenstein local ring, a module has G-dimension zero if and only if it is a maximal Cohen-Macaulay module. Hence it is natural to expect that modules of G-dimension zero over an arbitrary local ring behave similarly to maximal Cohen-Macaulay modules over a Gorenstein local ring.

A Cohen-Macaulay local ring is said to be of finite Cohen-Macaulay representation type if it has only finitely many non-isomorphic indecomposable maximal Cohen-Macaulay modules. Under a few assumptions, Gorenstein local rings of finite Cohen-Macaulay representation type have been classified completely, and it is known that all non-isomorphic indecomposable maximal

Received December 3, 2003; received in final form July 9, 2004.
2000 *Mathematics Subject Classification.* Primary 13D05. Secondary 16G60.

Cohen-Macaulay modules over them can be described concretely; see [7] for the details.

Thus we are interested in non-Gorenstein local rings which have only finitely many non-isomorphic indecomposable modules of G-dimension zero, and particularly in determining all non-isomorphic indecomposable modules of G-dimension zero over such rings.

Now, we form the following conjecture:

CONJECTURE 1.1. Let R be a non-Gorenstein local ring. Suppose that there exists a non-free R-module of G-dimension zero. Then there exist infinitely many non-isomorphic indecomposable R-modules of G-dimension zero.

This conjecture is against our expectation that modules of G-dimension zero over an arbitrary local ring behave similarly to maximal Cohen-Macaulay modules over a Gorenstein local ring. Indeed, let S be a d-dimensional non-regular Gorenstein local ring of finite Cohen-Macaulay representation type. (Such a ring does exist; see [7].) Then the dth syzygy module of the residue class field of S is a non-free maximal Cohen-Macaulay S-module. Hence the above conjecture does not necessarily hold without the assumption that R is non-Gorenstein.

For a local ring R, we denote by $\operatorname{mod} R$ the category of finitely generated R-modules, and by $\mathcal{G}(R)$ the full subcategory of $\operatorname{mod} R$ consisting of all R-modules of G-dimension zero. We conjecture that even the following statement that is stronger than Conjecture 1.1 is true. (It can be seen from the proof of [5, Theorem 2.9] that Conjecture 1.2 implies Conjecture 1.1.)

CONJECTURE 1.2. Let R be a non-Gorenstein local ring. Suppose that there exists a non-free R-module in $\mathcal{G}(R)$. Then the category $\mathcal{G}(R)$ is not contravariantly finite in $\operatorname{mod} R$.

In [4] and [5], it is proved that Conjecture 1.2 is true if R is henselian and has depth at most one:

THEOREM 1.3 ([4, Theorem 1.2], [5, Theorem 2.8]). *Let $(R, \mathfrak{m}, k)$ be a henselian non-Gorenstein local ring. Suppose that there exists a non-free R-module in $\mathcal{G}(R)$. If the depth of R is zero (resp. one), then k (resp. $\mathfrak{m}$) does not admit a $\mathcal{G}(R)$-precover, and hence $\mathcal{G}(R)$ is not contravariantly finite in* $\operatorname{mod} R$.

The purpose of this paper is to prove that Conjecture 1.2 is true if R is henselian and has depth two:

THEOREM 1.4. *Let R be a henselian non-Gorenstein local ring of depth two. Suppose that there exists a non-free R-module in $\mathcal{G}(R)$. Then the category $\mathcal{G}(R)$ is not contravariantly finite in* $\operatorname{mod} R$.

Under the assumptions of Theorem 1.4, take a non-split exact sequence

$$0 \to R \to M \to \mathfrak{m} \to 0,$$

where $\mathfrak{m}$ is the unique maximal ideal of R. (Such an exact sequence exists because $\operatorname{Ext}_R^1(\mathfrak{m}, R) \neq 0$.) Then it can be proved that the R-module M does not admit a $\mathcal{G}(R)$-precover, and hence $\mathcal{G}(R)$ is not contravariantly finite in $\operatorname{mod} R$.

In Section 2, we will state some definitions and auxiliary results necessary to prove the theorem. The proof of the theorem is given in Section 3.

2. Background material

In this section, we provide some background material. Throughout this section, let $(R, \mathfrak{m}, k)$ be a commutative noetherian local ring. All R-modules in this section are assumed to be finitely generated.

First, we recall the definition of G-dimension. We denote by $\operatorname{mod} R$ the category of finitely generated R-modules. Put $M^* = \operatorname{Hom}_R(M, R)$ for an R-module M.

Definition 2.1.

(1) We denote by $\mathcal{G}(R)$ the full subcategory of $\operatorname{mod} R$ consisting of all R-modules M satisfying the following three conditions.
   (i) The natural homomorphism $M \to M^{**}$ is an isomorphism.
   (ii) $\operatorname{Ext}_R^i(M, R) = 0$ for every $i > 0$.
   (iii) $\operatorname{Ext}_R^i(M^*, R) = 0$ for every $i > 0$.

(2) Let M be an R-module. If n is a non-negative integer such that there is an exact sequence

$$0 \to G_n \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0$$

of R-modules with $G_i \in \mathcal{G}(R)$ for every i, $0 \leq i \leq n$, then we say that M has *G-dimension at most* n, and write $\operatorname{G-dim}_R M \leq n$. If such an integer n does not exist, then we say that M has *infinite G-dimension*, and write $\operatorname{G-dim}_R M = \infty$.

Of course, if an R-module M has G-dimension at most n, but does not have G-dimension at most $n-1$, then we say that M has G-dimension n and write $\operatorname{G-dim}_R M = n$.

Let M be an R-module. We denote by $\Omega^n M$ the nth syzygy module of M, and set $\Omega M = \Omega^1 M$. If $F_1 \xrightarrow{\partial} F_0 \to M \to 0$ is the minimal free presentation of M, then we denote by $\operatorname{Tr} M$ the cokernel of the dual homomorphism $\partial^* : F_0^* \to F_1^*$. G-dimension is a homological invariant for modules sharing a lot of properties with projective dimension. We state here just those properties that will be used later.

Proposition 2.2.

(1) *The following conditions are equivalent.*
   (i) *R is Gorenstein.*
   (ii) $\mathrm{G}\text{-}\dim_R M < \infty$ *for any R-module M.*
   (iii) $\mathrm{G}\text{-}\dim_R k < \infty$.
(2) *Let M, N be R-modules. Then* $\mathrm{G}\text{-}\dim_R(M \oplus N) = \sup\{\mathrm{G}\text{-}\dim_R M, \mathrm{G}\text{-}\dim_R N\}$.
(3) *If an R-module M belongs to $\mathcal{G}(R)$, then so do M^*, ΩM, $\mathrm{Tr}\, M$, and any direct summand of M.*
(4) *Let $0 \to L \to M \to N \to 0$ be an exact sequence of R-modules. If L and N belong to $\mathcal{G}(R)$, then so does M.*

The proof of this proposition and other properties of G-dimension are stated in detail in [2, Chapter 3,4] and [3, Chapter 1].

Now we introduce the notion of a cover of a module.

Definition 2.3. Let $\mathcal{X}$ be a full subcategory of $\mathrm{mod}\, R$.

(1) Let $\phi : X \to M$ be a homomorphism from $X \in \mathcal{X}$ to $M \in \mathrm{mod}\, R$.
   (i) We call ϕ an *$\mathcal{X}$-precover* of M if for any homomorphism $\phi' : X' \to M$ with $X' \in \mathcal{X}$ there exists a homomorphism $f : X' \to X$ such that $\phi' = \phi f$.
   (ii) Assume that ϕ is an $\mathcal{X}$-precover of M. We call ϕ an *$\mathcal{X}$-cover* of M if any endomorphism f of X with $\phi = \phi f$ is an automorphism.
(2) The category $\mathcal{X}$ is said to be *contravariantly finite* if every $M \in \mathrm{mod}\, R$ has an $\mathcal{X}$-precover.

An $\mathcal{X}$-precover (resp. an $\mathcal{X}$-cover) is often called a right $\mathcal{X}$-approximation (resp. a minimal right $\mathcal{X}$-approximation).

Proposition 2.4 ([5, Remark 2.6]). *Let $\mathcal{X}$ be a full subcategory of* $\mathrm{mod}\, R$ *which is closed under direct summands, and let*

$$0 \to N \xrightarrow{\psi} X \xrightarrow{\phi} M$$

be an exact sequence of R-modules, where ϕ is an $\mathcal{X}$-precover of M. Suppose that R is henselian. Then there exists a direct summand L of N satisfying the following conditions:

(i) *$\psi(L)$ is a direct summand of X.*
(ii) *Let N' (resp. X') be the complement of L (resp. $\psi(L)$) in N (resp. X), and let*

$$0 \to N' \xrightarrow{\psi'} X' \xrightarrow{\phi'} M$$

be the induced exact sequence. Then ϕ' is an $\mathcal{X}$-cover of M.

For R-modules M, N, we define a homomorphism

$$\lambda_M(N) : M \otimes_R N \to \mathrm{Hom}_R(M^*, N)$$

of R-modules by $\lambda_M(N)(m \otimes n)(f) = f(m)n$ for $m \in M,\ n \in N$ and $f \in M^*$.

3. Proof of the theorem

Now, let us prove our theorem.

Proof of Theorem 1.4. Let $(R, \mathfrak{m}, k)$ be a henselian non-Gorenstein local ring of depth two. Then, since $\mathrm{Ext}^1_R(\mathfrak{m}, R) \cong \mathrm{Ext}^2_R(k, R) \neq 0$, we have a non-split exact sequence

$$\sigma : 0 \to R \to M \to \mathfrak{m} \to 0. \tag{1}$$

Dualizing this, we obtain an exact sequence

$$0 \to \mathfrak{m}^* \to M^* \to R^* \xrightarrow{\eta} \mathrm{Ext}^1_R(\mathfrak{m}, R).$$

Note that, by definition, the connecting homomorphism η sends $\mathrm{id}_R \in R^*$ to the element $s \in \mathrm{Ext}^1_R(\mathfrak{m}, R)$ corresponding to the exact sequence σ. Since σ does not split, s is a non-zero element of $\mathrm{Ext}^1_R(\mathfrak{m}, R)$. Hence η is a non-zero map. Noting that $\mathrm{Ext}^1_R(\mathfrak{m}, R) \cong \mathrm{Ext}^2_R(k, R)$, we see that the image of η is annihilated by $\mathfrak{m}$. Also noting that $\mathfrak{m}^* \cong R^* \cong R$, we get an exact sequence

$$0 \to R \to M^* \to \mathfrak{m} \to 0. \tag{2}$$

Claim 1. *The modules* $\mathrm{Hom}_R(G, M)$ *and* $\mathrm{Hom}_R(G, M^*)$ *belong to* $\mathcal{G}(R)$ *for every non-free indecomposable module* $G \in \mathcal{G}(R)$.

Proof. Applying the functor $\mathrm{Hom}_R(G, -)$ to the exact sequence (1) gives an exact sequence

$$0 \to G^* \to \mathrm{Hom}_R(G, M) \to \mathrm{Hom}_R(G, \mathfrak{m}) \to \mathrm{Ext}^1_R(G, R).$$

Since G is non-free and indecomposable, any homomorphism from G to R factors through $\mathfrak{m}$, and hence $\mathrm{Hom}_R(G, \mathfrak{m}) \cong G^*$. Also, since $G \in \mathcal{G}(R)$, we have $\mathrm{Ext}^1_R(G, R) = 0$. Thus Proposition 2.2.4 implies that $\mathrm{Hom}_R(G, M) \in \mathcal{G}(R)$. The same argument for the exact sequence (2) shows that $\mathrm{Hom}_R(G, M^*) \in \mathcal{G}(R)$. □

We shall prove that the module M cannot have a $\mathcal{G}(R)$-precover. Suppose that M has a $\mathcal{G}(R)$-precover. Then M has a $\mathcal{G}(R)$-cover $\pi : X \to M$ by Proposition 2.4. Since $R \in \mathcal{G}(R)$, any homomorphism from R to M factors through π. Hence π is a surjective homomorphism. Setting $N = \mathrm{Ker}\ \pi$, we get an exact sequence

$$0 \to N \xrightarrow{\theta} X \xrightarrow{\pi} M \to 0, \tag{3}$$

where θ is the inclusion. We see from Proposition 2.2.3, 2.2.4, and Wakamatsu's Lemma [6, Lemma 2.1.1] that $\mathrm{Ext}_R^i(G,N)=0$ for any $G\in\mathcal{G}(R)$ and any $i>0$. Dualizing the exact sequence (3), we obtain an exact sequence

$$0\to M^*\xrightarrow{\pi^*}X^*\xrightarrow{\theta^*}N^*.$$

Put $C=\mathrm{Im}(\theta^*)$ and let $\mu:X^*\to C$ be the surjection induced by θ^*.

Claim 2. *The homomorphism μ is a $\mathcal{G}(R)$-precover of C.*

Proof. Fix a non-free indecomposable module $G\in\mathcal{G}(R)$. Applying the functors $G\otimes_R -$ and $\mathrm{Hom}_R(G^*,-)$ to the exact sequence (3) yields a commutative diagram

$$\begin{array}{ccc}
 & & 0 \\
 & & \downarrow \\
G\otimes_R N & \xrightarrow{\lambda_G(N)} & \mathrm{Hom}_R(G^*,N) \\
{\scriptstyle G\otimes_R\theta}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}_R(G^*,\theta)} \\
G\otimes_R X & \xrightarrow{\lambda_G(X)} & \mathrm{Hom}_R(G^*,X) \\
{\scriptstyle G\otimes_R\pi}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}_R(G^*,\pi)} \\
G\otimes_R M & \xrightarrow{\lambda_G(M)} & \mathrm{Hom}_R(G^*,M) \\
\downarrow & & \downarrow \\
0 & & \mathrm{Ext}_R^1(G^*,N)
\end{array}$$

with exact columns. Noting that $\mathrm{Tr}\,G\in\mathcal{G}(R)$ by Proposition 2.2.3, we see from [2, Proposition (2.6)] that $\mathrm{Ker}\,\lambda_G(N)\cong\mathrm{Ext}_R^1(\mathrm{Tr}\,G,N)=0$ and $\mathrm{Coker}\,\lambda_G(N)\cong\mathrm{Ext}_R^2(\mathrm{Tr}\,G,N)=0$. This means that $\lambda_G(N)$ is an isomorphism. It follows from the commutativity of the above diagram that the homomorphism $G\otimes_R\theta$ is injective. Also, we have $\mathrm{Ext}_R^1(G^*,N)=0$ because $G^*\in\mathcal{G}(R)$ by Proposition 2.2.3. Thus we obtain a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & G\otimes_R N & \xrightarrow{G\otimes_R\theta} & G\otimes_R X & \longrightarrow & G\otimes_R M & \longrightarrow & 0 \\
 & & {\scriptstyle \lambda_G(N)}\downarrow{\scriptstyle\cong} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}_R(G^*,N) & \longrightarrow & \mathrm{Hom}_R(G^*,X) & \longrightarrow & \mathrm{Hom}_R(G^*,M) & \longrightarrow & 0
\end{array}$$

with exact rows. Dualizing this diagram induces a commutative diagram

$$\begin{array}{ccccc} \mathrm{Hom}_R(G^*,X)^* & \longrightarrow & \mathrm{Hom}_R(G^*,N)^* & \longrightarrow & \mathrm{Ext}^1_R(\mathrm{Hom}_R(G^*,M),R) \\ \downarrow & & {\scriptstyle (\lambda_G(N))^*}\downarrow\cong & & \downarrow \\ (G\otimes_R X)^* & \xrightarrow{(G\otimes_R\theta)^*} & (G\otimes_R N)^* & \longrightarrow & \mathrm{Ext}^1_R(G\otimes_R M,R) \end{array}$$

with exact rows. Since $\mathrm{Hom}_R(G^*,M) \in \mathcal{G}(R)$ by Claim 1, we have $\mathrm{Ext}^1_R(\mathrm{Hom}_R(G^*,M),R)=0$. From the above commutative diagram it is seen that $(G\otimes_R\theta)^*$ is a surjective homomorphism. Note that there is a natural commutative diagram

$$\begin{array}{ccc} (G\otimes_R X)^* & \xrightarrow{(G\otimes_R\theta)^*} & (G\otimes_R N)^* \\ \downarrow\cong & & \downarrow\cong \\ \mathrm{Hom}_R(G,X^*) & \xrightarrow{\mathrm{Hom}_R(G,\theta^*)} & \mathrm{Hom}_R(G,N^*) \end{array}$$

with isomorphic vertical maps. Therefore the homomorphism $\mathrm{Hom}_R(G,\theta^*)$ is also surjective, and so is the homomorphism $\mathrm{Hom}_R(G,\mu):\mathrm{Hom}_R(G,X^*)\to \mathrm{Hom}_R(G,C)$. It is easy to see from this that μ is a $\mathcal{G}(R)$-precover of C. □

According to Claim 2 and Proposition 2.4, we have direct sum decompositions $M^*=Y\oplus L,\ X^*=\pi^*(Y)\oplus Z$, and an exact sequence

$$0\to L\to Z\xrightarrow{\nu} C\to 0,$$

where ν is a $\mathcal{G}(R)$-cover of C. Since Y is isomorphic to the direct summand $\pi^*(Y)$ of X^*, Proposition 2.2.3 implies that $Y\in\mathcal{G}(R)$. Wakamatsu's Lemma yields $\mathrm{Ext}^1_R(G,L)=0$ for any $G\in\mathcal{G}(R)$.

Claim 3. *The module* $\mathrm{Hom}_R(G,Y)$ *belongs to* $\mathcal{G}(R)$ *for any* $G\in\mathcal{G}(R)$.

Proof. We may assume that G is non-free and indecomposable. The module $\mathrm{Hom}_R(G,Y)$ is isomorphic to a direct summand of $\mathrm{Hom}_R(G,M^*)$. Since the module $\mathrm{Hom}_R(G,M^*)$ is an object of $\mathcal{G}(R)$ by Claim 1, so is the module $\mathrm{Hom}_R(G,Y)$ by Proposition 2.2.3. □

Here, by the assumption of the theorem, we have a non-free indecomposable module $W\in\mathcal{G}(R)$. There is an exact sequence

$$0\to\Omega W\to F\to W\to 0$$

such that F is a free module. Applying the functor $\mathrm{Hom}_R(-,Y)$ to this exact sequence, we get an exact sequence

$$0\to\mathrm{Hom}_R(W,Y)\to\mathrm{Hom}_R(F,Y)\to\mathrm{Hom}_R(\Omega W,Y)\to\mathrm{Ext}^1_R(W,Y)\to 0.$$

Since $\mathrm{Hom}_R(W,Y)$, $\mathrm{Hom}_R(F,Y)$, and $\mathrm{Hom}_R(\Omega W,Y)$ belong to $\mathcal{G}(R)$ by Claim 3, the R-module $\mathrm{Ext}^1_R(W,Y)$ has G-dimension at most two, so in particular it has finite G-dimension.

On the other hand, there are isomorphisms

$$\begin{aligned}\mathrm{Ext}^1_R(W,Y) &\cong \mathrm{Ext}^1_R(W,Y)\oplus \mathrm{Ext}^1_R(W,L)\\ &\cong \mathrm{Ext}^1_R(W,M^*)\\ &\cong \mathrm{Ext}^1_R(W,\mathfrak{m}),\end{aligned}$$

where the last isomorphism is induced by the exact sequence (2). Applying the functor $\mathrm{Hom}_R(W,-)$ to the exact sequence

$$0\to\mathfrak{m}\to R\to k\to 0$$

and noting that $\mathrm{Hom}_R(W,\mathfrak{m})\cong W^*$ because W is a non-free indecomposable module, we obtain an isomorphism $\mathrm{Ext}^1_R(W,\mathfrak{m})\cong \mathrm{Hom}_R(W,k)$, and hence $\mathrm{Ext}^1_R(W,Y)$ is a non-zero k-vector space. Therefore Proposition 2.2.1 and 2.2.2 say that R is Gorenstein, contrary to the assumption of our theorem. This contradiction proves that the R-module M does not have a $\mathcal{G}(R)$-precover, which establishes our theorem. □

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