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ISOMETRIC IMMERSIONS IN CODIMENSION TWO OF WARPED PRODUCTS INTO SPACE FORMS

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ABSTRACT. We provide a local classification of isometric immersions $f\colon L^p \times_\rho M^n \to \mathbb{Q}_c^{p+n+k}$ in codimensions $k = 1, 2$ of warped products of Riemannian manifolds into space forms, under the assumptions that $n \geq k+1$ and that $N^{p+n} = L^p \times_\rho M^n$ has no points with the same constant sectional curvature c as the ambient space form.

1. Introduction

A basic decomposition theorem due to Nölker [16] states that an isometric immersion $f\colon N^{p+n} = L^p \times_\rho M^n \to \mathbb{Q}_c^\ell$ of a warped product of connected Riemannian manifolds with warping function $\rho \in C^\infty(L^p)$ into a complete simply-connected space form of constant sectional curvature c is a *warped product of isometric immersions* (see [16] or Section 1 for the precise definition of this concept) whenever its second fundamental form $\alpha\colon TN \times TN \to T^\perp N$ satisfies

$$\alpha(X, V) = 0 \quad \text{for all} \quad X \in TL \quad \text{and} \quad V \in TM.$$

This generalizes a well-known result for isometric immersions of Riemannian products into Euclidean space due to Moore [14] as well as its extension by Molzan [13] for nonflat ambient space forms; see also [18].

It is a natural problem to understand the possible cases in which the isometric immersion f may fail, locally or globally, to be a warped product of isometric immersions. In high codimensions the warped product structure of the manifold does not seem to place enough restrictions on the isometric immersion in order to make possible a complete classification in either case. Even in the much more restrictive situation of Riemannian products, a successful local analysis has only been carried out in the case in which the codimension is two and the first factor is one-dimensional; see [3]. This was used therein to characterize isometric immersions $f\colon L^p \times M^n \to \mathbb{R}^{p+n+2}$ of complete Riemannian products none of whose factors is everywhere flat (see Remark 35

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below), carrying through the global results previously obtained by Moore [14] and Alexander-Maltz [1]. Based on earlier work due to Moore, such submanifolds were shown in [1] to split as products of hypersurfaces under the global assumption that they do not carry an Euclidean strip.

The main goal of this paper is to provide a local classification of isometric immersions $f\colon L^p \times_\rho M^n \to \mathbb{Q}_c^{p+n+k}$ in codimensions $k = 1, 2$, under the assumptions that $n \geq k+1$ and that $N^{p+n} = L^p \times_\rho M^n$ has no points with the same constant sectional curvature c as the ambient space form. In the case of codimension $k = 1$, we prove that f must be a warped product of isometric immersions. In codimension $k = 2$ we show that only two other possibilities may arise. Namely, either f is a composition of a warped product of isometric immersions into $\mathbb{Q}_c^{p+n+1}$ with a local isometric immersion of $\mathbb{Q}_c^{p+n+1}$ into $\mathbb{Q}_c^{p+n+2}$ or N^{p+n} is a Riemannian manifold of a special type that admits a second decomposition as a warped product with respect to which f splits as a warped product of isometric immersions. We leave the precise statement for Section 3, where we also state its corresponding version for the case of Riemannian products.

We give examples showing that the restriction on the dimension of M^n is necessary. As for the hypothesis that N^{p+n} has no points with constant sectional curvature c, we observe that for manifolds with constant sectional curvature the assumption that they be warped products places no further restrictions on them since any such manifold can be realized as a warped product in many possible ways; see the discussion on warped product representations of space forms in Section 2. Therefore, for Riemannian manifolds with constant sectional curvature our problem reduces to classifying *all* isometric immersions in codimension two of such manifolds into space forms. In this regard, recall that if $f\colon U \subset \mathbb{Q}_{\tilde{c}}^n \to \mathbb{Q}_c^{n+2}$, $n \geq 4$, is an isometric immersion, then $\tilde{c} \geq c$ and, for $\tilde{c} > c$, away from the set of umbilical points the immersion must be locally a composition of the umbilical inclusion of $\mathbb{Q}_{\tilde{c}}^n$ into $\mathbb{Q}_c^{n+1}$ with a local isometric immersion of $\mathbb{Q}_c^{n+1}$ into $\mathbb{Q}_c^{n+2}$ (see [10], [12] and [7]). In fact, the latter statement can be derived from our main theorem, but in that result we exclude from our analysis the case of local isometric immersions of $\mathbb{Q}_c^n$ into $\mathbb{Q}_c^{n+2}$ with the same constant sectional curvature. A local description of these isometric immersions when $c = 0$ was given in [6].

As striking applications of our main result, we obtain that if L^p is a Riemannian manifold no open subset of which can be isometrically immersed into $\mathbb{Q}_c^{p+1}$, then any isometric immersion $f\colon L^p \times_\rho M^n \to \mathbb{Q}_c^{p+n+2}$, $n \geq 3$, is either a cylindrical submanifold of Euclidean space or a rotational submanifold. Moreover, if L^p cannot be locally isometrically immersed in $\mathbb{Q}_c^{p+2}$, then $L^p \times_\rho M^n$ cannot be locally isometrically immersed in $\mathbb{Q}_c^{p+n+2}$ either for whatever Riemannian manifold M^n of dimension $n \geq 3$ and warping function ρ.

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2. Preliminaries

In this section we establish our notation and state some basic facts on warped products of Riemannian manifolds and their isometric immersions into the standard real space forms.

Given a a vector bundle E over a Riemannian manifold N, we denote by $\Gamma(E)$ the set of all locally defined smooth sections of E. If $N = L \times M$ is a product manifold, we denote by $\mathcal{H}$ and $\mathcal{V}$ the *horizontal* and *vertical* subbundles of TN, that is, the distributions on N corresponding to the product foliations determined by L and M, respectively. Elements of $\Gamma(\mathcal{H})$ will always be denoted by the letters X, Y, Z, and those in $\Gamma(\mathcal{V})$ by the letters U, V, W. The same applies to individual tangent vectors. A vector field $X \in \Gamma(\mathcal{H})$ (resp., $V \in \Gamma(\mathcal{V})$) is said to be the *lift* of a vector field $\tilde{X} \in \Gamma(TL)$ (resp., $\tilde{V} \in \Gamma(TM)$) if $\pi_{L*}X = \tilde{X} \circ \pi_L$ (resp., $\pi_{M*}V = \tilde{V} \circ \pi_M$), where $\pi_L \colon L \times M \to L$ (resp., $\pi_M \colon L \times M \to M$) is the canonical projection onto L (resp., M). We denote the set of all lifts of vector fields in L (resp., M) by $\mathcal{L}(L)$ (resp., $\mathcal{L}(M)$), and we always denote vector fields in L and M with a tilde and use the same letters without the tilde to represent their lifts to N.

If L and M are Riemannian manifolds with Riemannian metrics $\langle\ ,\ \rangle_L$ and $\langle\ ,\ \rangle_M$, respectively, the *warped product* $N = L \times_\rho M$ with *warping function* $\rho \in C^\infty(L)$ is the product manifold $L \times M$ endowed with the *warped product metric*

$$\langle\ ,\ \rangle = {\pi_L}^*\langle\ ,\ \rangle_L + (\rho \circ \pi_L)^2 {\pi_M}^*\langle\ ,\ \rangle_M.$$

We always assume N to be connected. The Levi-Civita connections of N, L and M are related by (cf. [17])

$$\nabla_X Y \ \text{ is the lift of } \ \nabla^L_{\tilde{X}}\tilde{Y}, \tag{1}$$

$$\nabla_X V = \nabla_V X = -\langle X, \eta\rangle V, \tag{2}$$

$$(\nabla_V W)_{\mathcal{V}} \ \text{ is the lift of } \ \nabla^M_{\tilde{V}}\tilde{W}, \tag{3}$$

$$(\nabla_V W)_{\mathcal{H}} = \langle V, W\rangle \eta, \tag{4}$$

where $X, Y \in \mathcal{L}(L)$, $V, W \in \mathcal{L}(M)$ and $\eta = -\operatorname{grad}(\log \rho \circ \pi_L)$. Here and in the sequel, writing a vector field with a vector subbundle as a subscript indicates taking the section of that vector subbundle obtained by orthogonally projecting the vector field pointwise onto the corresponding fiber of the subbundle. Observe that the formula $\nabla_X V = -\langle X, \eta\rangle V$ (resp., $\nabla_V X = -\langle X, \eta\rangle V$) is tensorial in X (resp., V), hence it also holds for horizontal (resp., vertical) vector fields that are not necessarily lifts. On the other hand, it characterizes vertical (resp., horizontal) vector fields that are lifts.

Recall that a vector subbundle E of TN is called *totally geodesic* or *autoparallel* if $\nabla_X Y \in \Gamma(E)$ for all $X, Y \in \Gamma(E)$. It is called *totally umbilical* if there exists a vector field $\eta \in \Gamma(E^\perp)$ such that $(\nabla_X Y)_{E^\perp} = \langle X, Y\rangle \eta$ for all $X, Y \in \Gamma(E)$. If, in addition, the so called *mean curvature normal* η of E satisfies $(\nabla_X \eta)_{E^\perp} = 0$ for all $X \in \Gamma(E)$, then E is said to be *spherical*. A totally umbilical vector subbundle E of TN is automatically integrable, and its leaves are totally umbilical submanifolds of N. If E is totally geodesic or spherical then the leaves are totally geodesic or spherical submanifolds of N, respectively. By a *spherical* submanifold we mean a totally umbilical submanifold whose mean curvature vector field is parallel in the normal connection.

It follows from (1) and (4), respectively, that $\mathcal{H}$ is totally geodesic and that $\mathcal{V}$ is totally umbilical with mean curvature normal $\eta = -\operatorname{grad}(\log \rho \circ \pi_L)$. Moreover, since η is a gradient vector field and $\mathcal{H}$ is totally geodesic we have

$$\langle \nabla_V \eta, X\rangle = \langle \nabla_X \eta, V\rangle = 0$$

for all $X \in \Gamma(\mathcal{H})$ and $V \in \Gamma(\mathcal{V})$, and hence $\mathcal{V}$ is spherical. The following extension due to Hiepko of the well-known decomposition theorem of de Rham shows that these properties characterize warped products.

THEOREM 1 ([11]). *Let N be a Riemannian manifold and let $TN = \mathcal{H} \oplus \mathcal{V}$ be an orthogonal decomposition into nontrivial vector subbundles such that $\mathcal{H}$ is totally geodesic and $\mathcal{V}$ is spherical. Then, for every point $z_0 \in N$ there exist an isometry Ψ of a warped product $L \times_\rho M$ onto a neighborhood of z_0 in N such that $\Psi(L \times \{x\})$ and $\Psi(\{y\} \times M)$ are integral manifolds of $\mathcal{H}$ and $\mathcal{V}$, respectively, for all $y \in L$ and $x \in M$. Moreover, if N is simply connected and complete then the isometry Ψ can be taken onto all of N.*

Given a warped product $N = L \times_\rho M$, the lift of the curvature tensor ${}^M R$ of M to N is the tensor whose value at $E_1, E_2, E_3 \in T_z N$ is the unique vector in $\mathcal{V}_z$ that projects to ${}^M R(\pi_{M*}E_1, \pi_{M*}E_2)\pi_{M*}E_3$ in $T_{\pi_M(z)}M$. The lift of the curvature tensor ${}^L R$ of L is similarly defined. Then the curvature tensors of L, M and N are related by

$$R(X,Y)Z = {}^L R(X,Y)Z, \tag{5}$$

$$R(X,Y)V = R(V,W)X = 0, \tag{6}$$

$$R(X,U)V = \langle U, V\rangle(\nabla_X \eta - \langle \eta, X\rangle \eta), \tag{7}$$

$$R(V,W)U = {}^M R(V,W)U - \|\eta\|^2(\langle W,U\rangle V - \langle V,U\rangle W). \tag{8}$$

Since $\nabla_X \eta - \langle \eta, X\rangle\eta \in \mathcal{H}$ because $\mathcal{H}$ is totally geodesic, all the information of (7) is contained in

$$\langle R(X,V)W, Y\rangle = \langle V, W\rangle\langle \nabla_X \eta - \langle X, \eta\rangle \eta, Y\rangle. \tag{9}$$

The starting point for the proof of the main results of this paper is the observation that the curvature relations (5) to (8) impose several restrictions on the second fundamental form $\alpha\colon\ TN \times TN \to T^{\perp}N$ of an isometric immersion $f\colon\ N \to \mathbb{Q}_c^\ell$ when combined with the Gauss equation for f.

PROPOSITION 2. *Let $f\colon\ L \times_\rho M \to \mathbb{Q}_c^\ell$ be an isometric immersion of a warped product. Then the curvature-like tensor*

$$
\begin{aligned}
(10)\quad C(E_1,E_2,E_3,E_4) &:= \langle R(E_1,E_2)E_3,E_4\rangle - c\langle (E_1\wedge E_2)E_3,E_4\rangle \\
&= \langle \alpha(E_1,E_4),\alpha(E_2,E_3)\rangle - \langle \alpha(E_1,E_3),\alpha(E_2,E_4)\rangle
\end{aligned}
$$

satisfies

$$C(X,V,W,Y) = \langle V,W\rangle\langle \nabla_X\eta - \langle X,\eta\rangle\eta - cX, Y\rangle, \tag{11}$$

$$C(X,Y,V,Z) = 0, \tag{12}$$

$$C(X,Y,V,W) = 0, \tag{13}$$

$$C(X,U,V,W) = 0. \tag{14}$$

We now introduce the notion of a warped product of isometric immersions into $\mathbb{Q}_c^\ell$ which plays a fundamental role in this paper. This relies on the warped product representations of $\mathbb{Q}_c^\ell$, that is, isometries of warped products onto open subsets of $\mathbb{Q}_c^\ell$. All such isometries were described by Nölker for warped products with arbitrarily many factors; see [16] for details. In particular, any isometry of a warped product with two factors onto an open subset of $\mathbb{Q}_c^\ell$ arises as a restriction of an explicitly constructible isometry

$$\Psi\colon\ V^{\ell-m}(\subset \mathbb{Q}_c^{\ell-m}) \times_\sigma \mathbb{Q}_{\tilde c}^m \to \mathbb{Q}_c^\ell$$

onto an open dense subset of $\mathbb{Q}_c^\ell$, where $\mathbb{Q}_{\tilde c}^m$ is a complete spherical submanifold of $\mathbb{Q}_c^\ell$ and $V^{\ell-m}$ is an open subset of the unique totally geodesic submanifold $\mathbb{Q}_c^{\ell-m}$ of $\mathbb{Q}_c^\ell$ (of constant sectional curvature c if $\ell - m \geq 2$) whose tangent space at some point $\bar z \in \mathbb{Q}_{\tilde c}^m$ is the orthogonal complement of the tangent space of $\mathbb{Q}_{\tilde c}^m$ at $\bar z$. The isometry Ψ is, in fact, completely determined by the choice of $\mathbb{Q}_{\tilde c}^m$ and of a point $\bar z \in \mathbb{Q}_{\tilde c}^m$, and it is called the *warped product representation* of $\mathbb{Q}_c^\ell$ determined by $(\bar z, \mathbb{Q}_{\tilde c}^m)$. If $c \neq 0$, we consider the standard model of $\mathbb{Q}_c^\ell$ as a complete spherical submanifold of $\mathbb{O}^{\ell+1}$, where $\mathbb{O}^{\ell+1}$ denotes either the Euclidean space $\mathbb{R}^{\ell+1}$ if $c > 0$ or the Lorentzian space $\mathbb{L}^{\ell+1}$ if $c < 0$. Then, for $c \neq 0$ the warping function σ is the restriction to $V^{\ell-m}$ of the height function $z \mapsto \langle z, a\rangle$ in $\mathbb{O}^{\ell+1}$, where $-a$ is the mean curvature vector of $\mathbb{Q}_{\tilde c}^m$ in $\mathbb{O}^{\ell+1}$ at $\bar z$. Similarly, if $c = 0$ then $\sigma(z) = 1 + \langle z - \bar z, a\rangle$, where $-a$ is the mean curvature vector of $\mathbb{Q}_{\tilde c}^m$ in $\mathbb{Q}_c^\ell = \mathbb{R}^\ell$ at $\bar z$. In every case $\langle a, a\rangle = \tilde c$.

DEFINITION. Let $\Psi\colon\ V^{\ell-m} \times_\sigma \mathbb{Q}_{\tilde c}^m \to \mathbb{Q}_c^\ell$ be a warped product representation, let $h_1\colon\ L \to V^{\ell-m}$ and $h_2\colon\ M \to \mathbb{Q}_{\tilde c}^m$ be isometric immersions, and let $\rho = \sigma\circ h_1$. Then the isometric immersion $f = \Psi\circ(h_1 \times h_2)\colon\ N = L \times_\rho M \to \mathbb{Q}_c^\ell$ is called the *warped product of the isometric immersions h_1 and h_2 determined by Ψ.*

$$
\begin{tikzcd}
V^{\ell-m} & \times_\sigma & \mathbb{Q}_{\tilde c}^{m} \arrow[drr, "\Psi"] & & \\
L \arrow[u, "h_1"] & \times_{\rho=\sigma\circ h_1} & M \arrow[u, "h_2"] \arrow[rr, "f=\Psi\circ(h_1\times h_2)"'] \arrow[r, phantom, "\circlearrowleft"] & & \mathbb{Q}_c^{\ell}
\end{tikzcd}
$$

EXAMPLE 3. If $N = L \times_\rho M$ is not a Riemannian product and h_2 is an isometry, then f is called a *rotational submanifold* with profile h_1. This means that $V^{\ell-m}$ is a half-space of a totally geodesic submanifold $\mathbb{Q}_c^{\ell-m} \subset \mathbb{Q}_c^\ell$ bounded by a totally geodesic submanifold $\mathbb{Q}_c^{\ell-m-1}$ and $f(N)$ is the submanifold of $\mathbb{Q}_c^\ell$ generated by the action on $h_1(L)$ of the subgroup of isometries of $\mathbb{Q}_c^\ell$ that leave $\mathbb{Q}_c^{\ell-m-1}$ invariant.

EXAMPLE 4. If $N = L \times_\rho M$ is not a Riemannian product and $h_1\colon L \to V^{\ell-m}$ is a local isometry then, for $c = 0$, we have that $f(N)$ is contained in the product of an Euclidean factor $\mathbb{R}^{\ell-m-1}$ with a cone in $\mathbb{R}^{m+1}$ over h_2. If $c \neq 0$, then $f(N)$ is the union of open subsets of the totally geodesic submanifolds of $\mathbb{Q}_c^\ell$ through the points of $h_2(M) \subset \mathbb{Q}_{\tilde c}^m$ whose tangent spaces at the points of $h_2(M)$ are the normal spaces of $\mathbb{Q}_{\tilde c}^m$ in $\mathbb{Q}_c^\ell$.

Notice that any warped product of isometric immersions in codimension one must be as in one of the preceding examples. In codimension two only a third possibility arises, namely, the case in which both h_1 and h_2 are hypersurfaces.

Important special cases of warped products of isometric immersions arise as follows. Let $\mathbb{Q}_{c_1}^{\ell_1}$ and $\mathbb{Q}_{c_2}^{\ell_2}$ be complete spherical submanifolds of $\mathbb{Q}_c^\ell$ through a fixed point $\bar z \in \mathbb{Q}_c^\ell$ whose tangent spaces at $\bar z$ are orthogonal and whose mean curvature vectors ψ_1 and ψ_2 at $\bar z$ satisfy $\langle \psi_1, \psi_2\rangle = -c$ and ψ_1 (resp., ψ_2) is orthogonal to $T_{\bar z}\mathbb{Q}_{c_2}^{\ell_2}$ (resp., $T_{\bar z}\mathbb{Q}_{c_1}^{\ell_1}$). Let $\Psi\colon V^{\ell-\ell_2} \times_\sigma \mathbb{Q}_{c_2}^{\ell_2} \to \mathbb{Q}_c^\ell$ be the warped product representation of $\mathbb{Q}_c^\ell$ determined by $(\bar z, \mathbb{Q}_{c_2}^{\ell_2})$. Then $\mathbb{Q}_{c_1}^{\ell_1} \subset V^{\ell-\ell_2}$ and $\sigma \circ i = 1$, where $i\colon \mathbb{Q}_{c_1}^{\ell_1} \to V^{\ell-\ell_2}$ is the inclusion map. The warped product $\Psi \circ (i \times \mathrm{id})\colon \mathbb{Q}_{c_1}^{\ell_1} \times \mathbb{Q}_{c_2}^{\ell_2} \to \mathbb{Q}_c^\ell$ of the inclusion and the identity map is an isometric embedding called the isometric embedding of the Riemannian product $\mathbb{Q}_{c_1}^{\ell_1} \times \mathbb{Q}_{c_2}^{\ell_2}$ into $\mathbb{Q}_c^\ell$ as an *extrinsic Riemannian product*.

The structure of the second fundamental form of a warped product of isometric immersions is described in the following result.

PROPOSITION 5. *Let $N = L \times_\rho M$ and $\bar N = \bar L \times_{\bar\rho} \bar M$ be warped product manifolds, and let $F\colon L \to \bar L$ and $G\colon M \to \bar M$ be isometric immersions with $\rho = \bar\rho \circ F$. Then $f = F \times G\colon N \to \bar N$ is an isometric immersion and at $z = (y, x) \in N$ we have*

(i) $\pi_{\bar L *} f_* T_z N = F_* T_y L$, $\pi_{\bar L *} T_z^\perp N = T_y^\perp L$, $\pi_{\bar M *} f_* T_z N = G_* T_x M$, $\pi_{\bar M *} T_z^\perp N = T_x^\perp M$.

(ii) $(\operatorname{grad}\bar{\rho}\,(F(y)))_{T_y^\perp L} = \operatorname{grad}\bar{\rho}\,(F(y)) - F_* \operatorname{grad}\rho\,(y)$.

(iii) *The second fundamental forms of F, G and f are related by*

$$\pi_{\bar{M}*}\alpha^f(E_1,E_2) = \alpha^G(\pi_{M*}E_1, \pi_{M*}E_2), \tag{15}$$

$$\begin{aligned}\pi_{\bar{L}*}\alpha^f(E_1,E_2) = {} & \alpha^F(\pi_{L*}E_1, \pi_{L*}E_2) \\ & - \rho\,(y)\langle \pi_{M*}E_1, \pi_{M*}E_2\rangle (\operatorname{grad}\bar{\rho}\,(F(y)))_{T_y^\perp L}.\end{aligned} \tag{16}$$

Given an isometric immersion $f\colon\ N \to \bar{N}$, a normal vector $\zeta \in T_z^\perp N$ is called a *principal curvature normal vector* at z if the subspace

$$\Delta_\zeta(z) = \{T \in T_zN : \alpha(T,E) = \langle T,E\rangle\zeta \text{ for all } E \in T_zN\}$$

is nontrivial. In this case $\Delta_\zeta(z)$ is called the eigenspace corresponding to ζ. If $\zeta = 0$ then $\Delta(z) := \Delta_0(z)$ is called the *relative nullity subspace* of f at z.

Corollary 6. *Let f be an isometric immersion as in Proposition 5. Then at $z = (y,x) \in N$ we have:*

(i) *$\mathcal{V}_z \subset \Delta_\zeta(z)$ for a principal curvature normal vector $\zeta \in T_z^\perp N$ if and only if G is umbilical at x with mean curvature vector $\pi_{\bar{M}*}\zeta$ and $\pi_{\bar{L}*}\zeta = -\rho^{-1}(\operatorname{grad}\bar{\rho}\,)_{T_y^\perp L}$. In particular, we have that $\mathcal{V}_z \subset \Delta(z)$ if and only if G is totally geodesic at x and $(\operatorname{grad}\bar{\rho}\,)_{T_y^\perp L} = 0$.*

(ii) *$\mathcal{H}_z \subset \Delta(z)$ if and only if F is totally geodesic at y.*

Given a vector $a \neq 0$ in either $\mathbb{R}^\ell$ or $\mathbb{O}^{\ell+1}$, according as $c = 0$ or $c \neq 0$, let U be the vector field on $\mathbb{Q}_c^\ell$ defined by $U_z = a - c\langle a,z\rangle z$ and let $\mathcal{F}^a$ be the 1-dimensional totally geodesic distribution generated by U on the open dense subset $W^a = \{z \in \mathbb{Q}_c^\ell : U_z \neq 0\}$. Notice that $\mathbb{Q}_c^\ell \setminus W^a$ is empty for $c = 0$ as well as for $(c < 0, \langle a,a\rangle \geq 0)$, and contains one point for $(c < 0, \langle a,a\rangle < 0)$ and two points for $c > 0$. Observe also that for $c = 0$ (resp., $c \neq 0$) the vector field U is the gradient of the function $\sigma\colon\ \mathbb{Q}_c^\ell \to \mathbb{R}$ given by $\sigma(z) = 1 + \langle z - \bar{z}, a\rangle$ for a fixed $\bar{z} \in \mathbb{Q}_c^\ell$ (resp., $\sigma(z) = \langle z,a\rangle$), which was used in the definition of a warped product representation of $\mathbb{Q}_c^\ell$. We say that an isometric immersion $g\colon\ L^p \to \mathbb{Q}_c^\ell$ is *cylindrical with respect to a* if $g(L) \subset W^a$ and $\mathcal{F}^a$ is everywhere tangent to $g(L)$, or equivalently, if $U_{g(y)} = \operatorname{grad}\ \sigma(g(y))$ is nonzero and tangent to $g(L)$ for any $y \in L$. The last assertion in Corollary 6-(i) yields the following result.

Corollary 7. *Let $f = \Psi \circ (F \times G)\colon\ N = L \times_\rho M \to \mathbb{Q}_c^\ell$ be a warped product of isometric immersions, where $\Psi\colon\ V^{\ell-m} \times_\sigma \mathbb{Q}_{\tilde{c}}^m \to \mathbb{Q}_c^\ell$ is a warped product representation determined by $(\bar{z}, \mathbb{Q}_{\tilde{c}}^m)$. If G is totally geodesic and F is cylindrical with respect to the mean curvature vector $-a$ of $\mathbb{Q}_{\tilde{c}}^m$ at $\bar{z}$ in either $\mathbb{R}^\ell$ or $\mathbb{O}^{\ell+1}$, according as $c = 0$ or $c \neq 0$, then the vertical subbundle of TN is contained in the relative nullity subbundle of f. Conversely, if the vertical subbundle of TN is contained in the relative nullity subbundle of f then G*

is totally geodesic and $F|_U$ *is cylindrical with respect to the mean curvature vector* $-a$ *of* $\mathbb{Q}^m_{\tilde{c}}$ *at* $\bar{z}$*; here* U *is the open subset of* L *where* grad ρ *does not vanish.*

If $f\colon L \times_\rho M \to \mathbb{Q}^\ell_c$ is a warped product of isometric immersions, then it follows from Proposition 5-(ii) that at any point $z \in L \times M$ its second fundamental form satisfies

$$\alpha(X, V) = 0 \quad \text{for all} \;\; X \in \mathcal{H}_z \;\; \text{and} \;\; V \in \mathcal{V}_z. \tag{17}$$

The following theorem due to Nölker states that the converse is also true. Recall that the *spherical hull* of an isometric immersion $G\colon M \to \mathbb{Q}^\ell_c$ is the complete spherical submanifold of least dimension that contains $G(M)$.

Theorem 8 ([16]). *Let* $f\colon L \times_\rho M \to \mathbb{Q}^\ell_c$ *be an isometric immersion of a warped product whose second fundamental form satisfies condition* (17) *everywhere. For a fixed point* $(\bar{y}, \bar{x}) \in L \times_\rho M$ *with* $\rho(\bar{y}) = 1$*, let* $F\colon L \to \mathbb{Q}^\ell_c$ *and* $G\colon M \to \mathbb{Q}^\ell_c$ *be given by* $F(y) = f(y, \bar{x})$ *and* $G(x) = f(\bar{y}, x)$*, and let* $\mathbb{Q}^m_{\tilde{c}}$ *be the spherical hull of* G*. Then* $(f(\bar{y}, \bar{x}), \mathbb{Q}^m_{\tilde{c}})$ *determines a warped product representation* $\Psi\colon V^{\ell-m} \times_\sigma \mathbb{Q}^m_{\tilde{c}} \to \mathbb{Q}^\ell_c$ *such that* $F(L) \subset V^{\ell-m}$ *and* $f = \Psi \circ (F \times G)$*, where in the last equation* F *and* G *are regarded as maps into* $V^{\ell-m}$ *and* $\mathbb{Q}^m_{\tilde{c}}$*, respectively.*

The preceding theorem is also valid for isometric immersions of warped products with arbitrarily many factors (see [16]). It contains as a particular case the following result due to Molzan (cf. Corollary 17 of [16]), which is an extension to nonflat ambient space forms of the main lemma in [14].

Corollary 9 ([13]). *Let* $f\colon L \times M \to \mathbb{Q}^\ell_c$ *be an isometric immersion of a Riemannian product whose second fundamental form satisfies condition* (17) *everywhere. For a fixed point* $(\bar{y}, \bar{x}) \in L \times M$ *define* $F\colon L \to \mathbb{Q}^\ell_c$ *and* $G\colon M \to \mathbb{Q}^\ell_c$ *by* $F(y) = f(y, \bar{x})$ *and* $G(x) = f(\bar{y}, x)$*, and denote by* $\mathbb{Q}^{\ell_1}_{c_1}$ *and* $\mathbb{Q}^{\ell_2}_{c_2}$ *the spherical hulls of* $F(L)$ *and* $G(M)$*, respectively. Then* F *and* G *are isometric immersions and there exists an isometric embedding* $\Phi\colon \mathbb{Q}^{\ell_1}_{c_1} \times \mathbb{Q}^{\ell_2}_{c_2} \to \mathbb{Q}^\ell_c$ *as an extrinsic Riemannian product such that* $f = \Phi \circ (F \times G)$*, where in the last equation* F *and* G *are regarded as maps into* $\mathbb{Q}^{\ell_1}_{c_1}$ *and* $\mathbb{Q}^{\ell_2}_{c_2}$*, respectively.*

In applying Theorem 8 one must often be able to determine the dimension of the spherical hull of G. In the remainder of this section we develop a tool for computing this dimension.

Given an isometric immersion $g\colon M^n \to \mathbb{Q}^\ell_c$, a subbundle $\tilde{\mathcal{Z}}$ of the normal bundle of g is called *umbilical* if there exists $\theta \in \Gamma(\tilde{\mathcal{Z}})$ such that

$$(\alpha^g(E_1, E_2))_{\tilde{\mathcal{Z}}} = \langle E_1, E_2 \rangle \theta$$

for all $E_1, E_2 \in \Gamma(TM)$. We say that θ is the *principal curvature normal* of $\tilde{\mathcal{Z}}$. If $n \geq 2$ and the subbundle $\tilde{\mathcal{Z}}$ is parallel in the normal connection, then the Codazzi equations of g imply that the vector field θ is also parallel in the normal connection. In particular, it has constant length. If $g(M^n)$ is contained in a complete spherical submanifold $\mathbb{Q}_{\tilde{c}}^m$ of $\mathbb{Q}_c^\ell$ with dimension m and constant sectional curvature $\tilde{c}$, then the pulled-back subbundle $\tilde{\mathcal{Z}} = g^*T^\perp\mathbb{Q}_{\tilde{c}}^m$, where $T^\perp\mathbb{Q}_{\tilde{c}}^m$ is the normal bundle of $\mathbb{Q}_{\tilde{c}}^m$ in $\mathbb{Q}_c^\ell$, is an umbilical parallel subbundle of $T^\perp M$ of rank $\ell - m$. Conversely, we have the following result due to Yau.

PROPOSITION 10 ([20]). *Let $g\colon M^n \to \mathbb{Q}_c^\ell$, $n \geq 2$, be an isometric immersion. Assume that there exists an umbilical parallel subbundle $\tilde{\mathcal{Z}}$ of $T^\perp M$ with principal curvature normal θ and rank $\ell - m$. Then there exists a complete spherical submanifold $\mathbb{Q}_{\tilde{c}}^m$ of $\mathbb{Q}_c^\ell$ with dimension m and constant sectional curvature $\tilde{c} = c + \|\theta\|^2$ such that $g(M^n) \subset \mathbb{Q}_{\tilde{c}}^m$.*

As a consequence, the dimension of the spherical hull of an isometric immersion can be characterized as follows.

COROLLARY 11. *Let $g\colon M^n \to \mathbb{Q}_c^\ell$, $n \geq 2$, be an isometric immersion. Then the dimension of the spherical hull of g is m if and only if $\ell - m$ is the maximal rank of an umbilical parallel subbundle $\tilde{\mathcal{Z}}$ of $T^\perp M$. Moreover, the spherical hull of g has constant sectional curvature $\tilde{c} = c + \|\theta\|^2$, where θ is the principal curvature normal of $\tilde{\mathcal{Z}}$.*

COROLLARY 12. *Let $f\colon N^{p+n} = L^p \times_\rho M^n \to \mathbb{Q}_c^\ell$, $n \geq 2$, be an isometric immersion of a warped product whose second fundamental form satisfies* (17) *everywhere. Given $\bar{y} \in L$ with $\rho(\bar{y}) = 1$, let $G\colon M \to \mathbb{Q}_c^\ell$ be defined by $G = f \circ i_{\bar{y}}$, where $i_{\bar{y}}\colon M^n \to N^{p+n}$ given by $i_{\bar{y}}(x) = (\bar{y}, x)$ is the (isometric) inclusion of M^n into N^{p+n} as a leaf of the vertical subbundle $\mathcal{V}$. Then the spherical hull of G has dimension $m = \ell - p - k$, where k is the maximal rank of a parallel subbundle $\mathcal{Z}$ of $i_{\bar{y}}^*T^\perp N$ such that*

$$\alpha^f(i_{\bar{y}*}V, i_{\bar{y}*}W)_{\mathcal{Z}} = \langle V, W\rangle\theta \tag{18}$$

for some $\theta \in \Gamma(\mathcal{Z})$ and for all $V, W \in \Gamma(TM)$. If $\mathcal{Z}$ is such a subbundle, then $\theta \in \Gamma(\mathcal{Z})$ is parallel, hence has constant length. Moreover, the spherical hull of G has constant sectional curvature $c + \|\theta\|^2 + \|\operatorname{grad}\log\rho(\bar{y})\|^2$.

Proof. The normal bundle of $i_{\bar{y}}$ is $i_{\bar{y}}^*\mathcal{H}$, where $\mathcal{H}$ is the horizontal subbundle of TN, hence the normal bundle of G splits as

$$T_G^\perp M = i_{\bar{y}}^*T^\perp N \oplus f_*i_{\bar{y}}^*\mathcal{H}$$

and the second fundamental form of G splits accordingly as

$$\alpha^G(V, W) = \alpha^f(i_{\bar{y}*}V, i_{\bar{y}*}W) + \langle V, W\rangle f_*(\eta \circ i_{\bar{y}}), \tag{19}$$

where $\eta = -\operatorname{grad}(\log\rho \circ \pi_L)$ is the mean curvature normal of $\mathcal{V}$. In particular, it follows that $f_*i_{\bar{y}}^*\mathcal{H}$ is an umbilical subbundle of $T_G^\perp M$ with principal curvature normal $f_*(\eta \circ i_{\bar{y}})$. Moreover, using that the second fundamental form of f satisfies (17) it follows that $f_* i_{\bar y}^* \mathcal{H}$ is parallel in the normal connection of G. It is now easily seen that a subbundle $\mathcal{Z}$ of $i_{\bar y}^* T^\perp N$ is parallel and satisfies (18) if and only if $\mathcal{Z} \oplus f_* i_{\bar y}^* \mathcal{H}$ is a parallel umbilical subbundle of $T_G^\perp M$ with principal curvature normal $\theta + f_*(\eta \circ i_{\bar y})$. The conclusion follows from Corollary 11. □

In the sequel only the following two special cases of Corollary 12 will be needed, in which the assumptions in part (i) (resp., (ii)) easily imply that the vector subbundle $\mathcal{Z}$ equals $i_{\bar y}^* T^\perp N$ (resp., $\{0\}$).

Corollary 13. *Under the assumptions of Corollary 12 we have:*

(i) *If the vertical subbundle $\mathcal{V}$ is contained in the eigendistribution corresponding to a principal curvature normal ζ of f, then the spherical hull of G has dimension $m = n$ and constant sectional curvature $\tilde c = c + \|\zeta \circ i_{\bar y}\|^2 + \|\,\mathrm{grad}\,\log\,\rho(\bar y)\|^2$.*

(ii) *If there exists no local vector field $\bar\xi \in \Gamma(i_{\bar y}^* T^\perp N)$ such that $A^f_{\bar\xi} \circ i_{\bar y *} = \lambda i_{\bar y *}$ for some $\lambda \in C^\infty(M)$, then the spherical hull of G has dimension $m = \ell - p$ and constant sectional curvature $\tilde c = c + \|\,\mathrm{grad}\,\log\,\rho(\bar y)\|^2$.*

3. The results

Our main result provides a complete local classification of isometric immersions $f\colon L^p \times_\rho M^n \to \mathbb{Q}_c^{p+n+2}$ of a warped product under the assumptions that $n \geq 3$ and that $N^{p+n} = L^p \times_\rho M^n$ is free of points with constant sectional curvature c. Here and in the sequel it is always assumed that $p, n \geq 1$, and only further restrictions on those dimensions are explicitly stated.

Theorem 14. *Assume that a warped product $N^{p+n} = L^p \times_\rho M^n$ with $n \geq 3$ is free of points with constant sectional curvature c. Then for any isometric immersion $f\colon N^{p+n} \to \mathbb{Q}_c^{p+n+2}$ there exists an open dense subset of N^{p+n} each of whose points lies in an open product neighborhood $U = L_0^p \times M_0^n \subset L^p \times M^n$ such that one of the following possibilities holds:*

(i) *$f|_U$ is a warped product of isometric immersions with respect to a warped product representation $\Psi\colon V^{p+k_1} \times_\sigma \mathbb{Q}_{\tilde c}^{n+k_2} \to \mathbb{Q}_c^{p+n+2}$, $k_1 + k_2 = 2$.*

$$\begin{tikzcd}
V^{p+k_1} \times_\sigma \mathbb{Q}_{\tilde c}^{n+k_2} \arrow[drr, "\Psi"] & & \\
L_0^p \times_{\sigma\circ h_1} M_0^n \arrow[u, "h_1 \quad h_2"] \arrow[rr, "f|_U = \Psi \circ (h_1 \times h_2)"'] \arrow[ur, phantom, "\circlearrowleft"] & & \mathbb{Q}_c^{p+n+2}
\end{tikzcd}$$

(ii) *$f|_U$ is a composition $H \circ g$ of isometric immersions, where g is a warped product of isometric immersions $g = \Psi \circ (h_1 \times h_2)$ determined by a warped product representation $\Psi\colon V^{p+k_1} \times_\sigma \mathbb{Q}_{\tilde{c}}^{n+k_2} \to \mathbb{Q}_c^{p+n+1}$ with $k_1 + k_2 = 1$, and $H\colon W \to \mathbb{Q}_c^{p+n+2}$ is an isometric immersion of an open subset $W \supset g(U)$ of $\mathbb{Q}_c^{p+n+1}$.*

$$\begin{tikzcd}
V^{p+k_1} \times_\sigma \mathbb{Q}_{\tilde{c}}^{n+k_2} \arrow[r, "\Psi"] \arrow[dr, phantom, "\circlearrowright"] & W \subset \mathbb{Q}_c^{p+n+1} \arrow[d, "H"] \\
L_0^p \times_{\sigma\circ h_1} M_0^n \arrow[u, "h_1", "h_2"'] \arrow[r, "f|_U = H \circ \Psi \circ (h_1 \times h_2)"'] & \mathbb{Q}_c^{p+n+2}
\end{tikzcd}$$

(iii) *There exist open intervals $I, J \subset \mathbb{R}$ such that L_0^p, M_0^n, U split as $L_0^p = L_0^{p-1} \times_{\rho_1} I$, $M_0^n = J \times_{\rho_2} M_0^{n-1}$ and*

$$U = L_0^{p-1} \times_{\rho_1} ((I \times_{\rho_3} J) \times_{\bar{\rho}} M_0^{n-1}),$$

where $\rho_1 \in C^\infty(L_0^{p-1})$, $\rho_2 \in C^\infty(J)$, $\rho_3 \in C^\infty(I)$ and $\bar{\rho} \in C^\infty(I \times J)$ satisfy

$$\rho = (\rho_1 \circ \pi_{L_0^{p-1}})(\rho_3 \circ \pi_I) \quad \textit{and} \quad \bar{\rho} = (\rho_3 \circ \pi_I)(\rho_2 \circ \pi_J),$$

and there exist warped product representations

$$\Psi_1\colon V^{p-1} \times_{\sigma_1} \mathbb{Q}_{\tilde{c}}^{n+3} \to \mathbb{Q}_c^{p+n+2} \quad \textit{and} \quad \Psi_2\colon W^4 \times_{\sigma_2} \mathbb{Q}_{\tilde{c}}^{n-1} \to \mathbb{Q}_{\tilde{c}}^{n+3},$$

an isometric immersion $g\colon I \times_{\rho_3} J \to W^4$ and isometries $i_1\colon L_0^{p-1} \to W^{p-1} \subset V^{p-1} \subset \mathbb{Q}_c^{p-1}$ and $i_2\colon M^{n-1} \to W^{n-1} \subset \mathbb{Q}_{\tilde{c}}^{n-1}$ onto open subsets such that $f|_U = \Psi_1 \circ (i_1 \times (\Psi_2 \circ (g \times i_2)))$, $\bar{\rho} = \sigma_2 \circ g$ and $\rho_1 = \sigma_1 \circ i_1$. Moreover, L_0^p has constant sectional curvature c if $p \geq 2$.

$$\begin{tikzcd}
V^{p-1} \times_{\sigma_1} \mathbb{Q}_{\tilde{c}}^{n+3} \arrow[rrr, "\Psi_1"] & & & \mathbb{Q}_c^{p+n+2} \\
\cup & & & \\
W^{p-1} & & & \\
 & W^4 \times_{\sigma_2} \arrow[uuul, "\Psi_2"] \arrow[uuur, phantom, "\circlearrowright"] & \mathbb{Q}_{\tilde{c}}^{n-1} & \\
 & & \cup & \\
 & & W^{n-1} & \\
L_0^{p-1} \times_{\rho_1=\sigma_1\circ i_1} \arrow[uuuu, "i_1"] & ((I \times_{\rho_3} J) \arrow[uuu, "g"] & \times_{\bar{\rho}=\sigma_2\circ g} M_0^{n-1}) \arrow[u, "i_2"] & = U = L_0^p \times_\rho M_0^n \arrow[uuuuuu, "f|_U = \Psi_1 \circ (i_1 \times (\Psi_2 \circ (g \times i_2)))"']
\end{tikzcd}$$

In case (iii) the isometric immersion $g\colon I\times_{\rho_3} J\to W^4$ is neither a warped product $g=\Psi_3\circ(\alpha\times\beta)$, where $\Psi_3\colon V^{1+k_1}\times_{\sigma_3}\mathbb{Q}_{\hat c}^{1+k_2}\to\mathbb{Q}_{\tilde c}^4$ is a warped product representation with $k_1+k_2=2$ and $\alpha\colon I\to V^{1+k_1}$ and $\beta\colon J\to\mathbb{Q}_{\hat c}^{1+k_2}$ are unit speed curves with $\rho_3=\sigma_3\circ\alpha$, nor a composition $H\circ G$ of such a warped product $G=\Psi_3\circ(\alpha\times\beta)$, determined by a warped product representation $\Psi_3\colon V^{1+k_1}\times_{\sigma_3}\mathbb{Q}_{\hat c}^{1+k_2}\to\mathbb{Q}_{\tilde c}^3$ as before with $k_1+k_2=1$, and an isometric immersion H of an open subset $W\supset G(I\times J)$ into $\mathbb{Q}_{\tilde c}^4$. It would be interesting to exhibit an explicit example of such an isometric immersion. Notice that it must satisfy the additional condition $\sigma_2\circ g=(\rho_3\circ\pi_I)(\rho_2\circ\pi_J)$ for some $\rho_2\in C^\infty(J)$.

Cases (i)–(iii) are disjoint. In fact, we will prove that under the assumptions of the theorem there are three distinct possible structures for the second fundamental form of f, each of which corresponds to one of the cases in the statement.

Notice that the conclusion of the theorem remains unchanged under the apparently weaker assumption that the subset of points of N^{p+n} with constant sectional curvature c has empty interior.

Theorem 14 does not hold without the assumption that $n\geq 3$. In fact, we argue next that local isometric immersions of the round three-dimensional sphere $\mathbb{S}^3$ into $\mathbb{R}^5$ are generically as in neither of the cases in the statement with respect to any local decomposition of $\mathbb{S}^3$ as a warped product.

Example 15. It was shown in [8] (cf. Corollary 4 in [9]) that local isometric immersions of $\mathbb{S}^3$ into $\mathbb{R}^5$ that are *nowhere compositions* (i.e., on no open subset they are compositions of the umbilical inclusion into $\mathbb{R}^4$ with a local isometric immersion of $\mathbb{R}^4$ into $\mathbb{R}^5$) are in correspondence with solutions (V,h) on open simply connected subsets $U_0\subset\mathbb{R}^3$ of the nonlinear system of PDE's

$$(I)\quad\begin{cases}(i)\ \dfrac{\partial V_{ir}}{\partial u_j}=h_{ji}V_{jr}, & (ii)\ \dfrac{\partial h_{ik}}{\partial u_j}=h_{ij}h_{jk},\\[2mm] (iii)\ \dfrac{\partial h_{ij}}{\partial u_i}+\dfrac{\partial h_{ji}}{\partial u_j}+\displaystyle\sum_k h_{ki}h_{kj}+V_{i3}V_{j3}=0, & i\neq j\neq k\neq i,\end{cases}$$

called the generalized elliptic sinh-Gordon equation. Here $V\colon U_0\to\mathbb{O}_1(3)$ is a smooth map taking values in the group of orthogonal matrices with respect to the Lorentz metric of signature $(+,+,-)$ and $x\in U_0\mapsto h(x)$ is a smooth map such that $h(x)$ is an off-diagonal (3×3)-matrix for every $x\in U_0$. More precisely, for any such isometric immersion there exist a local system of coordinates (u_1,u_2,u_3), an orthonormal normal frame $\{\xi_1,\xi_2\}$ and matrix functions V and h as above such that

$$A_{\xi_r}X_i=V_{i3}^{-1}V_{ir}X_i,\ \ 1\leq r\leq 2,\ \ 1\leq i\leq 3, \tag{20}$$

and

$$\nabla_{\partial/\partial u_i} X_j = h_{ji} X_i, \ \ 1 \le i \ne j \le 3, \tag{21}$$

where X_i is a unit vector field with $\partial/\partial u_i = V_{i3} X_i$. The compatibility equations for f are equivalent to system (I). Conversely, any solution (V, h) of system (I) on an open simply connected subset $U_0 \subset \mathbb{R}^3$ gives rise to such an isometric immersion by means of the fundamental theorem of submanifolds.

By a theorem of Bourlet (see [4], Théorème VIII), there exists one and only one analytic solution (V, h) of system (I) in a neighborhood of an initial value $u_0 = (u_1^0, u_2^0, u_3^0)$ such that $V(u_0) \in \mathbb{O}_1(3)$ and such that V_i^k and h_{ij}, $i < j$ (resp., $i > j$) reduce to an arbitrarily given analytic function of u_i (resp., u_j) when the remaining variables take their initial values. Thus, for a generic local analytic solution (V, h) the functions h_{ij} are nowhere vanishing; see the last section of [9] for explicit isometric immersions with this property.

It follows easily from (20) that no such isometric immersion admits a normal vector field whose shape operator has rank one. In particular, it can not be as in case (ii). Also, if f is as in case (i) with respect to a decomposition $U = L^{k_1} \times_\rho M^{k_2}$ of U as a warped product, then we must have that $k_2 = 1$. In fact, otherwise the second fundamental form of f would be given by

$$\alpha(Y, Z) = \langle Y_1, Z_1 \rangle \eta_1 + \langle Y_2, Z_2 \rangle \eta_2$$

for some normal vector fields η_1, η_2 satisfying $\langle \eta_1, \eta_2 \rangle = 1 = \|\eta_2\|$, where Y_i, Z_i, $1 \le i \le 2$, are the components of Y, Z according to the product decomposition of U. This easily implies that the shape operator with respect to a normal vector field orthogonal to η_2 has rank one. Thus, the distribution tangent to the second factor is one-dimensional and invariant by all shape operators of f, and hence it must be spanned by one of the vector fields X_i, $1 \le i \le 3$, say, X_3. In particular, this implies that the distribution spanned by X_1, X_2 is totally geodesic, and hence the functions h_{31} and h_{32} vanish everywhere by (21). Finally, we claim that the same holds if f is as in case (iii). In effect, in this case U splits as a warped product

$$U = L^1 \times_\rho \mathbb{Q}_{\tilde{c}}^2 = L^1 \times_\rho (J \times_{\bar{\rho}} M^1) = (L^1 \times_\rho J) \times_{(\rho \circ \pi_{L^1})\, \bar{\rho}} M^1,$$

and f is a warped product $f = g \times i$ with respect to the last decomposition. Thus, we have again that the one-dimensional distribution tangent to M^1 is invariant by all shape operators of f, and the same argument used in the preceding case proves our claim. It follows that f is generically as in neither of the cases in Theorem 14 with respect to any local decomposition of $\mathbb{S}^3$ as a warped product.

We now discuss some further results. The case of hypersurfaces is interesting in its own right. Although it can be proved as a corollary of Theorem 14, it is easier to derive it as an immediate consequence of Theorem 8 and Proposition 23 of Section 3.

THEOREM 16. *Assume that a warped product $L^p \times_\rho M^n$, $n \geq 2$, has no points with constant sectional curvature c. Then any isometric immersion $f\colon L^p \times_\rho M^n \to \mathbb{Q}_c^{p+n+1}$ is a warped product $f = \Psi \circ (F \times G)$, where $\Psi\colon V^{p+k_1} \times_\sigma \mathbb{Q}_{\tilde{c}}^{n+k_2} \to \mathbb{Q}_c^{p+n+1}$ is a warped product representation with $k_1+k_2 = 1$ and $F\colon L^p \to V^{p+k_1}$, $G\colon M^n \to \mathbb{Q}_{\tilde{c}}^{n+k_2}$ are isometric immersions.*

Again, the preceding result is false if the assumption that $n \geq 2$ is dropped; rotation surfaces in $\mathbb{R}^3$ admit many isometric deformations into nonrotational surfaces (cf. [2]).

In deriving Theorem 14 we also obtain the following result for the case of Riemannian products, which extends Theorem 1 in [14] in the case of products with two factors. Therein, isometric immersions of Riemannian products with arbitrarily many factors into Euclidean space were shown to split as a product of isometric immersions under the assumptions that no factor has an open subset of flat points and that the codimension equals the number of factors. We point out that in the case of Riemannian products the factors may change the roles. This observation is applied several times throughout the paper.

THEOREM 17. *Let $f\colon L^p \times M^n \to \mathbb{Q}_c^{p+n+2}$ be an isometric immersion of a Riemannian product. If $c = 0$ assume that either L^p or M^n has dimension at least two and is free of flat points. If $c \neq 0$ assume that either $n \geq 3$ or $p \geq 3$. Then there exists an open dense subset of $L^p \times M^n$ each of whose points lies in an open product neighborhood $U = L_0^p \times M_0^n \subset L^p \times M^n$ such that one of the following possibilities holds:*

Case $c = 0$.

(i) *There exist an orthogonal decomposition $\mathbb{R}^{p+n+2} = \mathbb{R}^{p+k_1} \times \mathbb{R}^{n+k_2}$ with $k_1 + k_2 = 2$ and isometric immersions $h_1\colon L_0^p \to \mathbb{R}^{p+k_1}$ and $h_2\colon M_0^n \to \mathbb{R}^{n+k_2}$ such that $f|_U = h_1 \times h_2$.*

$$\begin{tikzcd}
\mathbb{R}^{p+k_1} & \times & \mathbb{R}^{n+k_2} & = & \mathbb{R}^{p+n+2} \\
L_0^p \arrow[u, "h_1"] & \times & M_0^n \arrow[u, "h_2"] \arrow[urr, "f|_U = h_1 \times h_2"'] & &
\end{tikzcd}$$

(ii) *There exist an orthogonal decomposition $\mathbb{R}^{p+n+1} = \mathbb{R}^{p+k_1} \times \mathbb{R}^{n+k_2}$, $k_1 + k_2 = 1$, and isometric immersions $h_1\colon L_0^p \to \mathbb{R}^{p+k_1}$, $h_2\colon M_0^n \to \mathbb{R}^{n+k_2}$ and $H\colon W \to \mathbb{R}^{p+n+2}$ of an open subset $W \supset (h_1 \times h_2)(U)$ of $\mathbb{R}^{p+n+1}$ such that $f|_U = H \circ (h_1 \times h_2)$.*

$$
\begin{tikzcd}
\mathbb{R}^{p+k_1} & \times & \mathbb{R}^{n+k_2} \supset W \arrow[drr, "H"] & & \\
L_0^p \arrow[u, "h_1"] & \times & M_0^n \arrow[u, "h_2"] \arrow[rr, "f|_U = H\circ(h_1\times h_2)"'] & \arrow[u, phantom, "\circlearrowright"] & \mathbb{R}^{p+n+2}
\end{tikzcd}
$$

Case $c \neq 0$.

(i) *There exist an embedding $\Phi\colon\ \mathbb{Q}_{c_1}^{p+k_1}\times\mathbb{Q}_{c_2}^{n+k_2} \to \mathbb{Q}_c^{p+n+2}$ as an extrinsic Riemannian product with $k_1+k_2=1$, and isometric immersions $h_1\colon\ L_0^p \to \mathbb{Q}_{c_1}^{p+k_1}$ and $h_2\colon\ M_0^n \to \mathbb{Q}_{c_2}^{n+k_2}$ such that $f|_U = \Phi\circ(h_1\times h_2)$.*

$$
\begin{tikzcd}
\mathbb{Q}_{c_1}^{p+k_1} & \times & \mathbb{Q}_{c_2}^{n+k_2} \arrow[r, "\Phi"] & \mathbb{Q}_c^{p+n+2} \\
L_0^p \arrow[u, "h_1"] & \times & M_0^n \arrow[u, "h_2"] \arrow[ur, "f|_U = \Phi\circ(h_1\times h_2)"'] \arrow[ur, phantom, "\circlearrowright", bend left] &
\end{tikzcd}
$$

(ii) *There exist an embedding $\Phi\colon\ \mathbb{Q}_{c_1}^{p}\times\mathbb{Q}_{c_2}^{n} \to \mathbb{Q}_c^{p+n+1}$ as an extrinsic Riemannian product, local isometries $i_1\colon\ L_0^p \to \mathbb{Q}_{c_1}^{p}$ and $i_2\colon\ M_0^n \to \mathbb{Q}_{c_2}^{n}$, and an isometric immersion $H\colon\ W \to \mathbb{Q}_c^{p+n+2}$ of an open subset $W \supset \Phi\circ(i_1\times i_2)(U)$ of $\mathbb{Q}_c^{p+n+1}$ such that $f|_U = H\circ\Phi\circ(i_1\times i_2)$.*

$$
\begin{tikzcd}
\mathbb{Q}_{c_1}^{p} & \times & \mathbb{Q}_{c_2}^{n} \arrow[rr, "\Phi"] & & W \subset \mathbb{Q}_c^{p+n+1} \arrow[d, "H"] \\
L_0^p \arrow[u, "i_1"] & \times & M_0^n \arrow[u, "i_2"] \arrow[rr, "f|_U = H\circ\Phi\circ(i_1\times i_2)"'] & \arrow[u, phantom, "\circlearrowright"] & \mathbb{Q}_c^{p+n+2}
\end{tikzcd}
$$

As an example showing that for $c \neq 0$ the assumption that either $n \geq 3$ or $p \geq 3$ is indeed necessary, we may take any local isometric immersion of $\mathbb{R}^3$ into $\mathbb{S}^5$ that is not a product $\alpha\times g\colon\ I\times V \to \mathbb{S}^1(r_1)\times\mathbb{S}^3(r_2)$, $\ r_1^2+r_2^2=1$, where $\alpha\colon\ I\to\mathbb{R}^2$ is a unit speed parametrization of an open subset of a circle of radius r_1 and $g\colon\ V\to\mathbb{S}^3(r_2)$ is an isometric immersion of an open subset $V\subset\mathbb{R}^2$. Recall that local isometric immersions of $\mathbb{R}^3$ into $\mathbb{S}^5$ were shown in [19] to be in correspondence with solutions on simply connected open subsets of $\mathbb{R}^3$ of the so-called generalized wave equation. As in the previous discussion on local isometric immersions of $\mathbb{S}^3$ into $\mathbb{R}^5$, one may easily argue that the class of local isometric immersions of $\mathbb{R}^3$ into $\mathbb{S}^5$ that are given as products as just described is only a rather special subclass of the whole class of such isometric immersions.

We now give precise statements of the applications of Theorem 14 referred to at the end of the introduction. Recall that an isometric immersion $F\colon L^p \to \mathbb{Q}_c^{p+m}$ is said to be *locally rigid* if it is rigid restricted to any open subset of L^p.

COROLLARY 18. *Let L^p be a Riemannian manifold, no open subset of which can be isometrically immersed in $\mathbb{Q}_c^{p+1}$. If $f\colon L^p \times_\rho M^n \to \mathbb{Q}_c^{p+n+2}$, $n \geq 3$, is an isometric immersion, then there exist a warped product representation $\Psi\colon V^{p+2} \times_\sigma \mathbb{Q}_{\tilde{c}}^n \to \mathbb{Q}_c^{p+n+2}$, an isometric immersion $F\colon L^p \to V^{p+2}$ and a local isometry $i\colon M^n \to \mathbb{Q}_{\tilde{c}}^n$ such that $f = \Psi \circ (F \times i)$. In particular, if L^p is a Riemannian manifold that admits a locally rigid isometric immersion $F\colon L^p \to \mathbb{Q}_c^{p+2}$, then the preceding conclusion holds and, in addition, the isometric immersion f is also locally rigid.*

Proof. By Theorem 14, any isometric immersion $f\colon L^p \times_\rho M^n \to \mathbb{Q}_c^{p+n+2}$, $n \geq 3$, must be locally as in one of the three cases described in its statement. However, under the assumption that L^p has no open subset that can be isometrically immersed in $\mathbb{Q}_c^{p+1}$, it follows that $f|_U$ can not be as in case (ii) on any open subset $U = L_0^p \times M_0^n \subset L^p \times M^n$, for there can not exist by that assumption any isometric immersion $h_1\colon L_0^p \to \mathbb{Q}_c^{p+k_1}$ with $0 \leq k_1 \leq 1$. Moreover, $f|_U$ can not be as in case (iii) either on any such open subset, for in that case L_0^p would have constant sectional curvature c, and hence it would admit locally an isometric immersion into $\mathbb{Q}_c^{p+1}$. Therefore f must be globally as in case (i). The last assertion is now clear. $\square$

We say that a Riemannian manifold can be locally isometrically immersed in $\mathbb{Q}_c^\ell$ if each point has an open neighborhood that admits an isometric immersion into $\mathbb{Q}_c^\ell$. Arguing in a similar way as in the proof of Corollary 18 yields the following result.

COROLLARY 19. *Let L^p be a Riemannian manifold that cannot be locally isometrically immersed in $\mathbb{Q}_c^{p+2}$. Then $L^p \times_\rho M^n$ can not be locally isometrically immersed in $\mathbb{Q}_c^{p+n+2}$ for any Riemannian manifold M^n of dimension $n \geq 3$ and any warping function ρ.*

In view of Nölker's result it is also natural to study isometric immersions of warped products into space forms in the light of the following definition.

DEFINITION. Let $f\colon L^p \times_\rho M^n \to \mathbb{Q}_c^\ell$ be an isometric immersion of a warped product. Given $z \in L^p \times_\rho M^n$, we set

$$\alpha(\mathcal{H}_z, \mathcal{V}_z) = \operatorname{span}\{\alpha(Y, V) : Y \in \mathcal{H}_z \text{ and } V \in \mathcal{V}_z\}.$$

We say that the immersion f at z is of *type*

(A) if $\dim\alpha(\mathcal{H}_z,\mathcal{V}_z)=0$, i.e., $\alpha(Y,V)=0$ for all $Y\in\mathcal{H}_z$ and $V\in\mathcal{V}_z$,
(B) if $\dim\alpha(\mathcal{H}_z,\mathcal{V}_z)=1$,
(C) if $\dim\alpha(\mathcal{H}_z,\mathcal{V}_z)\geq 2$.

Notice that type A is the case of Nölker's decomposition theorem. Therefore, it is a natural problem to determine the isometric immersions that are everywhere of type B. In the following section we obtain a complete solution to this problem in the codimension two case under the assumption that $n\geq 3$ (for the case of Riemannian products it is enough to assume that $p+n\geq 3$); see the two paragraphs before Proposition 27. The proof of Theorem 14 is then accomplished as follows: type C is excluded by Proposition 36, type A corresponding to Theorem 8 gives case (i), and type B splits into two subcases B_1 and B_2 handled in Propositions 27 and 31, respectively, which correspond to the cases (ii) and (iii). Similarly for the proof of Theorem 17: type C is excluded by Corollary 37, type A corresponding to Theorem 9 gives subcase (i) in both cases $c=0$ and $c\neq 0$, type B_1 handled in Corollary 30 gives subcase (ii) in either case, and type B_2 is excluded in either case by Corollary 34 and Corollary 32, respectively.

4. Immersions of type B

Our main goal in this section is to provide a complete local classification of isometric immersions $f\colon L^p\times_\rho M^n\to\mathbb{Q}_c^{p+n+2}$ that are everywhere of type B under the assumption that $n\geq 3$. A similar classification for the special case of isometric immersions of Riemannian products is also given, for which it is enough to assume $p+n\geq 3$.

First we determine the pointwise structure of the second fundamental forms of isometric immersions of type B, starting with some general facts that are valid in arbitrary codimension.

Lemma 20. *Let $f\colon L^p\times_\rho M^n\to\mathbb{Q}_c^\ell$ be an isometric immersion of a warped product. Assume that f is not of type A at a point $z\in N^{p+n}=L^p\times_\rho M^n$ and that for every $Y\in\mathcal{H}_z$ the linear map*

$$B_Y\colon \mathcal{V}_z\to T_z^\perp N,\quad V\mapsto\alpha(Y,V),\tag{22}$$

satisfies $\operatorname{rank}B_Y\leq 1$. *Then there exists a unit vector $e\in\mathcal{V}_z$, uniquely determined up to its sign, such that*

$$\alpha(Y,V)=\langle V,e\rangle\alpha(Y,e)\tag{23}$$

and

$$\alpha(V,W)-\langle V,e\rangle\langle W,e\rangle\alpha(e,e)\perp\alpha(\mathcal{H}_z,\mathcal{V}_z)\tag{24}$$

for all $Y\in\mathcal{H}_z$ and $V,W\in\mathcal{V}_z$.

Proof. Let $X \in \mathcal{H}_z$ be such that $\operatorname{rank} B_X = 1$. Then $\mathcal{D}(X) := \ker B_X$ has codimension 1 in $\mathcal{V}_z$. Let $e \in \mathcal{V}_z$ be one of the unit vectors perpendicular to $\mathcal{D}(X)$ and write $B_X e = \lambda\xi$, where $\lambda \neq 0$ and $\xi \in T_z^\perp N$ is a unit vector. Let $Y \in \mathcal{H}_z$ and $V \in \mathcal{D}(X)$ be arbitrary vectors. Then (13) implies

$$\begin{aligned}\langle B_X e, B_Y V\rangle &= \langle \alpha(X,e), \alpha(Y,V)\rangle = \langle \alpha(X,V), \alpha(Y,e)\rangle \\ &= \langle B_X V, \alpha(Y,e)\rangle = 0.\end{aligned} \tag{25}$$

Now consider the linear map B_{X+tY} for arbitrary $t \in \mathbb{R}$. By assumption its rank is at most 1. Therefore the vectors $B_{X+tY}e = \lambda\xi + tB_Y e$ and $B_{X+tY}V = tB_Y V$ are linearly dependent, and hence

$$\langle B_{X+tY}e, B_{X+tY}e\rangle\langle B_{X+tY}V, B_{X+tY}V\rangle - \langle B_{X+tY}e, B_{X+tY}V\rangle^2 = 0.$$

As the left hand side of this equation is a polynomial $\sum_{i=2}^4 a_i t^i$, its coefficients must vanish; in particular, because of (25) we obtain $0 = a_2 = \lambda^2\|B_Y V\|^2$. Hence $B_Y|_{\mathcal{D}(X)} = 0$, and (23) follows.

By means of (14) we derive

$$\langle \alpha(V,W), \alpha(Y,e)\rangle = \langle \alpha(Y,V), \alpha(e,W)\rangle = \langle V,e\rangle\langle \alpha(e,W), \alpha(Y,e)\rangle.$$

Applying this result to $\alpha(W,e)$ instead of $\alpha(V,W)$ we obtain

$$\langle \alpha(V,W) - \langle V,e\rangle\langle W,e\rangle\alpha(e,e), \alpha(Y,e)\rangle = 0,$$

which implies (24) in view of (23). $\square$

LEMMA 21. *Let $f\colon L^p \times_\rho M^n \to \mathbb{Q}_c^\ell$ be an isometric immersion of a warped product. Assume that f is of type B at $z \in N^{p+n} = L^p \times_\rho M^n$. Then there exist unique, up to their signs, unit vectors $X \in \mathcal{H}_z$, $e \in \mathcal{V}_z$ and $\xi \in T_z^\perp N$, and $\beta, \lambda, \gamma \in \mathbb{R}$ with $\lambda \neq 0$ such that*

$$\langle \alpha(Y,Z), \xi\rangle = \beta\langle Y,X\rangle\langle Z,X\rangle, \tag{26}$$

$$\alpha(Y,V) = \lambda\langle Y,X\rangle\langle V,e\rangle\xi, \tag{27}$$

$$\langle \alpha(V,W), \xi\rangle = \gamma\langle V,e\rangle\langle W,e\rangle, \tag{28}$$

$$\begin{aligned}&\langle \tilde{P}\alpha(Y,Z), \tilde{P}\alpha(V,W) - \langle V,W\rangle\tilde{P}\alpha(e,e)\rangle \\ &\qquad = (\beta\gamma - \lambda^2)\langle Y,X\rangle\langle Z,X\rangle\langle PV, PW\rangle,\end{aligned} \tag{29}$$

where $\tilde{P}\colon T_z^\perp N \to T_z^\perp N$ and $P\colon \mathcal{V}_z \to \mathcal{V}_z$ denote the orthogonal projections onto the subspaces $\{\xi\}^\perp \subset T_z^\perp N$ and $\{e\}^\perp \subset \mathcal{V}_z$, respectively. Moreover, if $N^{p+n} = L^p \times M^n$ is a Riemannian product then

$$\begin{aligned}&\langle \tilde{P}\alpha(Y,Z), \tilde{P}\alpha(V,W)\rangle + (\beta\gamma - \lambda^2)\langle Y,X\rangle\langle Z,X\rangle\langle V,e\rangle\langle W,e\rangle \\ &\qquad + c\langle Y,Z\rangle\langle V,W\rangle = 0.\end{aligned} \tag{30}$$

Proof. Let $\xi \in T_z^\perp N$ be a unit vector such that $\alpha(\mathcal{H}_z, \mathcal{V}_z) = \mathbb{R}\,\xi$. Given $Y \in \mathcal{H}_z$, then B_Y takes its values in $\mathbb{R}\,\xi$, and hence $\operatorname{rank} B_Y \leq 1$. Thus, we may apply Lemma 20. On the other hand, since the linear map $\mathcal{H}_z \to T_z^\perp N$

defined by $Y \mapsto \alpha(Y, e)$ also takes its values in $\mathbb{R}\,\xi$, it follows that, up to sign, there exists exactly one unit vector $X \in \mathcal{H}_z$ perpendicular to its kernel. Set $\gamma = \langle \alpha(e,e), \xi \rangle$, $\lambda = \langle \alpha(X,e), \xi \rangle$ and $\beta = \langle \alpha(X,X), \xi \rangle$. Notice that $\lambda \neq 0$ because f is of type B at z. We obtain (28) from (24), whereas (27) follows from (23) and $\alpha(Y,e) = \langle Y, X \rangle \alpha(X,e) = \lambda \langle Y, X \rangle \xi$. Using (12) we obtain

$$\lambda \langle \alpha(Y,Z), \xi \rangle = \langle \alpha(Y,Z), \alpha(X,e) \rangle = \langle \alpha(X,Z), \alpha(Y,e) \rangle = \lambda \langle Y, X \rangle \langle \alpha(X,Z), \xi \rangle,$$

and applying this result to $\alpha(Z,X)$ instead of $\alpha(Y,Z)$ we end up with (26).

We obtain from (11), (26), (27) and (28) that

$$\begin{aligned} &\langle V, W \rangle \langle \nabla_Y \eta - \langle Y, \eta \rangle \eta - cY, Z \rangle \\ &\qquad = \langle \alpha(Y,Z), \alpha(V,W) \rangle - \langle \alpha(Y,W), \alpha(Z,V) \rangle \\ &\qquad = (\beta\gamma - \lambda^2) \langle Y, X \rangle \langle Z, X \rangle \langle V, e \rangle \langle W, e \rangle + \langle \tilde{P}\alpha(Y,Z), \tilde{P}\alpha(V,W) \rangle. \end{aligned}$$

This yields (30) if $N^{p+n} = L^p \times M^n$ is a Riemannian product. In the general case, putting $W = V = e$ we get

$$\langle \nabla_Y \eta - \langle Y, \eta \rangle \eta - cY, Z \rangle = (\beta\gamma - \lambda^2) \langle Y, X \rangle \langle Z, X \rangle + \langle \tilde{P}\alpha(Y,Z), \tilde{P}\alpha(e,e) \rangle.$$

The two preceding equations yield

$$\begin{aligned} &\langle \tilde{P}\alpha(Y,Z), \tilde{P}\alpha(V,W) - \langle V, W \rangle \tilde{P}\alpha(e,e) \rangle \\ &\qquad = (\beta\gamma - \lambda^2) \langle Y, X \rangle \langle Z, X \rangle (\langle V, W \rangle - \langle V, e \rangle \langle W, e \rangle), \end{aligned}$$

which coincides with (29). □

Remark and Definition 22. Equations (26), (27) and (28) are equivalent to

$$A_\xi Y = \langle Y, X \rangle (\beta X + \lambda e), \quad A_\xi V = \langle V, e \rangle (\lambda X + \gamma e), \tag{31}$$

and

$$\tilde{P}\alpha(Y,V) = 0. \tag{32}$$

In particular, it follows from (31) that the rank of A_ξ at z is either 1 or 2, according as $\beta\gamma - \lambda^2$ is zero or not. We say accordingly that f is of *type* B_1 or of *type* B_2 at z.

We now show that in the case of hypersurfaces $f\colon L^p \times_\rho M^n \to \mathbb{Q}_c^{p+n+1}$, $n \geq 2$, only types A and B_1 can occur pointwise.

Proposition 23. *Let $f\colon L^p \times_\rho M^n \to \mathbb{Q}_c^{p+n+1}$, $n \geq 2$, be an isometric immersion of a warped product. Then, at any point $z \in N^{p+n} = L^p \times_\rho M^n$ either f is of type A or of type B_1. Moreover, in the latter case N^{p+n} has constant sectional curvature c at z.*

Proof. Assume that f is not of type A at z. Since $n \geq 2$, we may choose a unit vector $V \in \{e\}^\perp \subset \mathcal{V}_z$. Applying (29) for $W = V$ and $Z = Y = X$, and using that $\tilde{P} = 0$, it follows that $\beta\gamma - \lambda^2 = 0$. Therefore f is of type B_1 at z. The last assertion follows from the Gauss equation of f. □

Theorem 16 now follows by putting together the preceding result and Theorem 8. For Riemannian products we obtain the following corollary.

COROLLARY 24. *Let $f\colon L^p \times M^n \to \mathbb{Q}_c^{p+n+1}$ be an isometric immersion of a Riemannian product. Assume that $p+n \geq 3$ and, if $c = 0$, that either L^p or M^n, say, the latter, has dimension at least two and is free of flat points. Then f is of type A everywhere and we have:*

(i) *If $c = 0$ there exist an orthogonal decomposition $\mathbb{R}^{p+n+1} = \mathbb{R}^p \oplus \mathbb{R}^{n+1}$, a local isometry $i\colon L^p \to \mathbb{R}^p$ and an isometric immersion $h\colon M^n \to \mathbb{R}^{n+1}$ such that $f = i \times h$.*

(ii) *If $c \neq 0$ there exist an embedding $\Phi\colon \mathbb{Q}_{c_1}^p \times \mathbb{Q}_{c_2}^n \to \mathbb{Q}_c^{p+n+1}$ as an extrinsic Riemannian product and local isometries $i_1\colon L^p \to \mathbb{Q}_{c_1}^p$ and $i_2\colon M^n \to \mathbb{Q}_{c_2}^n$ such that $f = \Phi \circ (i_1 \times i_2)$.*

From now on we consider isometric immersions $f\colon L^p \times_\rho M^n \to \mathbb{Q}_c^{p+n+2}$. Assume that f is of type B at a point $z \in N^{p+n} = L^p \times_\rho M^n$ and let $X, e, \xi, \beta, \lambda$ and γ be as in Lemma 21. Choose one of the unit vectors $\tilde{\xi} \in T_z^\perp N$ perpendicular to ξ, and define the symmetric bilinear forms

$$\begin{aligned} \tilde{\beta}\colon \mathcal{H}_z \times \mathcal{H}_z \to \mathbb{R}, &\quad (Y, Z) \mapsto \langle \alpha(Y, Z), \tilde{\xi} \rangle \\ \tilde{\gamma}\colon \mathcal{V}_z \times \mathcal{V}_z \to \mathbb{R}, &\quad (V, W) \mapsto \langle \alpha(V, W), \tilde{\xi} \rangle. \end{aligned}$$

Set also $\tilde{\beta}_0 = \tilde{\beta}(X, X)$, $\tilde{\gamma}_0 = \tilde{\gamma}(e, e)$ and $\tilde{\delta}_0 := \tilde{\beta}_0\tilde{\gamma}_0 + \beta\gamma - \lambda^2$. Then Lemma 21 can be strengthened as follows.

PROPOSITION 25. *Let $f\colon L^p \times_\rho M^n \to \mathbb{Q}_c^{p+n+2}$ be an isometric immersion. Assume that f is of type B at $z \in N^{p+n} = L^p \times_\rho M^n$. With the preceding notations we have:*

(i) *If f is of type B_1 at z, then one of the following (not exclusive) possibilities holds:*

$$\tilde{\gamma}(V, W) = \langle V, W \rangle \tilde{\gamma}_0 \qquad \textit{or} \qquad \tilde{\beta} = 0.$$

(ii) *If $n \geq 2$ and f is of type B_2 at z, then*

$$\tilde{\beta}(Y, Z) = \langle Y, X \rangle \langle Z, X \rangle \tilde{\beta}_0 \quad \textit{with} \quad \tilde{\beta}_0 \neq 0, \ \textit{and} \tag{33}$$

$$\tilde{\beta}_0\, \tilde{\gamma}(V, W) = \langle V, W \rangle \tilde{\delta}_0 - (\beta\gamma - \lambda^2) \langle V, e \rangle \langle W, e \rangle. \tag{34}$$

Proof. Equation (29) now reads

$$\tilde{\beta}(Y, Z)(\tilde{\gamma}(V, W) - \langle V, W \rangle \tilde{\gamma}_0) = (\beta\gamma - \lambda^2) \langle Y, X \rangle \langle Z, X \rangle \langle PV, PW \rangle. \tag{35}$$

If $\beta\gamma-\lambda^2=0$ then the preceding equation proves assertion (i). Choosing $Y=Z=X$ in (35) yields

$$\tilde{\beta}_0\left(\tilde{\gamma}(V,W)-\langle V,W\rangle\tilde{\gamma}_0\right)=(\beta\gamma-\lambda^2)\langle PV,PW\rangle. \tag{36}$$

Using (36), $n\geq 2$ and $\beta\gamma-\lambda^2\neq 0$ we derive $\tilde{\beta}_0\neq 0$ and from (35) it follows that $\tilde{\beta}(Y,Z)=\langle Y,X\rangle\langle Z,X\rangle\tilde{\beta}_0$. Finally, (36) also yields (34). □

Taking into account (30) we have the following additional information in the case of Riemannian products.

COROLLARY 26. *Let $f\colon L^p\times M^n\to\mathbb{Q}_c^{p+n+2}$ be an isometric immersion of a Riemannian product. Assume that f is of type B at $z\in L^p\times M^n$. Then, with the preceding notations, we have that $\tilde{\delta}_0=-c$ and, in addition:*

(i) *If f is of type B_1 at z, then*

$$\tilde{\beta}(Y,Z)=\langle Y,Z\rangle\tilde{\beta}_0,\quad \tilde{\gamma}(V,W)=\langle V,W\rangle\tilde{\gamma}_0\quad \textit{and}\quad \tilde{\beta}_0\tilde{\gamma}_0+c=0.$$

(ii) *If $p,n\geq 2$ and f is of type B_2 at z, then $c=0$,*

$$\tilde{\beta}(Y,Z)=\langle Y,X\rangle\langle Z,X\rangle\tilde{\beta}_0,\quad \textit{and}\quad \tilde{\gamma}(V,W)=\langle V,e\rangle\langle W,e\rangle\tilde{\gamma}_0. \tag{37}$$

In the remainder of this section we make a detailed study of isometric immersions $f\colon L^p\times_\rho M^n\to\mathbb{Q}_c^{p+n+2}$, $n\geq 2$, that are everywhere of type B. In this case we may choose smooth unit vector fields X, e and ξ (and hence a smooth unit normal vector field $\tilde{\xi}$ orthogonal to ξ), and smooth functions β,λ and γ that satisfy pointwise the conditions of Lemma 21. In Propositions 27 and 31 below we classify isometric immersions of types B_1 and B_2, respectively, the latter only for $n\geq 3$. In Corollary 33 we determine the special subclass of isometric immersions of type B_2 for which $\tilde{\delta}_0$ in (34) is everywhere vanishing. Isometric immersions of type B_1 of Riemannian products with dimension $p+n\geq 3$ are classified in Corollary 30. In Corollary 32 we show that there exists no isometric immersion $f\colon L^p\times M^n\to\mathbb{Q}_c^{p+n+2}$, $p+n\geq 3$, of type B_2 if $c\neq 0$ and in Corollary 34 we classify such isometric immersions for $c=0$. This yields a local classification of isometric immersions of type B in codimension two of warped products $L^p\times_\rho M^n$ for which $n\geq 3$, as well as of Riemannian products $L^p\times M^n$ for which $p+n\geq 3$.

PROPOSITION 27. *Let $f\colon N^{p+n}=L^p\times_\rho M^n\to\mathbb{Q}_c^{p+n+2}$, $n\geq 2$, be an isometric embedding of type B_1. Then f is a composition $H\circ g$ of isometric immersions, where $g=\Psi\circ(h_1\times h_2)$ is a warped product of isometric immersions determined by a warped product representation $\Psi\colon V^{p+k_1}\times_\sigma\mathbb{Q}_{\tilde{c}}^{n+k_2}\to\mathbb{Q}_c^{p+n+1}$, $k_1+k_2=1$, and $H\colon W\to\mathbb{Q}_c^{p+n+2}$ is an isometric immersion of an open subset $W\supset g(N^{p+n})$ of $\mathbb{Q}_c^{p+n+1}$ (see the diagram in Theorem 14-(ii)).*

Proof. Set $E_0 = (\beta X + \lambda e)/\|\beta X + \lambda e\|$. Since $\beta\gamma - \lambda^2 = 0$ we have that $\lambda(\beta X + \lambda e) = \beta(\lambda X + \gamma e)$, and hence (31) yields

$$A_\xi E = (\beta + \gamma)\langle E, E_0\rangle E_0 \quad \text{for all } \; E \in \Gamma(TN). \tag{38}$$

Observe that $\beta + \gamma \neq 0$ because $\beta\gamma - \lambda^2 > 0$. The Gauss equation for f and the fact that A_ξ has rank 1 imply that $A_{\tilde{\xi}}$ satisfies the Gauss equation for an isometric immersion of N^{p+n} into $\mathbb{Q}_c^{p+n+1}$. We claim that it also satisfies the Codazzi equation for such an isometric immersion. Define a connection one-form ω on N^{p+n} by $\omega(E) = \langle \nabla_E^\perp \xi, \tilde{\xi}\rangle$. By the Codazzi equation for f we have

$$(\nabla_{E_1} A_{\tilde{\xi}})E_2 - (\nabla_{E_2} A_{\tilde{\xi}})E_1 = \omega(E_2)A_\xi E_1 - \omega(E_1)A_\xi E_2. \tag{39}$$

The following fact and (38) imply that the right hand side of (39) vanishes, and the claim follows.

FACT 28. *The one-form ω satisfies $\omega(E) = \langle E, E_0\rangle\omega(E_0)$ for all $E \in \Gamma(TN)$, or equivalently, $\omega(E) = 0$ for all $E \in \Gamma(\ker A_\xi)$.*

In proving Fact 28 it is useful to observe that

$$\ker A_{\xi(z)} = \{E_0(z)\}^\perp = \operatorname{span}\{\lambda\langle V, e\rangle Y - \beta\langle Y, X\rangle V \colon\ Y \in \mathcal{H}_z, V \in \mathcal{V}_z\} \tag{40}$$

$$= \operatorname{span}\{\lambda\langle Y, X\rangle V - \gamma\langle V, e\rangle Y \colon\ Y \in \mathcal{H}_z, V \in \mathcal{V}_z\}. \tag{41}$$

By Proposition 25-(i), at each point $z \in N^{p+n}$ either $A_{\tilde{\xi}}|_{\mathcal{V}_z} = \tilde{\gamma}_0 \operatorname{id}$, where id denotes the identity tensor, or $A_{\tilde{\xi}}|_{\mathcal{H}_z} = 0$. Since $\omega(E)$ is a continuous function, it suffices to prove that $\omega(E)(z) = 0$ at points $z \in N^{p+n}$ that are contained in an entire neighborhood $U \subset N^{p+n}$ in which one of the preceding possibilities holds everywhere.

Case $A_{\tilde{\xi}}|_{\mathcal{V}} = \tilde{\gamma}_0 \operatorname{id}$. For $Y \in \mathcal{L}(L^p)$ and $V, W \in \mathcal{L}(M^n)$ we obtain using (2) and (4) that

$$\langle(\nabla_Y A_{\tilde{\xi}})V - (\nabla_V A_{\tilde{\xi}})Y, W\rangle = \langle \nabla_Y A_{\tilde{\xi}} V - \nabla_V A_{\tilde{\xi}} Y - A_{\tilde{\xi}}[Y, V], W\rangle \tag{42}$$

$$= \langle \nabla_Y(\tilde{\gamma}_0 V), W\rangle + \langle A_{\tilde{\xi}} Y, \nabla_V W\rangle \tag{43}$$

$$= (Y(\tilde{\gamma}_0) + \langle (A_{\tilde{\xi}} - \tilde{\gamma}_0 \operatorname{id})Y, \eta\rangle)\langle V, W\rangle. \tag{44}$$

On the other hand, by the Codazzi equation we have

$$\begin{aligned}\langle(\nabla_Y A_{\tilde{\xi}})V - (\nabla_V A_{\tilde{\xi}})Y, W\rangle &= \langle \omega(V)A_\xi Y - \omega(Y)A_\xi V, W\rangle \\ &= \langle W, e\rangle\omega(\lambda\langle Y, X\rangle V - \gamma\langle V, e\rangle Y),\end{aligned}$$

where in the second equality we have used (31). Thus

$$(Y(\tilde{\gamma}_0) + \langle (A_{\tilde{\xi}} - \tilde{\gamma}_0 \operatorname{id})Y, \eta\rangle)\langle V, W\rangle = \langle W, e\rangle\omega(\lambda\langle Y, X\rangle V - \gamma\langle V, e\rangle Y) \tag{45}$$

for all $Y \in \mathcal{L}(L^p)$ and $V, W \in \mathcal{L}(M^n)$. As these equations are tensorial, they are also valid for arbitrary horizontal (resp., vertical) vector fields Y (resp., V, W). In particular, if we apply (45) for $W = V$ orthogonal to e we obtain that the expression between parentheses on the left-hand-side vanishes. Then, for $W = e$ this yields

$$\omega(\lambda\langle Y, X\rangle V - \gamma\langle V, e\rangle Y) = 0,$$

thus proving Fact 28 by (41) in this case.

Case $A_{\tilde{\xi}}|_{\mathcal{H}} = 0$. Using (1) we obtain for $Y, Z \in \mathcal{L}(L^p)$ and $V \in \mathcal{L}(M^n)$ that

$$\begin{aligned}\langle(\nabla_Y A_{\tilde{\xi}})V, Z\rangle &= \langle\nabla_Y A_{\tilde{\xi}}V, Z\rangle - \langle A_{\tilde{\xi}}\nabla_Y V, Z\rangle\\ &= Y\langle A_{\tilde{\xi}}V, Z\rangle - \langle A_{\tilde{\xi}}V, \nabla_Y Z\rangle - \langle\nabla_Y V, A_{\tilde{\xi}}Z\rangle = 0\end{aligned}$$

and, analogously, that

$$\langle(\nabla_V A_{\tilde{\xi}})Y, Z\rangle = 0.$$

Using again that these equations are tensorial, we can replace Z by the vector field X. We obtain from (31) and the Codazzi equation that

$$\omega(\lambda\langle V, e\rangle Y - \beta\langle Y, X\rangle V) = 0,$$

which by (40) proves Fact 28 also in this case.

It follows from Theorem 5' in [7] and the assumption that f is an embedding that f is a composition $f = H \circ g$, where $g\colon N^{p+n} \to \mathbb{Q}_c^{p+n+1}$ is an isometric immersion such that $A_\delta^g = A_{\tilde{\xi}}$ for some unit normal vector field δ of g, and $H\colon W \to \mathbb{Q}_c^{p+n+2}$ is an isometric immersion of an open subset $W \subset \mathbb{Q}_c^{p+n+1}$ containing $g(N^{p+n})$. Moreover, since $A_\delta^g = A_{\tilde{\xi}}$ satisfies $A_\delta^g|_{\mathcal{V}_z} = \tilde{\gamma}_0\,\mathrm{id}$ or $A_\delta^g|_{\mathcal{H}_z} = 0$ at any $z \in N^{p+n}$, it follows that g is of type A, and the conclusion follows from Theorem 8. □

Remark 29. By Proposition 27, if $f\colon N^{p+n} = L^p \times_\rho M^n \to \mathbb{Q}_c^{p+n+2}$, $n \geq 2$, is an isometric embedding of type B_1, then it must satisfy one of the conditions in Proposition 25-(i) *everywhere*. Moreover, for $n \geq 1$ we have that both conditions hold simultaneously if and only if the isometric immersion $g\colon N^{p+n} \to \mathbb{Q}_c^{p+n+1}$ satisfies $A_\delta^g|_{\mathcal{V}} = \tilde{\gamma}_0\,\mathrm{id}$ and $A_\delta^g|_{\mathcal{H}} = 0$. By Corollary 6, this is the case if and only if $g = \Psi \circ (h_1 \times h_2)$ with h_1 totally geodesic and h_2 a local isometry, where $\Psi\colon V^{p+1} \times_\sigma \mathbb{Q}_{\tilde{c}}^n \to \mathbb{Q}_c^{p+n+1}$ is a warped product representation determined by $(\mathbb{Q}_{\tilde{c}}^n, \bar{z})$. Furthermore, if in addition $\tilde{\gamma}_0 = 0$, that is, g is totally geodesic, and $\operatorname{grad}\rho$ has no zeros, then Corollary 7 implies that h_1 must be cylindrical with respect to a, where $-a$ is the mean curvature vector of $\mathbb{Q}_{\tilde{c}}^n$ at $\bar{z}$ in either $\mathbb{R}^{p+n+1}$ or $\mathbb{O}^{p+n+2}$, according as $c = 0$ or $c \neq 0$. Conversely, if h_1 is totally geodesic and cylindrical with respect to a and h_2 is a local isometry, then $g = \Psi \circ (h_1 \times h_2)$ is totally geodesic.

By using the main lemma in [14] or Corollary 9, according as $c = 0$ or $c \neq 0$, instead of Theorem 8, we obtain the following result for isometric immersions of Riemannian products.

Corollary 30. *Let $f\colon L^p \times M^n \to \mathbb{Q}_c^{p+n+2}$, $p+n \geq 3$, be an isometric embedding of type B_1 of a Riemannian product.*

(i) *If $c = 0$, then there exist an orthogonal decomposition $\mathbb{R}^{p+n+2} = \mathbb{R}^{p+k_1} \times \mathbb{R}^{n+k_2}$ with $k_1 + k_2 = 1$ and isometric immersions $h_1\colon L_0^p \to \mathbb{R}^{p+k_1}$, $h_2\colon M_0^n \to \mathbb{R}^{n+k_2}$ and $H\colon W \to \mathbb{R}^{p+n+2}$ of an open subset $W \supset (h_1 \times h_2)(U)$ of $\mathbb{R}^{p+n+1}$ such that $f|_U = H \circ (h_1 \times h_2)$ (see the diagram in Case $c = 0$—*(ii) *of Theorem 17).*

(ii) *If $c \neq 0$ there exist an isometric embedding $\Phi\colon \mathbb{Q}_{c_1}^p \times \mathbb{Q}_{c_2}^n \to \mathbb{Q}_c^{p+n+1}$ as an extrinsic Riemannian product, local isometries $i_1\colon L^p \to \mathbb{Q}_{c_1}^p$ and $i_2\colon M^n \to \mathbb{Q}_{c_2}^n$, and an isometric immersion $H\colon W \to \mathbb{Q}_c^{p+n+2}$ of an open subset $W \supset \Phi \circ (i_1 \times i_2)(L^p \times M^n)$ of $\mathbb{Q}_c^{p+n+1}$ such that $f = H \circ \Phi \circ (i_1 \times i_2)$ (see the diagram in Case $c \neq 0$—*(ii) *of Theorem 17).*

We now consider isometric immersions $f\colon L^p \times_\rho M^n \to \mathbb{Q}_c^{p+n+2}$ of type B_2. In the following statement, in order not to have to consider separately the cases $p = 1$ and $p \geq 2$, we agree that in the first case all information related to the splitting $L^p = L^{p-1} \times_{\rho_1} I$ should be disregarded.

Proposition 31. *Let $f\colon L^p \times_\rho M^n \to \mathbb{Q}_c^{p+n+2}$ be an isometric immersion of type B_2 and assume that $n \geq 3$. Then locally we have: L^p and M^n split as warped products $L^p = L^{p-1} \times_{\rho_1} I$ and $M^n = J \times_{\rho_2} M^{n-1}$, where $I, J \subset \mathbb{R}$ are open intervals, and*

$$N^{p+n} = L^{p-1} \times_{\rho_1} ((I \times_{\rho_3} J) \times_{\bar{\rho}} M^{n-1}),$$

where $\rho_1 \in C^\infty(L^{p-1})$, $\rho_2 \in C^\infty(J)$, $\rho_3 \in C^\infty(I)$ and $\bar{\rho} \in C^\infty(I \times J)$ satisfy

$$\rho = (\rho_1 \circ \pi_{L^{p-1}})(\rho_3 \circ \pi_I) \ \text{ and } \ \bar{\rho} = (\rho_3 \circ \pi_I)(\rho_2 \circ \pi_J),$$

and there exist warped product representations

$$\Psi_1\colon V^{p-1} \times_{\sigma_1} \mathbb{Q}_{\tilde{c}}^{n+3} \to \mathbb{Q}_c^{p+n+2} \quad \text{and} \quad \Psi_2\colon W^4 \times_{\sigma_2} \mathbb{Q}_{\tilde{c}}^{n-1} \to \mathbb{Q}_{\tilde{c}}^{n+3},$$

isometries $i_1\colon L^{p-1} \to W^{p-1} \subset V^{p-1} \subset \mathbb{Q}_c^{p-1}$ and $i_2\colon M^{n-1} \to W^{n-1} \subset \mathbb{Q}_{\tilde{c}}^{n-1}$ onto open subsets, and an isometric immersion $g\colon I \times_{\rho_3} J \to W^4$ of type B_2 such that $\bar{\rho} = \sigma_2 \circ g$, $\rho_1 = \sigma_1 \circ i_1$ and $f = \Psi_1 \circ (i_1 \times (\Psi_2 \circ (g \times i_2)))$. Moreover, L^p has constant sectional curvature c if $p \geq 2$ (see the diagram in Theorem 14-(iii)).

Proof. We have by (32), (33) and (34) that

$$A_{\tilde{\xi}} Y = \tilde{\beta}_0 \langle Y, X\rangle X \quad \text{and} \quad A_{\tilde{\xi}} V = \tilde{b} V + (\tilde{\gamma}_0 - \tilde{b})\langle V, e\rangle e, \tag{46}$$

where $\tilde{b} = \tilde{\beta}_0^{-1}\, \tilde{\delta}_0$. On the other hand, we have that A_ξ is given by (31). Thus, for the relative nullity subspace $\Delta(z)$ at $z \in N^{p+n}$ there are two possibilities:

$$\Delta(z) = \begin{cases} \{X\}^\perp \subset \mathcal{H} & \text{if } \tilde{\delta}_0(z) \neq 0, \\ (\{X\}^\perp \subset \mathcal{H}) \oplus (\{e\}^\perp \subset \mathcal{V}) & \text{if } \tilde{\delta}_0(z) = 0. \end{cases}$$

In the remainder of this proof the letters T and S will always denote vector fields in $\Gamma(\{X\}^\perp \subset \mathcal{H})$ and $\Gamma(\{e\}^\perp \subset \mathcal{V})$, respectively. We also denote by (A_δ, u, v, w) taking the w-component of the Codazzi equation for A_δ and the vectors u, v.

We first prove that if $p \geq 2$ then L^p splits locally as $L^p = L^{p-1} \times_{\rho_1} I$, where I is an open interval, and that $\rho = (\rho_1 \circ \pi_{L^{p-1}})(\rho_3 \circ \pi_I)$ for some functions $\rho_1 \in C^\infty(L^{p-1})$ and $\rho_3 \in C^\infty(I)$. We point out that this fact also holds if $n = 1, 2$. As a first step, we show that the vector field X is the lift of a vector field $\tilde{X} \in \Gamma(TL)$. For that, we must prove that

$$\nabla_V X = -\langle X, \eta \rangle V. \tag{47}$$

Notice that $(A_{\tilde{\xi}}, X, V, T)$ reads

$$\langle \nabla_X A_{\tilde{\xi}} V - A_{\tilde{\xi}} \nabla_X V - \nabla_V A_{\tilde{\xi}} X + A_{\tilde{\xi}} \nabla_V X, T \rangle = \langle \omega(V) A_\xi X - \omega(X) A_\xi V, T \rangle.$$

Then, by means of (31) and (46) we obtain

$$\langle \nabla_V X, T \rangle = 0. \tag{48}$$

On the other hand, by means of (4),

$$\langle \nabla_V X, W \rangle = -\langle \nabla_V W, X \rangle = -\langle V, W \rangle \langle X, \eta \rangle. \tag{49}$$

Using also that $\langle \nabla_V X, X \rangle = 0$, for X has unit length, we obtain (47) from (48) and (49). Notice that $\tilde{X}$ has unit length, because

$$\langle \tilde{X}, \tilde{X} \rangle_L \circ \pi_L = \langle \pi_{L*} X, \pi_{L*} X \rangle_L = \langle X, X \rangle_N = 1.$$

We show next that the distribution $\{\tilde{X}\}^\perp$ is totally geodesic in L^p. In effect, for any $\tilde{T}_1, \tilde{T}_2 \in \Gamma(\{\tilde{X}\}^\perp)$ we have

$$\langle \nabla^L_{\tilde{T}_1} \tilde{T}_2, \tilde{X} \rangle_L \circ \pi_L = \langle \pi_{L*} \nabla_{T_1} T_2, \pi_{L*} X \rangle_L = \langle \nabla_{T_1} T_2, X \rangle_N = 0,$$

where the last equality follows from the fact that the relative nullity distribution Δ is totally geodesic and $T_1, T_2 \in \Gamma(\Delta)$, whereas $X \in \Gamma(\Delta^\perp)$.

Our next step is to prove that the vector field $\zeta = \eta - \langle X, \eta \rangle X \in \Gamma(\{X\}^\perp)$ is the lift of a vector field $\tilde{\zeta} \in \Gamma(\{\tilde{X}\}^\perp)$, i.e., $\nabla_V \zeta = -\langle \zeta, \eta \rangle V$. This follows from

$$\langle \nabla_V \zeta, W \rangle = -\langle \zeta, \nabla_V W \rangle = -\langle V, W \rangle \langle \eta, \zeta \rangle, \quad \langle \nabla_V \zeta, X \rangle = -\langle \zeta, \nabla_V X \rangle = 0,$$

where we have used (47), and from

$$\langle \nabla_V \zeta, T \rangle = \langle \nabla_V \eta, T \rangle - V(\langle X, \eta \rangle) \langle X, T \rangle - \langle X, \eta \rangle \langle \nabla_V X, T \rangle = 0,$$

where we have used (47) for the last term and that $\mathcal{V}$ is spherical for the first term.

Our final step is to show that the distribution $\{\tilde{X}\}$ is spherical with mean curvature vector $\tilde{\zeta}$. We have from (A_ξ, X, e, T) and $\lambda \neq 0$ that $\langle \nabla_X X, T\rangle = \langle \nabla_e e, T\rangle = \langle \eta, T\rangle$, and hence

$$\nabla^L_{\tilde{X}} \tilde{X} \circ \pi_L = \pi_{L*} \nabla_X X = \pi_{L*}\zeta = \tilde{\zeta}.$$

On the other hand, we obtain from (11) for $Y = T$ and $V = W \neq 0$ that

$$\langle \nabla_X \eta, T\rangle = \langle X, \eta\rangle\langle T, \eta\rangle.$$

Thus,

$$\langle \nabla_X \zeta, T\rangle = \langle \nabla_X \eta, T\rangle - \langle X, \eta\rangle\langle \nabla_X X, T\rangle = 0,$$

and therefore,

$$\langle \nabla^L_{\tilde{X}} \tilde{\zeta}, \tilde{T}\rangle_L \circ \pi_L = \langle \pi_{L*}\nabla_X \zeta, \pi_{L*} T\rangle_L = \langle \nabla_X \zeta, T\rangle_N = 0 \;\text{ for }\quad \tilde{T} \in \Gamma(\{\tilde{X}\}^\perp),$$

which completes the proof of the step.

By Theorem 1, we have that locally L^p splits as $L^p = L^{p-1} \times_{\rho_1} I$, where I is an open interval and $\tilde{\zeta} = -\operatorname{grad}\log(\rho_1 \circ \pi_{L^{p-1}})$. In particular, the lift ζ of $\tilde{\zeta}$ to N^{p+n} is

$$\zeta = -\operatorname{grad}\log(\rho_1 \circ \pi_{L^{p-1}} \circ \pi_{L^p}).$$

Since we also have $\eta = -\operatorname{grad}\log(\rho \circ \pi_L)$, we obtain $\langle X, \eta\rangle X = \eta - \zeta = -\operatorname{grad}\log(\hat{\rho} \circ \pi_L)$ with $\hat{\rho} = \rho(\rho_1 \circ \pi_{L^{p-1}})^{-1} \in C^\infty(L^p)$. Moreover, since

$$(\tilde{T}(\log\hat{\rho})) \circ \pi_{L^p} = T(\log(\hat{\rho} \circ \pi_{L^p})) = -\langle X, \eta\rangle\langle X, T\rangle = 0,$$

it follows that there exists $\rho_3 \in C^\infty(I)$ such that $\hat{\rho} = \rho_3 \circ \pi_I$.

Let us now prove that locally M^n also splits as $M^n = J \times_{\rho_2} M^{n-1}$, where J is an open interval. First, we obtain from $(A_{\tilde{\xi}}, Y, e, S)$ and $\tilde{b} \neq \tilde{\gamma}_0$ that $\langle \nabla_Y e, S\rangle = 0$. Since also $\langle \nabla_Y e, e\rangle = 0$, for e has unit length, and $\langle \nabla_Y e, Z\rangle = -\langle \nabla_Y Z, e\rangle = 0$, we have

$$\nabla_Y e = 0. \tag{50}$$

It follows that

$$\nabla_Y(\rho \circ \pi_L)e = Y(\rho \circ \pi_L)e = -\langle Y, \eta\rangle(\rho \circ \pi_L)e.$$

This implies that $(\rho \circ \pi_L)e$ is the lift of a vector field $\tilde{e} \in \Gamma(M^n)$. Notice that $\tilde{e}$ is a unit vector field, for

$$\langle \tilde{e}, \tilde{e}\rangle_M \circ \pi_M = \langle \pi_{M*}(\rho e), \pi_{M*}(\rho e)\rangle_M = \rho^{-2}\langle \rho e, \rho e\rangle_N = 1.$$

Thus, in order to show that locally M^n splits as claimed, by Theorem 1 it suffices to prove that the distribution $\{\tilde{e}\}$ is totally geodesic and that $\{\tilde{e}\}^\perp$ is spherical.

We obtain from (A_ξ, X, e, S) and (50) that

$$\langle \nabla_e e, S\rangle = 0. \tag{51}$$

In particular, it follows that $\nabla_e e = \eta$. We conclude that the distribution $\{\tilde{e}\}$ is totally geodesic from

$$\langle \nabla^M_{\tilde{e}} \tilde{e}, \tilde{S}\rangle_M \circ \pi_M = \langle \pi_{M*}\nabla_{\rho e}\rho e, \pi_{M*}S\rangle_M = \langle \nabla_e e, S\rangle_N = 0$$

for any $\tilde{S} \in \Gamma(\{\tilde{e}\}^\perp)$. On the other hand, we obtain from $(A_{\tilde{\xi}}, S_1, e, S_2)$ that

$$\langle \nabla_{S_1} S_2, e\rangle = \varphi\langle S_1, S_2\rangle,$$

where $\varphi = (\tilde{b} - \tilde{\gamma}_0)^{-1} e(\tilde{b})$. Thus, the distribution $\{e\}^\perp \subset \mathcal{V}$ is totally umbilical. Moreover,

$$\begin{aligned}\langle \nabla^M_{\tilde{S}_1} \tilde{S}_2, \tilde{e}\rangle_M \circ \pi_M &= \langle \pi_{M*}\nabla_{S_1} S_2, \pi_{M*}\rho e\rangle_M = \rho^{-1}\langle \nabla_{S_1} S_2, e\rangle_N \\ &= \rho^{-1}\varphi\langle S_1, S_2\rangle_N = \rho\varphi\langle \tilde{S}_1, \tilde{S}_2\rangle_M \circ \pi_M.\end{aligned}$$

The preceding equality implies that there exists $\tilde{\varphi} \in C^\infty(M^n)$ such that $\rho\varphi = \tilde{\varphi} \circ \pi_M$ and that the distribution $\{\tilde{e}\}^\perp$ is totally umbilical with mean curvature normal $\tilde{\varphi}\tilde{e}$. *In particular, if* $\tilde{b}$*, or equivalently,* $\tilde{\delta}_0$*, vanishes on an open subset* $L_0^p \times M_0^n \subset N^{p+n}$*, then* $\{\tilde{e}\}^\perp$ *is a totally geodesic distribution in* M_0^n. In the general case, in order to show that $\{\tilde{e}\}^\perp$ is spherical, it remains to prove that $\tilde{S}(\tilde{\varphi}) = 0$ or, equivalently, that $S(\varphi) = 0$. First, using that $\mathcal{V}$ is umbilical and invariant by $A_{\tilde{\xi}}$, and that $\lambda \neq 0$, we obtain from $(A_{\tilde{\xi}}, e, S, X)$ that $\nabla^\perp_S \tilde{\xi} = 0$. Now, choosing linearly independent sections $S_1, S_2 \in \Gamma(\{e\}^\perp)$ we obtain from $(A_{\tilde{\xi}}, S_1, S_2, S_1)$ that $S(\tilde{b}) = 0$. *We point out that the assumption that* $n \geq 3$ *is only used here. In particular, if* $\tilde{\delta}_0$ *is everywhere vanishing then it is enough to assume that* $n \geq 2$. Using (51) we obtain from $(A_{\tilde{\xi}}, e, S, e)$ that $S(\tilde{\gamma}_0) = 0$. Since $\nabla_e S \in \{e\}^\perp$ and $\nabla_S e \in \{e\}^\perp$, as follows from (51) and the fact that $\mathcal{V}$ is totally umbilical, then

$$Se(\tilde{b}) = eS(\tilde{b}) + \nabla_e S(\tilde{b}) - \nabla_S e(\tilde{b}) = 0,$$

and hence $S(\varphi) = 0$. Therefore, locally N^{p+n} splits as

$$N^{p+n} = L^{p-1} \times_{\rho_1} M^{n+1} \ \text{ with } \ M^{n+1} := M^2 \times_{\bar{\rho}} M^{n-1} \text{ and } M^2 := I \times_{\rho_3} J,$$

where $J \subset \mathbb{R}$ is an open interval and $\bar{\rho} = (\rho_3 \circ \pi_I)(\rho_2 \circ \pi_J)$. Notice that f is of type A with respect to this decomposition of N^{p+n}. We claim that there exist a warped product representation $\Psi_1\colon V^{p-1} \times_{\sigma_1} \mathbb{Q}_{\tilde{c}}^{n+3} \to \mathbb{Q}_c^{p+n+2}$, an isometric immersion $\tilde{G}\colon M^{n+1} \to \mathbb{Q}_{\tilde{c}}^{n+3}$ and a local isometry $i_1\colon L^{p-1} \to V^{p-1}$ such that $\rho_1 = \sigma_1 \circ i_1$ and $f = \Psi_1 \circ (i_1 \times \tilde{G})$.

Fix $\bar{y} \in L^{p-1}$ with $\rho_1(\bar{y}) = 1$ and let $i_{\bar{y}}\colon M^{n+1} \to N^{n+p}$ be the (isometric) inclusion of M^{n+1} into N^{n+p} as a leaf of the vertical subbundle $\bar{\mathcal{V}}$ according to the latter decomposition of N^{n+p}. Define $G\colon M^{n+1} \to \mathbb{Q}_c^{p+n+2}$ by $G = f \circ i_{\bar{y}}$. By Corollary 13-(ii), in order to show that the spherical hull of G has dimension $n+3$, it suffices to prove that for no point $z \in N^{n+p}$ there exists a unit vector $\bar{\xi} \in T_z^\perp N$ such that $A_{\bar{\xi}}|_{\bar{\mathcal{V}}_z}\colon \bar{\mathcal{V}}_z \to \bar{\mathcal{V}}_z$ is a multiple of the identity tensor. Write $\bar{\xi} = \cos\theta\xi + \sin\theta\tilde{\xi}$. Then $\langle A_{\bar{\xi}}X, e\rangle = 0$ and $\langle A_{\bar{\xi}}X, X\rangle = \langle A_{\bar{\xi}}e, e\rangle = \langle A_{\bar{\xi}}S, S\rangle$ for any unit vector $S \in (\{e\}_z^\perp \subset \mathcal{V}_z)$ if and only if

$$\lambda\cos\theta = 0 \quad \text{and} \quad \tilde{\beta}_0\sin\theta + \beta\cos\theta = \gamma\cos\theta + \tilde{\gamma}_0\sin\theta = \tilde{b}\sin\theta.$$

Since $\lambda \neq 0$, we obtain that $\tilde{\gamma}_0 = \tilde{b}$, a contradiction to the fact that $\beta\gamma - \lambda^2 \neq 0$. Our claim then follows from Theorem 8 by letting $\mathbb{Q}_{\tilde{c}}^{n+3}$ be the spherical hull of G and defining $\tilde{G}\colon M^{n+1} \to \mathbb{Q}_{\tilde{c}}^{n+3}$ by $G = j \circ \tilde{G}$, where j is the inclusion of $\mathbb{Q}_{\tilde{c}}^{n+3}$ into $\mathbb{Q}_c^{p+n+2}$.

We now study the isometric immersion $\tilde{G}\colon M^{n+1} \to \mathbb{Q}_{\tilde{c}}^{n+3}$. First observe that the second fundamental form of $G = j \circ \tilde{G}$ is given by

$$\alpha_G(V, W) = \alpha_f(i_{\bar{y}*}V, i_{\bar{y}*}W) + \langle V, W\rangle f_*(\bar{\eta} \circ i_{\bar{y}}) \quad \text{for all} \quad V, W \in \Gamma(TM^{n+1}),$$

where $\bar{\eta}$ denotes the mean curvature normal of $\bar{\mathcal{V}}$. Let $\hat{\mathcal{V}}$ denote the vertical subbundle of TM^{n+1} according to the decomposition $M^{n+1} = M^2 \times_{\bar{\rho}} M^{n-1}$. Using that $A_\xi^f|_{i_{\bar{y}*}\hat{\mathcal{V}}} = 0$ and that $A_{\tilde{\xi}}^f|_{i_{\bar{y}*}\hat{\mathcal{V}}} = \tilde{b}\,\mathrm{id}$, where id denotes the identity tensor, it follows that

$$\alpha_G(V, \bar{V}) = \langle V, \bar{V}\rangle((\tilde{b} \circ i_{\bar{y}})(\tilde{\xi} \circ i_{\bar{y}}) + f_*(\bar{\eta} \circ i_{\bar{y}})) \text{ for all } V \in \Gamma(TM^{n+1}),\ \bar{V} \in \hat{\mathcal{V}}.$$

Therefore $\hat{\eta} = (\tilde{b} \circ i_{\bar{y}})(\tilde{\xi} \circ i_{\bar{y}}) + f_*(\bar{\eta} \circ i_{\bar{y}})$ is a principal curvature normal of G and $\hat{\mathcal{V}}$ is contained in the corresponding eigendistribution. Since j is umbilical, it follows that $\hat{\eta}_{T\mathbb{Q}_{\tilde{c}}^{n+3}}$ is a principal curvature normal of $\tilde{G}$ with the same eigendistribution as $\hat{\eta}$. By means of Corollary 13-(i) and Theorem 8, we conclude that there exist a warped product representation $\Psi_2\colon W^4 \times_{\sigma_2} \mathbb{Q}_{\tilde{c}}^{n-1} \to \mathbb{Q}_{\tilde{c}}^{n+3}$, an isometric immersion $g\colon M^2 \to W^4$ and a local isometry $i_2\colon M^{n-1} \to \mathbb{Q}_{\tilde{c}}^{n-1}$ such that $\bar{\rho} = \sigma_2 \circ g$ and $\tilde{G} = \Psi_2 \circ (g \times i_2)$. Moreover, since the second fundamental form of g is determined by the restriction of α_G to the horizontal subbundle $\hat{\mathcal{H}}$ of TM^{n+1} according to the decomposition $M^{n+1} = M^2 \times_{\bar{\rho}} M^{n-1}$, and hence by the restriction of α_f to span$\{X, e\}$ (see formula (15)), it follows that g is of type B_2.

Finally, since $\{X\}^\perp \subset \mathcal{H}$ is contained in Δ, we obtain that the curvature-like tensor C defined in Proposition 2 satisfies

$$C(Y_1, Y_2, Y_3, Y_4) = 0 \quad \text{for all } \ Y_1, Y_2, Y_3, Y_4 \in \Gamma(\mathcal{H}).$$

The last assertion then follows from the fact that, for a fixed $\bar{x} \in M^n$, the inclusion $i_{\bar{x}}\colon L^p \to N^{p+n}$ given by $i_{\bar{x}}(y) = (y, \bar{x})$ is a totally geodesic isometric immersion. $\square$

Corollary 32. *Let $N^{p+n} = L^p \times M^n$ be a Riemannian product of dimension $p+n \geq 3$. Then there exists no isometric immersion $f\colon N^{p+n} \to \mathbb{Q}_c^{p+n+2}$ of type B_2 if $c \neq 0$.*

Proof. We may assume $p \geq 2$. It follows from Proposition 31 that locally L^p splits as a Riemannian product $L^p = L^{p-1} \times I$. The statement now follows from the fact that L^p has constant sectional curvature c. $\square$

In order to complete the classification of isometric immersions $f\colon L^p \times M^n \to \mathbb{Q}_c^{p+n+2}$ of type B of Riemannian products of dimension $p+n \geq 3$, it remains to determine those that are of type B_2 for $c = 0$. Observe that for such isometric immersions equation (34) in Proposition 25 holds with $\tilde{\delta}_0 = 0$ (see Corollary 26). In the following result we solve the more general problem of classifying isometric immersions of type B_2 of *warped* products satisfying this condition.

Corollary 33. *Let $f\colon N^{p+n} = L^p \times_\rho M^n \to \mathbb{Q}_c^{p+n+2}$, $n \geq 2$, be an isometric immersion of type B_2 for which $\tilde{\delta}_0$ vanishes everywhere. Then $c \leq 0$, N^{p+n} has constant sectional curvature c and one of the following holds locally:*

(i) *If $c = 0$ then L^p and M^n split as Riemannian products $L^p = L^{p-1} \times I$ and $M^n = J \times M^{n-1}$, where $I, J \subset \mathbb{R}$ are open intervals, and there exist isometries $i_1\colon L^{p-1} \to U \subset \mathbb{R}^{p-1}$ and $i_2\colon M^{n-1} \to V \subset \mathbb{R}^{n-1}$ onto open subsets and an isometric immersion $g\colon I \times J \to \mathbb{R}^4$ such that $f = i_1 \times g \times i_2$.*

$$
\begin{tikzcd}
\mathbb{R}^{p-1} & \times\ \mathbb{R}^4\ \times & \mathbb{R}^{n-1} & = \mathbb{R}^{p+n+2} \\
U \arrow[u, phantom, "\cup" description] & & V \arrow[u, phantom, "\cup" description] & \\
L^{p-1} \arrow[u, "i_1"'] & \times\ (I \times J) \arrow[uu, "g"'] \times & M^{n-1} \arrow[u, "i_2"'] & = N^{p+n} \arrow[uu, "f = i_1 \times g \times i_2"']
\end{tikzcd}
$$

(ii) *If $c < 0$ then L^p splits as a warped product $L^p = L^{p-1} \times_{\rho_1} I$, M^n splits as a Riemannian product $M^n = J \times M^{n-1}$, where $I, J \subset \mathbb{R}$ are open intervals and $\rho = \rho_1 \circ \pi_{L^{p-1}}$, and there exist a warped product representation $\Psi\colon V^{p-1} \times_\sigma \mathbb{R}^{n+3} \to \mathbb{Q}_c^{p+n+2}$, isometries $i_1\colon L^{p-1} \to U \subset V^{p-1}$ and $i_2\colon M^{n-1} \to W \subset \mathbb{R}^{n-1}$ onto open subsets, and an isometric immersion $g\colon I \times J \to \mathbb{R}^4$ such that $f = \Psi \circ (i_1 \times (g \times i_2))$ and $\rho_1 = \sigma \circ i_1$.*

$$
\begin{tikzcd}
V^{p-1} \times_\sigma (\mathbb{R}^4 \times \mathbb{R}^{n-1}) \arrow[ddr, "\Psi"] & \\
U \times W & \\
L^{p-1} \times_{\rho_1} ((I \times J) \times M^{n-1}) \arrow[r, "f = \Psi \circ (i_1 \times (g \times i_2))"'] \arrow[u, "i_1,\ g,\ i_2"] \arrow[ur, phantom, "\circlearrowright"] & \mathbb{Q}_c^{p+n+2}
\end{tikzcd}
$$

Proof. First observe that for the statement of Proposition 31 to hold in this case it is enough to require that $n \geq 2$, as follows from the second italicized statement in its proof. In order to prove that N^{p+n} has constant sectional curvature c, we must show that the curvature-like tensor C defined in Proposition 2 vanishes identically. Since the relative nullity distribution of f is $\Delta = (\operatorname{span}\{X, e\})^{\perp}$, we have that $C(E_1, E_2, E_3, E_4) = 0$ whenever two of the vectors E_1, E_2, E_3, E_4 belong to $(\operatorname{span}\{X, e\})^{\perp}$. Thus it remains to show that $C(X, e, e, X) = 0$, because C is a curvature-like tensor. But this follows from (31), (46) and the assumption that $\tilde{\delta}_0 = 0$.

We have from Proposition 31 that locally L^p and M^n split as warped products $L^p = L^{p-1} \times_{\rho_1} I$, $M^n = J \times_{\rho_2} M^{n-1}$, and $N^{p+n} = L^{p-1} \times_{\rho_1} ((I \times_{\rho_3} J) \times_{\bar{\rho}} M^{n-1})$, where $I, J \subset \mathbb{R}$ are open intervals and $\rho_1 \in C^{\infty}(L^{p-1})$, $\rho_2 \in C^{\infty}(J)$, $\rho_3 \in C^{\infty}(I)$ and $\bar{\rho} \in C^{\infty}(I \times J)$ satisfy

$$\rho = (\rho_1 \circ \pi_{L^{p-1}})(\rho_3 \circ \pi_I) \;\text{ and }\; \bar{\rho} = (\rho_3 \circ \pi_I)(\rho_2 \circ \pi_J).$$

But now $\langle X, \eta \rangle = \langle \nabla_S S, X \rangle = 0$, because Δ is totally geodesic. Hence we may assume that $\rho_3 = 1$ (recall the proof of Proposition 31), and therefore $\rho = \rho_1 \circ \pi_{L^{p-1}}$. On the other hand, by the first italicized statement in the proof of Proposition 31, the distribution $\{\tilde{e}\}^{\perp}$ in M^n is now totally geodesic, and hence $\rho_2 = 1$, which implies that also $\bar{\rho} = 1$. Summing up, we have

$$N^{p+n} = L^{p-1} \times_{\rho_1} M^{n+1}, \;\text{ with }\; M^{n+1} := M^2 \times M^{n-1} \text{ and } \; M^2 := I \times J.$$

Now, for a fixed point $\bar{y} \in L^p$ with $\rho(\bar{y}) = 1$, let $i_{\bar{y}}\colon M^n \to N^{p+n}$ denote the (isometric) inclusion of M^n into N^{p+n} as a leaf of $\mathcal{V}$. The second fundamental form of $i_{\bar{y}}$ is $\alpha_{i_{\bar{y}}}(V, W) = \langle V, W \rangle (\eta \circ i_{\bar{y}})$ for all $V, W \in \Gamma(TM^n)$, where $\eta = -\operatorname{grad} \log(\rho \circ \pi_L)$ is the mean curvature normal of $\mathcal{V}$. Since N^{p+n} has constant sectional curvature c, it follows from the Gauss equation for $i_{\bar{y}}$ that M^n has constant sectional curvature $c + \|\eta \circ i_{\bar{y}}\|^2 = c + \|\operatorname{grad} \log \rho(\bar{y})\|^2$. We conclude from the fact that $M^n = J \times M^{n-1}$ is a Riemannian product that it must be flat, hence M^{n-1} must be flat when $n \geq 3$ and $c + \|\operatorname{grad} \log \rho(\bar{y})\|^2 = 0$. Now choose any other point $y^* \in L^p$ and modify the warped product representation of N^{n+p} so that the modified warping function ρ^* satisfies $\rho^*(y^*) = 1$. By this modification $\eta = -\operatorname{grad} \log(\rho \circ \pi_L)$ does not change. Therefore the preceding argument yields $\|\operatorname{grad} \log \rho(y^*)\|^2 = -c$.

We now distinguish the two possible cases:

Case $c = 0$. Here $\eta = -\operatorname{grad}(\log \rho \circ \pi_{L^p})$ vanishes, hence $\rho = 1$, consequently also $\rho_1 = 1$ and therefore $N^{p+n} = L^{p-1} \times M^2 \times M^{n-1}$, with L^{p-1} and M^{n-1} flat and $M^2 = I \times J$. Using that $\Delta = (\operatorname{span}\{X, e\})^{\perp}$ is the relative nullity distribution of f, the main lemma in [14] implies that f splits as

$$f = i_1 \times g \times i_2\colon\; L^{p-1} \times M^2 \times M^{n-1} \to \mathbb{R}^{p-1} \times \mathbb{R}^4 \times \mathbb{R}^{n-1} = \mathbb{R}^{p+n+2},$$

where $i_1\colon\; L^{p-1} \to U \subset \mathbb{R}^{p-1}$ and $i_2\colon\; M^{n-1} \to V \subset \mathbb{R}^{n-1}$ are isometries onto open subsets, and $g\colon\; M^2 \to \mathbb{R}^4$ is an isometric immersion.

Case $c<0$. For $G=f\circ i_{\bar y}$ as in the proof of Proposition 31, we have from Corollary 13-(ii) that its spherical hull $\mathbb{Q}_{\tilde c}^{n+3}$ has constant sectional curvature $\tilde c = c+\|\operatorname{grad}\log\rho(\bar y)\|^2=0$. Therefore $\mathbb{Q}_{\tilde c}^{n+3}$ is a horosphere $\mathbb{R}^{n+3}\subset\mathbb{Q}_c^{p+n+2}$. Let $\tilde G\colon M^{n+1}\to\mathbb{R}^{n+3}=\mathbb{Q}_{\tilde c}^{n+3}$ be such that $G=j\circ\tilde G$, where j denotes the inclusion of $\mathbb{Q}_{\tilde c}^{n+3}$ into $\mathbb{Q}_c^{p+n+2}$. Using that the vertical subbundle of M^{n+1} corresponding to the splitting $M^{n+1}=M^2\times M^{n-1}$ is contained in the relative nullity distribution of $\tilde G$, the conclusion now follows from the main lemma in [14] applied to $\tilde G\colon M^2\times M^{n-1}\to\mathbb{R}^{n+3}$. ☐

Corollary 34. *Let $N^{p+n}=L^p\times M^n$ be a Riemannian product of dimension $p+n\geq 3$. Then any isometric immersion $f\colon N^{p+n}\to\mathbb{R}^{p+n+2}$ of type B_2 is locally given as in Corollary 33-(i).*

Remark 35. By making use of global arguments from [1], a complete description of the possible cases in which an isometric immersion $f\colon L^p\times M^n\to\mathbb{R}^{p+n+2}$, $p\geq 2$ and $n\geq 2$, of a Riemannian product of complete nonflat Riemannian manifolds may fail locally to be a product of isometric immersions was given in [3]. Namely, it was shown therein that there exists an open dense subset of $L^p\times M^n$ each of whose points lies in an open product neighborhood $U_0=L_0^p\times M_0^n$ restricted to which f is either (i) a product of isometric immersions, (ii) an isometric immersion of type B_2 given as in Corollary 33 - (i), or (iii) an isometric immersion of type B_1 of the following special type: either L_0^p or M_0^n, say, the latter, splits as $M_0^n=I\times\mathbb{R}^{n-1}$, the manifold L_0^p is free of flat points and $f|_{U_0}$ splits as

$$f|_{U_0}=F\times\mathrm{id}\colon (L_0^p\times I)\times\mathbb{R}^{n-1}\to\mathbb{R}^{p+3}\times\mathbb{R}^{n-1}=\mathbb{R}^{p+n+2}.$$

Moreover, $F\colon L_0^p\times I\to\mathbb{R}^{p+3}$ is a composition $F=H\circ\tilde F$, where

$$\tilde F=G\times i\colon L_0^p\times I\to\mathbb{R}^{p+1}\times\mathbb{R}=\mathbb{R}^{p+2}$$

is a cylinder over a hypersurface $G\colon L_0^p\to\mathbb{R}^{p+1}$, and $H\colon W\to\mathbb{R}^{p+3}$ is an isometric immersion of an open subset $W\supset\tilde F(L_0^p\times I)$ of $\mathbb{R}^{p+2}$.

We take the opportunity to point out that the main theorem in [3] misses the conditions that L_0^p is free of flat points and that $\tilde F$ is a cylinder $\tilde F=G\times i$, which follow from Corollary 24.

5. Immersions of type C

The aim of this section is to prove the following pointwise result for isometric immersions of type C.

Proposition 36. *Let $f\colon L^p\times_\rho M^n\to\mathbb{Q}_c^{p+n+2}$ be an isometric immersion of a warped product. If f is of type C at $z\in N^{p+n}=L^p\times_\rho M^n$ and $n\geq 3$, then N^{p+n} has constant sectional curvature c at z.*

Proof. We must prove that the curvature like tensor C on T_zN defined in Proposition 2 vanishes identically. Two possible cases may occur:

Case 1. For every $X \in \mathcal{H}_z$ the linear map B_X defined in (22) satisfies $\operatorname{rank} B_X \leq 1$.

Case 2. There exists $X \in \mathcal{H}_z$ such that $\operatorname{rank} B_X = 2$.
We first prove the following facts.

(i) In Case 1, for any $X \in \mathcal{H}_z$ with $\operatorname{rank} B_X = 1$ we have

$$\alpha(E,V) = 0 \quad \text{for all } \ V \in \mathcal{D}(X) = \ker B_X \ \text{ and } \ E \in T_zN, \tag{52}$$

that is, $\mathcal{D}(X)$ is contained in the relative nullity subspace Δ of f at z.
(ii) In Case 2, condition (52) is true for any $X \in \mathcal{H}_z$ such that $\operatorname{rank} B_X = 2$.

Proof of (i). Here Lemma 20 applies and we have by (23) that $\mathcal{D}(Y) = \{e\}^{\perp}$ for any $Y \in \mathcal{H}_z$ with $\operatorname{rank} B_Y = 1$. In particular, this shows that (52) is satisfied for any $E \in \mathcal{H}_z$. On the other hand, (24) yields $\alpha(V,W) = \langle V,e\rangle\langle W,e\rangle\alpha(e,e)$, and hence (52) also holds for any $E \in \mathcal{V}_z$.

Proof of (ii). If $\operatorname{rank} B_X = 2$, i.e., $B_X(\mathcal{V}_z) = T_z^{\perp}N$, then (52) is equivalent to

$$\langle \alpha(E,V), \alpha(X,W)\rangle = 0$$

for all $V \in \mathcal{D}(X)$, $W \in \mathcal{V}_z$ and $E \in T_zN$. But this follows from (13) for $E \in \mathcal{H}_z$ and from (14) for $E \in \mathcal{V}_z$.

We now prove that

$$C(E_1,V_1,V_2,E_2) = 0 \quad \text{for all } \ E_1,E_2 \in T_zN \ \text{ and } \ V_1,V_2 \in \mathcal{V}_z. \tag{53}$$

We obtain from (14) that (53) holds whenever one of the vectors E_1, E_2 lies in $\mathcal{H}_z$ and the other in $\mathcal{V}_z$. On the other hand, if we are in Case 1 (resp., Case 2) and $X \in \mathcal{H}_z$ satisfies $\operatorname{rank} B_X = 1$ (resp., $\operatorname{rank} B_X = 2$), then it follows from (52) that (53) is also satisfied if any of the vectors E_1, E_2, V_1 or V_2 belongs to $\mathcal{D}(X)$. In particular, $C(E_1,V,V,E_2) = 0$ holds for any $V \in \mathcal{D}(X)$. Notice that $\mathcal{D}(X) \neq \{0\}$ by our assumption that $n \geq 3$. Applying (11) for $0 \neq W = V \in \mathcal{D}(X)$ yields

$$\nabla_Y \eta - \langle Y,\eta\rangle\eta - cY = 0 \ \text{ for any } \ Y \in \mathcal{H}_z. \tag{54}$$

Notice that if N^{p+n} is a Riemannian product then this implies that $c = 0$. Moreover, since for $c = 0$ this equation holds automatically for Riemannian products, in this case it is enough to assume that either $n \geq 2$ or $p \geq 2$.

Therefore, (53) is satisfied for all $E_1, E_2 \in \mathcal{H}_z$ and $V_1, V_2 \in \mathcal{V}_z$. This completes the proof of (53) in Case 1 and shows that in Case 2 it remains to prove that (53) is satisfied if E_1, E_2, V_1, V_2 all belong to the two-dimensional subspace $\mathcal{D}(X)^\perp$. Since C is a curvature like tensor, this will follow once we prove the existence of an orthonormal basis $\{e_1, e_2\}$ of $\mathcal{D}(X)^\perp$ such that

$$C(e_1, e_2, e_2, e_1) = 0. \tag{55}$$

In order to prove (55) take an orthonormal basis e_1, e_2 of $\mathcal{D}(X)^\perp$ such that e_1 is one of the two points on the unit circle S^1 in $\mathcal{D}(X)^\perp$ where $\phi\colon S^1 \to \mathbb{R}$ given by $\phi(V) = \|B_X V\|^2$ assumes its maximum value. Differentiating $\psi(t) = \phi(\cos te_1 + \sin te_2)$ yields

$$0 = \psi'(0) = 2\langle B_X e_1, B_X e_2\rangle.$$

Thus, there exist an orthonormal basis $\{\xi_1, \xi_2\}$ of $T_z^\perp N$ and positive real numbers λ_1, λ_2 such that $B_X e_r = \lambda_r \xi_r$ for $r = 1, 2$. We have from (14) that

$$\lambda_1\langle\alpha(e_2, e_r), \xi_1\rangle = \lambda_2\langle\alpha(e_1, e_r), \xi_2\rangle, \quad r = 1, 2. \tag{56}$$

Using that (53) holds for $E_1 = E_2 = X$, $V_1 = e_s$ and $V_2 = e_t$, $1 \leq s, t \leq 2$, we obtain

$$\langle\alpha(X, X), \alpha(e_r, e_r)\rangle = \langle\alpha(X, e_r), \alpha(X, e_r)\rangle = \lambda_r^2,$$
$$\langle\alpha(X, X), \alpha(e_1, e_2)\rangle = \langle\alpha(X, e_1), \alpha(X, e_2)\rangle = 0.$$

Setting $\gamma_{st}^r = \langle\alpha(e_s, e_t), \xi_r\rangle = \gamma_{st}^r$, $\alpha_r = \langle\alpha(X, X), \xi_r\rangle$ and $D = \gamma_{11}^1\gamma_{22}^2 - \gamma_{11}^2\gamma_{22}^1$, where $1 \leq s, t \leq 2$, it follows that

$$a_1\gamma_{rr}^1 + a_2\gamma_{rr}^2 = \lambda_r^2 \quad \text{and} \quad a_1\gamma_{12}^1 + a_2\gamma_{12}^2 = 0. \tag{57}$$

If we compute Da_1 and Da_2 from the first equation in (57) and put the result into the second we obtain an equation which because of (56) is equivalent to

$$\lambda_1\lambda_2(\langle\alpha(e_1, e_1), \alpha(e_2, e_2)\rangle - \langle\alpha(e_1, e_2), \alpha(e_1, e_2)\rangle) = 0,$$

and this gives (55) and concludes the proof of (53).

Because of Proposition 2 it remains to show that

$$C(Y_1, Y_2, Y_3, Y_4) = 0 \quad \text{for all } \; Y_1, Y_2, Y_3, Y_4 \in \mathcal{H}_z. \tag{58}$$

We divide the proof into the same two cases considered before.

Case 1. By Lemma 20 the linear map $\mathcal{H}_z \to T_z^\perp N$, $\;Y \mapsto \alpha(Y, e)$ is surjective. Hence $p \geq 2$, and we can take $X_1, X_2 \in \mathcal{H}_z$ such that the vectors $\xi_j := \alpha(X_j, e)$, $j = 1, 2$, form an orthonormal normal basis and $\alpha(Z, e) = 0$ for all $Z \in \{X_1, X_2\}^\perp$. We have from (12) and (13) that

$$0 = C(Z, X_j, e, E) = \langle A_{\xi_j} Z, E\rangle$$

for all $Z \in \{X_1, X_2\}^\perp$ and any $E \in T_z N$. Therefore, $A_{\xi_j} Z = 0$ for all $Z \in \{X_1, X_2\}^\perp$, that is, $\{X_1, X_2\}^\perp \subset \Delta(z)$. Hence (58) holds whenever $Y_i \in \{X_1, X_2\}^\perp$ for some $1 \leq i \leq 4$. Thus, it remains to prove that

$$C(X_1, X_2, X_2, X_1) = 0. \tag{59}$$

Because of (53) we have

$$\langle \alpha(X_i, X_j), \alpha(e,e)\rangle = \langle \xi_i, \xi_j\rangle. \tag{60}$$

We now may assume that $\alpha(e,e) = a\,\xi_2$ with $a \neq 0$. We obtain from (60) that

$$\langle A_{\xi_2} X_1, X_2\rangle = 0 \tag{61}$$

and

$$a\langle A_{\xi_2} X_j, X_j\rangle = 1, \quad j = 1, 2.$$

Since $a \neq 0$, it follows from the last equation that

$$\langle A_{\xi_2} X_1, X_1\rangle = \langle A_{\xi_2} X_2, X_2\rangle. \tag{62}$$

In addition, (12) yields

$$0 = C(X_i, X_j, e, X_i) = \langle A_{\xi_j} X_i, X_i\rangle - \langle A_{\xi_i} X_i, X_j\rangle, \quad i \neq j. \tag{63}$$

Then (59) follows from (61), (62) and (63), and the proof is completed in this case.

Case 2. Since (55) holds, it follows from a result of E. Cartan ([5]; cf. Theorem 1 in [15]), that we may choose nonzero vectors v_1, v_2 in $\mathcal{D}(X)^\perp$ (not necessarily orthogonal) such that $\langle \alpha(v_1, v_1), \alpha(v_2, v_2)\rangle = 0$ and $\alpha(v_1, v_2) = 0$. From (53) we have

$$0 = C(Z, v_1, v_2, Y) = -\langle \alpha(Z, v_2), \alpha(Y, v_1)\rangle.$$

Therefore, the subspaces $\alpha(\mathcal{H}_z, v_1), \alpha(\mathcal{H}_z, v_2)$ are orthogonal lines spanned by $\eta_1 = \alpha(X, v_1)$ and $\eta_2 = \alpha(X, v_2)$, respectively, which we may assume to have unit length by rescaling v_1 and v_2 if necessary. In particular, the kernel $\mathcal{H}_j$ of the linear map $F_j\colon \mathcal{H}_z \to T_z^\perp N$ given by $F_j(Y) = \alpha(Y, v_j)$, $j = 1, 2$, has codimension one in $\mathcal{H}_z$. On the other hand, for $Y \in \mathcal{H}_j$ and any $Z \in \mathcal{H}_z$ we obtain using (12) that

$$\langle A_{\eta_j} Y, Z\rangle = \langle \alpha(Y, Z), \alpha(X, v_j)\rangle = \langle \alpha(Y, v_j), \alpha(X, Z)\rangle = \langle F_j(Y), \alpha(X, Z)\rangle = 0.$$

Let Z_j be a unit vector in $\mathcal{H}_z$ orthogonal to $\mathcal{H}_j$, $j = 1, 2$. Then

$$\begin{aligned}
C(Y_1, Y_2, Y_3, Y_4) &= \langle \alpha(Y_1, Y_4), \alpha(Y_2, Y_3)\rangle - \langle \alpha(Y_1, Y_3), \alpha(Y_2, Y_4)\rangle \\
&= \sum_{j=1}^{2} (\langle A_{\eta_j} Y_1, Y_4\rangle \langle A_{\eta_j} Y_2, Y_3\rangle - \langle A_{\eta_j} Y_1, Y_3\rangle \langle A_{\eta_j} Y_2, Y_4\rangle) \\
&= \sum_{j=1}^{2} \langle Y_1, Z_j\rangle \langle Y_2, Z_j\rangle (\langle A_{\eta_j} Z_j, Y_4\rangle \langle A_{\eta_j} Z_j, Y_3\rangle \\
&\qquad - \langle A_{\eta_j} Z_j, Y_3\rangle \langle A_{\eta_j} Z_j, Y_4\rangle) = 0. \qquad \square
\end{aligned}$$

Proposition 36 and the italicized statement in its proof yield the following result for the case of Riemannian products.

Corollary 37. *Let* $f\colon L^p \times M^n \to \mathbb{Q}_c^{p+n+2}$ *be an isometric immersion of a Riemannian product. Assume that* f *is of type* C *at* $z \in N^{p+n} = L^p \times M^n$*. Then we have:*

(i) *If either* $p \geq 3$ *or* $n \geq 3$ *then* $c = 0$.
(ii) *If* $p + n \geq 3$ *and* $c = 0$ *then* N^{p+n} *is flat at* z.

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