FURTHER STUDY OF A SIMPLE PDE

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ABSTRACT. This essentially self-contained paper continues the study of a simple PDE in which an unorthodox sign in the spacial boundary condition destroys the usual Minimum Principle. The long-term behavior of solutions with time-parameter set $(0, \infty)$ is established, and this clarifies in analytic terms the characterization of non-negative solutions which had been obtained previously by probabilistic methods. The paper then studies by direct methods bounded 'ancient' solutions in which the time-parameter set is $(-\infty, 0)$. In the final section, Martin-boundary theory is used to describe all non-negative ancient solutions in the most interesting case. The relevant Green kernel density behaves rather strangely, exhibiting two types of behavior in relation to scaling of its arguments. The Martin kernel density, a *ratio* of Green kernel densities, behaves more sensibly. Doob h-transforms illuminate the structure. As a somewhat surprising consequence of our Martin-boundary analysis, we find that non-negative solutions to our parabolic-looking equation satisfy an elliptic-type Harnack principle.

1. Introduction and summary

For each $(\mu, \sigma) \in \mathbb{R}^2$, consider the boundary value problem[1]

$$(1.1) \qquad \dot{u} = \frac{1}{2}u'' + \mu u' \quad \text{with boundary condition} \quad \dot{u}(t, 0) = \sigma u'(t, 0)$$

in some time-space region of the form $I \times [0, \infty)$, where I is an interval in $\mathbb{R}$. We emphasize that, throughout, our solutions u to (1.1) are continuously differentiable, once in time and twice in space, in the *whole* of $I \times [0, \infty)$, including the spacial boundary $I \times \{0\}$.

In [7], we proved the following result about the Cauchy initial value problem for (1.1). In its statement, U denotes the set of $u \in C^{1,2}\big((0, \infty) \times [0, \infty); \mathbb{R}\big)$

[1]We use $\dot{u}$ and u' to denote, respectively, the time and spacial derivatives $\partial_t u$ and $\partial_x u$ of u. In addition, we follow the *probabilist's* convention of taking time t as the first variable and space x as the second.

which are bounded in $(0, 1] \times [0, \infty)$ and satisfy

$$\sup_{(t,x) \in [T_1, T_2] \times [0, \infty)} |u(t,x)| \vee |\dot{u}(t,x)| \vee |u'(t,x)| \vee |u''(t,x)| < \infty$$

for all $0 < T_1 < T_2 < \infty$. In addition, we use F to stand for the set of bounded $f : [0, \infty) \to \mathbb{R}$ which are continuous on $(0, \infty)$, but *not* necessarily at 0.

THEOREM 1.1. *If $u \in U$ satisfies (1.1) and $\lim_{t \searrow 0} u(t, x)$ exists for each $x \in (0, \infty)$, then $\lim_{t \searrow 0} u(t, 0)$ exists. Moreover, for each $f \in F$, there exists a unique $u = u_f \in U$ which satisfies (1.1) in $(0, \infty) \times [0, \infty)$ and the initial conditions that, as $t \searrow 0$, $u(t, 0) \to f(0)$ and $u(t, \cdot) \restriction (0, \infty) \to f \restriction (0, \infty)$ uniformly on compact subsets of $(0, \infty)$.*

As an easy dividend from Theorem 1.1, we can define a semigroup $\{\mathbf{Q}_t^{\sigma,\mu} : t \geq 0\}$ of operators on F by $\mathbf{Q}_t^{\sigma,\mu} f = u_f(t, \cdot)$. However, unless $\sigma \geq 0$, $\{\mathbf{Q}_t^{\sigma,\mu} : t \geq 0\}$ will not be a Markov semigroup. In fact, we have shown in [7] that the situation is the one summarized next. In this statement and throughout, for given $(\sigma, \mu) \in (-\infty, 0) \times \mathbb{R}$, we use $J_{\sigma,\mu}$ to denote the function $2|\sigma| e^{2(\sigma \wedge \mu)x}$ of $x \in (0, \infty)$ and write $\langle J_{\sigma,\mu}, f \rangle$ for $\int_{(0,\infty)} f(x) J_{\sigma,\mu}(x) \, dx$.

THEOREM 1.2.   *When $\sigma \geq 0$, $\{\mathbf{Q}_t^{\sigma,\mu} : t \geq 0\}$ is a conservative, Markov semigroup. On the other hand, if $\sigma < 0$, then $u_f \geq 0$ if and only if $f \geq 0$ and $f(0) \geq \langle J_{\sigma,\mu}, f \rangle$. Moreover, if $f(0) = \langle J_{\sigma,\mu}, f \rangle$, then $u_f(t, 0) = \langle J_{\sigma,\mu}, u_f(t, \cdot) \rangle$ for all $t > 0$. In particular, if $F(\sigma, \mu) := \{f \in F : f(0) = \langle J_{\sigma,\mu}, f \rangle\}$ and $\mathbf{P}_t^{\sigma,\mu} := \mathbf{Q}_t^{\sigma,\mu} \restriction F(\sigma, \mu)$, then $\{\mathbf{P}_t^{\sigma,\mu} : t \geq 0\}$ is a Markov, contraction semigroup which is conservative if and only if $\mu \geq \sigma$.*

The probabilistic interpretation of $\{\mathbf{P}_t^{\sigma,\mu}\}$ is explained in [7].

Comments on the proof of Theorem 1.2 are made in Section 2. In Section 3, we study the limiting behavior of $u_f(t, \cdot)$ as $t \nearrow \infty$, and thereby clarify Theorem 1.2. In Section 4, we study by direct methods the bounded 'ancient' solutions of our PDE, that is, bounded solutions defined on $(-\infty, 0) \times [0, \infty)$. In Section 5, we use Martin-boundary theory to construct all non-negative ancient solutions in the most interesting (and most difficult) 'balanced' case when $\mu = \sigma < 0$. The behavior of the relevant Green kernel density is rather weird, but, fortunately, the Martin kernel density behaves rather sensibly.

We emphasize that our PDE concerns a case with only one spacial boundary point. Section 2 of [7] on the Markov-chain case provides clues to what to expect in general. The two-point boundary analogue of the problem studied here has been studied in [8] and [3]. Paper [3] uses methodology of the sort used here although the presence of a second boundary point introduces some interesting complications. Paper [8] employs indefinite inner products.

2. Some notation and methods from [7]

The initial assertion in Theorem 1.2, the one when $\sigma \geq 0$, is a simple application of the minimum principle. To understand the role of $J_{\sigma,\mu}$ when $\sigma < 0$, suppose $u \in U$ solves (1.1), use a little integration by parts to see that

$$\frac{d}{dt}\big(u(t,0) - \langle J_{\sigma,\mu}, u(t,\,\cdot\,)\rangle\big) = C(\mu,\sigma)\big(u(t,0) - \langle J_{\sigma,\mu}, u(t,\,\cdot\,)\rangle\big),$$

where $C(\mu,\sigma) := 2|(\mu - \sigma)\sigma|$ if $\sigma \leq \mu$, $C(\mu,\sigma) = 0$ if $\sigma \geq \mu$, and conclude that

$$(2.1) \qquad \big(u(t,0) - \langle J_{\sigma,\mu}, u(t,\,\cdot\,)\rangle\big) = e^{C(\mu,\sigma)(t-s)}\big(u(s,0) - \langle J_{\sigma,\mu}, u(s,\,\cdot\,)\rangle\big)$$

for $s < t$. In particular, for $t \in [0, \infty)$,

$$(2.2) \qquad \big(u_f(t,0) - \langle J_{\sigma,\mu}, u_f(t,\,\cdot\,)\rangle\big) = e^{C(\mu,\sigma)t}\big(f(0) - \langle J_{\sigma,\mu}, f\rangle\big),$$

which shows that $F(\sigma,\mu)$ is invariant under $\{\mathbf{Q}_t^{\sigma,\mu} : t \geq 0\}$. Next, introduce

$$(2.3) \qquad Q_\mu(t,x,y) = e^{\mu(y-x) - \frac{\mu^2 t}{2}}\big(g(t, x-y) - g(t, x+y)\big),$$

$$(2.4) \qquad q_\mu(t,x) = xe^{-\mu x - \frac{\mu^2 t}{2}}\frac{g(t,x)}{t},$$

where $g(t,x) := (2\pi t)^{-\frac{1}{2}}\exp\left(-\frac{x^2}{2t}\right)$ is the centered Gauss kernel with variance t. Then it is an easy matter to check that if $u \in C^{1,2}\big((0,T] \times [0,\infty); \mathbb{R}\big)$ satisfies $\dot{u} = \frac{1}{2}u'' + \mu u'$ with initial condition $\lim_{t \searrow 0} u(t,x) = f(x)$ for each $x \in (0,\infty)$ and growth condition

$$\sup_{(t,x) \in (0,T] \times [0,\infty)} e^{-\lambda x}|u(t,x)| < \infty \quad \text{for some } \lambda < \infty,$$

then, for $(t,x) \in (0,T] \times [0,\infty)$,

$$(2.5) \qquad u(t,x) = \int_0^\infty Q_\mu(t,x,y)f(y)\,dy + \int_0^t q_\mu(t-\tau, x)u(\tau,0)\,d\tau.$$

Applying (2.5) to u_f and taking into account (2.2), we now see that

$$(2.6) \qquad u_f(t,x) = \int_0^\infty Q_\mu(t,x,y)f(y)\,dy$$

$$+ \big(f(0) - \langle J_{\sigma,\mu}, f\rangle\big)\int_0^t q_\mu(t-\tau,x)e^{C(\mu,\sigma)\tau}\,d\tau$$

$$+ \int_0^t q_\mu(t-\tau,x)\langle J_{\sigma,\mu}, u_f(\tau,\,\cdot\,)\rangle\,d\tau.$$

In particular, if $f \geq 0$ and $f(0) \geq \langle J_{\sigma,\mu}, f\rangle$, then (2.5) implies that

$$u_f(t,x) \geq \int_0^t q_\mu(t-\tau,x)\langle J_{\sigma,\mu}, u_f(\tau,\cdot)\rangle\,d\tau,$$

from which it is a simple matter to conclude first that $\langle J_{\sigma,\mu}, u_f(t, \cdot) \rangle \geq 0$ and then that $u_f \geq 0$.

We now assume that $\sigma < 0$. In [7], we based our derivation of the necessity assertion in the second part of Theorem 1.2 on the probabilistic interpretation of solutions to (1.1). Namely, let $\{B(t) : t \geq 0\}$ be a standard, $\mathbb{R}$-valued Brownian motion and define

$$(2.7) \qquad\qquad L_\mu(t) = \max\{(B(\tau) + \mu\tau)^- : \tau \in [0,t]\}$$

and

$$(2.8) \qquad\qquad X_\mu(t) = B(t) + \mu t + L_\mu(t)$$

so that X_μ is a reflected BM with drift μ and L_μ is its local time at 0. Finally, define

$$(2.9) \qquad \Psi_{\sigma,\mu}(t) := |\sigma|^{-1} L_\mu(t) - t, \qquad \rho_{\sigma,\mu} = \inf\{t > 0 : \Psi_{\sigma,\mu} \leq 0\}.$$

Then, with probability 1, $\rho_{\sigma,\mu} > 0$ and, for any solution $u \in C^{1,2}\big((0,\infty) \times [0,\infty); \mathbb{R}\big)$ to (1.1),

$$t \rightsquigarrow u\big(\Psi_{\sigma,\mu}(t \wedge \rho_{\sigma,\mu}), X_\mu(t \wedge \rho_{\sigma,\mu})\big)$$

will be a local martingale. The proof in [7] of the necessity of $f(0) \geq \langle J_{\sigma,\mu}, f \rangle$ turned on the equality

$$\mathbb{E}\big[f\big(X_\mu(\rho_{\sigma,\mu})\big), \rho_{\sigma,\mu} < \infty\big] = \langle J_{\sigma,\mu}, f \rangle$$

and the observation that the local martingale determined by u_f would be a supermartingale if $u_f \geq 0$. As was pointed out in [5], the same line of reasoning leads to the following lemma.

LEMMA 2.1. *Suppose that $\sigma < 0$. If $\mu < \sigma$, then u_f is always bounded on $(0,\infty) \times [0,\infty)$ and $\lim_{t \to \infty} u_f(t, \cdot) = 0$ uniformly on compacts if and only if $f(0) = \langle J_{\sigma,\mu}, f \rangle$.*
If $\mu \geq \sigma$, then u_f is bounded on $(0,\infty) \times [0,\infty)$ if and only if $f(0) = \langle J_{\sigma,\mu}, f \rangle$.

In the following section we will give another proof of both the necessity in the second part of Theorem 1.2 and of Lemma 2.1, one which yields sharper results and removes all of the probability.

3. Limits as $t \nearrow \infty$

Everything in this section comes from the following statement.

LEMMA 3.1.   *Let $\sigma < 0$ and $\mu \in \mathbb{R}$ be given, and use $v_{\sigma,\mu}$ to denote the unique solution $u_{\mathbf{1}_{\{0\}}}$, described in Theorem 1.1, to (1.1) with initial value $\mathbf{1}_{\{0\}}$. Then $v_{\sigma,\mu} \geq 0$ and, as $t \nearrow \infty$,*

(3.1) $$\mu < \sigma \implies v_{\sigma,\mu}(t,x) \to \frac{\mu}{\mu - \sigma},$$

(3.2) $$\mu = \sigma \implies t^{-1} v_{\sigma,\mu}(t,x) \to 2\sigma^2,$$

(3.3) $$\mu > \sigma \implies e^{-2|(\mu-\sigma)\sigma|t} v_{\sigma,\mu}(t,x) \to \frac{\mu - 2\sigma}{\mu - \sigma} e^{-2(\mu-\sigma)x},$$

the limits being uniform over x in compact subsets of $[0, \infty)$.

Proof. By the sufficiency assertion in the second part of Theorem 1.2, we know that $v_{\sigma,\mu}$ non-negative.

In order to find the behavior of $v_{\sigma,\mu}$ at infinity, we use the representation

(3.4) $$v_{\sigma,\mu} \;=\; \frac{u_f - u_{\tilde{f}}}{f(0) - \langle J_{\sigma,\mu}, f \rangle},$$

where f is chosen so that $f(0) \neq \langle J_{\sigma,\mu}, f \rangle$ and $\tilde{f} \in F(\sigma, \mu)$ is defined so that $\tilde{f} \upharpoonright (0,\infty) = f \upharpoonright (0,\infty)$ and $\tilde{f}(0) = \langle J_{\sigma,\mu}, f \rangle$. Of course, (3.4) is useful only if f is chosen so we know a lot about u_f and can control $u_{\tilde{f}}$.

When $\mu < \sigma$, we take $f = \mathbf{1}$, in which case $u_f = \mathbf{1}$ and $f(0) - \langle J_{\sigma,\mu}, f \rangle = 1 - \frac{\sigma}{\mu} = \frac{\mu-\sigma}{\mu}$. Now, $u_{\tilde{1}}(t,\cdot) = \mathbf{P}_t^{\sigma,\mu}\tilde{1}$, and since $\mu < \sigma$, $\{\mathbf{P}_t^{\sigma,\mu}\}$ is the transition semigroup of a process with finite lifetime. On combining this with (2.5), we see that, as $t \nearrow \infty$, $u_{\tilde{1}}(t,\cdot) \to 0$ uniformly on compacts, and so (3.4) with this choice of f leads to (3.1).

If $\mu > \sigma$ and $f(x) = e^{-2(\mu-\sigma)x}$, then $u_f(t,x) = e^{-2(\mu-\sigma)(x+\sigma t)}$ and $f(0) - \langle J_{\sigma,\mu}, f \rangle = \frac{\mu-\sigma}{\mu-2\sigma}$, and, because $\tilde{f} \in F(\sigma,\mu)$, Theorem 1.2 implies that $u_{\tilde{f}}$ is bounded. Thus, (3.3) is also proved.

One can treat the case when $\sigma = \mu < 0$ in a similar way, taking $u(t,x) = x + \sigma t$, $f(x) = x$, $\tilde{u}(t,\cdot) = \mathbf{P}_t^{\sigma,\sigma}\tilde{f}$. The problem is that since f is unbounded, this requires extensions of Theorems 1.1 and 1.2. Though these extensions may be established, one estimate requiring renewal theory, we now adopt a 'direct calculation' approach in which renewal theory plays an implicit part.

So suppose that $\mu = \sigma < 0$. Then, by (2.6),

(3.5) $$v_{\sigma,\sigma}(t,x) \;=\; \int_0^t q_\sigma(t-\tau, x)\big(1 + \langle J_{\sigma,\sigma}, v_{\sigma,\sigma}(\tau,\cdot)\rangle\big)\, d\tau.$$

Moreover, Lemma 3.4 of [7] shows that, for $t > 0$,

$$\sup_x v_{\sigma,\sigma}(t,x) \;\leq\; Ae^{At}$$

for some $A > 0$. Hence, for $\lambda > A$, we may introduce for each x the Laplace transform

$$\hat{v}_{\sigma,\sigma}(\lambda, x) := \int_0^\infty e^{-\lambda t} v_{\sigma,\sigma}(t, x)\, dt.$$

Now,

$$\hat{q}_\sigma(\lambda, x) := \int_0^\infty e^{-\lambda t} q_\sigma(t, x)\, dt = e^{-\left(\sqrt{\sigma^2 + 2\lambda} - |\sigma|\right)x},$$

whence

$$\langle J_{\sigma,\sigma}, \hat{q}_\sigma(\lambda, \cdot)\rangle = \frac{2|\sigma|}{\sqrt{\sigma^2 + 2\lambda} + |\sigma|}.$$

Equation (3.5) now implies that (for $\lambda > A$)

$$\langle J_{\sigma,\sigma}, \hat{v}_{\sigma,\sigma}(\lambda, \cdot)\rangle = \frac{2|\sigma|}{\sqrt{\sigma^2 + 2\lambda} + |\sigma|} \left(\frac{1}{\lambda} + \langle J_{\sigma,\sigma}, \hat{v}_{\sigma,\sigma}(\lambda, \cdot)\rangle\right),$$

whence, again from (3.5),

$$\hat{v}_{\sigma,\sigma}(\lambda, x) = \hat{v}_{\sigma,\sigma}(\lambda, 0) e^{-\left(\sqrt{\sigma^2 + 2\lambda} - |\sigma|\right)x},$$

where

$$\begin{aligned}
\hat{v}_{\sigma,\sigma}(\lambda, 0) &= \frac{\sqrt{\sigma^2 + 2\lambda} + |\sigma|}{\sqrt{\sigma^2 + 2\lambda} - |\sigma|} \times \frac{1}{\lambda} \\
&= \frac{\sigma^2}{\lambda^2} + \frac{|\sigma|^3}{\lambda^2 \sqrt{\sigma^2 + 2\lambda}} + \frac{2|\sigma|}{\lambda \sqrt{\sigma^2 + 2\lambda}} + \frac{1}{\lambda}.
\end{aligned}$$

If we write $\mathcal{L}^{-1}$ for 'inverse Laplace transform', then

$$\mathcal{L}^{-1}\left(\frac{1}{\lambda \sqrt{\sigma^2 + 2\lambda}}\right) = \int_0^t \frac{e^{-\frac{1}{2}\sigma^2 s}}{\sqrt{2\pi s}}\, ds$$

and

$$\mathcal{L}^{-1}\left(\frac{1}{\lambda^2 \sqrt{\sigma^2 + 2\lambda}}\right) = \frac{t}{|\sigma|} - t\int_t^\infty \frac{e^{-\frac{1}{2}\sigma^2 s}}{\sqrt{2\pi s}}\, ds - \int_0^t \frac{s^{\frac{1}{2}} e^{-\frac{1}{2}\sigma^2 s}}{\sqrt{2\pi}}\, ds.$$

We therefore have, as $t \to \infty$, $v_{\sigma,\sigma}(t, 0) \sim 2\sigma^2 t$, whence

$$t^{-1} v_{\sigma,\sigma}(t, x) = t^{-1} \int_0^t q_\sigma(\tau, x) v_{\sigma,\sigma}(t - \tau, 0)\, d\tau \;\to\; 2\sigma^2,$$

uniformly over compact x-intervals. $\square$

THEOREM 3.2.   *Assume that $\sigma < 0$. Given $f \in F$, set $\Delta_{\sigma,\mu}(f) = f(0) - \langle J_{\sigma,\mu}, f \rangle$. Then, as $t \nearrow \infty$,*

$$\mu < \sigma \implies u_f(t,x) \to \frac{\mu}{\mu - \sigma} \Delta_{\sigma,\mu}(f),$$

$$\mu = \sigma \implies t^{-1} u_f(t,x) \to 2\sigma^2 \Delta_{\sigma,\mu}(f),$$

$$\mu > \sigma \implies e^{-2|(\mu-\sigma)\sigma|t} u_f(t,x) \to \frac{\mu - 2\sigma}{\mu - \sigma} e^{-2(\mu-\sigma)x} \Delta_{\sigma,\mu}(f),$$

the limits being uniform over x in compact subsets of $[0, \infty)$. In particular,

$$\Delta_{\sigma,\mu}(f) \geq 0 \iff \limsup_{t \to \infty} u_f(t,x) \geq 0 \text{ for some } x \in [0, \infty)$$

$$\iff \liminf_{t \to \infty} u(t,x) \geq 0 \text{ for all } x \in [0, \infty).$$

Proof. There is hardly anything to do. Namely, with the notation that we used in Lemma 3.1 and its proof, we have that $u_f = \Delta_{\sigma,\mu}(f) v_{\sigma,\sigma} + u_{\tilde{f}}$. When $\mu < \sigma$, $u_{\tilde{f}}(t, \cdot) \to 0$ uniformly on compacts, and in general $u_{\tilde{f}}$ is bounded. Hence, the asserted conclusions follow immediately from (3.1)–(3.3). $\square$

4. Bounded, ancient solutions

In this section we will study bounded solutions to (1.1) which are "ancient" in the sense that they are solutions for $t \in (-\infty, 0)$. For our analysis of these solutions, we will require two 'regularity' lemmas, in the first of which our boundary condition plays no part.

LEMMA 4.1.   *Let $\mu \in \mathbb{R}$. Then there exists a $B < \infty$, which depends only on $|\mu|$, such that, for any $n \geq 1$ and $\varphi \in C_{\mathrm{b}}^{2n}([0, \infty); \mathbb{R})$,*

$$(4.1) \qquad \|\partial^{2n-1} \varphi\|_{\mathrm{u}} \vee \|\partial^{2n} \varphi\|_{\mathrm{u}} \ \leq \ B^n \sum_{m=0}^{n} \binom{n}{m} \|L_\mu^m \varphi\|_{\mathrm{u}},$$

where $L_\mu = \frac{1}{2}\partial_x^2 + \mu\partial_x$.

Proof. By Taylor's Theorem, if $\varphi \in C_{\mathrm{b}}^2([0, \infty); \mathbb{R})$, then

$$(4.2) \qquad \|\varphi'\|_{\mathrm{u}} \ \leq \ \frac{1}{\alpha}\|\varphi\|_{\mathrm{u}} + \alpha\|\varphi''\|_{\mathrm{u}}, \quad \alpha > 0,$$

$\| \cdot \|$ denoting the supremum (uniform) norm. Hence, for any $\alpha > 0$,

$$\|\varphi''\|_{\mathrm{u}} \ \leq \ 2\|L_\mu \varphi\|_{\mathrm{u}} + 2|\mu| \ \|\varphi'\|_{\mathrm{u}} \ \leq \ 2\|L_\mu \varphi\|_{\mathrm{u}} + \frac{2|\mu|}{\alpha}\|\varphi\|_{\mathrm{u}} + 2\alpha|\mu| \ \|\varphi''\|_{\mathrm{u}},$$

and so, by taking $\alpha = (4|\mu|)^{-1}$, we get

$$\|\varphi''\|_{\mathrm{u}} \ \leq \ 4\|L_\mu \varphi\|_{\mathrm{u}} + 16|\mu|^2\|\varphi\|_{\mathrm{u}} \ \leq \ B\big(\|L_\mu \varphi\|_{\mathrm{u}} + \|\varphi\|_{\mathrm{u}}\big),$$

where $B = 4 \vee (4|\mu|)^2$. Of course, by another application of (4.2), this leads to

$$\tag{4.3} \|\varphi'\|_{\mathrm{u}} \vee \|\varphi''\|_{\mathrm{u}} \leq B\big(\|L_\mu\varphi\|_{\mathrm{u}} + \|\varphi\|_{\mathrm{u}}\big),$$

for a slightly different choice of $B < \infty$. Note that (4.3) is (4.1) with $n = 1$. To get (4.1) in general, we work by induction on $n \geq 1$. Namely, assuming (4.1) for n, we have

$$\|\partial^{2n+1}\varphi\|_{\mathrm{u}} = \|\partial^{2n}\partial\varphi\|_{\mathrm{u}} \leq B^n \sum_{m=0}^{n} \binom{n}{m} \|L_\mu^m \partial\varphi\|_{\mathrm{u}}$$

$$= B^n \sum_{m=0}^{n} \binom{n}{m} \|\partial L_\mu^m \varphi\|_{\mathrm{u}} \leq B^{n+1} \sum_{m=0}^{n} \binom{n}{m}\big(\|L_\mu^{m+1}\varphi\|_{\mathrm{u}} + \|L_\mu^m\varphi\|_{\mathrm{u}}\big)$$

$$= B^{n+1} \sum_{m=0}^{n+1} \binom{n+1}{m} \|L_\mu^m\varphi\|_{\mathrm{u}},$$

and similarly for $\|\partial^{2(n+1)}\varphi\|_{\mathrm{u}}$. $\qquad\square$

LEMMA 4.2. *Let $-\infty < a < b < \infty$, and suppose that $u \in C^{1,2}\big([a,b] \times [0,\infty); \mathbb{R}\big)$ is bounded and satisfies (1.1). Then, $u(b, \cdot) \in C^\infty\big([0,\infty); \mathbb{R}\big)$, and there exists a number K in $[1,\infty)$, depending only on σ and μ, such that, for each $n \geq 0$,*

$$\tag{4.4} \|\partial_x^n u(b, \cdot)\|_{\mathrm{u}} \leq \left(\frac{Kn}{b-a}\right)^{\frac{n}{2}} e^{K(b-a)} \|u(a, \cdot)\|_{\mathrm{u}}.$$

In particular, if $u \in C^{1,2}((-\infty, 0) \times [0,\infty); \mathbb{R})$ is a bounded solution to (1.1), then $u \in C^\infty\big((-\infty, 0) \times [0,\infty); \mathbb{R}\big)$ and there is a number K_1 in $[1,\infty)$, depending only on σ and μ such that, for each $n \geq 0$,

$$\tag{4.5} \|u\|_{C_{\mathrm{b}}^{n,2n}((-\infty,0)\times[0,\infty);\mathbb{R})} \leq K_1^n \|u\|_{\mathrm{u}}.$$

Hence, such a u admits a unique continuation to $\mathbb{C}^2$ as an entire, analytic function.

Proof. Because of Lemma 3.4 in [7], (4.4) for $n \in \{0,1,2\}$ will follow once we show that, for each $a < T_1 < T_2 < b$, u' and u'' are bounded on $[T_1, T_2] \times [1,\infty)$. To this end, note that, by (2.5),

$$u(a + t, x) = \int_{(0,\infty)} Q_\mu(t, x, y)u(a, y)\, dy + \int_0^t q_\mu(t - \tau, x)u(a + \tau, 0)\, d\tau,$$

and so the required boundedness is trivial.

Given (4.4) for $n \in \{0,1,2\}$ and (4.1), we can complete the proof as follows. First observe that, because $u(t, \cdot) \in C_{\mathrm{b}}^2\big([0,\infty); \mathbb{R}\big)$ for each $t \in (a, b]$, we can use the semigroup property to check that, for any $a \leq s < t \leq b$, $L_\mu u(t) =$

$\dot{u}(t) = \mathbf{Q}_{t-s}^{\sigma,\mu} L_\mu u(s)$. Hence, by (4.4) for $n \in \{0, 1, 2\}$ and induction, we see that, for any $n \in \mathbb{N}$, $u \in C^{n,2n}\big((a, b] \times [0, \infty); \mathbb{R}\big)$ and

$$\|\partial_t^n u(b, \cdot)\|_{\mathrm{u}} \; = \; \|L_\mu^n u(b, \cdot)\|_{\mathrm{u}} \; \leq \; \left(\frac{An}{b-a}\right)^n e^{A(b-a)} \|u(a, \cdot)\|_{\mathrm{u}}$$

for an appropriate choice of $1 \leq A < \infty$ depending on σ and μ. After putting this together with (4.1), we get (4.4) for all $n \in \mathbb{N}$ with $K = A(1 + B)$.

Of course, to get (4.5) from (4.4), we take $a = b - n$.   □

THEOREM 4.3.   *If* $w \in C^{1,2}\big((-\infty, 0) \times [0, \infty); \mathbb{R}\big)$ *is a bounded solution to* $\dot{w} = \frac{1}{2}w'' + \mu w'$ *for some* $\mu \in \mathbb{R}$ *with* $w(t, 0) = 0$ *for all* $t < 0$, *then* $w(t, x) = C\big(1 - e^{-2\mu x}\big)$ *for some* $C \in \mathbb{R}$, *and* C *must be* 0 *if* $\mu \leq 0$.

Proof. Obviously, it suffices to check that $\dot{w} \equiv 0$. To this end, begin by observing that, for any $T > 0$ and $x > 0$,

$$w(t_2, x) - w(t_1, x) \; = \; \int_{(0, \infty)} Q_\mu(T, x, y)\big(w(t_2 - T, y) - w(t_1 - T, y)\big)\, dy.$$

Hence, since

$$\lim_{T \to \infty} \int_{(0, \infty)} Q_\mu(T, x, y) e^{-2\mu^+ y}\, dy = 0,$$

we will be done once we show that

$$\big|w(s_2, y) - w(s_1, y)\big| \; \leq \; 2\|w\|_{\mathrm{u}} e^{-2\mu^+ y} \quad \text{for all } s_1 < s_2 \leq 0 \text{ and } y \geq 0.$$

To this end, set $s = s_1$ and $\delta = s_2 - s_1$, let B_t and B_t' be a pair of independent Brownian motions starting from 0, define

$$\begin{aligned} \tau \; &:= \; \inf\{t \geq 0 : y + B_{t+\delta} + \mu(t + \delta)) = y + B_t' + \mu t\} \\ &= \; \inf\{t \geq 0 : B_t' - B_{t+\delta} = \mu\delta\}, \end{aligned}$$

and set $U_t := B_{t \wedge \tau}' + \big(B_{t+\delta} - B_{t \wedge \tau + \delta}\big)$. Then, because $\mathbb{P}(\tau < \infty) = 1$ and U_t is again a Brownian motion starting from 0 and $y + B_{t+\delta} + \mu(t + \delta) = y + U_t + \mu t$ on $\{\tau \leq t\}$, we have that

$$\big|w(s+\delta,y)-w(s,y)\big|$$

$$= \lim_{t\to\infty}\Big|\mathbb{E}\big[w\big(s-t,y+B_{t+\delta}+\mu(t+\delta)\big);\,\zeta_y^B>t+\delta\big]$$

$$-\ \mathbb{E}\big[w\big(s-t,y+U_t+\mu t\big);\,\zeta_y^U>t\big]\Big|$$

$$= \lim_{t\to\infty}\Big|\mathbb{E}\big[w\big(s-t,y+U_t+\mu t\big);\,\zeta_y^B>t+\delta\big]$$

$$-\ \mathbb{E}\big[w\big(s-t,y+U_t+\mu t\big);\,\zeta_y^U>t\big]\Big|$$

$$\le\ \|w\|_{\mathrm{u}}\Big(\mathbb{P}\big(\zeta_y^B>t+\delta\,;\,\zeta_y^U\le t\big)\ +\ \mathbb{P}\big(\zeta_y^B\le t+\delta\,;\,\zeta_y^U>t\big)\Big)$$

$$\le\ 2\|w\|_{\mathrm{u}}\mathbb{P}(\zeta_y^B<\infty),$$

where ζ_y^B and ζ_y^U are, respectively, the first times when $y+B_t+\mu t$ and $y+U_t+\mu t$ hit 0. Finally, since $\mathbb{P}(\zeta_y^B<\infty)=e^{-2\mu^+ y}$, this completes the proof. $\qquad\square$

THEOREM 4.4. *If $u\in C^{1,2}\big((-\infty,0)\times[0,\infty);\mathbb{R}\big)$ is a bounded solution to $\dot u=\frac12 u''+\mu u'$ for some $\mu\in\mathbb{R}$ and if $\dot u(t,0)=\sigma u'(t,0)$ for some $\sigma\in\mathbb{R}$ and all $t<0$, then $u(t,x)=A+Be^{-2(\mu-\sigma)(\sigma t+x)}$ for some $A,\,B\in\mathbb{R}$, where B is 0 if either $\sigma\ge\mu$ or $0<\sigma<\mu$.*

Proof. Set $w=\sigma u'-\dot u$. By Theorem 4.3 and Lemma 4.2, we know that $w(t,x)=C\big(1-e^{-2\mu x}\big)$ for some $C\in\mathbb{R}$ and that $C=0$ when $\mu\le 0$. Hence, if $\tilde u(\xi,\eta)=u(\sigma\eta-\xi,\eta+\sigma\xi)$ for $(\xi,\eta)\in\mathbb{R}^2$ satisfying $\sigma\eta-\xi\le 0$ and $\eta+\sigma\xi\ge 0$, then $\partial_\xi\tilde u(\xi,\eta)=C\big(1+(2\mu)^{-1}e^{-2\mu(\eta+\sigma\xi)}\big)$, where $C=0$ unless $\mu>0$. But, even when $\mu>0$, it is easy to check that only if $C=0$ can this be consistent with u being bounded. Hence, we now know that $\tilde u(\xi,\eta)=\psi(\eta)$, equivalently, that $u(t,x)=\psi(\sigma t+x)$ for some twice continuously differentiable ψ either on $\mathbb{R}$ if $\sigma\ge 0$ or on $[0,\infty)$ if $\sigma<0$. But in either case, the fact that $\dot u=\frac12 u''+\mu u'$ leads to the existence of $A,\,B\in\mathbb{R}$ such that $\psi(\alpha)=A+Be^{-2(\mu-\sigma)\alpha}$, and so $u(t,x)=A+Be^{-2(\mu-\sigma)(x+\sigma t)}$. Finally, because u is bounded, B must be 0 if either $\mu<\sigma$ or $\mu>\sigma>0$, and it may be taken to be 0 if $\mu=\sigma$. $\qquad\square$

COROLLARY 4.5. *Again let u be as in Theorem 4.4. If either $\sigma\le\mu$ or $\mu>\sigma>0$, then $u\equiv 0$ if $u(t,x)=0$ for some $(t,x)\in(-\infty,0)\times[0,\infty)$. If $\sigma<0$ and $\mu>\sigma$, then $u\equiv 0$ if $u(t,x)=0=u'(t,x)$ for some $(t,x)\in(-\infty,0)\times[0,\infty)$.*

Remarks. Our first proof of Theorem 4.4 was for the case when $\mu<0$ (yes 0, not σ!). For that case, we established a functional equation for the Laplace transform of the function $(t,x)\rightsquigarrow u(-t,x)$. An analytic-extension argument typical of Wiener-Hopf theory then helped clinch the result. The

methods we have presented in this section are much more naturally related to the context of our problem, deal with all cases, and, as we have seen, lead to other interesting results.

5. Non-negative ancient solutions

At (considerable) extra cost, one can use Martin-boundary theory to obtain a characterization of all *non-negative* ancient solutions, and deduce from this the results of the previous section. We shall deal only with the most interesting (and most difficult) case when $\mu = \sigma < 0$. Because our formulae are rather complicated, we shall avoid one parameter by assuming from now on that

$$\mu = \sigma = -1.$$

The process (Ψ, X) **on** $\mathbb{R} \times [0, \infty)$. We consider the process (Ψ, X) on $\mathbb{R} \times [0, \infty)$ where B is a Brownian motion starting at 0 and

$$
\begin{aligned}
L_t &:= \sup_{s \leq t}\{(X_0 + B_t - t)^-\}, \\
X_t &:= X_0 + B_t - t + L_t, \\
\Psi_t &:= \Psi_0 + L_t - t.
\end{aligned}
$$

We write $\mathbb{P}^{\psi, x}$ for the law of (Ψ, X) when $\Psi_0 = \psi$ and $X_0 = x$. As usual, $\mathbb{E}^{\psi, x}$ denotes the corresponding expectation.

The infinitesimal generator of (Ψ, X) is

$$\mathcal{A} = -\partial_\psi + \frac{1}{2}\partial_x^2 - \partial_x,$$

functions in the domain of $\mathcal{A}$ satisfying

$$\partial_\psi f + \partial_x f = 0 \text{ when } x = 0.$$

We make a 't to ψ' switch in our PDE, and from now on regard a solution of our PDE as a function $u(\psi, x)$ satisfying $\mathcal{A}u = 0$, the boundary condition being understood as implicit here. Thus we are interested in $\mathcal{A}$-*harmonic functions*. In accordance with our previous conventions, we insist that a function u in the domain of $\mathcal{A}$ be $C^{1,2}$ in the domain in question including at the spacial boundary.

THEOREM 5.1.   *A non-negative $\mathcal{A}$-harmonic function u on $\mathbb{R} \times [0, \infty)$ is constant.*

Proof. This is just a use of recurrence. Let (ψ, x) and (η, y) be points of $\mathbb{R} \times [0, \infty)$. Start (Ψ, X) at (ψ, x). Since $u(\Psi, X)$ is a non-negative local martingale, it is a supermartingale. Take a small neighbourhood N of (η, y) on which $u > u(\eta, y) - \varepsilon$. We need only show that $T := \inf\{t : (\Psi_t, X_t) \in N\}$ is almost surely finite, for then $u(\psi, x) \geq u(\eta, y) - \varepsilon$. Let $V := \inf\{t : \Psi_t < \min(\psi, \eta) - 1\}$. Then, as is shown in [7], the times greater than V at which $\Psi_2 = \eta$ form a sequence $\tau_1, \tau_2, \ldots$ of finite stopping times at each of which

$X = 0$. (See §5.1, Corollary 5.3 and part (D) of Section 6 of [7].) Let S be the open subset $\{\xi : (\eta, \xi) \in N\}$. Now the event E_k that for some $t \in [\tau_k, \tau_{k+1})$, $(\Psi_t, X_t) \in \{\eta\} \times S$ has probability $\int_S J_{-1,-1}(y)dy$ where $J_{-1,-1}(y) = 2e^{-2y}$. Moreover, the events $E_1, E_2, \ldots$ are independent. Hence, by the second Borel-Cantelli lemma, T is indeed almost surely finite. □

Discussion. Separation of variables quickly shows that when $\mu \neq \sigma$, the cone of non-negative $\mathcal{A}$-harmonic functions is infinite-dimensional. We have just seen that when $\mu = \sigma = -1$, that dimension collapses to 1. This made us wonder about 'collapse of dimension' in regard to non-negative ancient solutions.

Martin-boundary theory has the important benefit of completely characterizing the set of non-negative $\mathcal{A}$-harmonic functions on $E := (-\infty, 0) \times [0, \infty)$. Doob h-transforms really illuminate this characterization. There *is* a reflection of a 'partial collapse of dimension' in the curious *identification* of boundary points which we shall see later.

The previous sentence may be clarified by the following remarks. The Green's function (or 'Green kernel density') for $\frac{1}{2}\Delta$ (probabilists' normalization!) on $\mathbb{R}^3$ is $G(\mathbf{x}, \mathbf{y}) = (2\pi\|\mathbf{x} - \mathbf{y}\|)^{-1}$. Let $\mathbf{x}_0$ be a point in $\mathbb{R}^3$. Then the Martin kernel for $\frac{1}{2}\Delta$ on $\mathbb{R}^3$ relative to reference point $\mathbf{x}_0$ is

$$\kappa(\mathbf{x}, \mathbf{y}) := \frac{G(\mathbf{x}, \mathbf{y})}{G(\mathbf{x}_0, \mathbf{y})} = \frac{\|\mathbf{x}_0 - \mathbf{y}\|}{\|\mathbf{x} - \mathbf{y}\|}.$$

Now $\kappa(\cdot, \mathbf{y})$ converges pointwise to the constant function $\mathbf{1}$ if $\mathbf{y}$ tends to infinity in *any* manner. This implies that all points on the sphere at infinity must be identified into a single Martin boundary point, and that every non-negative $(\frac{1}{2}\Delta)$-harmonic function on $\mathbb{R}^3$ is a multiple of $\mathbf{1}$.

The resolvent kernel density for (Ψ, X) on $\mathbb{R} \times [0, \infty)$. The transition semigroup of (Ψ, X) on $\mathbb{R} \times [0, \infty)$ is the sum of two parts. First, with $\zeta_0 := \inf\{t : X_t = 0\}$, we have the obvious

$$\mathbb{P}^{\psi,x}(\Psi_t = \eta, \ X_t \in dy; \ t < \zeta_0)$$

$$= \begin{cases} 0 & \text{if } t \neq \psi - \eta, \\ Q^0(\psi - \eta, x, y)dy & \text{if } t = \psi - \eta, \end{cases}$$

Q^0 denoting Q_{-1}. Moreover, after some calculation, we find that, with q denoting q_{-1},

$$\mathbb{P}^{\psi,x}(\Psi_t \in d\eta, \ X_t \in dy; \ t > \zeta_0)/d\eta dy$$

$$= \begin{cases} 0 & \text{if } t < \psi - \eta, \\ 2e^{-2y}q(t, x + y + \eta - \psi + t) & \text{if } t > \psi - \eta. \end{cases}$$

This follows from equation (I.13.10) of Rogers and Williams [4] and the probabilistically obvious convolution result

$$\int_0^t q(s,a)q(t-s,b)\,ds = q(t,a+b) \qquad (a,b,t>0).$$

The resolvent density of (Ψ, X) on $\mathbb{R} \times [0,\infty)$ is therefore

$$r_\lambda(\psi,x;\eta,y) \;=\; I_{\{\psi-\eta>0\}}e^{-\lambda(\psi-\eta)}Q^0(\psi-\eta,x,y)$$

$$+\, 2e^{-2y} \int_{(\psi-\eta)^+}^\infty e^{-\lambda t}q(t,x+y+\eta-\psi+t)\,dt.$$

The Green and Martin kernel densities for (Ψ, X) killed on exiting the set $E := (-\infty,0) \times [0,\infty)$. What we really want is the resolvent density $r_\lambda^{0,0}$ of the process (Ψ, X) within $E := (-\infty,0) \times [0,\infty)$ killed at the first time that Ψ approaches 0: in other words, killed at the first time $\tau_{0,0}$ the process (Ψ, X) approaches $(0,0)$. The description of (Ψ, X) in terms of the driving Brownian motion B makes it clear that the first time that (Ψ, X) started from (ψ, x) (where $\psi < 0$) approaches $(0,0)$ has the same distribution as the time B started from $x - \psi$ takes to hit 0.

Please note that from now on, ψ and η will denote negative numbers, and x and y will denote non-negative numbers. We have

$$f_\lambda(\psi,x) \;:=\; \mathbb{E}^{\psi,x}e^{-\lambda\tau_{0,0}} \;=\; e^{-(x-\psi)\sqrt{2\lambda}},$$

and then,

$$r_\lambda^{0,0}(\psi,x;\eta,y) = r_\lambda(\psi,x;\eta,y) - f_\lambda(\psi,x)r_\lambda(0,0;\eta,y).$$

A key formula. It is crucial that

$$(5.1) \qquad r_\lambda^{0,0}(\psi,x;\eta,y) = I_{\{\psi-\eta>0\}}e^{-\lambda(\psi-\eta)}Q^0(\psi-\eta,x,y)$$
$$+\, 2e^{-2y}(Z_1 - Z_2),$$

where

$$Z_1 \;=\; \int_{(\psi-\eta)^+}^\infty e^{-\lambda t}\frac{(x-\psi+y+\eta+t)}{\sqrt{2\pi t^3}}\exp\left\{-\frac{(x-\psi+y+\eta)^2}{2t}\right\}\,dt$$

and

$$Z_2 \;=\; e^{-(x-\psi)\sqrt{2\lambda}}\int_{|\eta|}^\infty e^{-\lambda t}\frac{(y+\eta+t)}{\sqrt{2\pi t^3}}\exp\left\{-\frac{(y+\eta)^2}{2t}\right\}\,dt.$$

The *Green kernel density* is

$$(5.2) \qquad G(\psi,x;\eta,y) \;=\; \lim_{\lambda\searrow 0} r_\lambda^{0,0}(\psi,x;\eta,y).$$

Choose a point (ψ_0, x_0) of $E = (-\infty, 0) \times [0, \infty)$. The *Martin kernel density with reference point* (ψ_0, x_0) is

$$(5.3) \qquad \kappa(\psi, x; \eta, y) \;=\; \frac{G(\psi, x; \eta, y)}{G(\psi_0, x_0; \eta, y)}.$$

Martin(-Doob-Hunt-Kunita-Watanabe)-boundary theory tells us in particular that any extremal non-negative $\mathcal{A}$-harmonic function u with $u(\psi_0, x_0) = 1$ must be of the form

$$u(\psi, x) \;=\; \lim \kappa(\psi, x; \eta, y)$$

as (η, y) converges to a Martin boundary point.

For Martin-Doob-Hunt-Kunita-Watanabe boundary theory, see the fine account of the classical case in Doob [1] and of the general case in Meyer [2]. Part 4 of Chapter III of Rogers and Williams [4] contains an introduction to the subject, and also explains in a simple setting how to derive the theory from that of Ray semigroups, something Meyer does for the general case with his usual mastery.

Case 1: the case when $\psi < \eta$. Suppose that $\psi < \eta$. The facts that our process started from (ψ, x) can hit (η, y) only via $(\eta, 0)$ and that it *must* hit $(\eta, 0)$ mean that for $\psi < \eta$,

$$G(\psi, x; \eta, y) \;=\; G(\eta, 0; \eta, y).$$

In particular, if $\eta > \max(\psi_0, \psi)$ then $\kappa(\psi, x; \eta, y) = 1$. Hence if (ψ, x) remains fixed but (η, y) varies so that $\eta \to 0$ (y varying in arbitrary manner), then

$$(5.4) \qquad\qquad\qquad \lim \kappa(\psi, x; \eta, y) \;=\; 1.$$

Case 2: the case when $\psi > \eta$ and $y + \eta \geq 0$. The following theorem describes the results for this case.

THEOREM 5.2. *Suppose that (ψ, x) is fixed, and (η, y) varies in such a way that*

$$(5.5) \qquad\qquad\qquad \eta \to -\infty \ \text{and} \ y + \eta \geq 0.$$

(a) *If*

$$(5.6) \qquad\qquad \eta \to -\infty \ \text{and} \ y/|\eta| \to 1 + c \ \text{where} \ c \in (0, \infty)$$

then

$$c^2 (2\pi)^{\frac{1}{2}} e^{2y} e^{\frac{(y+\eta)^2}{2|\eta|}} G(\psi, x; \eta, y) \to h_{1+c}(\psi, x),$$

where with $\theta_c := \frac{1}{2}c^2 + c = \frac{1}{2}c(2 + c),$

$$(5.7) \qquad h_{1+c}(\psi, x) \;:=\; 4(c + 1) \left[1 - e^{\theta_c \psi - cx} \right] + c^2 e^{\theta_c \psi} \left[e^{(2+c)x} - e^{-cx} \right].$$

We note that h_{1+c} is clearly non-negative, and it is immediately verified that h_{1+c} satisfies our PDE: it is $\mathcal{A}$-harmonic.

(b) *If $|\eta|^{-\frac{1}{2}}(y+\eta) \to \infty$ and $|\eta|^{-1}(y+\eta) \to 0$, then*

$$\kappa(\psi, x; \eta, y) \to \frac{x - \psi}{x_0 - \psi_0} \quad \left(= \lim_{c \searrow 0} \frac{h_{1+c}(\psi, x)}{h_{1+c}(\psi_0, x_0)} \right).$$

(c) *If $|\eta|^{-1}(y+\eta) \to \infty$, then*

$$(2\pi)^{\frac{1}{2}} e^{2y} e^{\frac{(y+\eta)^2}{2|\eta|}} G(\psi, x; \eta, y) \sim \frac{4y|\eta|}{(y+\eta)^2},$$

so that

$$\kappa(\psi, x; \eta, y) \to 1 \quad \left(= \lim_{c \nearrow \infty} \frac{h_{1+c}(\psi, x)}{h_{1+c}(\psi_0, x_0)} \right).$$

(d) *If $|\eta|^{-\frac{1}{2}}(y+\eta) \to w \in [0, \infty)$, then*

$$G(\psi, x; \eta, y) \sim 4(x - \psi) e^{-2y} \int_w^\infty \frac{e^{-\frac{1}{2}t^2}}{\sqrt{2\pi}} \, dt,$$

whence

$$\kappa(\psi, x; \eta, y) \to \frac{x - \psi}{x_0 - \psi_0}.$$

Important remarks. Note that in part (b) we speak of the *ratio κ*, not of the individual G values in its numerator and denominator.

We shall see that if we drop hypothesis (5.5), then part (d) is true for all w in $[-\infty, \infty)$.

First estimates. Suppose that $y + \eta \geq 0$. If the integral defining Z_1 at (5.1) were from 0 to ∞, then Z_1 would equal

$$\left(1 + \frac{1}{\sqrt{2\lambda}} \right) e^{-(x - \psi + y + \eta)\sqrt{2\lambda}}.$$

If the integral defining Z_2 at (5.1) were from 0 to ∞, then Z_2 would equal the same thing. Hence we have the following lemma.

LEMMA 5.3. *If $0 > \psi > \eta$ and $y + \eta \geq 0$, then*

$$G(\psi, x; \eta, y)$$

$$= \frac{e^{-(y-x)-\frac{1}{2}(\psi-\eta)}}{\sqrt{2\pi(\psi-\eta)}} \exp\left\{ -\frac{(x^2+y^2)}{2(\psi-\eta)} \right\} \times 2 \sinh \frac{xy}{\psi - \eta}$$

$$+ 2e^{-2y} \left[\int_0^{|\eta|} \frac{(y+\eta+t)}{\sqrt{2\pi t^3}} \exp\left\{ -\frac{(y+\eta)^2}{2t} \right\} \, dt \right.$$

$$\left. - \int_0^{|\eta|-|\psi|} \frac{(x - \psi + y + \eta + t)}{\sqrt{2\pi t^3}} \exp\left\{ -\frac{(x - \psi + y + \eta)^2}{2t} \right\} \, dt \right].$$

To obtain asymptotics on G, we use the following lemma.

LEMMA 5.4.   *Suppose that a and b vary so that*

$$a \to \infty, \ b > a \ \text{and} \ \frac{a(b+2a)}{b(b-a)^2} \to 0.$$

Then

$$I := \int_0^a t^{-3/2}(b-a+t)e^{-\frac{(b-a)^2}{2t}}\, dt \ \sim \ \frac{2a^{\frac{1}{2}}b}{(b-a)^2}e^{-\frac{(b-a)^2}{2a}}.$$

Proof. Two applications of integration by parts show that, for $0 < a < b$,

$$
\begin{aligned}
I &= \frac{2a^{\frac{1}{2}}b}{(b-a)^2}e^{-\frac{(b-a)^2}{2a}} - \frac{1}{(b-a)^2}\int_0^a t^{-\frac{1}{2}}(b-a+3t)e^{-\frac{(b-a)^2}{2t}}\, dt \\
&\geq \frac{2a^{\frac{1}{2}}b}{(b-a)^2}e^{-\frac{(b-a)^2}{2a}} - \frac{2a^{3/2}(b+2a)}{(b-a)^4}e^{-\frac{(b-a)^2}{2a}} \\
&= \frac{2a^{\frac{1}{2}}b}{(b-a)^2}e^{-\frac{(b-a)^2}{2a}}\left(1 - \frac{a(b+2a)}{b(b-a)^2}\right).
\end{aligned}
$$

The lemma follows. $\qquad\square$

Proof of parts (a)–(c) of Theorem 5.2. The previous two lemmas imply that as $\eta \to -\infty$ and $|\eta|^{-\frac{1}{2}}(y+\eta) \to \infty$, $(2\pi)^{\frac{1}{2}}e^{2y+\frac{(y+\eta)^2}{2|\eta|}}G(\psi,x;\eta,y)$ is asymptotic to

$$
\left(\frac{|\eta|}{\psi-\eta}\right)^{\frac{1}{2}}\exp\left(x - \frac{\psi}{2} + \frac{y^2\psi}{2|\eta|(\psi-\eta)} - \frac{x^2}{2(\psi-\eta)}\right)\left(e^{\frac{xy}{\psi-\eta}} - e^{-\frac{xy}{\psi-\eta}}\right)
$$
$$
+ \frac{4y|\eta|}{(y+\eta)^2}
$$
$$
- \frac{4(y+x)(\psi-\eta)^{\frac{1}{2}}|\eta|^{\frac{1}{2}}}{(y+x+\eta-\psi)^2}\exp\left(\frac{(y+\eta)^2\psi}{2|\eta|(\psi-\eta)} - \frac{(y+\eta)(x-\psi)}{\psi-\eta} - \frac{(x-\psi)^2}{2(\psi-\eta)}\right).
$$

Hence, if $\eta \to -\infty$ and $\frac{y+\eta}{|\eta|} \to c \in (0,\infty)$, then

$$
\begin{aligned}
c^2(2\pi)^{\frac{1}{2}}&e^{2y+\frac{(y+\eta)^2}{2|\eta|}}G(\psi,x;\eta,y) \\
&\to c^2\exp\left(x - \frac{1}{2}\psi + \frac{1}{2}(c+1)^2\psi\right)\left(e^{(c+1)x} - e^{-(c+1)x}\right) + 4(c+1) \\
&\qquad - 4(c+1)\exp\left(\frac{1}{2}c^2\psi - c(x-\psi)\right) \\
&= h_{1+c}(\psi,x).
\end{aligned}
$$

Part (a) of the theorem is proved; and the same line of reasoning yields parts (b) (dealing with κ as a ratio) and (c). $\qquad\square$

An estimate on Q^0: for fixed ψ, x, as $\eta \to -\infty$,

(5.8)

$$
e^{2y} Q^0(\psi - \eta, x, y)
$$

$$
= e^{2y - (y-x) - \frac{1}{2}(\psi - \eta)} \exp \left\{ -\frac{(x^2 + y^2)}{2\pi(\psi - \eta)} \right\} \frac{1}{\sqrt{2\pi(\psi - \eta)}} \times 2 \sinh \frac{xy}{\psi - \eta}
$$

$$
= O \left((2\pi|\eta|)^{-\frac{1}{2}} \exp \left(y - \frac{1}{2}|\eta| - (\frac{1}{2} y^2/|\eta|) \right) \right)
$$

$$
= O \left((2\pi|\eta|)^{-\frac{1}{2}} \exp \left(-\frac{1}{2}(y - |\eta|)^2/|\eta| \right) \right) = O \left(|\eta|^{-\frac{1}{2}} \right).
$$

The point of this is to show that in the situation covered by part (d) (and at certain later stages in the paper), the Q^0 part of the Green kernel density does not affect the asymptotics of that density.

Proof of part (d) of the theorem. Suppose first that

$$
\eta \to -\infty \text{ and } |\eta|^{-\frac{1}{2}}(y + \eta) \to w \in (0, \infty).
$$

The nicest way to deal with this case is to use the identity

(5.9) $\displaystyle \int_0^a t^{-\frac{3}{2}}(b - a + t)e^{-\frac{(b-a)^2}{2t}} \, dt = 2 \int_{a^{-\frac{1}{2}}(b-a)}^{\infty} \left(1 + \frac{b-a}{t^2} \right) e^{-\frac{t^2}{2}} \, dt.$

This implies that

$$
4^{-1}(2^{-1}\pi)^{\frac{1}{2}} e^{2y} \left[G(\psi, x; \eta, y) - Q^0(\psi, x; \eta, y) \right]
$$

$$
= \int_{|\eta|^{-\frac{1}{2}}(y+\eta)}^{(\psi-\eta)^{-\frac{1}{2}}(y+\eta+x-\psi)} \left(1 + \frac{y+\eta}{t^2} \right) e^{-\frac{1}{2}t^2} \, dt
$$

$$
- (x - \psi) \int_{(\psi-\eta)^{-\frac{1}{2}}(y+\eta+x-\psi)}^{\infty} t^{-2} e^{-\frac{1}{2}t^2} \, dt
$$

$$
\to (x - \psi) \left(w^{-1} e^{-\frac{1}{2}w^2} - \int_w^{\infty} t^{-2} e^{-\frac{1}{2}t^2} \right) = (x - \psi) \left(\int_w^{\infty} e^{-\frac{1}{2}t^2} \, dt \right),
$$

as required. The Q^0 term may be ignored because of (5.8).

To deal with the case when $w = 0$, we have to work just a little bit harder. Integration by parts applied to (5.9) implies that

$$
\frac{1}{2} \int_0^a t^{-\frac{3}{2}}(b - a + t)e^{-\frac{(b-a)^2}{2t}} \, dt
$$

$$
= [1 - (b - a)] \int_{a^{-\frac{1}{2}}(b-a)}^{\infty} e^{-\frac{t^2}{2}} \, dt + \sqrt{a} e^{-\frac{(b-a)^2}{2a}} \qquad (0 < a < b).
$$

When we substitute this in the formula for G in Lemma 5.3, we find that, provided $y + \eta > 0$,

$$4^{-1}(2^{-1}\pi)^{\frac{1}{2}} e^{2y} \left[G(\psi, x; \eta, y) - Q^0(\psi, x; \eta, y) \right]$$

$$= [1 - (y + \eta)] \int_{|\eta|^{-\frac{1}{2}}(y+\eta)}^{(\psi-\eta)^{-\frac{1}{2}}(y+\eta+x-\psi)} e^{-\frac{t^2}{2}} dt$$

$$+ (x - \psi) \int_{(\psi-\eta)^{-\frac{1}{2}}(y+\eta+x-\psi)}^{\infty} e^{-\frac{t^2}{2}} dt$$

$$+ |\eta|^{\frac{1}{2}} e^{-\frac{(y+\eta)^2}{2|\eta|}} - (\psi - \eta)^{\frac{1}{2}} e^{-\frac{(y+\eta+x-\psi)^2}{2(\psi-\eta)}}.$$

The first term on the right-hand side is easily shown to converge to 0 when, as we now assume,

$$\eta \to -\infty, \quad y + \eta > 0, \quad |\eta|^{-\frac{1}{2}}(y + \eta) \to 0.$$

The second term on the right-hand side gives the answer we want. The only real 'threat' comes from the term

$$-\frac{(y + \eta)(x - \psi)}{\psi - \eta} = -\frac{y + \eta}{(\psi - \eta)^{\frac{1}{2}}} \frac{x - \psi}{(\psi - \eta)^{\frac{1}{2}}}$$

in the exponential, but because this term is $o(|\eta|^{-\frac{1}{2}})$, it causes no trouble.

If $\eta \to -\infty$ and $y + \eta = 0$, a simple direct argument echoed in the discussion following (5.13) below, establishes the result.

Theorem 5.2 is proved. $\qquad \square$

Case 3: the case when $\psi > \eta$ and $y + \eta < 0$. We have the following theorem.

THEOREM 5.5. *Suppose that $\eta \to -\infty$ and y varies so that $y + \eta \leq 0$. Then for all ψ, x, we have*

$$\kappa(\psi, x; \eta, y) \to \frac{x - \psi}{x_0 - \psi_0}.$$

Suppose further that $|\eta|^{-\frac{1}{2}}(y + \eta) \to w \in [-\infty, 0]$. Then in exact extension of part (d) of Theorem 5.2,

$$G(\psi, x; \eta, y) \sim 4(x - \psi)e^{-2y} \int_w^{\infty} \frac{e^{-\frac{1}{2}t^2}}{\sqrt{2\pi}} dt.$$

Proof. We do not (yet) have a uniform way of dealing with the various cases. We study the function G when, as we now assume,

(5.10) $\qquad \eta < \psi, \qquad y + \eta < 0, \qquad |y + \eta| > x - \psi.$

We look first at the part of Z_1 which would diverge if $\lambda = 0$, namely,

$$V_1 \;:=\; \int\limits_{\psi-\eta}^{\infty} e^{-\lambda t} \frac{t}{\sqrt{2\pi t^3}} \exp\left\{ -\frac{(|y+\eta| - (x-\psi))^2}{2t} \right\} dt$$

$$= \int_0^\infty - \int_0^{|\eta|-|\psi|}$$

$$= (2\lambda)^{-\frac{1}{2}} \exp\left\{ -[|y+\eta| - (x-\psi)]\sqrt{2\lambda} \right\} - \int_0^{|\eta|-|\psi|} .$$

Similarly, we have for the 'divergent' part of Z_2,

$$V_2 \;:=\; e^{-(x-\psi)\sqrt{2\lambda}} \int\limits_{|\eta|}^{\infty} e^{-\lambda t} \frac{1}{\sqrt{2\pi t}} \exp\left\{ -\frac{(y+\eta)^2}{2t} \right\} dt$$

$$= (2\lambda)^{-\frac{1}{2}} \exp\{ -[|y+\eta| + (x-\psi)]\sqrt{2\lambda}\}$$

$$- e^{-(x-\psi)\sqrt{2\lambda}} \int\limits_0^{|\eta|} e^{-\lambda t} \frac{1}{\sqrt{2\pi t}} \exp\left\{ -\frac{(y+\eta)^2}{2t} \right\} dt.$$

We now see that

$$(5.11) \qquad D(\psi, x; \eta, y) \;:=\; \lim_{\lambda \searrow 0} (V_1 - V_2) \;=\; 2(x - \psi)$$

$$+ \int\limits_0^{|\eta|} \frac{1}{\sqrt{2\pi t}} \exp\left\{ -\frac{(y+\eta)^2}{2t} \right\} dt$$

$$- \int\limits_0^{|\eta|-|\psi|} \frac{1}{\sqrt{2\pi t}} \exp\left\{ -\frac{(|y+\eta| - (x-\psi))^2}{2t} \right\} dt.$$

To get $Z_1 - Z_2$, we have to add to this the 'non-divergent' part

$$- \int_{|\eta|-|\psi|}^{\infty} \frac{|y+\eta| - (x-\psi)}{\sqrt{2\pi t^3}} \exp\left\{ -\frac{(|y+\eta| - (x-\psi))^2}{2t} \right\} dt$$

$$+ \int_{|\eta|}^{\infty} \frac{|y+\eta|}{\sqrt{2\pi t^3}} \exp\left\{ -\frac{|y+\eta|^2}{2t} \right\} dt.$$

Now, if the range of integration in each of these integrals were $(0, \infty)$, then each would equal 1. We therefore find that

$$(5.12) \qquad\qquad Z_1 - Z_2 \;=\; D(\psi, x; \eta, y) + Y_1 - Y_2,$$

where

$$Y_1 \;=\; \int_0^{|\eta|-|\psi|} \frac{|y+\eta| - (x-\psi)}{\sqrt{2\pi t^3}} \exp\left\{ -\frac{(|y+\eta| - (x-\psi))^2}{2t} \right\} dt$$

and

$$Y_2 = \int_0^{|\eta|} \frac{|y+\eta|}{\sqrt{2\pi t^3}} \exp\left\{-\frac{|y+\eta|^2}{2t}\right\} dt.$$

We could now use our previous estimation techniques to prove the theorem when $w \in (-\infty, 0)$, but we skip this step here.

The case when $w = -\infty$ is easily dealt with.

The case when $w = 0$ has to be divided up further. We remark only on one new aspect. Suppose that

(5.13) $\eta \to -\infty, \qquad y + \eta \le 0, \qquad |y+\eta| < x - \psi.$

One finds this time that the first term in the analogue of equation (5.11) is now $2|y+\eta|$ rather than $2(x-\psi)$. Under assumption (5.13) with, in addition, $y + \eta \to b$, we have

$$D(\psi, x; \eta, y) - 2|y+\eta|$$

$$\to \int_0^\infty \frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{|b|^2}{2t}\right\} dt$$

$$- \int_0^\infty \frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{[x-\psi-|b|]^2}{2t}\right\} dt$$

$$= x - \psi - 2|b|,$$

so that, indeed, $D(\psi, x; \eta, y) \to x - \psi$!

Our proof of Theorem 5.5 is complete.   □

The Martin boundary for our process. We construct a compactification F of $E = (-\infty, 0) \times [0, \infty)$ in two steps. First we extend E to its natural 'Euclidean' compactification H got by adjoining to E the line $\{0\} \times [0, \infty)$ and the quarter-circle at infinity with angular parts in $[\frac{1}{2}\pi, \pi]$. Now produce F by identifying all points of $\{0\} \times [0, \infty)$ into a single boundary point γ and also identifying all points of the circle at infinity with angular parts in $[3/4\pi, \pi]$ into a single boundary point α. For $c > 0$, denote by β_{1+c} the point where the line $x = (1+c)|\psi|$ hits the quarter-circle at infinity.

A concrete description. The first stage may be thought of as imbedding E as a subset of $\tilde{H} := \{(r, \theta); 0 \le r \le 1; \frac{1}{2}\pi \le \theta \le \pi\}$ via the map

$$E \ni (r, \theta) \mapsto (r/(1+r), \theta).$$

Now condense $\tilde{H}$ to $\tilde{F}$ via the mapping

$$(r, \theta) \mapsto \left(4/\pi(\theta - \frac{1}{2}\pi)r, \theta\right) \text{ on } \{(r, \theta) \in \tilde{H} : \frac{1}{2}\pi \le \theta \le 3/4\pi\},$$

and

$$(r, \theta) \mapsto (r, 3/4\pi + (1-r)(\theta - 3/4\pi)) \text{ on } \{(r, \theta) \in \tilde{H} : 3/4\pi \le \theta \le \pi\}.$$

But it is much better to think in the terms first described, leaving E as it was; and this we now do.

The kernel κ on $E \times F$. As (η, y) converges to a boundary point β of $F \setminus E$,

$$\kappa(\psi, x; \eta, y) \to \kappa(\psi, x; \beta)$$

where

$$\kappa(\psi, x; \gamma) = 1, \qquad \kappa(\psi, x; \alpha) = \frac{x - \psi}{x_0 - \psi_0},$$

$$\kappa(\psi, x; \beta_{1+c}) = \frac{h_{1+c}(\psi, x)}{h_{1+c}(\psi_0, x_0)}.$$

The kernel κ is continuous from $E \times F$ to $[0, \infty]$. The set F is therefore a Martin compactification of E for κ. Note that if one regards $(-\infty, 0) \times \{0\}$ properly, E is imbedded as an open subset of F, and that the $\kappa(\cdot, \cdot; \beta)$ functions are different for different points of F, facts relevant to Section IV.1 of Meyer [2].

For our process, we know resolvent kernel densities, the Green kernel density, etc, *explicitly.* We can therefore verify directly all of the conditions necessary for the key theorems in Meyer's very abstract treatment. Meyer could well have written here too: '*il faut ... un cours de six mois sur les définitions. Que peut on y faire?*' It is not easy to understand either the definitions or the results of [2] unless one is steeped in the Strasbourg theory. For Meyer, it *mattered* that one has the right hypotheses (no pun intended) and for someone with his technical mastery of probabilistic potential theory (much of it his creation, building on work of Doob, Hunt, and others), that was fine. For us lesser mortals, it's a struggle.

The integral representation. One can now apply Section IV.3 of [2] to obtain the following theorem. We write

$$B := F \setminus E = \{\alpha, \gamma\} \cup \{\beta_{1+c} : c > 0\}.$$

THEOREM 5.6. *If u is a non-negative $\mathcal{A}$-harmonic function on $E = (-\infty, 0) \times [0, \infty)$ then there is a unique measure ν on Borel subsets of B such that*

$$u = K\nu,$$

which means that (transferring ν between $\{\beta_{1+c} : c \in (0, \infty)\}$ and $(0, \infty)$ in the obvious way)

$$u(\psi, x) = \int_B \kappa(\psi, x; \beta)\nu(d\beta)$$

$$= \nu(\{\gamma\}) + \nu(\{\alpha\})\frac{x - \psi}{x_0 - \psi_0} + \int_0^\infty \frac{h_{1+c}(\psi, x)}{h_{1+c}(\psi_0, x_0)}\nu(dc).$$

We have $\nu(B) = u(\psi_0, x_0)$.

It is now better to regard $h_{1+c}(\psi, x)$ as

$$(5.14) \quad h_{1+c}(\psi, x) = 4(c+1) - (c+2)^2 e^{-c[x+\frac{1}{2}(2+c)|\psi|]} + c^2 e^{(2+c)[x-\frac{1}{2}c|\psi|]}.$$

We can show that the representing measure ν for h_{1+b} (where $b > 0$) is concentrated at β_b by using the following facts: for $c > b$, the ratio $h_{1+c}(\psi, x)/h_{1+b}(\psi, x)$ tends to infinity when $x \to \infty$ and ψ is fixed; while, for $c < b$, this ratio tends to infinity when $\psi \to -\infty$ and $x = \frac{1}{2} b |\psi|$. That $\mathbf{1}$ and $x - \psi$ lie on extremal rays of the cone of non-negative $\mathcal{A}$-harmonic functions (in other words that they are minimal non-negative $\mathcal{A}$-harmonic) is easily shown. Thus, each $\kappa(\cdot, \cdot\,; \beta)$ is minimal non-negative $\mathcal{A}$-harmonic, and (therefore) each is necessary for the representation theorem.

Note that the theorem implies that if u is a *bounded* ancient solution, then u is constant.

A digression about Harnack's Principle. Interesting dividends of Theorem 5.6 are the observations

(a) For each compact $K \subset (-\infty, 0) \times [0, \infty)$ there is a $C_K \in [1, \infty)$ such that, for any non-negative, ancient solution u to (1.1) with $\sigma = -1 = \mu$,

$$u(s, x) \leq C_K u(t, y) \quad \text{for all } (s, x), (t, y) \in K.$$

(b) Given any $(s, x) \in (-\infty, 0) \times [0, \infty)$, the set of non-negative, ancient solutions to (1.1) with $\sigma = -1 = \mu$ which satisfy $u(s, x) \leq 1$ is a compact subset with respect to the topology of uniform convergence on compact subsets.

The property in (a) is a Harnack's principle, and the one in (b) is a standard sort of conclusion which one expects from a Harnack principle. What is interesting, and perhaps surprising, is the form of the Harnack principle in (a). Namely, familiar Harnack's principles for parabolic equations tell us that the size of $u(s, x)$ is controlled by that of $u(t, y)$ when $s < t$. What one expects is that this control breaks down as $s \nearrow t$ and is lost when $t \geq s$. Indeed, it is a relatively simple task to show that $\frac{u(t,y)}{u(s,x)}$ can be arbitrarily large when $t \geq s$ and u is a non-negative, ancient solution to (1.1) with $\sigma \geq 0$ and any $\mu \in \mathbb{R}$.

The explanation for the Harnack's principle in (a) is most readily found in the probabilistic picture provided by the process (Ψ_t, X_t). Namely, even though Ψ_t can only decrease as long as X_t stays away from 0, it can, and will, increase as soon as X_t hits 0. As we will outline below, without much difficulty one can convert this insight into a proof of (a) which does not depend on knowing Theorem 5.6. In fact, although we will continue here with the assumption that $\sigma = -1 = \mu$, it should be evident that the argument which follows applies equally well for all $\mu \in \mathbb{R}$ and $\sigma < 0$.

Let u be a non-negative solution to (1.1), with $\sigma = -1 = \mu$, in a domain of the form $(a - L, a + L) \times [0, \infty)$ for some $a \in \mathbb{R}$ and $L > 0$. We will show that,

for each $\ell \in (0, L)$ and $R > 0$, there is a constant $C(\ell, L, R) \in (1, \infty)$, such that $u(s, x) \le C(\ell, L, R)u(t, y)$ for all $(s, x), (t, y) \in [a - \ell, a + \ell] \times [0, R]$. To this end, set $\delta = \frac{L - \ell}{4}$, $b_\pm = a \pm \ell \pm \delta$, and $c = a + \ell + 3\delta = b_+ + 2\delta$. Clearly, for any $(\psi, x) \in [a - \ell, a + \ell] \times [0, \infty)$,

$$(5.15) \quad u(\psi, x) \ \ge \ \mathbb{E}^{\psi, x}\big[u(\Psi_{\delta \wedge \tau_{b_+, 0}}, X_{\delta \wedge \tau_{b_+, 0}})\big] \ \ge \ u(b_+, 0)\mathbb{P}^{\psi, x}(\tau_{b_+, 0} \le \delta),$$

where $\tau_{b_+, 0}$ is the hitting time of $(b_+, 0)$ by the process (Ψ, X). Because the maximium rate of decrease of Ψ is 1, $\Psi_t \le c - t$ implies $\tau_{c, 0} > t$, whence we have for any $t \le \psi - a + L$,

$$u(\psi, x) \ \ge \ \mathbb{E}^{\psi, x}\big[u(\Psi_t, X_t); \tau_{c, 0} > t\big] \ \ge \ \mathbb{E}^{\psi, x}\big[u(\Psi_t, X_t); \Psi_t \le c - t\big]$$

$$\ge \ \int_y \int_{\eta \in [\psi - t, c - t]} q(t, x + y + \eta - \psi + t)u(\eta, y) \, d\eta dy.$$

Hence, after averaging this over $t \in (\delta, \psi - a + L)$ and taking into account (5.15), we see that there is an $\varepsilon(\ell, L, R) > 0$ such that

$$(5.16) \qquad u(\psi, x) \ \ge \ \varepsilon(\ell, L, R) \left(u(b_+, 0) + \iint_{[b_-, b_+] \times [0, 2R]} u(\eta, y) \, d\eta dy \right)$$

for all $(\psi, x) \in [a - \ell, a + \ell] \times [0, R]$. Next, choose a smooth bump function $\rho : \mathbb{R} \longrightarrow [0, 1]$ so that $\rho = 1$ on $(-\infty, 3R/2)$ and $\rho = 0$ on $[2R, \infty)$, and define v via $v(\psi, x) = \rho(x)u(\psi, x)$. Then, for $(\psi, x) \in [a - \ell, a + \ell] \times [0, R]$ and $t \in (0, \delta)$, we have

$$v(\psi, x) \ = \ \mathbb{E}^{\psi, x}\big[v(\Psi_{t \wedge \tau_{b_+, 0}}, X_{t \wedge \tau_{b_+, 0}})\big] - \mathbb{E}^{\psi, x}\left[\int_0^{t \wedge \tau_{b_+, 0}} \mathcal{A}v(\Psi_s, X_s)\right]$$

$$\le \ u(b_+, 0) + \mathbb{E}^{\psi, x}\big[u(\Psi_t, X_t); \tau_{b_+, 0} > t\big]$$

$$- \int_0^t \mathbb{E}^{\psi, x}\big[\mathcal{A}v(\Psi_s, X_s); \tau_{b_+ 0} > s\big] \, ds.$$

Note that

$$\mathbb{E}^{\psi, x}\big[\mathcal{A}v(\Psi_s, X_s); \tau_{b_+ 0} > s\big] \ = \ \int_{(0, \infty)} Q^0(s, x, y)\mathcal{A}v(\psi - s, y) \, dy$$

$$+ \iint_{(\psi - s, b_+) \times [0, \infty)} \hat{Q}\big(s, (\psi, x), (\eta, y)\big)\mathcal{A}v(\eta, y) \, d\eta dy,$$

where

$$\hat{Q}\big(s, (\psi, x), (\eta, y)\big) \ = \ q(t, x + y + \eta - \psi + t)\mathbf{1}_{[\psi - s, b_+]}(\eta)$$

$$- \mathbf{1}_{[\psi - s, b_+]}(\eta)\mathbb{E}^{\psi, x}\big[q(t - \tau_{b_+, 0}, x + y + \eta - \psi + t - \tau_{b_+, 0});$$

$$\tau_{b_+, 0} \le t - (\eta - \psi)^-\big].$$

Hence, because $\mathcal{A}v = (\mathcal{A}\rho)u + \rho'u'$ and $\rho' = 0$ off $[3R/2, 2R]$, after integration by parts one can find a $C(R) < \infty$ for which

$$u(\psi, x) \leq u(b_+, 0) + C(R) \int_{[0,2R]} \left(u(\psi - t, y) + \int_{b_-}^{b_+} u(\eta, y)\, d\eta \right) dy$$

for all $(\psi, x) \in [a - \ell, a + \ell]$ and $t \in (0, \delta]$. Thus, after averaging this over $t \in (0, \delta]$ and combining the result with (5.16), we arrive at the desired conclusion.

Turning to the compactness property in (b), let $a \in \mathbb{R}$ and $0 < \ell < L$ be as in the preceding discussion, and consider the class $\mathcal{U}(\psi_0, x_0)$ of non-negative solutions u to (1.1), with $\sigma = -1 = \mu$, on $(a - L, a + L) \times [0, \infty)$ which satisfy $u(\psi_0, x_0) \leq 1$. By the above, we know that, for each $R > x_0$, $u \upharpoonright [a - \ell, a + \ell] \times [0, 2R] \leq C(\ell, L, 2R)$ for all $u \in \mathcal{U}(\psi_0, x_0)$. Thus, if we define v from u as we did before and proceed, as we did there, to represent $u(\psi, x)$ in terms of integrals against the kernels Q^0 and $\hat{Q}$, it is an easy matter to deduce the equicontinuity of $\{u \upharpoonright [a - \ell, a + \ell] \times [0, R] : u \in \mathcal{U}(\psi_0, x_0)\}$ from its boundedness.

Before closing this digression, it may be worth pointing out that there is another direction in which (a) and (b) can be localized. Namely, we do not need to know that u is a solution on the whole of $[a - L, a + L] \times [0, \infty)$ in order to get to the preceding conclusions about $u \upharpoonright [a - \ell, a + \ell] \times [0, R]$. Indeed, it should be clear that the only place where we used that u is a solution outside of $[a - L, a + L] \times [0, 2R]$ was in the first step, when we were getting our lower bound. However, we could have avoided this by stopping the process (Ψ_t, X_t) at the first time either Ψ_t hits c or X_t hits $2R$. On the other hand, because of the essential role played by the boundary condition at $x = 0$, it seems that we should not be able to generalize much further.

Doob h-transforms. Doob proved a wonderful result which, for our context, reads as follows (see T9, page 99, of [2]). Let h be a positive $\mathcal{A}$-harmonic function with representing measure ν. Let $\mathcal{A}^h$ be the Doob h-transformed operator

$$\mathcal{A}^h = h^{-1}\mathcal{A}h = \mathcal{A} + (h'/h)\partial_x.$$

Note that, because it is first-order, the boundary condition for (functions in the domain of) $\mathcal{A}^h$ is the same as that for $\mathcal{A}$. Let (Ψ^h, X^h) be a process with generator $\mathcal{A}^h$, killed on exiting E. We should say more precisely that (Ψ^h, X^h) has resolvent density

(5.17) $r_\lambda^h(\psi, x; \eta, y) = h(\psi, x)^{-1} r_\lambda^{0,0}(\psi, x; \eta, y) h(\eta, y).$

Then

(5.18)   if $h(0, 0) = 0$, then (Ψ^h, X^h) has infinite lifetime and

$$\lim_{t \to \infty} (\Psi_t^h, X_t^h) \text{ exists in } B,$$

and, with $\mathbb{P}^{\psi,x,h}$ denoting the law of (Ψ^h, X^h) starting at (ψ, x) and B_1 denoting a Borel subset of B,

$$(5.19) \qquad \mathbb{P}^{\psi,x,h}\left(\lim_{t\to\infty}(\Psi_t^h, X_t^h) \in B_1\right) \;=\; h(\psi,x)^{-1}\int_{B_1}\kappa(\psi,x;\beta)\nu(d\beta).$$

This gives an explicit statement of the uniqueness of the representing measure ν of h: we know that for (ψ, x) in E, we have $\kappa(\psi, x; \beta) > 0$ for every β in B; so we can reverse the Radon-Nikodým statement (5.19) to obtain (for every (ψ, x) in E and) for every Borel subset B_2 of B,

$$\nu(B_2) \;=\; h(\psi,x)\int_{B_2}\kappa(\psi,x;\beta)^{-1}\mathbb{P}^{\psi,x,h}\left(\lim_{t\to\infty}(\Psi_t^h, X_t^h) \in d\beta\right).$$

In particular, if $h = h_\beta := \kappa(\cdot,\cdot;\beta)$, then, since each h_β is minimal, we have, from (5.19),

$$(5.20) \qquad \text{for } \beta \text{ in } F \setminus E, \quad \left(\Psi_t^{h_\beta}, X_t^{h_\beta}\right) \to \beta \text{ almost surely } \mathbb{P}^{\psi,x,h}.$$

This needs special interpretation when $\beta = \gamma$ and $h = \mathbf{1}$. See Case 1 below.

Invariance. Suppose that $\beta \in B \setminus \{\gamma\}$ and that $h = \kappa(\cdot,\cdot;\beta)$. The fact that (Ψ^h, X^h) has infinite lifetime implies a stronger property of h than that h is $\mathcal{A}$-harmonic: namely, it says that h is *invariant* under the transition semigroup of (Ψ, X), or again that $h(\Psi_{t\wedge\tau_{0,0}}, X_{t\wedge\tau_{0,0}})$ is a *true* martingale, not just a *local* martingale. [[[To illustrate in a simple context, suppose just for the moment that $\mathcal{A}$ is the generator of Brownian motion B^0 on $(0,\infty)$ killed on approaching 0. Then $\mathbf{1}$ is harmonic for $\mathcal{A}$ but obviously not invariant for the non-conservative semigroup of B^0 (the supermartingale in question here being 1 up to the time of hitting 0, and 0 thereafter because the process is at a coffin state where all functions are 0). On the other hand, the function x *is* invariant for that semigroup. Rephrasing this in terms of $\mathcal{A}^x$ which is the generator $\frac{1}{2}\partial_x^2 + x^{-1}\partial_x$ of 3-dimensional Bessel process, we have that $\mathbf{1}$ is invariant under the semigroup of Bes(3), whereas (Helms-Johnson example!) $1/x$ is $\mathcal{A}^x$-harmonic but not invariant.]]]

Of course, one can establish invariance for our problem by showing that for $\beta \in B \setminus \{\gamma\}$ and $h = \kappa(\cdot,\cdot;\beta)$ we have

$$\int\int \lambda r_\lambda^{0,0}(\psi,x;\eta,y)h(\eta,y)\,d\eta dy \;=\; h(\psi,x).$$

We leave that little exercise in integration to the reader!

Discussion of the above results. Here, our aim is to indicate how some pieces of the jigsaw fit together, rather than to give full proofs. We omit 'almost surely' qualifying phrases: they would become tiresome.

Case 1: the boundary point γ**.** This is a somewhat trivial case. We have $h_\gamma(\psi, x) = 1$, $\mathcal{A}^h = \mathcal{A}$, and (Ψ^h, X^h) hits $(0,0)$ in finite time. Recall that γ is a fusion of all points on the '$\psi = 0$' axis.

Case 2: Suppose now that $h = h_{1+b}$ **for some** $b \in (0, \infty)$**.** The submartingale-problem formulation of the law of (Ψ^h, X^h) would be that (i) Ψ^h is a process of finite variation and (ii) for $f \in C^{1,2}(E)$ satisfying $(\partial_\psi + \partial_x)f \geq 0$ at the spacial boundary,

$$f(\Psi^h_t, X^h_t) - \int_0^t (\mathcal{A}^h f)(\Psi^h_s, X^h_s)ds$$

is a local submartingale up to explosion time (escape time from E). We are not going discuss existence and uniqueness of solution of this (somewhat unusual) submartingale problem in the manner of Stroock and Varadhan [6].

Working with $f(\psi, x) = x - \psi$ and $f(\psi, x) = (x - \psi)^2$, we find that

$$B^h_t := (X^h_t - \Psi^h_t) - (x - \psi) - \int_0^t \frac{h'(\Psi^h_s, X^h_s)}{h(\Psi^h_s, X^h_s)} ds$$

defines a Brownian motion of the natural filtration of (Ψ^h, X^h).

Now set

$$L^h_t := \Psi^h_t - \psi + t.$$

Since $(\partial_\psi + \partial_x)\psi = 1$ and $\mathcal{A}(\psi) = -1$, the process $\Psi^h_t + t$ is a continuous submartingale; and since it is of also of finite variation, L^h_t is a non-decreasing process. Let N be an open subset of $\mathbb{R}^2$ with $\overline{N}$ a compact subset of $(-\infty, 0) \times (0, \infty)$. Let $f(\psi, x) = \psi$ on N and extend f to be $C^{1,2}$ on $E := (-\infty, 0) \times [0, \infty)$ satisfying our boundary condition. Since $\mathcal{A}^h f = -1$ on N, we must have that L^h is constant on every component of $(0, \infty) \setminus \{t : X^h_t = 0\}$.

We have

$$X^h_t = x + B^h_t - t + \int_0^t \frac{h'(\Psi^h_s, X^h_s)}{h(\Psi^h_s, X^h_s)} ds + L^h_t,$$

and therefore (see the discussion of Skorokhod's equation in Subsection V.6 of [4])

$$L^h_t = \sup_{s \leq t} \left\{ \left(x + B^h_s - s + \int_0^s \frac{h'(\Psi^h_r, X^h_r)}{h(\Psi^h_r, X^h_r)} dr \right)^- \right\},$$

all as expected. Since $\Psi^h_t \geq \psi - t$, Ψ^h cannot explode to $-\infty$ in finite time.

Now, $\mathcal{A}^h(1/h) = 0$, so $M^h_t := 1/h(\Psi^h_t, X^h_t)$ defines a non-negative local martingale, and so must converge to a limit. Note that because $h(0,0) = 0$, the process (Ψ^h, X^h) never approaches $(0,0)$ and therefore, because of the local-time description, Ψ^h never approaches 0.

Since the local martingale M^h converges to a limit, its quadratic-variation process is bounded for each ω. Thus,

$$\int_0^\infty \left\{ \frac{h'(\Psi_t^h, X_t^h)}{h(\Psi_t^h, X_t^h)^2} \right\}^2 dt < \infty.$$

Looking at the explicit form of h at (5.14) shows that the limit of the local martingale M^h must be 0 and that $X_t^h - \frac{1}{2}b|\Psi_t^h| \to \infty$, whence $X_t^h \to \infty$. But the local time at 0 therefore eventually stops growing and $\Psi_t^h + t$ tends to a finite limit. Hence, indeed, $\Psi_t^h \to -\infty$.

Continue to assume that $h = h_{1+b}$. Because $h_{1+c}(\Psi^h, X^h)/h_{1+b}(\Psi^h, X^h)$ is a non-negative local martingale and so must converge to a limit, we see on looking at the dominant exponential terms in the explicit formulae for h_{1+c} and h_{1+b} that

$$\limsup \left\{ (2+c)[X_t^h - \frac{1}{2}c|\Psi_t^h|\,] - (2+b)[X_t^h - \frac{1}{2}b|\Psi_t^h|\,] \right\} \leq 0,$$

that is,

$$\limsup(c-b)\left\{ X_t^h - [\frac{1}{2}(b+c)+1]\,|\Psi_t^h| \right\} \leq 0.$$

We therefore have

$$\limsup X_t^h/|\Psi_t^h| \leq 1+b, \qquad \liminf X_t^h/|\Psi_t^h| \geq 1+b,$$

the first [second] by taking a sequence of c-values converging down [up] to b. We have shown that

for $b \in (0, \infty)$ and $h = h_{1+b}$, $\Psi_t^h + t \to$ finite limit and $t^{-1}X_t^h \to 1+b$,

establishing (5.20) for Case 2.

Case 3: the boundary point α. Here we gain a better understanding of the 'collapse of dimension' associated with α. The facts that

$$\kappa(\cdot, \cdot; \alpha) = \lim_{c \searrow 0} \kappa(\cdot, \cdot; \beta_{1+c})$$

and that $\kappa(\psi, x; \eta, y) \to (x-\psi)/(x_0-\psi_0)$ when $\eta \to -\infty$ and $y/|\eta| \to 1$ might have tempted us into thinking that if, as we now assume, $h(\psi, x) = x - \psi$, then $X_t^h/|\Psi_t^h| \to 1$. However, we now show that

(5.21)   for $h(\psi, x) = x - \psi$, we have $\Psi_t^h \to -\infty$ and $X_t^h/|\Psi_t^h| \to 0$.

So it is not that surprising that α is (as it were) a fusion of all β_r with $0 \leq r \leq 1$.

We can describe the h-transformed process started from (ψ, x) as follows. Let $\tilde{R}^{x-\psi}$ be a 3-dimensional Bessel process starting at $x - \psi$. Define

$$
\tilde{L}_t := \sup_{s \leq t}\{(\psi + \tilde{R}_s^{x-\psi} - s)^-\},
$$

$$
\tilde{X}_t := \psi + \tilde{R}_t^{x-\psi} - t + \tilde{L}_t,
$$

$$
\tilde{\Psi}_t := \psi + \tilde{L}_t - t.
$$

Then $(\tilde{\Psi}, \tilde{X})$ has the same law as (Ψ^h, X^h).

We focus on one typical sample path, and suppress the 'ω' symbols which attach in particular to the t_n and s_n. Suppose that there exist a finite positive K and a sequence (t_n) with $t_n \nearrow \infty$ such that $\tilde{\Psi}(t_n) - \psi \geq -K$. We may assume that each t_n is greater than K. We have $\tilde{L}(t_n) \geq t_n - K$, so there exists $s_n \leq t_n$ with $(\psi + \tilde{R}^{x-\psi}(s_n) - s_n)^- \geq t_n - K$. Thus $s_n - \tilde{R}^{x-\psi}(s_n) - \psi \geq t_n - K$, so $s_n \to \infty$ and $\tilde{R}^{x-\psi}(s_n) \leq K - \psi$, a contradiction, since $\tilde{R}^{x-\psi}$ drifts to ∞. We have shown that $\tilde{\Psi}_t \to -\infty$.

We now prove the second part of (5.21). For $c > 0$,

$$
\frac{h_{1+c}(\tilde{\Psi}_t, \tilde{X}_t)}{\tilde{X}_t - \tilde{\Psi}_t}
$$

is a non-negative local martingale, hence a supermartingale, and so must (almost surely) converge to a limit. The explicit form of h_{1+c} now forces it to be true that $\tilde{X}_t/|\tilde{\Psi}_t| \to 0$; for if $\limsup \tilde{X}_t/|\tilde{\Psi}_t| \geq \varepsilon > 0$ and $c < 2\varepsilon$, then

$$
\limsup \frac{h_{1+c}(\tilde{\Psi}_t, \tilde{X}_t)}{\tilde{X}_t - \tilde{\Psi}_t} = \infty,
$$

a contradiction.

Doob-conditioning. For $\beta \in B$ and $h_\beta := \kappa(\cdot, \cdot; \beta)$, the law $\mathbb{P}^{\psi, x, h_\beta}$ is regarded as the law of (Ψ, X) 'Doob-conditioned' to converge to β. The reader should check, assuming (5.19), that 'everything tallies' by showing that if h has representation $h = K\nu$, then, for an event Λ on the path-space of (Ψ, X), we have

$$
\mathbb{P}^{\psi, x, h}(\Lambda) = \int_B \mathbb{P}^{\psi, x, h_\beta}(\Lambda) \mathbb{P}^{\psi, x, h}\left(\lim_{t \to \infty}(\Psi_t^h, X_t^h) \in d\beta\right).
$$

Of course, the limit is counted as being equal to γ if (Ψ^h, X^h) hits $(0, 0)$ in finite time.

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