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ON A CONJECTURE ON ALGEBRAS THAT ARE LOCALLY EMBEDDABLE INTO FINITE DIMENSIONAL ALGEBRAS

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Abstract. The notion of an algebra that is locally embeddable into finite dimensional algebras (LEF) and the notion of an LEF group was introduced by Gordon and Vershik in [1]. M. Ziman proved in [5] that the group algebra of a group G is an LEF algebra if and only if G is an LEF group. He conjectured that an algebra generated as a vector space by a multiplicative subgroup G of its invertible elements is an LEF algebra if and only if G is an LEF group. In this paper we give a characterization of the invertible elements of an LEF algebra and use it to construct a counterexample to this conjecture.

Fix an arbitrary field $\mathbb{K}$ and consider all vector spaces and algebras as $\mathbb{K}$-vector spaces and $\mathbb{K}$-algebras, respectively. An algebra A is said to be locally embeddable into finite dimensional algebras (LEF) if for every finite subset M of A there exists a finite dimensional algebra B and a vector space monomorphism $\varphi : [M] \to B$ such that $\varphi(x)\varphi(y) = \varphi(xy)$ for all $x, y \in M$ with $xy \in M$. Here, $[M]$ denotes the vector subspace of A generated by M. In an analoguous manner, a group G is said to be locally embeddable into finite groups (LEF) if for every finite subset M of G there exists a finite group H and an injective map $\varphi : M \to H$ such that $\varphi(x)\varphi(y) = \varphi(xy)$ for all $x, y \in M$ with $xy \in M$. These notions were first introduced by E.I. Gordon and A.M. Vershik in [1]. Gordon and Vershik raised the question whether the group algebra $A = \mathbb{K}[G]$ of a group G is LEF if and only if G is LEF. This question was answered positively by M. Ziman in [5]. He conjectured the following generalization:

($*$) *Let an algebra A be generated as a vector space by a subgroup G of its group of (multiplicatively) invertible elements. Then A is LEF if and only if G is LEF.*

The condition that A be LEF is sufficient for G to be LEF by Corollary 1 of [5]. The converse, however, is not true as we shall show in this note. Without danger of confusion, the neutral element of any occurring multiplicative group

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is denoted by 1. The group of invertible elements of an algebra A is denoted by A^{-1}. For LEF algebras A, we give a characterization of A^{-1}.

PROPOSITION. *In an LEF-algebra A the set of left invertible elements equals the set of right invertible elements and thus equals A^{-1}.*

Proof. Let $r, l \in A$ be such that $lr = 1$, and let $M = \{1, r, l, rl\}$. Then there exists a vector space monomorphism $\varphi : [M] \to B$, where B is a finite dimensional algebra, such that $\varphi(x)\varphi(y) = \varphi(xy)$ for all $x, y \in M$ with $xy \in M$. Without loss, we may assume that B is generated as an algebra by $\varphi(M)$. From the equations $\varphi(m)\varphi(1) = \varphi(m) = \varphi(1)\varphi(m)$ for $m \in M$ it follows that $\varphi(1) = 1$ in B. In a finite dimensional algebra, left invertible and right invertible elements are well known to be identical. In particular, $\varphi(l)\varphi(r) = \varphi(lr) = \varphi(1) = 1$ implies that $\varphi(l)$ and $\varphi(r)$ are inverse to each other. Hence $\varphi(rl) = \varphi(r)\varphi(l) = 1 = \varphi(1)$. By the injectivity of φ, we obtain $rl = 1$. □

The remainder of this paper is devoted to the construction of a counterexample to conjecture $(*)$. Our aim is to construct a non-LEF algebra A and an LEF subgroup G of A^{-1} such that $A = [G]$. The idea is to find an algebra containing two elements r and l such that $lr = 1$ but $rl \neq 1$. Probably the most straightforward candidates for r and l are the right shift and the left shift operators on a sequence space. To be precise, let $\mathbb{K} = \mathbb{C}$ be the field of complex numbers. Let

$$L^1 = \left\{ v = (v_1, v_2, \dots) \mid v_j \in \mathbb{C},\ \|v\|_1 = \sum_{j \in \mathbb{N}} |v_j| < \infty \right\}$$

be the Banach space of all absolutely convergent series in $\mathbb{C}$, with addition and scalar multiplication defined componentwise. The algebra

$$\mathcal{L} = \left\{ T \in \mathrm{End}_{\mathbb{C}}(L^1)\ ,\ \|T\|_{\mathrm{Op}} = \sup\{ \|Tv\|_1, v \in L^1, \|v\|_1 = 1\} < \infty \right\}$$

of all continuous endomorphisms of L^1, together with the operator norm $\|.\|_{\mathrm{Op}}$, is a Banach algebra (see [4]). Define $r, l \in \mathcal{L}$ by

$$r(v_1, v_2, \dots) = (0, v_1, v_2, \dots) \quad \text{and} \quad l(v_1, v_2, v_3, \dots) = (v_2, v_3, \dots)$$

for $v = (v_1, v_2, \dots) \in L^1$ (note that $\|r\|_{\mathrm{Op}} = \|l\|_{\mathrm{Op}} = 1$). Clearly $lr = 1$ but $rl \neq 1$ since l is not injective.

The next step is to look for an LEF-group $G \subseteq \mathcal{L}^{-1}$ for which $[G]$ contains r and l. In $\mathcal{L}$, the elements $1 + \frac{1}{2}r$ and $1 + \frac{1}{2}l$ lie within the open 1-ball around $1 \in \mathcal{L}$, so they are invertible ([4, Theorem 10.7]). This is indeed the reason for choosing L^1 as the underlying sequence space for $\mathcal{L}$. For other sequence spaces we considered, the "obvious" choices $1 + r$ and $1 + l$ were not both invertible. A good guess for G is the group $G = \langle 1 + \frac{1}{2}r, 1 + \frac{1}{2}l \rangle \subseteq \mathcal{L}^{-1}$. Here, $\langle\,.\,\rangle$ denotes the subgroup of a group generated by the elements enclosed in the brackets. The algebra $[G]$ contains the elements $r = 2 \cdot (1 + \frac{1}{2}r) - 2 \cdot 1$ and $l = 2 \cdot (1 + \frac{1}{2}l) - 2 \cdot 1$. By the proposition, $[G]$ is not LEF. It merely remains to check that G is an LEF group. Unfortunately, we have been unable to do this. Instead, we extended the above construction as follows:

Example. Let $\mathbb{X} = \mathbb{C}(x)$ and $\mathbb{Y} = \mathbb{C}(y)$ be the fields of rational functions in the indeterminants x and y, respectively. Define $\mathcal{M} = \mathbb{X} * \mathbb{Y}$ as the free product of the algebras $\mathbb{X}$ and $\mathbb{Y}$. Consider the direct product $\mathcal{A} = \mathcal{L} \times \mathcal{M}$, where $\mathcal{L}$ has been defined above. In $\mathcal{A}$ choose the invertible elements

$$x_1 = (1, x)\ ,\ x_2 = (1 + \tfrac{1}{2}r, \tfrac{1}{2} + x)\ ,\ y_1 = (1, y)\ ,\ y_2 = (1 + \tfrac{l}{2}l, \tfrac{1}{2} + y)\ \ .$$

Let $X = \langle x_1, x_2 \rangle$, $Y = \langle y_1, y_2 \rangle$, and $G = \langle X, Y \rangle \subseteq \mathcal{A}^{-1}$. The groups X and Y are abelian. Let $\pi : \mathcal{A} = \mathcal{L} \times \mathcal{M} \to \mathcal{M}$ be the canonical projection onto the second component of $\mathcal{A}$, let $X' = \pi(X)$, $Y' = \pi(Y)$. For an element $g \in X$ there are integers n_1, n_2 such that $g = x_1^{n_1} x_2^{n_2}$. Thus $\pi(g) = x^{n_1}(\frac{1}{2} + x)^{n_2}$ lies within $\mathbb{C}$ iff $n_1 = n_2 = 0$ and $g = 1$. In other words, $\pi|_X : X \to X'$ is an isomorphism of free abelian groups, $X' \subseteq \mathbb{X}$ and $X' \cap \mathbb{C} = \{1\}$. Mutatis mutandis, the same reasoning applies to Y. By the construction of $\mathcal{M}$ this implies that X' and Y' together generate their free product of groups, $\langle X', Y' \rangle = X' * Y'$. By the universal property of free products ([3, 6.2]), the group homomorphisms $\pi|_X^{-1} : X' \to X \subseteq G$ and $\pi|_Y^{-1} : Y' \to Y \subseteq G$ extend to a homomorphism $\varphi : X' * Y' \to G$ with the property $\varphi|_{X'} = \pi|_X^{-1}$ and $\varphi|_{Y'} = \pi|_Y^{-1}$. The composition $\varphi\pi : G \to G$ induces the identity mapping on X and Y. As G is generated by X and Y, $\varphi\pi$ is the identity on G. This implies that π is one-to-one. Clearly π is also onto $X' * Y'$, proving that G is isomorphic to $X' * Y'$. The groups X' and Y' are residually finite, because free abelian groups generally have this property (e.g., [3, Ex.4.2.15]). By a theorem of Grünberg ([2, Theorem 9.14]), the free product of residually finite groups is residually finite. It follows that $X' * Y'$ is residually finite. So G is residually finite, which by [1] means that it is LEF.

If we now let $A = [G]$, we have $G \subseteq A^{-1}$ and the algebra A contains the elements $\bar{r} = 2x_2 - 2x_1 = (2 + r, 1 + 2x) - (2, 2x) = (r, 1)$ and $\bar{l} = 2y_2 - 2y_1 = (l, 1)$. So $\bar{l}\bar{r} = 1$, but $\bar{r}\bar{l} \neq 1$. By the proposition, A is not LEF. This shows that conjecture $(*)$ is not true.

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