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RANDOM PERTURBATIONS OF TWO-DIMENSIONAL PSEUDOPERIODIC FLOWS

RICHARD B. SOWERS

ABSTRACT. We consider a random perturbation of a pseudoperiodic flow on $\mathbb{R}^2$. The structure of such flows has been studied by Arnol'd; it contains regions where there are local Hamiltonians, and an ergodic region. Under an appropriate change of time, we identify a reduced model as the strength of the random perturbation tends to zero (along a certain subsequence). In the Hamiltonian region, arguments of Freidlin and Wentzell are used to identify a limiting graph-valued process. The ergodic region is reduced to a single point, which is "sticky". The identification of the glueing conditions which rigorously describe this stickiness follows from a perturbed test-function analysis in the ergodic region.

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1. Introduction

An important technique in the analysis of many physical systems is the circle of ideas known as *model reduction*; i.e., the development of rigorous methods to replace, often in some limiting regime, a complicated system by a simpler, or lower-dimensional one. We study here a problem of model reduction for a diffusively-perturbed pseudoperiodic flow on the 2-dimensional torus.

Arnol'd [Arn91] (see also [SK92]) identified the general structure of such flows; he showed that there is a partition of the torus into an *ergodic* region, and a collection of *traps*. Inside each of the traps, the flow can be described via a local Hamiltonian. Our interest is how small diffusive perturbations cause transitions between the traps and the ergodic class.

Since our interest is the effect of *small* noise, we have a separation of scales. The fast variable is the position within orbits of the dominant dynamical system; an *angle*. The slow variable distinguishes between orbits; an *action*. The theory of averaging (in this case, stochastic averaging) suggests that we look for closed dynamics of the action variable. The effective coefficients of these closed dynamics are given, informally, by fixing the slow variable and taking long-time averages in the fast variable. In the simplest cases, when all orbits are periodic, the space of action variables is usually diffeomorphic to a line, and is formally given by taking the quotient with respect to the action of the fast orbits. When there are bifurcations in the topology of the orbits, the notion of *chain equivalence* is the correct way to include the effect of small perturbations; then the action variable in general takes values in a graph, or more generally, a stratified space [FW94], [FW98], [FW99], [NPR05], [Sow02]. The formal asymptotic goal is to show that when the trajectories of the original randomly perturbed dynamical system are projected onto the space of action variables, i.e., the space of chain-equivalent classes, they asymptotically (under an appropriate change of time scale), tend to a Markov process (on the space of chain-equivalent classes). The interesting part is the effect of bifurcations, which create different strata; at the chain-equivalent set representing these strata, gluing conditions must be imposed.

The focus of this paper is the effect of the ergodic class. Sooner or later, the diffusive perturbations will push the randomly-perturbed trajectory into the ergodic class. The ergodicity will then take the particle everywhere in the ergodic class, and eventually it will exit back into a trap. A quantification of this effect was conjectured in [Fre96, p. 74]. Chain equivalence collapses the whole ergodic class to a single point, and the conjecture is that the limiting process is *sticky* at this point, with a computable stickiness coefficient. Our goal is to show that in a certain weak sense, this is true.

2. Problem statement and main result

We wish to construct a diffusively-perturbed pseudoperiodic flow on the two-dimensional torus. To do so, let's start with a *pseudoperiodic Hamiltonian* on $\mathbb{R}^2$. Let $\langle \cdot, \cdot \rangle_{\mathbb{R}^2}$ be the standard inner product on $\mathbb{R}^2$.

ASSUMPTION 2.1. Let $\mathsf{H} \in C^\infty(\mathbb{R}^2)$ and $\boldsymbol{\omega} = (\omega_1, \omega_2) \in \mathbb{R}^2$ be such that: firstly, H is Morse, secondly, ω_1 and ω_2 are incommensurable (i.e., $\langle \boldsymbol{\omega}, K\rangle_{\mathbb{R}^2} \neq 0$ for all $K \in \mathbb{Z}^2 \setminus \{(0,0)\} \subset \mathbb{R}^2$), and thirdly, $\mathsf{H}(x+K) = \mathsf{H}(x) + \langle \boldsymbol{\omega}, K\rangle_{\mathbb{R}^2}$ for all $x \in \mathbb{R}^2$ and $K \in \mathbb{Z}^2 \subset \mathbb{R}^2$.

Define[1] the vector field

$$(1) \qquad (Ч_e \varphi)(x) \stackrel{\text{def}}{=} \left(\frac{\partial \mathsf{H}}{\partial x_2} \frac{\partial \varphi}{\partial x_1} - \frac{\partial \mathsf{H}}{\partial x_1} \frac{\partial \varphi}{\partial x_2} \right)(x)$$

for all $\varphi \in C^\infty(\mathbb{R}^2)$ and $x = (x_1, x_2) \in \mathbb{R}^2$ (i.e., $Ч_e$ is the symplectic or skew gradient of H).

We now want to add diffusivity, albeit in a periodic way.

NOTATION 2.2. Define for notational convenience

$$C_p^\infty(\mathbb{R}^2) \stackrel{\text{def}}{=} \left\{ f \in C^\infty(\mathbb{R}^2) : f(x+K) = f(x) \text{ for all } x \in \mathbb{R}^2 \text{ and } K \in \mathbb{Z}^2 \right\}.$$

Note that $\partial \mathsf{H}/\partial x_1$ and $\partial \mathsf{H}/\partial x_2$ are both in $C_p^\infty(\mathbb{R}^2)$. Thus $Ч C_p^\infty(\mathbb{R}^2) \subset C_p^\infty(\mathbb{R}^2)$.

ASSUMPTION 2.3 (Diffusion Generator and Bracket). Let $\mathscr{L}_e$ be a second-order partial differential operator of the form

$$(\mathscr{L}_e f)(x) \stackrel{\text{def}}{=} \frac{1}{2} \sum_{i,j \in \{1,2\}} a_{i,j}^{(2)}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_{i \in \{1,2\}} a_i^{(1)}(x) \frac{\partial f}{\partial x_i}(x)$$

[1] We shall attach the superscript e when referring to the Euclidean space $\mathbb{R}^2$, which we endow with the standard metric and symplectic form; see Subsection 3.2.

for all $f \in C^2(\mathbb{R}^2)$ and $x = (x_1, x_2) \in \mathbb{R}^2$, where the $a^{(2)}_{i,j}$'s and $a^{(1)}_i$'s are in $C^\infty_p(\mathbb{R}^2)$. We require for simplicity that $\mathscr{L}_e$ be strongly elliptic; i.e., that

$$\sum_{i,j\in\{1,2\}} a^{(2)}_{i,j}(x)\frac{\partial f}{\partial x_i}(x)\frac{\partial f}{\partial x_2}(x) > 0 \tag{2}$$

for all $x = (x_1, x_2) \in \mathbb{R}^2$ and all $f \in C^1(\mathbb{R}^2)$ with $df(x) \neq 0$.

Then $\mathscr{L}C^\infty_p(\mathbb{R}^2) \subset C^\infty_p(\mathbb{R}^2)$.

Since the $\partial\mathsf{H}/\partial x_i$'s, $a^{(1)}_i$'s, and $a^{(2)}_{i,j}$'s are all in $C^\infty_p(\mathbb{R}^2)$, we can now move to the two dimensional torus $\mathbb{T} \stackrel{\text{def}}{=} \mathbb{R}^2/\mathbb{Z}^2$. Let $\mathfrak{t} : \mathbb{R}^2 \to \mathbb{T}$ be the standard covering map; i.e., $\mathfrak{t}(x) = x + \mathbb{Z}^2$ for all $x \in \mathbb{R}^2$. We then define the vector field Ч and the second-order operator $\mathscr{L}$ by requiring that

$$(Ч\varphi)(\mathfrak{t}(x)) = (Ч_e(\varphi\circ\mathfrak{t}))(x) \qquad \text{and} \qquad (\mathscr{L}\varphi)(\mathfrak{t}(x)) = (\mathscr{L}_e(\varphi\circ\mathfrak{t}))(x) \tag{3}$$

for all $\varphi \in C^\infty(\mathbb{T})$ and all $x \in \mathbb{R}^2$.

We will consider the Markov process on $\mathbb{T}$ whose generator is

$$\mathscr{L}^\varepsilon \stackrel{\text{def}}{=} \varepsilon^{-2}Ч + \mathscr{L}$$

(with domain $\mathscr{D}(\mathscr{L}^\varepsilon) \supset C^2(\mathbb{T})$). We will construct this Markov process in a canonical way, via the martingale problem [EK86], [SV79]. Define the event space $\Omega \stackrel{\text{def}}{=} C([0,\infty);\mathbb{T})$. Define the coordinate functions $X_t(\omega) \stackrel{\text{def}}{=} \omega(t)$ for all $t \geq 0$ and all $\omega \in \Omega$. For each $t \geq 0$, define $\mathscr{F}_t \stackrel{\text{def}}{=} \sigma\{X_s;\, 0 \leq s \leq t\}$ and define a sigma-algebra on Ω by $\mathscr{F} \stackrel{\text{def}}{=} \bigvee_{t\geq 0}\mathscr{F}_t$. We can now define our principal objects of interest.

Definition 2.4 (Original Martingale Problem). Fix $x_\circ \in \mathbb{T}$. For each $\varepsilon > 0$, let $\mathbb{P}^\varepsilon \in \mathscr{P}(C([0,\infty);\mathbb{T}))$ be a solution to the martingale problem with generator $\mathscr{L}^\varepsilon$ whose domain contains $C^2(\mathbb{T})$ (as a dense subset), and initial condition $\delta_{x_\circ}$. Let $\mathbb{E}^\varepsilon$ be the corresponding expectation operator. This means the following. Firstly, that $\mathbb{P}^\varepsilon\{X_0 = x_\circ\} = 1$. Secondly, that if we fix $f \in C^2(\mathbb{T})$, $0 \leq r_1 < r_2 \cdots < r_n \leq s < t$ and $\{\varphi_j;\, j = 1, 2 \dots n\} \subset C_b(\mathbb{T})$, then

$$\mathbb{E}^\varepsilon\left[\left\{f(X_t) - f(X_s) - \int_s^t (\mathscr{L}^\varepsilon f)(X_u)du\right\}\prod_{j=1}^n \varphi_j(X_{r_j})\right] = 0.$$

In other words, $\mathbb{P}^\varepsilon$ is the law of the stochastic differential equation

$$dY^\varepsilon_t = \frac{1}{\varepsilon^2}Ч(Y^\varepsilon_t)dt + \check{a}_0(Y^\varepsilon_t)dt + \sum_{i\in\{1,2\}}\check{a}_i(Y^\varepsilon_t)\circ dW^i_t \qquad t > 0 \tag{4}$$

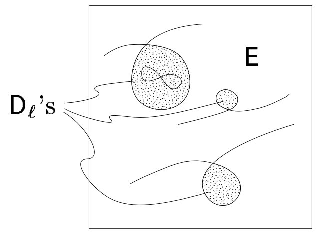


FIGURE 1. Pseudoperiodic Flow

where W^1 and W^2 are two independent standard Wiener processes, and where $\check{a}_0$, $\check{a}_1$, and $\check{a}_2$ are smooth vector fields on $\mathbb{T}$ such that (in Hörmander form) $\frac{1}{2}\sum_{i\in\{1,2\}} \check{a}_i^2 + \check{a}_0 = \mathscr{L}$.

The generator $\mathscr{L}^\varepsilon$ is, of course, a speeded-up version of the operator $\text{Ч} + \varepsilon^2\mathscr{L}$ (to get the corresponding stochastic differential equation, change (4) as follows: remove the $1/\varepsilon^2$ from the Ч term, put ε^2 in front of $\check{a}_0$, and ε in front of $\check{a}_1$ and $\check{a}_2$). The operator $\text{Ч} + \varepsilon^2\mathscr{L}$ represents a combination of motion along the integral curves of Ч and small random perturbations. The change in time scale stems from a desire to see how diffusive perturbations cause motion *across* the orbits of Ч.

Let now $\mathfrak{z}$ be the flow of diffeomorphisms of $\mathbb{T}$ defined by

$$\begin{aligned} \dot{\mathfrak{z}}_t(x) &\stackrel{\text{def}}{=} \text{Ч}(\mathfrak{z}_t(x)), \qquad t \geq 0, \\ \mathfrak{z}_0(x) &= x. \end{aligned} \qquad x \in \mathbb{T} \tag{5}$$

The novelty of our problem comes from the structure of $\mathfrak{z}$, which Arnol'd [Arn91] identified. There is a partition of $\mathbb{T}$ into a finite collection $\{\mathsf{D}_\ell;\, \ell \in \Lambda\}$ of closed *traps* (Λ is simply the index set) and an open *ergodic* set E. Both E and each of the D_ℓ's is invariant under $\mathfrak{z}$. The interior of each trap is diffeomorphic to the open unit disk in $\mathbb{R}^2$, and $\partial\mathsf{D}_\ell$ is a homoclinic orbit of $\mathfrak{z}$ with fixed point $\mathfrak{x}_\ell$. Furthermore, for each trap D_ℓ, there is an $\mathsf{H}_{T,\ell} \in C^\infty(\mathbb{T})$ (a Hamiltonian) such that $\mathsf{H}_{T,\ell} \equiv 0$ on $\partial\mathsf{D}_\ell$ and such that Ч is the symplectic gradient of $\mathsf{H}_{T,\ell}$ on D_ℓ (i.e., $\text{Ч}(\mathfrak{t}(x)) = T\mathfrak{t}(\bar{\nabla}_e(\mathsf{H}_{T,\ell}\circ \mathfrak{t}))(x)$ for all $x \in \mathfrak{t}^{-1}(\mathsf{D}_\ell)$; see Subsection 3.2). In E, the orbits of $\mathfrak{z}$ are dense. See Figure 1.

We want to use this description of $\mathfrak{z}$ to understand the asymptotics of the $\mathbb{P}^\varepsilon$-law of X. In particular, we want to understand the effect of the ergodic class E and conversely ignore the nature of the $\mathsf{H}_{T,\ell}$'s *within* the D_i's. For each $\eta > 0$ and $\ell \in \Lambda$, let $\partial_\eta\mathsf{D}_\ell$ be the connected component of $\{x \in \mathsf{D}_\ell : |\mathsf{H}_{T,\ell}(x)| < \eta\}$ which contains $\mathfrak{x}_\ell$.

DEFINITION 2.5. Let $\hbar > 0$ be such for each $\ell \in \Lambda$, $\partial_\hbar\mathsf{D}_\ell$ contains only one fixed point of $\mathfrak{z}$, namely $\mathfrak{x}_\ell$. Define then $\text{Đ}_\ell \stackrel{\text{def}}{=} \overline{\partial_\hbar \mathsf{D}} \setminus \partial\mathsf{D}_\ell$ for all $\ell \in \Lambda$ (see Figure 2).

We now define the set $\mathbf{S}$ and stopping time $\mathfrak{e}$ by

$$\mathbf{S} \stackrel{\text{def}}{=} \mathsf{E} \cup \bigcup_{\ell \in \Lambda} \overline{\mathsf{Đ}_\ell} \qquad \text{and} \qquad \mathfrak{e} \stackrel{\text{def}}{=} \inf\{t \geq 0;\, X_t \notin \mathbf{S}^\circ\}.$$

We are then interested in the $\mathbb{P}^\varepsilon$-law of X up to time $\mathfrak{e}$.

A general tool is considering small diffusive perturbations of conservative systems in the notion of *chain equivalence* relative to $\mathfrak{z}$ (see [Con78] and [Rob99]). We note that $\mathbf{S}$ is invariant under $\mathfrak{z}$ of (5). For a positive integer N, $\delta \in (0,1)$ and $T \in (0,\infty)$, we say that there is an (N,δ)-chain of time T from $x \in \mathbf{S}$ to $y \in \mathbf{S}$ if there is a sequence $(z_j;\, j = 1, 2, \dots, N)$ of points in $\mathbf{S}$ and a sequence $0 = t_0 < t_1 \cdots < t_N = T$ of times such that $z_0 = x$ and $z_N = y$ and such that $\|\mathfrak{z}_{t_j - t_{j-1}}(z_{j-1}) - z_j\| < \delta$ for all $1 \leq j \leq N$. We say that $x \Rightarrow y$, where x and y are in $\mathbf{S}$, if there is a positive integer N such that for each $\delta \in (0,1)$ and $T \in (0,\infty)$, there is an (N,δ)-chain of time $T' \in (T,\infty)$ from x to y. We say that $x \sim y$ if $x \Rightarrow y$ and $y \Rightarrow x$. We note that $x \sim x$ for each $x \in \mathbf{S}$, and that $\sim$ is an equivalence relation on $\mathbf{S}$. Define $\mathbb{M} \stackrel{\text{def}}{=} \mathbf{S}/\sim$ and endow $\mathbb{M}$ with the quotient topology defined by $\sim$. If $x \in \mathbf{S}$, we let $[x] \stackrel{\text{def}}{=} \{y \in \mathbf{S} : y \sim x\}$ be the equivalence class of x (the chain components of $\mathbf{S}$) and we define $\mathfrak{m}(x) \stackrel{\text{def}}{=} [x]$. Then

$$\mathbb{M} = \bigcup_{\ell \in \Lambda} \Gamma_\ell \cup [\mathsf{E}] \cup \bigcup_{\ell \in \Lambda} \circledast_\ell \tag{6}$$

where $\Gamma_\ell = \mathfrak{m}(\mathsf{Đ}_\ell^\circ)$ and $\circledast_\ell = \mathfrak{m}\{x \in \partial \mathsf{Đ}_\ell : |\mathsf{H}_{T,\ell}(x)| = \hbar\}$ for all $\ell \in \Lambda$, and $[\mathsf{E}]$ is a single point (we note that $[\mathsf{E}] = \overline{\mathsf{E}}$). It is easy to see that for each $\ell \in \Lambda$, Γ_ℓ is a one-dimensional open C^∞ manifold (diffeomorphic to $(0,1)$), and the points $[\mathsf{E}]$ and $\circledast_\ell$ are the limits of points in Γ_ℓ. This makes $\mathbb{M}$ into a *stratified space* [GM88] if we enforce the ordering $[\mathsf{E}] \prec \Gamma_\ell$ and $\circledast_\ell \prec \Gamma_\ell$ for all $\ell \in \Lambda$ (see also [Sow02] for another example of a Markov process on a stratified space resulting from averaging). We note that (6) represents $\mathbb{M}$ as a disjoint union of open manifolds and a collection of boundary points; the set $\Gamma_\Lambda \stackrel{\text{def}}{=} \bigcup_{\ell \in \Lambda} \Gamma_\ell$ consists of the open manifolds. Note that $\Gamma_\Lambda = \mathfrak{m}^{-1}(\mathbf{S}^\circ \setminus \overline{\mathsf{E}})$, and that $\mathbf{S}^\circ \setminus \overline{\mathsf{E}} = \bigcup_{\ell \in \Lambda} \mathsf{Đ}_\ell^\circ$.

We also note that there is a homeomorphism between Γ and a "wye" (see Figure 2). Let $\{\mathbf{v}_\ell;\, \ell \in \Lambda\}$ be a collection of unit vectors in $\mathbb{R}^2$ such that for each distinct pair ℓ and ℓ' of elements of Λ, $\mathbf{v}_\ell$ and $\mathbf{v}_{\ell'}$ are linearly independent. Define $\sqsupset : \mathbb{M} \to \mathbb{R}^2$ by

$$\sqsupset([x]) \stackrel{\text{def}}{=} \begin{cases} |\mathsf{H}_{T,\ell}(x)|\mathbf{v}_\ell & \text{if } x \in \overline{\mathsf{Đ}_\ell} \text{ for } \ell \in \Lambda \\ \mathbf{0}_e & \text{if } x \in \mathsf{E} \end{cases} \tag{7}$$

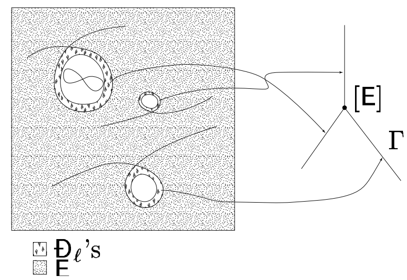


FIGURE 2. Chain Equivalence Reduction

where $\mathbf{0}_e$ is the origin of $\mathbb{R}^2$. The image of $\sqsupset$ is the "spider" $\{\mathbf{0}_e\}\cup\bigcup_{\ell\in\Lambda}(0,\hbar]\mathbf{v}_\ell$. Since $\sqsupset$ is a homeomorphism into $\mathbb{R}^2$, we can define the metric

$$d^\dagger(\mathfrak{m}(x),\mathfrak{m}(y)) \overset{\text{def}}{=} \|\sqsupset(x)-\sqsupset(y)\|_e \qquad x,y\in\mathbf{S} \tag{8}$$

where $\|\cdot\|_e$ is the standard metric on $\mathbb{R}^2$. It is now easy to see that $\mathbb{M}$ is in fact Polish.

Define now $X^M_t \overset{\text{def}}{=} [X_{t\wedge\mathfrak{e}}]$ for all $t\geq 0$ and define the probability measure

$$\boxed{\mathbb{P}^{\varepsilon,\dagger}(A)\overset{\text{def}}{=}\mathbb{P}^\varepsilon\{X^M\in A\};\qquad A\in\mathscr{B}(C([0,\infty),\mathbb{M}));}$$

i.e., $\mathbb{P}^{\varepsilon,\dagger}$ is the law of the projection of the process $t\mapsto X_{t\wedge\mathfrak{e}}$ onto $C([0,\infty),\mathbb{M})$.

It will not be hard to show

PROPOSITION 2.6 (Tightness). *The $\mathbb{P}^{\varepsilon,\dagger}$'s are tight in the Prohorov topology on $\mathscr{P}(C([0,\infty);\mathbb{M}))$.*

The proof will be in Section 4. Thus it is appropriate to investigate the existence and uniqueness of the limit $\lim_{\varepsilon\to 0}\mathbb{P}^{\varepsilon,\dagger}$, this limit being in the Prohorov topology. We want to show that in certain cases, this limit exists along a subsequence, and can be identified as a certain Markov process. We note that since $[X]$ records only part of the location of X, $\mathbb{P}^{\varepsilon,\dagger}$ is not Markovian for $\varepsilon>0$. Our goal is to show that as $\varepsilon\to 0$, the limit *is* Markovian (and thus that the discarded information can be replaced via *effective* coefficients).

As long as X^M stays in Γ_Λ, it should tend to a process with *averaged* coefficients. We define a linear *averaging* operator $\mathcal{A}: C(\mathbf{S}^\circ\setminus\overline{\mathsf{E}})\to C(\Gamma_\Lambda)$. For $\varphi\in C(\mathbf{S}^\circ\setminus\overline{\mathsf{E}})$, define

$$(\mathcal{A}\varphi)([x]) = \frac{\int_{z\in[x]}\varphi(z)\|\text{Ч}(z)\|^{-1}\mathscr{H}^1(dz)}{\int_{z\in[x]}\|\text{Ч}(z)\|^{-1}\mathscr{H}^1(dz)} \tag{9}$$

for all $[x]\in\Gamma_\Lambda$, where $\|\cdot\|$ is the standard metric on $T\mathbb{T}$ and where $\mathscr{H}^1$ is standard 1-dimensional Hausdorff measure on $\mathbb{T}$ (we will later need to average in E, but we will develop the notation for that later). If $\varphi \in C(\mathbf{S}^\circ \setminus \overline{\mathsf{E}})$, then

$$(\mathcal{A}\varphi)([x]) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \varphi(\mathfrak{z}_s(x))ds, \qquad x \in \mathbf{S}^\circ \setminus \overline{\mathsf{E}}.$$

Define an averaged diffusive operator $(\mathscr{L}_{\text{ave}}f)([x]) \stackrel{\text{def}}{=} (\mathcal{A}(\mathscr{L}(f\circ\mathfrak{m})))([x])$ for all $x \in \mathbb{M}\setminus\mathsf{E}$ and $f \in C^2(\Gamma_\Lambda)$ (since Γ_Λ is a C^∞ manifold, $f\circ\mathfrak{m} \in C^2(\mathbf{S}^\circ \setminus \overline{\mathsf{E}})$). We then expect that the limiting dynamics of X^M will be given by the generator $\mathscr{L}_{\text{ave}}$ as long as it remains in Γ_Λ. It will be killed at the $\circledast_\ell$'s, so we should also impose the requirement that $(\mathscr{L}_{\text{ave}}f)(\circledast_\ell) = 0$ for all $\ell \in \Lambda$.

The remaining, and most interesting, question is the limiting behavior at $[\mathsf{E}]$. The following was conjectured in [Fre96, p. 74]. Since $\mathscr{L}_{\text{ave}}$ is a nondegenerate elliptic operator on $C^2(\Gamma_\Lambda)$ we can consequently define the nonnegative bilinear form $\langle\cdot,\cdot\rangle_{\text{ave}}$ on $T^*\Gamma_\Lambda$ by

$$\langle df, dg\rangle_{\text{ave}}([x]) = (\mathscr{L}_{\text{ave}}(fg))([x]) - f([x])(\mathscr{L}_{\text{ave}}g)([x]) - g([x])(\mathscr{L}_{\text{ave}}f)([x])$$

for all f and g in $C^2(\Gamma_\Lambda)$ and all $[x] \in \Gamma_\Lambda$. We next define *area* functions for each Γ_ℓ. Let $\mathscr{H}^2$ be the standard 2-dimensional Hausdorff measure on $\mathbb{T}$. For each $\ell \in \Lambda$ and each $[x] \in \Gamma_\ell$, define

$$\eth_\ell([x]) \stackrel{\text{def}}{=} \mathscr{H}^2\{z \in \mathsf{Đ}_\ell : |\mathsf{H}_{T,\ell}(z)| \le |\mathsf{H}_{T,\ell}(x)|\}. \tag{10}$$

We then define the glueing operator $\mathscr{G}_\ell f \stackrel{\text{def}}{=} \lim_{[x]\to[\mathsf{E}],[x]\in\Gamma_\ell} \langle df, d\eth_\ell\rangle_{\mathscr{L},\text{ave}}([x])$ if this limit exists.

LEMMA 2.7. *Fix $f \in C^2(\Gamma_\Lambda)$ such that $\lim_{\substack{[x]\to[\mathsf{E}]\\ [x]\in\Gamma_\Lambda}}(\mathscr{L}_{\text{ave}}f)([x])$ exists. Then $\mathscr{G}_\ell f$ is well-defined for each $\ell \in \Lambda$.*

Proof. Same as that of [Sow03, Lemma 1.5]. □

DEFINITION 2.8 (Limiting Domain and Generator). Define

$$\mathscr{A}^\dagger \stackrel{\text{def}}{=} \Big\{(f,g) \in C(\mathbb{M})\times C(\mathbb{M}) : f \in C^2(\Gamma_\Lambda),\ g = \mathscr{L}_{\text{ave}}f \text{ on } \Gamma_\Lambda,$$
$$2g([\mathsf{E}])\mathscr{H}^2(\mathsf{E}) = \sum_{\ell\in\Lambda}\mathscr{G}_\ell f, \text{ and } g(\circledast_\ell) = 0 \text{ for all } \ell \in \Lambda\Big\}.$$

The requirement that

$$2g([\mathsf{E}])\mathscr{H}^2(\mathsf{E}) = \sum_{\ell\in\Lambda}\mathscr{G}_\ell f \tag{11}$$

is called the *glueing condition*. It means that $[\mathsf{E}]$ is "sticky" (see [HL81]); i.e., that, asymptotically, X^M spends positive time at $[\mathsf{E}]$. See also Remarks 2.13 of [Sow02] and Remark 1.7 of [Sow03] for some motivational comments.

To formalize the theory surrounding $\mathscr{A}^\dagger$, let's next make the usual setup on the event space $\Omega^\dagger \stackrel{\text{def}}{=} C([0,\infty);\mathbb{M})$. Define the coordinate functions $X^\dagger_t(\omega) \stackrel{\text{def}}{=} \omega(t)$ for all $t \ge 0$ and all $\omega \in \Omega^\dagger$. For each $t \ge 0$, define $\mathscr{F}^\dagger_t \stackrel{\text{def}}{=} \sigma\{X^\dagger_s; 0 \le s \le t\}$ and define a sigma-algebra on $\Omega^\dagger$ by $\mathscr{F}^\dagger \stackrel{\text{def}}{=} \bigvee_{t\ge 0}\mathscr{F}^\dagger_t$.

PROPOSITION 2.9. *The operator $\mathscr{A}^\dagger$ generates a strongly continuous, positive, contraction semigroup on $C(\mathbb{M})$; i.e., there is a unique $\mathbb{P}^\dagger \in \mathscr{P}(\Omega^\dagger)$ which solves the martingale problem with generator $\mathscr{A}^\dagger$ and initial distribution $\delta_{[x_\circ]}$. In other words, there is a unique $\mathbb{P}^\dagger \in \mathscr{P}(\Omega^\dagger)$ such that $\mathbb{P}^\dagger\{X^\dagger_0 = [x_\circ]\} = 1$ and such that for $(f,g) \in \mathscr{A}^\dagger$, $0 \le r_1 < r_2 \cdots < r_n \le s < t$, and $\{\varphi^\dagger_j; j = 1,2\dots n\} \subset C(\mathbb{M})$,*

$$\mathbb{E}^\dagger\left[\left\{f(X^\dagger_t) - f(X^\dagger_s) - \int_s^t g(X^\dagger_u)du\right\}\prod_{j=1}^n \varphi^\dagger_j(X^\dagger_{r_j})\right] = 0, \tag{12}$$

where $\mathbb{E}^\dagger$ denotes the expectation operator associated with $\mathbb{P}^\dagger$.

The proof will be in Section 4.

To properly state our results, we need some restrictions on the ratio

$$\varrho \stackrel{\text{def}}{=} \frac{\omega_1}{\omega_2},$$

which is by assumption irrational. First, let's construct the continued-fraction expansion of ϱ. Set

$$\lfloor z \rfloor \stackrel{\text{def}}{=} \max\{j \in \mathbb{Z} : j \le z\} \quad \text{and} \quad \iota(z) \stackrel{\text{def}}{=} z - \lfloor z \rfloor \tag{13}$$

for all $z \in \mathbb{R}$. Set $\mathbb{k}_1 \stackrel{\text{def}}{=} \varrho$ and recursively define[2] $\mathbb{k}_{n+1} \stackrel{\text{def}}{=} \frac{1}{\iota(\mathbb{k}_n)}$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, we then define $\bar{\mathbb{k}}_n \stackrel{\text{def}}{=} \lfloor \mathbb{k}_n \rfloor$. For each $j \in \mathbb{N}$, define $[[j]] \stackrel{\text{def}}{=} j$, and if we have defined a number $[[j_1, j_2 \dots j_N]] > 0$ for all $(j_1, j_2 \dots j_N) \in \mathbb{N}^N$ for some $N \in \mathbb{N}$, we then define $[[j_1, j_2 \dots j_{N+1}]] \stackrel{\text{def}}{=} j_1 + 1/[[j_2, \dots j_{N+1}]]$ for all $(j_1, j_2 \dots j_{N+1}) \in \mathbb{N}^{N+1}$; this will of course be positive. We then define

$$\varrho_N \stackrel{\text{def}}{=} [\bar{\mathbb{k}}_1, \bar{\mathbb{k}}_2 \dots \bar{\mathbb{k}}_N] \tag{14}$$

for each nonnegative integer N. This is the continued-fraction expansion of ϱ. We can write ϱ_N of (14) as $\varrho_N = \mathsf{a}^{(n)}_N/\mathsf{a}^{(d)}_N$ where $\mathsf{a}^{(n)}_N$ and $\mathsf{a}^{(d)}_N$ are relatively prime integers; then $\mathsf{a}^{(d)}_N \nearrow \infty$. For each positive integer N, define also for future reference

$$\boldsymbol{\nu}_N \stackrel{\text{def}}{=} \varrho - \varrho_N;$$

[2] $\mathbb{N} \stackrel{\text{def}}{=} \{1, 2 \dots\}$.

then for all positive integers N,

$$|\boldsymbol{\nu}_N| \le \frac{1}{\mathsf{a}_N^{(d)}\mathsf{a}_{N+1}^{(d)}}. \tag{15}$$

Our main theorem is

THEOREM 2.10. *Fix $\gamma > 0$. Assume that ϱ is such that*

$$\lim_{N\to\infty}\left(\left(\mathsf{a}_N^{(d)}\right)^{721/14+\gamma}/\mathsf{a}_{N+1}^{(d)}\right) = 0. \tag{16}$$

Define

$$\varepsilon_N \stackrel{\text{def}}{=} \left(1/\mathsf{a}_N^{(d)}\right)^{105/4+\gamma/2} \tag{17}$$

for all $N \in \mathbb{N}$. Then $\lim_{N\to\infty}\mathbb{P}^{\varepsilon_N,\dagger} = \mathbb{P}^\dagger$.

Some of the reason why complicated exponents appear is given in Remark 9.6. We can in fact consider sequences other than that given by (17); see Lemma 9.5 and Remark 9.6. Finally, we admit that the set of ρ's for which (16) holds is *very* small, and that the sequence of ε's for which we can get the desired result is also very small. New approaches will probably be needed for the full result.

Not surprisingly, the proof is a bit complicated. Our thoughts are organized as follows. First, we will prove the tightness and uniqueness claims of Proposition 2.6 and 2.9. Both of these results are straightforward. The hard part is convergence, which takes up most of our efforts. This starts in Section 5.

Dolgopyat and Koralov [DK] have a different approach to a related problem.

3. Notation

Here we collect some useful facts.

For A and B subsets of some topological space, we adapt the usual notation that $A \subset\subset B$ if $\bar{A}$ is a compact subset of B°.

3.1. Brackets. We first note that the generators $\mathscr{L}_e$ and $\mathscr{L}$ define *brackets*—sections of $T^*\mathbb{R}^2 \otimes T^*\mathbb{R}^2$ and $T^*\mathbb{T} \otimes T^*\mathbb{T}$—by requiring that

$$\begin{gathered}\langle df, dg\rangle_e(x) = (\mathscr{L}_e(fg))(x) - f(x)(\mathscr{L}_e g)(x) - g(x)(\mathscr{L}_e f)(x),\\ f, g \in C^2(\mathbb{R}^2),\ x \in \mathbb{R}^2,\\ \langle df, dg\rangle(x) = (\mathscr{L}(fg))(x) - f(x)(\mathscr{L} g)(x) - g(x)(\mathscr{L} f)(x),\\ f, g \in C^2(\mathbb{T}),\ x \in \mathbb{T}.\end{gathered}$$

Then $\langle df, dg\rangle(\mathfrak{t}(x)) = \langle d(f\circ\mathfrak{t}), d(g\circ\mathfrak{t})\rangle_e(x)$ for all f and g in $C^1(\mathbb{T})$ and all $x \in \mathbb{R}^2$, and $\langle df, dg\rangle_{\text{ave}}([x]) = (\mathcal{A}\langle d(f\circ\mathfrak{m}), d(g\circ\mathfrak{m})\rangle)(x)$ for all f and g in $C^1(\Gamma_\Lambda)$ and all $x \in \bigcup_{\ell\in\Lambda} \text{Đ}_\ell$. The brackets naturally appear when applying

$\mathscr{L}_e$ or $\mathscr{L}$ to compositions of functions; e.g., for $f \in C^2(\mathbb{T})$ and $\Phi \in C^\infty(\mathbb{R})$, we have that

$$\mathscr{L}(\Phi \circ f)(x) = \dot{\Phi}(f(x))(\mathscr{L}f)(x) + \frac{1}{2}\ddot{\Phi}(f(x)) \langle df, df \rangle (x)$$

for all $x \in \mathbb{T}$. We also note that the nondegeneracy assumption of (2) is exactly that $\langle df, df \rangle_e (x) > 0$ for all $x \in \mathbb{R}^2$ and $f \in C^1(\mathbb{R})$ such that $df(x) \neq 0$.

3.2. Euclidean notation. Let's also recall some PDE notation about functions on subsets of Euclidean spaces.

DEFINITION 3.1. Suppose that F is a subset of some $\mathbb{R}^d$. For any fixed nonnegative integer k, we say that $\varphi \in C^k(F)$ if φ has all derivatives of order k and less in F° and all of these derivatives are uniformly continuous on F° (in the relative topology inherited from $\mathbb{R}^2$); this is tantamount to requiring that the limits of all of these derivatives exist at each point in ∂F.

Let's next develop some Euclidean tools, keeping in mind the notational convention set down in footnote 1. We let $\mathbf{0}_e = (0,0)$ be the origin of $\mathbb{R}^2$ (as we did in (7)). Let $(\cdot,\cdot)_e$ be the standard Euclidean metric on $T\mathbb{R}^2$, let ∇_e be the standard Euclidean gradient operator, let $\omega_e \stackrel{\text{def}}{=} dx_1 \wedge dx_2$ be the standard symplectic form on $T^*\mathbb{R}^2$, and let $\bar{\nabla}_e$ be the standard Euclidean symplectic gradient operator; e.g., $\mathsf{Ч}_e = \bar{\nabla}_e \mathsf{H}$. Also, define $\|x\|_e \stackrel{\text{def}}{=} \sqrt{x_1^2 + x_2^2}$ (as we did in (8)) and $\mathbf{n}(x) \stackrel{\text{def}}{=} x_1^2 + x_2^2$ for all $x = (x_1, x_2) \in \mathbb{R}^2$. We have already defined the Hausdorff measures $\mathscr{H}^1$ and $\mathscr{H}^2$ on $\mathbb{T}$. Similarly, let $\mathscr{H}_e^1$ and $\mathscr{H}_e^2$ be, respectively, 1- and 2-dimensional Hausdorff measure on $\mathbb{R}^2$; i.e., then for any open $\mathcal{O}$ of $\mathbb{R}^2$ on which $\mathfrak{t}|_{\mathcal{O}}$ is a diffeomorphism, $\mathscr{H}^n(\mathfrak{t}(A)) = \mathscr{H}_e^n(A)$ for all subsets A of $\mathcal{O}$ and $n \in \{1, 2\}$.

The natural Riemannian metric $(\cdot,\cdot)$ and symplectic form $\omega(\cdot,\cdot)$ on $\mathbb{T}$ are defined from $(\cdot,\cdot)_e$ and ω_e by requiring that $(T\mathfrak{t}X, T\mathfrak{t}Y) = (X, Y)_e$ and $\omega(T\mathfrak{t}X, T\mathfrak{t}Y) = \omega_e(X, Y)$ for all X and Y in $T_x\mathbb{R}^2$ for any $x \in \mathbb{R}^2$. Thus $\omega(\mathsf{Ч}, X) = X\mathsf{H}_{T,\ell}$ on D_ℓ for any vector field X on $\mathbb{T}$. Secondly, for any $x \in \mathfrak{t}^{-1}(\mathsf{D}_\ell)$ and any $V \in T_x\mathbb{R}^2$,

$$\begin{aligned}\omega(\mathsf{Ч}(\mathfrak{t}(x)), T\mathfrak{t}V) = T\mathfrak{t}V\mathsf{H}_{T,\ell} = V(\mathsf{H}_{T,\ell} \circ \mathfrak{t}) &= \omega_e(\bar{\nabla}_e(\mathsf{H}_{T,\ell} \circ \mathfrak{t})(x), V) \\ &= \omega((T\mathfrak{t}\bar{\nabla}_e(\mathsf{H}_{T,\ell} \circ \mathfrak{t}))(x), T\mathfrak{t}V),\end{aligned}$$

so $\mathsf{Ч}(\mathfrak{t}(x)) = T\mathfrak{t}\bar{\nabla}_e(\mathsf{H}_{T,\ell} \circ \mathfrak{t})(x)$, as we claimed in the discussion following (5).

3.3. Graph-valued notation. First, note that the $\mathsf{Đ}_\ell$'s can be divided into two types, *wells* and *peaks*. We will call $\mathsf{Đ}_\ell$ a *well* if $\mathsf{H}_{T,\ell}$ is negative in $\mathsf{Đ}_\ell$. Conversely, we will call $\mathsf{Đ}_\ell$ a *peak* if $\mathsf{H}_{T,\ell}$ is positive in $\mathsf{Đ}_\ell$. Let Λ_W be the collection of indices in Λ such that $\mathsf{Đ}_\ell$ is a well, and let Λ_P be the collection of indices in Λ such that $\mathsf{Đ}_\ell$ is a peak.

If $\ell \in \Lambda_P$, define $\mathcal{I}_\ell \overset{\text{def}}{=} (0,\hbar)$, and if $\ell \in \Lambda_W$, define $\mathcal{I}_\ell \overset{\text{def}}{=} (-\hbar,0)$. Fix $f \in C(\Gamma_\Lambda)$. Then for each $\ell \in \Lambda$, we let $f_\ell \in C(\mathcal{I}_\ell)$ be such that $f([x]) = f_\ell(\mathsf{H}_{T,\ell}(x))$ for all $x \in \mathsf{Đ}^\circ_\ell$.

Let now $\boldsymbol{\sigma}$ and $\boldsymbol{\beta}$ in $C^\infty(\mathbb{T})$ be such that

$$\boldsymbol{\sigma}(\mathfrak{t}(x)) = \langle d\mathsf{H}, d\mathsf{H}\rangle (x) \qquad \text{and} \qquad \boldsymbol{\beta}(\mathfrak{t}(x)) \overset{\text{def}}{=} (\mathscr{L}\mathsf{H})(x) \tag{18}$$

for all $x \in \mathbb{R}^2$. We then define $\boldsymbol{\sigma}_M$ and $\boldsymbol{\beta}_M$ in $C(\Gamma_\Lambda)$ as $\boldsymbol{\sigma}_M \overset{\text{def}}{=} \mathcal{A}\boldsymbol{\sigma}$ and $\boldsymbol{\beta}_M = \mathcal{A}\boldsymbol{\beta}$ (using (9)). If $f \in C^2(\Gamma_\Lambda)$, define $(\mathscr{L}^\dagger_\ell f)(h) \overset{\text{def}}{=} \frac{1}{2}\boldsymbol{\sigma}_{M,\ell}(h)\ddot{f}_\ell(h) + \boldsymbol{\beta}_{M,\ell}(h)\dot{f}_\ell(h)$ for each $h \in \mathcal{I}_\ell$. Also, define

$$\mathrm{G}_\ell \overset{\text{def}}{=} \int_{z\in\partial\mathsf{D}_\ell} \frac{\boldsymbol{\sigma}(z)}{\|\mathsf{Ч}(z)\|}\mathscr{H}^1(dz) \tag{19}$$

for all $\ell \in \Lambda$. Define also $\bar{\mathrm{G}}_\ell \overset{\text{def}}{=} \mathrm{G}_\ell$ if $\ell \in \Lambda_P$ and $\bar{\mathrm{G}}_\ell \overset{\text{def}}{=} -\mathrm{G}_\ell$ if $\ell \in \Lambda_W$.

It is fairly easy to see that if $f \in C(\mathbb{M})$, then $(f,g) \in \mathscr{A}^\dagger$ for some $g \in C(\mathbb{M})$ if and only if

(a.i) for each $\ell \in \Gamma$, $f_\ell \in C(\overline{\mathcal{I}_\ell})$ and $f_\ell(0) \overset{\text{def}}{=} \lim_{\substack{h\to 0\\ h\in\mathcal{I}_\ell}} f_\ell(h)$ is the same for all $\ell \in \Lambda$,

(a.ii) $f_\ell \in C^2(\mathcal{I}_\ell)$ for each $\ell \in \Lambda$,

(a.iii) for each $\ell \in \Lambda$, $\mathscr{L}^\dagger_\ell f_\ell \in C(\overline{\mathcal{I}_\ell})$, and $\lim_{\substack{|h|\to\hbar\\ h\in\mathcal{I}_\ell}} \mathscr{L}^\dagger_\ell f_\ell(h) = 0$, and $(\mathscr{L}^\dagger_\ell f_\ell)(0) \overset{\text{def}}{=} \lim_{\substack{h\to 0\\ h\in\mathcal{I}_\ell}} \mathscr{L}^\dagger_\ell f_\ell(h)$ is the same for all $\ell \in \Lambda$,

(a.iv) we have that

$$2(\mathscr{L}^\dagger f)([\mathsf{E}])\mathscr{H}^2(\mathsf{E}) = \sum_{\ell\in\Lambda} \bar{\mathrm{G}}_\ell \lim_{\substack{h\to 0\\ h\in\mathcal{I}_\ell}} \dot{f}_\ell(h),$$

where $(\mathscr{L}^\dagger f)([\mathsf{E}])$ is the common value of the $(\mathscr{L}^\dagger_\ell f_\ell)(0)$'s.

Lemma 2.7 tells us that if (a.i)–(a.iii) are true, then the limits in (a.iv) exists; thus the requirement (a.iv) simply asserts that these limits are related in a certain way. If (a.i)–(a.iv) are true, then $(f,g) \in \mathscr{A}^\dagger$, where g is uniquely defined by requiring that $g([x]) = (\mathscr{L}^\dagger_\ell f_\ell)(\mathsf{H}_{T,\ell}(x))$ for each x in each $\mathsf{Đ}^\circ_\ell$; we note that (a.iii) ensures that g is well-defined at $[\mathsf{E}]$ and the $\circledast_\ell$'s.

3.4. Covering of $\mathbb{T}$ by $\mathbb{R}^2$. We define the usual metric on $\mathbb{T}$; for x and y in $\mathbb{R}^2$, define $d_{\mathbb{T}}(\mathfrak{t}(x),\mathfrak{t}(y)) \overset{\text{def}}{=} \inf_{k\in\mathbb{Z}^2} \|x-y-k\|_{\mathbb{R}^2}$. Define $\check{\mathfrak{t}}_\circ : \mathbb{T} \to [0,1)^2$ by setting $\check{\mathfrak{t}}_\circ((x_1,x_2)+\mathbb{Z}^2) \overset{\text{def}}{=} (\iota(x_1),\iota(x_2))$ for all $(x_1,x_2) \in \mathbb{R}^2$. Then $\mathfrak{t}\circ\check{\mathfrak{t}}_\circ$ is the identity map on $\mathbb{T}$. For each $\ell \in \Lambda$, define $\mathfrak{x}^e_\ell \overset{\text{def}}{=} \check{\mathfrak{t}}_\circ(\mathfrak{x}_\ell)$.

We next lift $\mathsf{H}_{T,\ell}$ back to H.

LEMMA 3.2. *Let $\mathcal{O}$ be a connected component of $\mathfrak{t}^{-1}(\mathsf{D}^\circ_\ell)$. Then $\mathsf{H}(x) - \mathsf{H}(y) = \mathsf{H}_{T,\ell}(\mathfrak{t}(x)) - \mathsf{H}_{T,\ell}(\mathfrak{t}(y))$ for all x and y in $\mathcal{O}$.*

Proof. For any $x \in \mathcal{O}$ and $V \in T_x\mathbb{R}^2$,

$$(20) \quad V\mathsf{H} = \omega_e(\bar{\nabla}_e \mathsf{H}(x), V) = \omega(T\mathfrak{t}\bar{\nabla}_e \mathsf{H}(x), T\mathfrak{t}V) = \omega(\mathsf{Ч}(\mathfrak{t}(x)), T\mathfrak{t}V)$$
$$= \omega(\bar{\nabla}\mathsf{H}_{T,\ell}(\mathfrak{t}(x)), T\mathfrak{t}V) = T\mathfrak{t}V\mathsf{H}_{T,\ell} = V(\mathsf{H}_{T,\ell} \circ \mathfrak{t});$$

thus $d\mathsf{H} = d(\mathsf{H}_{T,\ell} \circ \mathfrak{t})$ on $\mathcal{O}$. Since D°_ℓ is diffeomorphic to the open unit disk in $\mathbb{R}^2$, it is arcwise connected.

Fix now x and y in $\mathcal{O}$. Since D°_ℓ is arcwise connected, there is a piecewise differentiable continuous map $\gamma : [0,1] \to \mathsf{D}^\circ_\ell$ such that $\gamma(0) = \mathfrak{t}(y)$ and $\gamma(1) = \mathfrak{t}(x)$. Since $\mathfrak{t} : \mathfrak{t}^{-1}(\mathsf{D}^\circ_\ell) \to \mathsf{D}^\circ_\ell$ is a covering map, there is a lift $\gamma^e : [0,1] \to \mathfrak{t}^{-1}(\mathsf{D}^\circ_\ell)$. By definition of $\mathcal{O}$, it is fairly easy to see that in fact $\gamma^e([0,1]) \subset \mathcal{O}$. By (20), we then have that

$$\mathsf{H}(x) - \mathsf{H}(y) = \int_{t=0}^1 d\mathsf{H}(\dot{\gamma}^e_t)dt = \int_{t=0}^1 d\mathsf{H}_{T,\ell}(\dot{\gamma}_t)dt = \mathsf{H}_{T,\ell}(\mathfrak{t}(x)) - \mathsf{H}_{T,\ell}(\mathfrak{t}(y)),$$

which is the desired result. □

LEMMA 3.3. *There is a disjoint collection $\{\mathcal{D}_\ell,\ \ell \in \Lambda\}$ of open subsets of $\mathbb{T}$ such that for each $\ell \in \Lambda$, $\mathsf{D}_\ell \subset\subset \mathcal{D}_\ell$ and such that $\mathfrak{t}$ is evenly covered over $\mathcal{D}_\ell$.*

Proof. We start by collecting together some ideas from Arnol'd [Arn91]. Fix $\ell \in \Lambda$. Let $\mathcal{O}^e_\ell$ be a connected component of $\mathfrak{t}^{-1}(\mathsf{D}^\circ_\ell)$. By Lemma 3.2, we see that H is bounded on $\mathcal{O}^e_\ell$, so by the proof of [Arn91, Lemma 3], $\mathcal{O}^e_\ell$ is bounded. Since D°_ℓ is locally connected, $\mathfrak{t}(\mathcal{O}^e_\ell) = \mathsf{D}^\circ_\ell$, and hence $\mathfrak{t}^{-1}(\mathsf{D}^\circ_\ell) = \mathcal{O}^e_\ell + \mathbb{Z}^2$. Since the open unit disk and hence D°_ℓ is retractible to a point, all of the $\mathcal{O}^e_\ell + k$'s are disjoint. Since the D_ℓ's are disjoint, $\mathcal{O}^e_\ell$ and $\mathcal{O}^e_{\ell'} + k$ must be disjoint for all distinct ℓ and ℓ in Λ and all $k \in \mathbb{Z}^2$. We next claim that the constants $d_1 \stackrel{\text{def}}{=} \min_{\ell\in\Lambda} \operatorname{dist}_{\mathbb{R}^2}\left(\mathcal{O}^e_\ell, (\mathcal{O}^e_\ell + \mathbb{Z}^2) \setminus \mathcal{O}^e_\ell\right)$ and $d_2 \stackrel{\text{def}}{=} \min_{\ell,\,\ell'\in\Lambda,\ell'\neq\ell} \operatorname{dist}_{\mathbb{R}^2}\left(\mathcal{O}^e_\ell, \mathcal{O}^e_{\ell'} + \mathbb{Z}^2\right)$ are positive; as usual,

$$\operatorname{dist}_{\mathbb{R}^2}(A,B) \stackrel{\text{def}}{=} \inf\left\{\|x-y\|_{\mathbb{R}^2} : x \in A \text{ and } y \in B\right\}$$

for any two subsets A and B of $\mathbb{R}^2$. Recall that the $\mathcal{O}^e_\ell$'s are bounded. Also note that $\partial\mathcal{O}^e_\ell \subset \mathfrak{t}^{-1}(\partial\mathsf{D}_\ell)$, and thus that $\partial\mathcal{O}^e_\ell$ must be contained in the set of critical lines of H. If $d_1 = 0$, then $\partial\mathcal{O}^e_\ell \cap (\partial\mathcal{O}^e_\ell + k) \neq \emptyset$ for some $\ell \in \Lambda$ and $k \in \mathbb{Z}^2 \setminus \{0\}$. If $d_2 = 0$, then $\partial\mathcal{O}^e_\ell \cap (\partial\mathcal{O}^e_{\ell'} + k) \neq \emptyset$ for some distinct ℓ and ℓ' in Λ and some $k \in \mathbb{Z}^2$. In either case, two critical lines of H must intersect, contradicting the Morse assumption, and thus proving that d_1 and d_2 are positive.

Take now $\delta < \frac{1}{2}\min\{d_1, d_2\}$, and define

$$\mathcal{D}^e_\ell \stackrel{\text{def}}{=} (\mathcal{O}^e_\ell)^\delta = \left\{x \in \mathbb{R}^2 : \|x-y\|_{\mathbb{R}^2} < \delta \text{ for some } y \in \mathcal{O}^e_\ell\right\}$$

and set $\mathcal{D}_\ell \stackrel{\text{def}}{=} \mathfrak{t}(\mathcal{D}^e_\ell)$. Clearly the $\mathcal{D}_\ell$'s are open and disjoint, and, since $\mathcal{D}_\ell$ is open and contains D_ℓ, $\mathsf{D}_\ell \subset\subset \mathcal{D}_\ell$. Also, $\mathfrak{t} : \mathcal{D}^e_\ell \to \mathcal{D}_\ell$ is surjective and $\mathfrak{t}$ is a local diffeomorphism on $\mathcal{D}_\ell^e$. To show that $\mathfrak{t}$ is evenly covered over each $\mathcal{D}_\ell$, we need to show that $\mathfrak{t}$ is injective on $\mathcal{D}_\ell^e$. If $\mathfrak{t}(x) = \mathfrak{t}(y)$ for some distinct x and y in $\mathcal{D}_\ell^e$, then $x \in \mathcal{D}_\ell^e \cap (\mathcal{D}_\ell^e + k) \neq \emptyset$ for some $k \in \mathbb{Z}^2 \setminus \{0\}$. But then $\operatorname{dist}(\mathcal{O}_\ell^e, \mathcal{O}_\ell^e + k) < 2\delta < d_1$. This contradicts the positivity of d_1, finishing the proof. □

Finally, for $\varphi : \mathbb{T} \to \mathbb{R}$, define

$$\|D\varphi(\mathfrak{t}(x))\| \overset{\text{def}}{=} \max_{i\in\{1,2\}} \left| \frac{\partial(\varphi \circ \mathfrak{t})}{\partial x_i}(x) \right|,$$
$$\|D^2\varphi(\mathfrak{t}(x))\| \overset{\text{def}}{=} \max_{i,j\in\{1,2\}} \left| \frac{\partial^2(\varphi \circ \mathfrak{t})}{\partial x_i \partial x_j}(x) \right|$$

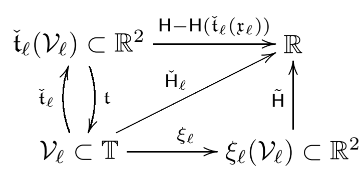
for all $x = (x_1, x_2) \in \mathbb{R}^2$ such that φ is C^∞ at $\mathfrak{t}(x)$.

3.5. Local coordinates. Let's now construct some coordinate charts near the $\mathfrak{x}$'s. Define $\tilde{\mathsf{H}}(x_1, x_2) \overset{\text{def}}{=} x_1 x_2$ for all $(x_1, x_2) \in \mathbb{R}^2$.

LEMMA 3.4. *Fix $\ell \in \Lambda$. There is a connected open neighborhood $\tilde{\mathcal{U}}$ of $\mathbf{0}_e$ (in $\mathbb{R}^2$) and, for each $\ell \in \Lambda$, a map $\tilde{\phi}_\ell : \tilde{\mathcal{U}} \to \mathbb{T}$ such that*

(b.i) *$\tilde{\phi}_\ell$ is a diffeomorphism from $\tilde{\mathcal{U}}$ to $\tilde{\phi}_\ell(\tilde{\mathcal{U}})$,*
(b.ii) *$\mathfrak{t}$ is evenly covered over $\tilde{\phi}_\ell(\tilde{\mathcal{U}})$,*
(b.iii) *$\tilde{\phi}_\ell(\mathbf{0}_e) = \mathfrak{x}_\ell$,*
(b.iv) *$\tilde{\mathsf{H}} \circ \tilde{\phi}_\ell^{-1} \circ \mathfrak{t} - \mathsf{H}$ is locally constant on $\mathfrak{t}^{-1}(\tilde{\phi}_\ell(\tilde{\mathcal{U}}))$,*
(b.v) *$\tilde{\phi}_\ell(\tilde{\mathcal{U}}) \subset\subset \mathcal{D}_\ell$ and $\tilde{\mathcal{U}} \subset\subset (-1,1)^2$.*

Proof. Since $\mathfrak{t}$ is a covering map, there is a connected neighborhood $\mathcal{O}_\ell$ of $\mathfrak{x}_\ell$ and a connected component $\mathcal{O}_\ell^e$ of $\mathfrak{t}^{-1}(\mathcal{O}_\ell)$ such that $\mathfrak{t}\big|_{\mathcal{O}_\ell^e}$ is a diffeomorphism. Let $\check{\mathfrak{t}}_\ell : \mathcal{O}_\ell \to \mathcal{O}_\ell^e$ be its inverse, and define $\check{\mathsf{H}}_\ell(x) \overset{\text{def}}{=} \mathsf{H}(\check{\mathfrak{t}}_\ell(x)) - \mathsf{H}(\check{\mathfrak{t}}_\ell(\mathfrak{x}_\ell))$ for all $x \in \mathcal{O}_\ell$. By Lemma 2.8 in [Sow05], there is a connected neighborhood $\mathcal{V}_\ell$ of $\mathfrak{x}_\ell$ which is contained in $\mathcal{O}_\ell$ and a map $\xi_\ell : \mathcal{V}_\ell \to \mathbb{R}^2$ such that ξ_ℓ is a diffeomorphism from $\mathcal{V}_\ell$ to $\xi_\ell(\mathcal{V}_\ell)$, $\xi_\ell(\mathfrak{x}_\ell) = \mathbf{0}_e$, $\xi_\ell \circ \mathfrak{t}$ is orientation-preserving, and such that $\tilde{\mathsf{H}} \circ \xi_\ell = \check{\mathsf{H}}_\ell$ on $\mathcal{V}_\ell$. The commutative diagram is thus



Since $\mathcal{V}_\ell \subset \mathcal{O}$, $\mathfrak{t}^{-1}(\mathcal{V}_\ell) = \check{\mathfrak{t}}_\ell(\mathcal{V}_\ell) + \mathbb{Z}^2$, this being a decomposition of $\mathfrak{t}^{-1}(\mathcal{V}_\ell)$ into disjoint components. For any $x \in \check{\mathfrak{t}}_\ell(\mathcal{V}_\ell)$ and any $K \in \mathbb{Z}^2$,

$$\begin{aligned}\tilde{\mathsf{H}}(\xi_\ell(\mathfrak{t}(x+K))) - \mathsf{H}(x+K) &= \tilde{\mathsf{H}}(\xi_\ell(\mathfrak{t}(x))) - \mathsf{H}(x) - \langle \omega, K\rangle_{\mathbb{R}^2} \\ &= \check{\mathsf{H}}_\ell(\mathfrak{t}(x)) - \mathsf{H}(x) - \langle \omega, K\rangle_{\mathbb{R}^2} = -\mathsf{H}(\check{\mathfrak{t}}_\ell(\mathfrak{x}_\ell)) - \langle \omega, K\rangle_{\mathbb{R}^2},\end{aligned}$$

proving that $\tilde{\mathsf{H}}\circ\xi_\ell\circ\mathfrak{t}-\mathsf{H}$ is locally constant on $\mathfrak{t}^{-1}(\mathcal{V}_\ell)$.

To complete the proof, let $\tilde{\mathcal{U}}$ be a connected neighborhood of $\mathbf{0}_e$ such that $\tilde{\mathcal{U}}\subset\subset(-1,1)^2\cap\bigcap_{\ell\in\Lambda}\xi_\ell(\mathcal{V}_\ell\cap\mathcal{D}_\ell)$ and set $\tilde{\phi}_\ell\stackrel{\text{def}}{=}\xi_\ell^{-1}$ on $\tilde{\mathcal{U}}$. Since $\tilde{\phi}_\ell(\tilde{\mathcal{U}})\subset\mathcal{V}_\ell\subset\mathcal{O}_\ell$, $\mathfrak{t}$ is evenly covered over $\tilde{\phi}_\ell(\tilde{\mathcal{U}})$. □

For convenience define $\phi_\ell\stackrel{\text{def}}{=}\tilde{\phi}_\ell^{-1}$ on $\mathcal{U}_\ell\stackrel{\text{def}}{=}\tilde{\phi}_\ell(\tilde{\mathcal{U}})$; i.e.,

$$\tilde{\mathcal{U}}\subset\mathbb{R}^2 \underset{\phi_\ell}{\overset{\tilde{\phi}_\ell}{\rightleftarrows}} \mathcal{U}_\ell\subset\mathbb{T}\ .$$

Let's next push various things through the ϕ_ℓ's, using Euclidean geometry on $\mathbb{R}^2$ as a reference. Note that

$$((\bar{\nabla}_e\tilde{\mathsf{H}})f)(x)=x_1\frac{\partial f}{\partial x_1}(x)-x_2\frac{\partial f}{\partial x_2}(x)$$

for all $x=(x_1,x_2)\in\mathbb{R}^2$ and all $f\in C^1(\mathbb{R}^2)$. Fix next $\ell\in\Lambda$. Define the second-order operators $\tilde{\mathscr{L}}_\ell$ and $\tilde{\mathscr{L}}_\ell^\varepsilon$ on $C^2(\tilde{\mathcal{U}})$ and the bilinear form $\langle\cdot,\cdot\rangle_\ell^\sim$ on $T^*\tilde{\mathcal{U}}$ by the formulae $(\tilde{\mathscr{L}}_\ell f)(x)\stackrel{\text{def}}{=}(\mathscr{L}(f\circ\phi_\ell))(\tilde{\phi}_\ell(x))$, $(\tilde{\mathscr{L}}_\ell^\varepsilon f)(x)\stackrel{\text{def}}{=}(\mathscr{L}^\varepsilon(f\circ\phi_\ell))(\tilde{\phi}_\ell(x))$, and $\langle df,df\rangle_\ell^\sim(x)\stackrel{\text{def}}{=}(\tilde{\mathscr{L}}_\ell f^2)(x)-2f(x)(\tilde{\mathscr{L}}_\ell f)(x)$, which we require to hold for all $f\in C^2(\tilde{\mathcal{U}})$ and all $x\in\tilde{\mathcal{U}}$. We also define

$$\tilde{\mathrm{B}}_\ell\stackrel{\text{def}}{=}\frac{d(\tilde{\phi}_\ell^*\omega)}{d\omega_e}\quad\text{on }\tilde{\mathcal{U}}.$$

Then

$$T\phi_\ell\text{Ч}(\tilde{\phi}_\ell(x))=\frac{1}{\tilde{\mathrm{B}}_\ell(x)}\bar{\nabla}_e\tilde{\mathsf{H}}(x),\qquad x\in\tilde{\mathcal{U}}. \tag{21}$$

Indeed, fix $x\in\tilde{\mathcal{U}}$ and $x'\in\mathfrak{t}^{-1}(\tilde{\phi}_\ell(x))$. Then $\mathfrak{t}(x')=\tilde{\phi}_\ell(x)\in\tilde{\phi}_\ell(\tilde{\mathcal{U}})$, and $\phi_\ell(\mathfrak{t}(x'))=x$. Fix $V\in T_{x'}\mathbb{R}^2$; then

$$\begin{aligned}\omega_e(T\phi_\ell\text{Ч}(\tilde{\phi}_\ell(x)),T\phi_\ell T\mathfrak{t}V)&=\frac{1}{\tilde{\mathrm{B}}_\ell(x)}\omega(\text{Ч}(\mathfrak{t}(x')),T\mathfrak{t}V)\\&=\frac{1}{\tilde{\mathrm{B}}_\ell(x)}\omega(T\mathfrak{t}\bar{\nabla}_e\mathsf{H}(x'),T\mathfrak{t}V)=\frac{1}{\tilde{\mathrm{B}}_\ell(x)}\omega_e(\bar{\nabla}_e\mathsf{H}(x'),V)=\frac{1}{\tilde{\mathrm{B}}_\ell(x)}V\mathsf{H}\\&=\frac{1}{\tilde{\mathrm{B}}_\ell(x)}V(\tilde{\mathsf{H}}\circ\phi_\ell\circ\mathfrak{t})=\frac{1}{\tilde{\mathrm{B}}_\ell(x)}T\phi_\ell T\mathfrak{t}V\tilde{\mathsf{H}}\\&=\frac{1}{\tilde{\mathrm{B}}_\ell(x)}\omega_e(\bar{\nabla}_e\tilde{\mathsf{H}}(\phi_\ell(\mathfrak{t}(x')),T\phi_\ell T\mathfrak{t}V)=\frac{1}{\tilde{\mathrm{B}}_\ell(x)}\omega_e(\bar{\nabla}_e\tilde{\mathsf{H}}(x),T\phi_\ell T\mathfrak{t}V).\end{aligned}$$

Since $T\phi_\ell$ and $T\mathfrak{t}$ are full rank, $T\phi_\ell T\mathfrak{t}T_{x'}\mathbb{R}^2=T_x\mathbb{R}^2$, and we have (21). From (21), we have that

$$\left(\tilde{\mathscr{L}}_\ell^\varepsilon f\right)(x)=\frac{1}{\varepsilon^2\tilde{\mathrm{B}}_\ell(x)}(\bar{\nabla}_e\tilde{\mathsf{H}},\nabla_e f)_e(x)+(\tilde{\mathscr{L}}_\ell f)(x)$$

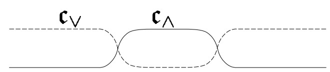


FIGURE 3. Cutoff Functions

for all $x \in \tilde{\mathcal{U}}$, $\varepsilon \in (0,1)$, $f \in C^2(\tilde{\mathcal{U}})$, and $\ell \in \Lambda$,

3.6. Useful functions. Next, let's fix some cutoff functions. Let $\mathfrak{c}_\wedge \in C^\infty(\mathbb{R};[0,1])$ be even and such that $\operatorname{supp} \mathfrak{c}_\wedge \subset (-2,2)$ and such that $\operatorname{supp}(1-\mathfrak{c}_\wedge) \subset \mathbb{R}\setminus[-1,1]$. We then define $\mathfrak{c}_\vee \stackrel{\text{def}}{=} 1-\mathfrak{c}_\wedge$; see Figure 3. Then $\mathfrak{c}_\vee(z)=0$ if $|z|\le 1$, and $\mathfrak{c}_\vee(z)=1$ if $|z|\ge 2$. Note that $\dot{\mathfrak{c}}_\vee = -\dot{\mathfrak{c}}_\wedge$ and $\ddot{\mathfrak{c}}_\vee = -\ddot{\mathfrak{c}}_\wedge$, and that both $\dot{\mathfrak{c}}_\wedge$ and $\ddot{\mathfrak{c}}_\wedge$ have support in $[-2,-1]\cup[1,2]$. Thus, there is a constant $K_{22}>0$ such that

$$|\dot{\mathfrak{c}}_\wedge(z)| + |\ddot{\mathfrak{c}}_\wedge(z)| \le K_{(22)}\mathfrak{c}_\wedge\left(\frac{z}{2}\right) \tag{22}$$

for all $z \in \mathbb{R}$. Define $\mathfrak{l}(x) \stackrel{\text{def}}{=} \ln(e+|x|^{-1})$ for all $x>0$. Then $\mathfrak{l} \ge 1$ and $\mathfrak{l}(x) \approx \ln\frac{1}{|x|}$ when $x \to 0$. Next, define $x^- \stackrel{\text{def}}{=} \min\{x,0\}$ for all $x \in \mathbb{R}$. For $x \in \mathbb{R}$, $x^- \le x$. Thus, for x and y in $\mathbb{R}$, $x+y \ge x^- + y^-$; since $x^-+y^- \le 0$, we in fact have that $x^-+y^- \le \min\{x+y,0\} = (x+y)^-$. Since $|y^-| = |y|\chi_{\{y\le 0\}} \le |y|$ for all $y \in \mathbb{R}$, we in fact have that

$$0 \le (x+y)^- \ge x^- - |y| \tag{23}$$

for all x and y in $\mathbb{R}$. Finally, define $\mathfrak{s}(x) \stackrel{\text{def}}{=} x/|x|$ for all $x \in \mathbb{R}\setminus\{0\}$, and for consistency, define $\mathfrak{s}(0) \stackrel{\text{def}}{=} 0$.

4. Tightness and uniqueness

4.1. Tightness. We first prove Proposition 2.6.

For each $\omega \in \Omega$, $\{X_t(\omega);\, t \ge 0\}$ is a continuous path in $\mathbb{T}$. Furthermore, the map $\omega \mapsto X(\omega)$ from Ω into $C([0,\infty);\mathbb{R}^2)$ is measurable (where we endow $C([0,\infty);\mathbb{R}^2)$ with the topology of uniform convergence on compacts).

Let's lift the process X to $\mathbb{R}^2$. Define $\Omega_\circ \stackrel{\text{def}}{=} \{\omega \in \Omega : X_0(\omega) = x_\circ\}$ (the point $x_\circ$ was chosen in Definition 2.4), and note that $\Omega_\circ \in \mathscr{F}_0$. Define $x^e_\circ \stackrel{\text{def}}{=} \check{\mathfrak{t}}_\circ(x_\circ)$. For $\omega \in \Omega_\circ$, let $\{X^e_t(\omega);\, t\ge 0\}$ be the lift of $\{X_t(\omega);\, t \ge 0\}$ to $\mathbb{T}$. For $\omega \in \Omega\setminus\Omega_\circ$, define $X^e_t(\omega) \stackrel{\text{def}}{=} x^e_\circ$ for all $t \ge 0$.

LEMMA 4.1. *The process $\{X^e_t;\, t\ge 0\}$ is adapted to $\{\mathscr{F}_t;\, t \ge 0\}$.*

Proof. Fix $T \ge 0$. It suffices to show that the set $A_\delta(\varphi^e) \stackrel{\text{def}}{=} \{\omega \in \Omega : \|X^e(\omega)-\varphi^e\|_{C([0,T];\mathbb{R}^2)} < \delta\}$ is in $\mathscr{F}_T$ for all $\varphi^e \in C([0,T];\mathbb{R}^2)$ and $\delta \in (0,1/2)$ (such sets are a base for the topology of $C([0,T];\mathbb{R}^2)$).

If $\|\varphi^e_0 - x^e_\circ\|_{\mathbb{R}^2} \geq \delta$, then $A_\delta(\varphi^e) = \emptyset \in \mathscr{F}_T$. Thus, we henceforth assume that $\|\varphi^e_0 - x^e_\circ\|_{\mathbb{R}^2} < \delta$. Then $A_\delta(\varphi^e) = (A_\delta(\varphi^e) \cap \Omega_\circ) \cup (A_\delta(\varphi^e) \setminus \Omega_\circ)$. If $\sup_{0\leq t\leq T} \|\varphi^e_t - x^e_\circ\|_{\mathbb{R}^2} \geq \delta$, $A_\delta(\varphi^e) \setminus \Omega_\circ = \emptyset$, while if $\sup_{0\leq t\leq T} \|\varphi^e_t - x^e_\circ\|_{\mathbb{R}^2} < \delta$, $A_\delta(\varphi^e) \setminus \Omega_\circ = \Omega$. In either case, $A_\delta(\varphi^e) \setminus \Omega_\circ \in \mathscr{F}_T$.

Set now $\varphi_t \stackrel{\text{def}}{=} \mathfrak{t}(\varphi^e_t)$ for all $t \in [0,T]$, and define the set

$$B_\delta(\delta) \stackrel{\text{def}}{=} \Omega_\circ \cap \{\omega \in \Omega : \sup_{0\leq t\leq T} d_{\mathbb{T}}(X_t(\omega), \varphi_t) < \delta\}.$$

We claim that $A_\delta(\varphi^e) \cap \Omega_\circ = B_\delta(\varphi)$. Since $d_{\mathbb{T}}(\mathfrak{t}(x), \mathfrak{t}(y)) < \|x-y\|_{\mathbb{R}^2}$ for all x and y in $\mathbb{R}^2$, $A_\delta(\varphi^e) \cap \Omega_\circ \subset B_\delta(\varphi)$. To show the reverse inclusion, fix $\omega \in B_\delta(\varphi)$, and define

$$I(\omega) \stackrel{\text{def}}{=} \{t \in [0,T] : \|X^e_t(\omega) - \varphi^e_t\|_{\mathbb{R}^2} < \delta\}.$$

Continuity implies that $I(\omega)$ is an open subset of $[0,T]$. If $t \in [0,T]$ is such $\|X^e_t(\omega) - \varphi^e_t\|_{\mathbb{R}^2} > \delta$, then by continuity it is in the interior of $[0,T] \setminus I(\omega)$. Consider finally $t \in [0,T]$ such that $\|X^e_t(\omega) - \varphi^e_t\|_{\mathbb{R}^2} = \delta$. Since $\delta < 1/2$, there is an open neighborhood $\mathcal{O}$ of t (in $[0,T]$) such that $\|X^e_s(\omega) - \varphi^e_s\|_{\mathbb{R}^2} < \frac{1}{2}$ for all $s \in \mathcal{O}$. For all $k \in \mathbb{Z}^2 \setminus \{0\}$,

$$\|X^e_s(\omega) - \varphi^e_s + k\|_{\mathbb{R}^2} \geq \|k\|_{\mathbb{R}^2} - \|X^e_s(\omega) - \varphi^e_s\|_{\mathbb{R}^2} \geq 1 - \frac{1}{2} = \frac{1}{2} > \|X^e_s(\omega) - \varphi^e_s\|_{\mathbb{R}^2},$$

so $\|X^e_s(\omega) - \varphi^e_s\|_{\mathbb{R}^2} = d_{\mathbb{T}}(X_s(\omega), \varphi_s) < \delta$ for all $s \in \mathcal{O}$. We conclude that $[0,T] \setminus I(\omega)$ is open (in the topology of $[0,T]$). Since by assumption $0 \in I(\omega)$, the connectedness of $[0,T]$ implies that $I(\omega) = [0,T]$, and hence that $\omega \in A_\delta(\varphi^e) \cap \Omega_\circ$, This implies that $B_\delta(\varphi) \subset A_\delta(\varphi^e) \cap \Omega_\circ$, and hence finishes the proof that $A_\delta(\varphi^e) \cap \Omega_\circ = B_\delta(\varphi)$.

Clearly $B_\delta(\varphi) \in \mathscr{F}_T$; collect things together, and the proof is complete. □

The main idea of the proof of tightness is that if X^M varies a lot, then so must $\mathsf{H}(X^e)$, but we can control the variation of $\mathsf{H}(X^e)$ by standard means.

Recall the metric on $\mathbb{M}$ defined by (8). Our first claim is the following.

LEMMA 4.2. *Fix $T > 0$ and $\mu > 0$. Then there is a $\mu' > 0$ such that for any $\eta \in (0,1)$,*

$$\left\{\omega \in \Omega_\circ : \sup_{\substack{0\leq s<t\leq T\\ |s-t|\leq\eta}} \left|\mathsf{H}(X^e_{s\wedge\mathfrak{e}(\omega)}(\omega)) - \mathsf{H}(X^e_{t\wedge\mathfrak{e}(\omega)}(\omega))\right| < \mu'\right\} \subset \left\{\omega \in \Omega_\circ : \sup_{\substack{0\leq s<t\leq T\\ |s-t|\leq\eta}} d^\dagger(X^M_s(\omega), X^M_t(\omega)) < \mu\right\}. \tag{24}$$

Proof. Assume not. Then for each $n \in \mathbb{N}$, there is an $\eta_n \in (0,1)$, an $\omega_n \in \Omega_\circ$ and an s_n and a t_n in $[0, T \wedge \mathfrak{e}(\omega_n)]$ such that $0 < s_n - t_n \le \eta_n$ and such that

$$\sup_{\substack{0 \le s' < t' \le T \wedge \mathfrak{e}(\omega_n) \\ |s'-t'| \le \eta_n}} |\mathsf{H}(X^e_{s'}(\omega_n)) - \mathsf{H}(X^e_{t'}(\omega_n))| < \frac{1}{n}, \tag{25}$$
$$d^\dagger(X^M_{s_n}(\omega_n), X^M_{t_n}(\omega_n)) \ge \mu.$$

Since $X_{s_n}(\omega_n)$ and $X_{t_n}(\omega_n)$ are in the compact set $\mathbf{S}$ for all n, we can extract a subsequence $\{n_k;\, k =\in \mathbb{N}\}$ of $\mathbb{N}$ such that $x^*_a \overset{\text{def}}{=} \lim_{k\to\infty} X^M_{s_{n_k}}(\omega_{n_k})$ and $x^*_b \overset{\text{def}}{=} \lim_{k\to\infty} X^M_{t_{n_k}}(\omega_{n_k})$ exist. We note that by continuity $d^\dagger([x^*_a], [x^*_b]) \ge \mu$.

The triangle inequality implies that $\max\{d^\dagger([x^*_a], [\mathsf{E}]), d^\dagger([x^*_b], [\mathsf{E}])\} \ge \mu/2$. We now define the open set

$$\mathcal{O} \overset{\text{def}}{=} \{z \in \mathbb{M} : d^\dagger(z, [x^*_a]) < \mu/5\} \cup \{z \in \mathbb{M} : d^\dagger(z, [x^*_b]) < \mu/5\} \cup \{z \in \mathbb{M} : d^\dagger(z, [\mathsf{E}]) < \mu/5\}.$$

Note that $\mathcal{O}$ is open and thus $\mathbb{M} \setminus \mathcal{O}$ is compact. We cover $\mathbb{M} \setminus \mathcal{O}$. Fix $z \in \mathbb{M} \setminus \mathcal{O}$. Then $z \in \mathfrak{m}(\text{Đ}_\ell)$ for some $\ell \in \Lambda$. It is easy to check that $\mathbb{M} \setminus \mathfrak{m}(\text{Đ}_\ell)$ is a closed set which does not contain z. Thus there is a $\mu_z \in \left(0, \frac{3\mu}{20}\right)$ such that $\mu_z < \operatorname{dist}^\dagger(z, \mathbb{M} \setminus \mathfrak{m}(\text{Đ}_\ell))$, where, as usual, $\operatorname{dist}^\dagger(x, A) \overset{\text{def}}{=} \inf_{y \in A} d^\dagger(x, y)$ for all points $x \in \mathbb{M}$ and all subsets A of $\mathbb{M}$. Define then $\mathcal{O}_z \overset{\text{def}}{=} \{z' \in \mathbb{M} : d^\dagger(z', z) < \mu_z/3\}$. By compactness there is thus a finite subset A of $\mathbb{M} \setminus \mathcal{O}$ such that $\mathbb{M} \setminus \mathcal{O} \subset \bigcup_{z \in A} \mathcal{O}_z$.

Now let $k \in \mathbb{N}$ be large enough that

$$d^\dagger(X^M_{s_{n_k}}(\omega_{n_k}), [x^*_a]) < \frac{\mu}{10} \qquad \text{and} \qquad d^\dagger(X^M_{t_{n_k}}(\omega_{n_k}), [x^*_b]) < \frac{\mu}{10}.$$

Then

$$d^\dagger(X^M_{t_{n_k}}(\omega_{n_k}), [x^*_a]) \ge d^\dagger([x^*_a], [x^*_b]) - d^\dagger(X^M_{t_{n_k}}(\omega_{n_k}), [x^*_b]) \ge \mu - \frac{\mu}{10} = \frac{9\mu}{10}.$$

Hence there is an $r^*_1 \in (s_{n_k}, t_{n_k})$ such that $d^\dagger(X^M_{r^*_1}(\omega_{n_k}), [x^*_a]) = \mu/4$. Thus $d^\dagger(X^M_{r^*_1}(\omega_{n_k}), [x^*_a]) = \mu/4 > \mu/5$ and

$$d^\dagger(X^M_{r^*_1}(\omega_{n_k}), [x^*_b]) \ge d^\dagger([x^*_a], [x^*_b]) - d^\dagger(X^M_{r^*_1}(\omega_{n_k}), [x^*_a]) \ge \mu - \frac{\mu}{4} = \frac{3\mu}{4} > \frac{\mu}{5},$$
$$d^\dagger(X^M_{r^*_1}(\omega_{n_k}), [\mathsf{E}]) \ge d^\dagger([x^*_a], [\mathsf{E}]) - d^\dagger(X^M_{r^*_1}(\omega_{n_k}), [x^*_a]) \ge \frac{\mu}{2} - \frac{\mu}{4} = \frac{\mu}{4} > \frac{\mu}{5}.$$

Thus $X^M_{r^*_1}(\omega_{n_k}) \in \mathbb{M} \setminus \mathcal{O}$ and thus is in $\mathcal{O}_z$ for some $z \in A$.

We consequently have that $d^\dagger(X^M_{r^*_1}(\omega_{n_k}), z) < \frac{\mu_z}{3}$, and since $z \in \mathbb{M} \setminus \mathcal{O}$,

$$d^\dagger(X^M_{t_{n_k}}(\omega_{n_k}), z) \ge d^\dagger([x^*_b], z) - d^\dagger(X^M_{t_{n_k}}(\omega_{n_k}), [x^*_b]) \ge \frac{\mu}{5} - \frac{\mu}{10} = \frac{\mu}{10} > \frac{2\mu_z}{3}.$$

Thus there is an $r_2^* \in (r_1^*, t_{n_k})$ such that $d^\dagger(X_s^M(\omega_{n_k}), z) < \frac{2\mu_z}{3}$ for all $s \in [r_1^*, r_2^*)$ and $d^\dagger(X_{r_2^*}^M(\omega_{n_k}), z) = \frac{2\mu_z}{3}$. Let $\ell \in \Lambda$ be such that $z \in \mathfrak{m}(\mathsf{Đ}_\ell)$, and let $x \in \mathsf{Đ}_\ell$ be such that $\mathfrak{m}(x) = z$. Then

$$\begin{aligned}
&\left|\mathsf{H}_{T,\ell}(X_{r_2^*}(\omega_{n_k})) - \mathsf{H}_{T,\ell}(X_{r_1^*}(\omega_{n_k}))\right| \\
&\quad \geq \left|\mathsf{H}_{T,\ell}(X_{r_2^*}(\omega_{n_k})) - \mathsf{H}_{T,\ell}(x)\right| - \left|\mathsf{H}_{T,\ell}(X_{r_1^*}(\omega_{n_k})) - \mathsf{H}_{T,\ell}(x)\right| \\
&\quad \geq d^\dagger(X_{r_2^*}^M(\omega_{n_k}), z) - d^\dagger(X_{r_1^*}^M(\omega_{n_k}), z) \geq \frac{2\mu_z}{3} - \frac{\mu_z}{3} \geq \frac{1}{3}\inf_{z' \in A} \mu_{z'}.
\end{aligned}$$

We can now move back to H. For all $s \in [r_1^*, r_2^*)$,

$$\begin{aligned}
&\operatorname{dist}^\dagger\left(X_s^M(\omega_{n_k}), \mathbb{M} \setminus \mathfrak{m}(\mathsf{Đ}_\ell)\right) \\
&\qquad \geq \operatorname{dist}^\dagger\left(z, \mathbb{M} \setminus \mathfrak{m}(\mathsf{Đ}_\ell)\right) - d^\dagger(X_s^M(\omega_{n_k}), z) \geq \mu_x - \frac{2\mu_x}{3} > 0.
\end{aligned}$$

Thus $X_s^M(\omega_{n_k}) \in \mathfrak{m}(\mathsf{Đ}_\ell)$ for all $s \in [r_1^*, r_2^*)$. Since $\mathfrak{m}^{-1}(\mathfrak{m}(\mathsf{Đ}_\ell)) = \mathsf{Đ}_\ell \subset \mathsf{D}_\ell^\circ$, $X_s(\omega_{n_k}) \in \mathsf{D}_\ell^\circ$ for all $s \in [r_1^*, r_2^*)$. Let $\mathcal{O}$ be a connected component of $\mathfrak{t}^{-1}(\mathsf{D}_\ell^\circ)$, and let $x^e \in \mathcal{O}$ be such that $\mathfrak{t}(x^e) = x$. Then necessarily $X_s^e(\omega_{n_k}) \in \mathcal{O}$ for all $s \in [r_1^*, r_2^*)$. By Lemma 3.2 and the continuity of H and $\mathsf{H}_{T,\ell}$,

$$\begin{aligned}
&\varliminf_{k\to\infty} \left|\mathsf{H}(X_{r_2^*}^e(\omega_{n_k})) - \mathsf{H}(X_{r_1^*}^e(\omega_{n_k}))\right| \\
&\qquad \geq \varliminf_{k\to\infty} \left|\mathsf{H}_{T,\ell}(X_{r_2^*}(\omega_{n_k})) - \mathsf{H}_{T,\ell}(X_{r_1^*}(\omega_{n_k}))\right| \geq \frac{1}{3}\inf_{z' \in A} \mu_{z'}.
\end{aligned}$$

This violates the first claim of (25), proving that our assumption is incorrect, and thus proving the lemma. □

We now can give the

Proof of Proposition 2.6. From (24), we have that

$$\begin{aligned}
(26) \quad &\sup_{0<\varepsilon<1} \mathbb{P}^\varepsilon\left\{\sup_{\substack{0\leq s<t\leq T\\ |s-t|\leq\eta}} d^\dagger(X_s^M, X_t^M) \geq \mu\right\} \\
&\qquad \leq \sup_{0<\varepsilon<1} \mathbb{P}^\varepsilon\left\{\sup_{\substack{0\leq s<t\leq T\\ |s-t|\leq\eta}} |\mathsf{H}(X_{s\wedge\mathfrak{e}}^e) - \mathsf{H}(X_{t\wedge\mathfrak{e}}^e)| \geq \mu'\right\}
\end{aligned}$$

for each $\eta \in (0,1)$ (use here the fact that $\mathbb{P}^\varepsilon(\Omega_\circ) = 1$ for all $\varepsilon \in (0,1)$). We now compute that

$$\mathsf{H}(X_{t\wedge\mathfrak{e}}^e) = \mathsf{H}(X_0^e) + \int_0^{t\wedge\mathfrak{e}} \boldsymbol{\beta}(X_s)ds + M_{t\wedge\mathfrak{e}}\,,$$

where M is a $\mathbb{P}^\varepsilon$-martingale with quadratic variation

$$\langle M\rangle_t = \int_0^t \boldsymbol{\sigma}(X_s)ds, \qquad t \geq 0.$$

Standard results imply that

$$\overline{\lim_{\eta\to 0}} \sup_{0<\varepsilon<1} \mathbb{P}^\varepsilon \left\{ \sup_{\substack{0\leq s<t\leq T \\ |s-t|\leq \eta}} |\mathsf{H}(X^e_{s\wedge\mathfrak{e}}) - \mathsf{H}(X^e_{t\wedge\mathfrak{e}})| \geq \mu' \right\} = 0.$$

Combining this and (26), we get that

$$\overline{\lim_{\eta\to 0}} \sup_{0<\varepsilon<1} \mathbb{P}^{\varepsilon,\dagger} \left\{ \sup_{\substack{0\leq s<t\leq T \\ |s-t|\leq \eta}} d^\dagger(X^\dagger_s, X^\dagger_t) \geq \mu \right\} = 0,$$

which is the claimed tightness. □

4.2. Uniqueness. We next prove Proposition 2.9. As usual, we endow $C(\mathbb{M})$ with the topology generated by the $\|\cdot\|_{C(\mathbb{M})}$ norm, and we define $\mathscr{D}(\mathscr{A}^\dagger) \overset{\text{def}}{=} \{f : (f,g) \in \mathscr{A}^\dagger\}$ and $\mathscr{R}(\lambda - \mathscr{A}^\dagger) \overset{\text{def}}{=} \{\lambda f - g : (f,g) \in \mathscr{A}^\dagger\}$. According to [EK86, Theorem 4.2.2], we need to prove three things: that $\mathscr{D}(\mathscr{A}^\dagger)$ is dense in $C(\mathbb{M})$, that $\mathscr{A}^\dagger$ is dissipative, and that $\mathscr{R}(\lambda - \mathscr{A}^\dagger)$ is dense in $C(\mathbb{M})$ for some $\lambda > 0$. We begin with

LEMMA 4.3. *The set $\mathscr{D}(\mathscr{A}^\dagger)$ is dense in $C(\mathbb{M})$.*

Proof. Fix $f \in C(\mathbb{M})$. We will kill the variation of f near $[\mathsf{E}]$ and the $\circledast_\ell$'s and replace the f_ℓ's by smooth approximations. Note that smooth functions which are flat near $[\mathsf{E}]$ and the $\circledast_\ell$'s are in $\mathscr{D}(\mathscr{A}^\dagger)$. For each $n \in \mathbb{N}$ and $\ell \in \Lambda$, let $f_{n,\ell} \in C^\infty_c(\mathcal{I}_\ell)$ be such that $|f_\ell(h) - f_{n,\ell}(h)| < 1/n$ for all $h \in \mathcal{I}_\ell$ such that $1/(2n) < |h| < \hbar - 1/(2n)$. For each $n \in \mathbb{N}$, define

$$\begin{aligned} f_n([x]) \overset{\text{def}}{=} \sum_{\ell\in\Lambda_P} \chi_{\text{Đ}_\ell}(x) &\{f_{n,\ell}(\mathsf{H}_{T,\ell}(x))\mathfrak{c}_\vee(n\mathsf{H}_{T,\ell}(x))\mathfrak{c}_\vee(n(\mathsf{H}_{T,\ell}(x) - \hbar)) \\ &+ f_\ell(\hbar)\mathfrak{c}_\wedge(n(\mathsf{H}_{T,\ell}(x) - \hbar))\} \\ - \sum_{\ell\in\Lambda_W} \chi_{\text{Đ}_\ell}(x) &\{f_{n,\ell}(\mathsf{H}_{T,\ell}(x))\mathfrak{c}_\vee(n\mathsf{H}_{T,\ell}(x))\mathfrak{c}_\vee(n(\mathsf{H}_{T,\ell}(x) + \hbar)) \\ &+ f_\ell(-\hbar)\mathfrak{c}_\wedge(n(\mathsf{H}_{T,\ell}(x) + \hbar))\} \\ + f([\mathsf{E}]) &\left\{1 - \sum_{\ell\in\Lambda} \chi_{\text{Đ}_\ell}(x)\mathfrak{c}_\vee(n\mathsf{H}_{T,\ell}(x))\right\} \end{aligned}$$

for each $x \in \mathbf{S}$ and each positive integer n. Note that $f_n \in C^2(\Gamma_\Lambda)$. Thus, we define $g_n([x]) \overset{\text{def}}{=} (\mathscr{L}_{\text{ave}} f_n)([x])$ if $x \in \Gamma_\Lambda$ and otherwise we set $g_n([x]) = 0$. It is fairly clear that $(f_n, g_n) \in \mathscr{A}^\dagger$ for all $n \in \mathbb{N}$ and that $\lim_{n\to\infty} \|f_n - f\|_{C(\mathbb{M})} = 0$. This means that we have approximated an arbitrarily-chosen element of $C(\mathbb{M})$ by elements of $\mathscr{D}(\mathscr{A}^\dagger)$. □

LEMMA 4.4. *The operator $\mathscr{A}^\dagger$ is dissipative.*

Proof. From [EK86, Lemma 4.2.1], it suffices to show the positive maximum principle. In other words, fix $(f, g) \in \mathscr{A}^\dagger$ and assume that $x^* \in \mathbb{M}$ is such that $f(x^*) = \max_{x\in\mathbb{M}} f(x) \geq 0$; then we must show that $g(x^*) \leq 0$.

If $x^* \in \bigcup_{\ell\in\Lambda} \circledast_\ell$, then by definition of $\mathscr{A}^\dagger$, $g(x^*) = 0$. Assume next that $x^* \in \Gamma_\Lambda$. Since $\mathscr{L}_{\text{ave}}$ is local and can be represented by a strongly elliptic operator on the Γ_ℓ's, we must here too have that $g(x^*) \leq 0$. Finally, assume that $x^* = [\mathsf{E}]$. By assumption that f attains its maximum at $x^* = [\mathsf{E}]$, we have that $\mathscr{G}_\ell f \leq 0$ for all $\ell \in \Lambda$ (use the same argument as in the proof of Lemma 4.4 of [Sow03]). Thus $2g([\mathsf{E}])\mathscr{H}^1(\mathsf{E}) = \sum_{\ell\in\Lambda} \mathscr{G}_\ell f \leq 0$, which gives us the desired result. □

Finally, we prove

LEMMA 4.5. *For each $\lambda > 0$, $\mathscr{R}(\lambda - \mathscr{A}^\dagger)$ is dense in $C(\mathbb{M})$.*

Proof. It suffices to fix a $\varphi \in C(\mathbb{M})$ such that $\varphi_\ell \in C^\infty(\overline{\mathcal{I}_\ell})$ for all $\ell \in \Lambda$, and a $\lambda > 0$ and find an $(f, g) \in \mathscr{A}^\dagger$ such that

$$g([x]) - \lambda f([x]) = \varphi([x]) \qquad x \in \mathbb{M} \tag{27}$$

(we have reversed signs to make the ensuing PDE's look more standard). We separately consider several PDE's on each of the $\mathcal{I}_\ell$'s. For each $\ell \in \Lambda$, define ϕ_ℓ, $\phi_{a,\ell}$, and $\phi_{b,\ell}$ as the solutions of the PDE's

$$\mathscr{L}_\ell^\dagger \phi_\ell - \lambda\phi_\ell = \varphi_\ell \quad \text{on } \mathcal{I}_\ell, \quad \lim_{\substack{h\to 0\\ h\in\mathcal{I}_\ell}} \phi_\ell(h) = 0, \quad \text{and} \quad \lim_{\substack{|h|\to\hbar\\ h\in\mathcal{I}_\ell}} \phi_\ell(h) = 0,$$

$$\mathscr{L}_\ell^\dagger \phi_{a,\ell} - \lambda\phi_{a,\ell} = 0 \quad \text{on } \mathcal{I}_\ell, \quad \lim_{\substack{h\to 0\\ h\in\mathcal{I}_\ell}} \phi_{a,\ell}(h) = 1, \quad \text{and} \quad \lim_{\substack{|h|\to\hbar\\ h\in\mathcal{I}_\ell}} \phi_{a,\ell}(h) = 0,$$

$$\mathscr{L}_\ell^\dagger \phi_{b,\ell} - \lambda\phi_{b,\ell} = 0 \quad \text{on } \mathcal{I}_\ell, \quad \lim_{\substack{h\to 0\\ h\in\mathcal{I}_\ell}} \phi_{b,\ell}(h) = 0, \quad \text{and} \quad \lim_{\substack{|h|\to\hbar\\ h\in\mathcal{I}_\ell}} \phi_{b,\ell}(h) = 1.$$

By standard results (see [Fel55] and [Fel57]), all of these PDE's have solutions; furthermore, by standard smoothness results [Eva98, Theorem 6.3.6], these solutions are infinitely smooth. Also, by standard arguments (see Lemma 1.5 of [Sow03]), all of the $\dot{\phi}_\ell(0)$'s, $\dot{\phi}_{a,\ell}(0)$'s, and $\dot{\phi}_{b,\ell}(0)$'s are all uniquely defined by continuity.

Let's next define some constants. Set $C_{\circledast,\ell} \stackrel{\text{def}}{=} -\frac{1}{\lambda}\varphi_\ell(\hbar)$ for $\ell \in \Lambda_P$ and set $C_{\circledast,\ell} \stackrel{\text{def}}{=} -\frac{1}{\lambda}\varphi_\ell(-\hbar)$ for $\ell \in \Lambda_W$. Next, define $C' \stackrel{\text{def}}{=} 2\lambda\mathscr{H}^2(\mathsf{E}) -$

$\sum_{\ell\in\Lambda} \bar{\mathrm{G}}_\ell \dot{\phi}_{a,\ell}(0)$. We claim that $C' \neq 0$. Indeed, consider the function

$$f_\circ([x]) = \sum_{\ell\in\Lambda} \chi_{\text{Đ}_\ell}(x)\phi_{a,\ell}(\mathsf{H}_{T,\ell}(x)) + \chi_{[\mathsf{E}]}([x]), \qquad x \in \mathbf{S}.$$

If $C' = 0$, then (as one can easily check) $(f_\circ, \lambda f_\circ) \in \mathscr{A}^\dagger$. By dissipativity, $0 = \|\lambda f_\circ - \lambda f_\circ\|_{C(\mathbb{M})} \geq \lambda \|f_\circ\|_{C(\mathbb{M})}$, so in fact $f_\circ \equiv 0$. Since clearly $f_\circ \not\equiv 0$, we must conclude that $C' \neq 0$ We now can define the last constant; set

$$C_0 \stackrel{\text{def}}{=} \frac{\sum_{\ell\in\Lambda} \bar{\mathrm{G}}_\ell \dot{\phi}_\ell(0) + \sum_{\ell\in\Lambda} C_{\circledast,\ell}\bar{\mathrm{G}}_\ell \dot{\phi}_{b,\ell}(0) - 2\varphi([\mathsf{E}])\mathscr{H}^2(\mathsf{E})}{C'}.$$

Finally, define

$$\tilde{f}([x]) = \sum_{\ell\in\Lambda} \chi_{\text{Đ}_\ell}(x)\{\phi_\ell(\mathsf{H}_{T,\ell}(x)) + C_0\phi_{a,\ell}(\mathsf{H}_{T,\ell}(x)) + C_{\circledast,\ell}\phi_{b,\ell}(\mathsf{H}_{T,\ell}(x))\} + C_0\chi_{[\mathsf{E}]}([x])$$

for all $[x] \in \mathbb{M}$, and define $\tilde{g} \stackrel{\text{def}}{=} \lambda\tilde{f} + \varphi$ We claim that (f, g) is in $\mathscr{A}^\dagger$ and is a solution of (27). We can easily check that $\tilde{f} \in C(\mathbb{M})$ and hence $\tilde{g} \in C(\mathbb{M})$. Clearly $\tilde{f} \in C^2(\Gamma_\Lambda)$ and $\mathscr{L}_{\text{ave}}\tilde{f} = \tilde{g}$ on Γ_Λ. Thirdly $\tilde{g}(\circledast_\ell) = \lambda C_{\circledast,\ell} + \varphi_\ell(\hbar) = 0$ if $\ell \in \Lambda_P$ and $\tilde{g}(\circledast_\ell) = \lambda C_{\circledast,\ell} + \varphi_\ell(-\hbar) = 0$ if $\ell \in \Lambda_W$. Finally, we can check that

$$2\mathscr{H}^2(\mathsf{E})\left\{\lambda C_0 + \varphi([\mathsf{E}])\right\} = \sum_{\ell\in\Lambda} \bar{\mathrm{G}}_\ell \dot{\phi}_\ell(0) + C_0 \sum_{\ell\in\Lambda} \bar{\mathrm{G}}_\ell \dot{\phi}_{a,\ell}(0) + \sum_{\ell\in\Lambda} C_{\circledast,\ell}\bar{\mathrm{G}}_\ell \dot{\phi}_{b,\ell}(0),$$

which implies that the glueing conditions hold. □

We have

Proof of Proposition 2.9. Use the above results and [EK86, Theorem 4.2.2]. □

5. Outline of the proof of convergence

We next organize the proof of Theorem 2.10. Intuitively, there are three things that we must understand:

(c.i) When the process is not near $[\mathsf{E}]$, we can use stochastic averaging.
(c.ii) The process "sticks" in $[\mathsf{E}]$ in some quantifiable way.
(c.iii) When the process leaves $[\mathsf{E}]$, it does so according to the stated glueing conditions.

We want to analytically extract these behaviors via arguments which start with Definition 2.4 and lead to (12). The hard work in proving Theorem 2.10 is in constructing various analytical characterizations of (c.i)–(c.iii). This will take up the body of this paper. To organize our reasoning, we will in this section simply state the main technical lemmas we need, and then put them together to prove Theorem 2.10.

Let's now fix once and for all $(f,g)\in\mathscr{A}^\dagger$. Then we have that

$$\left.\begin{aligned} f(\mathfrak{m}(x)) &= \sum_{\ell\in\Lambda} f_\ell(\mathsf{H}_{T,\ell}(x))\chi_{\mathsf{Đ}_\ell}(x) + f([\mathsf{E}])\chi_{\mathsf{E}}(x)\\ g(\mathfrak{m}(x)) &= \sum_{\ell\in\Lambda} g_\ell(\mathsf{H}_{T,\ell}(x))\chi_{\mathsf{Đ}_\ell}(x) + g([\mathsf{E}])\chi_{\mathsf{E}}(x). \end{aligned}\right\} \quad x\in\mathbf{S}$$

By definition of $\mathscr{A}^\dagger$, $f\circ\mathfrak{m}$ is C^2 on $\mathbf{S}\setminus[\mathsf{E}]$.

Remark 5.1. If $f\circ\mathfrak{m}\in C^2(\mathbf{S})$ (which in general will not occur unless the $\dot{f}_\ell(0)$'s are all identically zero and hence $g([\mathsf{E}])=0$), then

$$\mathbb{E}^\varepsilon\left[\left\{f(X^M_t) - f(X^M_s) - \int_{s\wedge\mathfrak{e}}^{t\wedge\mathfrak{e}} (\mathscr{L}^\varepsilon(f\circ\mathfrak{m}))(X_u)du\right\}\prod_{j=1}^n \varphi_j(X_{r_j})\right] = 0.$$

Thus

$$\begin{aligned} &\mathbb{E}^\varepsilon\left[\left\{f(X^M_t) - f(X^M_s) - \int_s^t g(X^M_u)du\right\}\prod_{j=1}^n \varphi_j(X_{r_j})\right]\\ &\qquad = \mathbb{E}^\varepsilon\left[\int_{s\wedge\mathfrak{e}}^{t\wedge\mathfrak{e}} \left\{(\mathscr{L}^\varepsilon(f\circ\mathfrak{m}))(X_u) - g(\mathfrak{m}(X_u))\right\}du\prod_{j=1}^n \varphi_j(X_{r_j})\right] \end{aligned}$$

and we want to show that the right-hand side tends to zero as $\varepsilon\to 0$. Away from $[\mathsf{E}]$, this should be standard stochastic averaging (item (c.i) above). The more interesting behavior is that near $[\mathsf{E}]$. When X^M hits $[\mathsf{E}]$, how long does it (asymptotically) stay there, and how does it (asymptotically) go back into one of the Γ_ℓ's? These behaviors cannot be identified by functions $f\in C(\mathbb{M})$ such that $f\circ\mathfrak{m}$ is smooth (similarly, one cannot distinguish between killed or reflected Brownian motion by looking only at functions which are locally constant near the origin). We need to look at more general $f\in C(\mathbb{M})$; in particular, we need to look at $f\in C(\mathbb{M})$ such that $\mathscr{L}_{\text{ave}}f$ is well-defined and continuous (particularly at $[\mathsf{E}]$) but which may have discontinuities in the first derivatives at $[\mathsf{E}]$ (for the general theory of diffusions on a bounded interval, see [EK86, Chapter 8.1] and [Man68]).

We will use a *perturbed test function* to make the above thoughts work. We will see that the glueing conditions *ensure that we can construct* such a family of perturbed test functions. The crucial result is the following. Define

$$f_{\text{outer}}(x) \overset{\text{def}}{=} f([\mathsf{E}]) + \sum_{\ell\in\Lambda}\dot{f}_\ell(0)\mathsf{H}_{T,\ell}(x)\chi_{\mathsf{Đ}_\ell}(x)$$

for all $x\in\mathbf{S}$. As we will see, this is a good approximation of $f\circ\mathfrak{m}$ near E.

Proposition 5.2 (Corrector Functions). *Assume that* (16) *and* (17) *hold. Then there is a sequence* $(\Psi^\varepsilon_A; \varepsilon \in (0,1))$ *of functions such that for each* $N \in \mathbb{N}$, $\Psi^{\varepsilon_N}_A + f_{\text{outer}}$ *is in* $C^2(\mathbf{S})$ *and such that*

$$\lim_{N\to\infty} \|\Psi^{\varepsilon_N}_A\|_{C(\mathbf{S})} = 0,$$

$$\lim_{N\to\infty} \mathbb{E}^{\varepsilon_N}\left[\left\{\int_{u=s\wedge\mathfrak{e}}^{t\wedge\mathfrak{e}} \{(\mathscr{L}^{\varepsilon_N}\Psi^{\varepsilon_N}_A)(X_u) - g([\mathsf{E}])\chi_{\mathsf{E}}(X_u)\}\,du\right\}^-\right] = 0 \tag{28}$$

for all $0 \le s < t$.

We will construct Ψ^ε_A at the end of Section 9.

Remark 5.3. Neglecting complications, we should roughly have that

$$\begin{aligned} M^{(f,g)}_t &\overset{\text{def}}{=} f(X^M_t) - \int_0^t g(X^M_s)ds \\ &= \Big\{(f\circ\mathfrak{m})(X_{t\wedge\mathfrak{e}}) + \Psi^\varepsilon_A(X_{t\wedge\mathfrak{e}}) \\ &\qquad - \int_0^{t\wedge\mathfrak{e}} (\mathscr{L}^\varepsilon(f\circ\mathfrak{m} + \Psi^\varepsilon_A))(X_u)du\Big\} - \Psi^\varepsilon_A(X_{t\wedge\mathfrak{e}}) \\ &\quad + \int_0^{t\wedge\mathfrak{e}} \{(\mathscr{L}^\varepsilon(f\circ\mathfrak{m}))(X_u) - (g\circ\mathfrak{m})(X_u)\}\,\chi_{\mathbf{S}\setminus\mathsf{E}}(X_u)du \\ &\quad + \int_0^{t\wedge\mathfrak{e}} \{(\mathscr{L}^\varepsilon\Psi^\varepsilon)(X_u) - g([\mathsf{E}])\chi_{\mathsf{E}}(X_u)\}\,du. \end{aligned} \tag{29}$$

Since $f\circ\mathfrak{m}$ is constant on E, $\mathscr{L}^\varepsilon(f\circ\mathfrak{m})$ vanishes there. The martingale problem ensures that the term in braces is a martingale. The first claim of (28) implies that Ψ^ε_A should be asymptotically negligible. Stochastic averaging should show that the third line is also small. Finally, the second claim of (28) should imply that the last line is asymptotically nonnegative. Thus $M^{(f,g)}$ is asymptotically a submartingale. Since $\mathscr{A}^\dagger$ is a vector space, it contains $(-f,-g)$. Thus the exact same arguments show that $M^{(-f,-g)} = -M^{(f,g)}$ is a submartingale, so $M^{(f,g)}$ is asymptotically a martingale (cf. the submartingale problem of [SV71]).

Notice that in contrast to standard applications of perturbed test function theory, where the perturbed test functions primarily help us average, here they also help to interpolate between a smooth state space and a singular one. Notice also that the perturbed test functions need not be unique; they need only *exist*.

To start to make the above thoughts precise, we would like to look at the test function $f\circ\mathfrak{m}+\Psi^\varepsilon_A$. To control the third line in (29), we need to average $\mathscr{L}(f\circ\mathfrak{m})$; to do so, we need $\mathscr{L}(f\circ\mathfrak{m})$ to be twice-differentiable. Since the definition of $\mathscr{A}^\dagger$ does not ensure that these higher derivatives exist, we first need to approximate f (as we did in [Sow02], [Sow03]).

LEMMA 5.4. *There is a collection* $\{f_\ell^\delta;\ \ell\in\Lambda, \delta\in(0,1)\}$ *of functions and a constant* $K>0$ *such that for each* $\ell\in\Lambda$,

(d.i) $f_\ell^\delta \in C^1(\overline{\mathcal{I}_\ell})\cap C^2(\mathcal{I}_\ell)$ *for all* $\delta\in(0,1)$,

(d.ii) $f_\ell^\delta(0)=f([\mathsf{E}])$ *and* $\dot f_\ell^\delta(0)=\dot f_\ell(0)$ *for all* $\delta\in(0,1)$,

(d.iii) $\lim_{\delta\to 0}\|f_\ell^\delta - f_\ell\|_{C(\mathcal{I}_\ell)}=0$,

(d.iv) $\lim_{\delta\to 0}\|\mathscr{L}_\ell^\dagger f_\ell^\delta - g_\ell\|_{C(\mathcal{I}_\ell)}=0$,

and such that for all $\delta\in(0,1)$ *and* $h\in\mathcal{I}_\ell$,

$$
\begin{aligned}
&\left|\frac{df_\ell^\delta}{dh}(h)\right|\le K,\quad \left|\frac{d^2 f_\ell^\delta}{dh^2}(h)\right|\le K\mathfrak{l}(h),\quad \left|\frac{d^3 f_\ell^\delta}{dh^3}(h)\right|\le \frac{K}{\delta|h|},\\
&\qquad\qquad \left|\frac{d^4 f_\ell^\delta}{dh^4}(h)\right|\le \frac{K}{\delta^2|h|^2}.
\end{aligned}
\tag{30}
$$

The proof, which requires some careful but standard mollification arguments, is at the end of this section.

Fix now an exponent $\varkappa\in(0,2/9)$ and define the relaxation parameter $\delta_\varepsilon \overset{\text{def}}{=} \varepsilon^\varkappa$ for all $\varepsilon\in(0,1)$. For ε in $(0,1)$, define now

$$
\begin{aligned}
F^\varepsilon(x) \overset{\text{def}}{=} &\sum_{\ell\in\Lambda} f_\ell^{\delta_\varepsilon}(\mathsf{H}_{T,\ell}(x))\chi_{\text{Đ}_\ell}(x)\mathfrak{c}_\vee\left(\frac{\mathsf{H}_{T,\ell}(x)}{\delta_\varepsilon}\right)\\
&\qquad + f_{\text{outer}}(x)\left\{1-\sum_{\ell\in\Lambda}\chi_{\text{Đ}_\ell}(x)\mathfrak{c}_\vee\left(\frac{\mathsf{H}_{T,\ell}(x)}{\delta_\varepsilon}\right)\right\} + \Psi_A^\varepsilon(x).
\end{aligned}
$$

LEMMA 5.5. *We have that* $\lim_{\varepsilon\to 0}\|f\circ\mathfrak{m} - F^\varepsilon\|_{C(\mathbf{S})}=0$.

Proof. Use the first claim in (28), (d.iii) of Lemma 5.4, and the continuity of f at $[\mathsf{E}]$. □

For all $x\in\mathbf{S}$ and $\varepsilon\in(0,1)$, we then have

$$
(\mathscr{L}^\varepsilon F^\varepsilon)(x) = g([x]) + \sum_{j=1}^{6}\mathscr{E}_{a,i}^\varepsilon(x),
$$

where

$$
\begin{aligned}
\mathscr{E}_{a,1}^\varepsilon(x) &\overset{\text{def}}{=} \sum_{\ell\in\Lambda}\left\{(\mathscr{L}(f_\ell^{\delta_\varepsilon}\circ\mathsf{H}_{T,\ell}))(x) - g_\ell(\mathsf{H}_{T,\ell}(x))\right\}\mathfrak{c}_\vee\left(\frac{\mathsf{H}_{T,\ell}(x)}{\delta_\varepsilon}\right)\chi_{\text{Đ}_\ell}(x),\\
\mathscr{E}_{a,2}^\varepsilon(x) &\overset{\text{def}}{=} (\mathscr{L}^\varepsilon\Psi_A^\varepsilon)(x) - g([\mathsf{E}])\chi_{\mathsf{E}}(x),\\
\mathscr{E}_{a,3}^\varepsilon(x) &\overset{\text{def}}{=} \sum_{\ell\in\Lambda}\{(\mathscr{L}f_{\text{outer}})(x) - g_\ell(\mathsf{H}_{T,\ell}(x))\}\mathfrak{c}_\wedge\left(\frac{\mathsf{H}_{T,\ell}(x)}{\delta_\varepsilon}\right)\chi_{\text{Đ}_\ell}(x),
\end{aligned}
$$

$$\mathscr{E}^\varepsilon_{a,4}(x) \overset{\text{def}}{=} -\sum_{\ell\in\Lambda} \frac{1}{\delta_\varepsilon}\dot{\mathfrak{c}}_\wedge\left(\frac{\mathsf{H}_{T,\ell}(x)}{\delta_\varepsilon}\right)\chi_{\mathsf{Đ}_\ell}(x)\left\{f^{\delta_\varepsilon}_\ell(\mathsf{H}_{T,\ell}(x)) - f_{\text{outer}}(x)\right\}\boldsymbol{\beta}(x),$$

$$\mathscr{E}^\varepsilon_{a,5}(x) \overset{\text{def}}{=} -\sum_{\ell\in\Lambda} \frac{1}{2\delta_\varepsilon^2}\ddot{\mathfrak{c}}_\wedge\left(\frac{\mathsf{H}_{T,\ell}(x)}{\delta_\varepsilon}\right)\chi_{\mathsf{Đ}_\ell}(x)\left\{f^{\delta_\varepsilon}_\ell(\mathsf{H}_{T,\ell}(x)) - f_{\text{outer}}(x)\right\}\boldsymbol{\sigma}(x),$$

$$\mathscr{E}^\varepsilon_{a,6}(x) \overset{\text{def}}{=} -\sum_{\ell\in\Lambda} \frac{1}{\delta_\varepsilon}\dot{\mathfrak{c}}_\wedge\left(\frac{\mathsf{H}_{T,\ell}(x)}{\delta_\varepsilon}\right)\chi_{\mathsf{Đ}_\ell}(x)\left\{\dot{f}^{\delta_\varepsilon}_\ell(\mathsf{H}_{T,\ell}(x))\left\langle d\mathsf{H}_{T,\ell}, d\mathsf{H}_{T,\ell}\right\rangle(x) - \left\langle df_{\text{outer}}, d\mathsf{H}_{T,\ell}\right\rangle(x)\right\}.$$

Let's bound the various error terms. The whole point of Proposition 5.2 was to bound $\mathscr{E}^\varepsilon_{a,2}$. We will take care of $\mathscr{E}^\varepsilon_{a,1}$ below in Lemma 5.11 by averaging. To start to take care of the remaining error terms (which are all concentrated near $\partial\mathsf{E}$), let's first state a residence-time result.

Lemma 5.6. *There is a $K_{5.6} > 0$ such that for all $t > 0$,*

$$\sum_{\ell\in\Lambda}\mathbb{E}^\varepsilon\left[\int_{u=0}^{t\wedge\mathfrak{e}} \mathfrak{c}_\wedge\left(\frac{\mathsf{H}_{T,\ell}(X_u)}{\delta}\right)\boldsymbol{\sigma}(X_u)\chi_{\mathsf{Đ}_\ell}(X_u)du\right] \le K_{5.6}(1+t)\delta,$$

$$\sum_{\ell\in\Lambda}\mathbb{E}^\varepsilon\left[\int_{u=0}^{t\wedge\mathfrak{e}} \mathfrak{c}_\wedge\left(\frac{\mathsf{H}_{T,\ell}(X_u)}{\delta}\right)\sqrt{\boldsymbol{\sigma}(X_u)}\chi_{\mathsf{Đ}_\ell}(X_u)du\right] \le K_{5.6}(1+t)\sqrt{\delta}$$

for all ε and δ in $(0,1)$.

The proof will be given in Section 8.

We can now prove

Lemma 5.7. *We have that*

$$\lim_{\varepsilon\to 0}\mathbb{E}^\varepsilon\left[\int_{u=0}^{t\wedge\mathfrak{e}}\left|\mathscr{E}^\varepsilon_{a,i}(X_u)\right|du\right] = 0$$

for all $i\in\{4,5,6\}$ and $t \ge 0$.

Proof. As in the proof of Lemmas 7.11 and 7.12 in [Sow03], we can find a constant $K > 0$ such that

$$|f^{\delta_\varepsilon}_\ell(\mathsf{H}_{T,\ell}(x)) - f_{\text{outer}}(x)| \le K|\mathsf{H}_{T,\ell}(x)|^2\mathfrak{l}(|\mathsf{H}_{T,\ell}(x)|),$$
$$|\dot{f}^{\delta_\varepsilon}_\ell(\mathsf{H}_{T,\ell}(x)) - \dot{f}_\ell(0)| \le K|\mathsf{H}_{T,\ell}(x)|\mathfrak{l}(|\mathsf{H}_{T,\ell}(x)|)$$

for all $\ell\in\Lambda$, $x\in\mathsf{Đ}_\ell$ and $\varepsilon\in(0,1)$. Recall (22). Thus there is a $K>0$ such that

$$|\mathscr{E}^\varepsilon_{a,4}(x)| \le K\delta_\varepsilon\mathfrak{l}(\varepsilon),$$

$$|\mathscr{E}^\varepsilon_{a,5}(x)| \le K\mathfrak{l}(\varepsilon)\sum_{\ell\in\Lambda}\mathfrak{c}_\wedge\left(\frac{\mathsf{H}_{T,\ell}(x)}{2\delta_\varepsilon}\right)\boldsymbol{\sigma}(x)\chi_{\mathsf{Đ}_\ell}(x),$$

$$|\mathscr{E}^{\varepsilon}_{a,6}(x)| \le K\mathfrak{l}(\varepsilon)\sum_{\ell\in\Lambda}\mathfrak{c}_\wedge\left(\frac{\mathsf{H}_{T,\ell}(x)}{2\delta_\varepsilon}\right)\boldsymbol{\sigma}(x)\chi_{\text{Đ}_\ell}(x)$$

for all ε in $(0,1)$, $\ell \in \Lambda$, and $x \in \text{Đ}_\ell$. We now use Lemma 5.6 to get the stated results. □

To bound $\mathscr{E}^{\varepsilon}_{a,3}$, we need a slightly stronger bound on residence time near $\partial\mathsf{E}$.

LEMMA 5.8. *There is a $K > 0$ such that for all $t > 0$,*

$$\sum_{\ell\in\Lambda}\mathbb{E}^{\varepsilon}\left[\int_{u=0}^{t\wedge\mathfrak{e}}\mathfrak{c}_\wedge\left(\frac{\mathsf{H}_{T,\ell}(X_u)}{\delta}\right)\chi_{\text{Đ}_\ell}(X_u)du\right] \le K(1+t)\left\{\delta + \left\{\varepsilon\mathsf{a}^{(d)}_N + \frac{1}{\varepsilon\mathsf{a}^{(d)}_{N+1}\mathsf{a}^{(d)}_N}\right\}^{2/3}\right\}\mathfrak{l}(\varepsilon)$$

for all ε and δ in $(0,1)$ and all $N \in \mathbb{N}$ such that $\delta > \varepsilon^{2/3}$.

The proof will be given in Section 8.

LEMMA 5.9. *We have that*

$$\lim_{N\to\infty}\mathbb{E}^{\varepsilon_N}\left[\int_{u=0}^{t\wedge\mathfrak{e}}\left|\mathscr{E}^{\varepsilon_N}_{a,3}(X_u)\right|du\right] = 0$$

for all $t \ge 0$.

Proof. $\mathscr{L}f_{\text{outer}}$ and g are bounded. Apply Lemma 5.8. Note that $\delta_\varepsilon > \varepsilon^{2/3}$. Also, note that

$$\lim_{N\to\infty}\varepsilon_N\mathsf{a}^{(d)}_N = \lim_{N\to\infty}\left(\frac{1}{\mathsf{a}^{(d)}_N}\right)^{105/4+\gamma-1} = \lim_{N\to\infty}\left(\frac{1}{\mathsf{a}^{(d)}_N}\right)^{101/4+\gamma} = 0,$$

$$\lim_{N\to\infty}\frac{1}{\varepsilon_N\mathsf{a}^{(d)}_{N+1}\mathsf{a}^{(d)}_N} = \lim_{N\to\infty}\frac{(\mathsf{a}^{(d)}_N)^{105/4+\gamma/2-1}}{\mathsf{a}^{(d)}_{N+1}} = \lim_{N\to\infty}\frac{(\mathsf{a}^{(d)}_N)^{101/4+\gamma/2}}{\mathsf{a}^{(d)}_{N+1}} = 0,$$

where in the second line we have used the fact that $101/4 < 721/14$. □

Next, let's average in the Đ_ℓ's. For $\varphi \in C^2(\mathbf{S}\setminus\mathsf{E})$ and $\lambda \in (0,1)$, define

$$\Phi^{\mathfrak{z},\lambda}_\varphi(x) \stackrel{\text{def}}{=} -\int_{t=0}^{\infty}e^{-\lambda t}\varphi(\mathfrak{z}_t(x))dt, \qquad x \in \mathbf{S}\setminus\mathsf{E}.$$

Then we have

LEMMA 5.10. *Fix $\varphi \in C^\infty(\mathbb{T})$. Then there is a $K > 0$ such that*

$$\left|\text{Ч}\Phi^{\mathfrak{z},\lambda}_\varphi - \{\varphi - \mathcal{A}\varphi\}\right| \le K\lambda\mathfrak{l}(|\mathsf{H}_{T,\ell}|),$$

$$|\Phi_\varphi^{\mathfrak{z},\lambda}| \le \frac{K}{\lambda}, \qquad \|D\Phi_\varphi^{\mathfrak{z},\lambda}\| \le \frac{K}{\lambda^2|\mathsf{H}_{T,\ell}|} \qquad \textit{and} \qquad \|D^2\Phi_\varphi^{\mathfrak{z},\lambda}\| \le \frac{K}{\lambda^3|\mathsf{H}_{T,\ell}|^2}$$

on Đ_ℓ *for all* $\ell \in \Lambda$.

The proof will be given in Section 8.5.

LEMMA 5.11. *We have that*

$$\lim_{\varepsilon\to 0}\mathbb{E}^\varepsilon\left[\left|\int_{u=s\wedge\mathfrak{e}}^{t\wedge\mathfrak{e}}\mathscr{E}^\varepsilon_{a,1}(X_u)du\right|\right] = 0$$

for all $0 \le s < t$.

Proof. For all ε in $(0,1)$ and $x \in \mathbb{T}$, we have that

(31)
$$\mathscr{E}^\varepsilon_{a,1}(x) = \text{I}^\varepsilon(x) + \sum_{\ell\in\Lambda}\left\{(\mathscr{L}^\dagger_\ell f_\ell^{\delta_\varepsilon})(\mathsf{H}_{T,\ell}(x)) - g_\ell(\mathsf{H}_{T,\ell}(x))\right\}\mathfrak{c}_\vee\left(\frac{\mathsf{H}_{T,\ell}(x)}{\delta_\varepsilon}\right)\chi_{\text{Đ}_\ell}(x),$$

where

$$\text{I}^\varepsilon(x) \stackrel{\text{def}}{=} \sum_{\ell\in\Lambda}\left\{(\mathscr{L}(f_\ell^{\delta_\varepsilon}\circ\mathsf{H}_{T,\ell})(x) - (\mathscr{L}^\dagger_\ell f_\ell^{\delta_\varepsilon})(\mathsf{H}_{T,\ell}(x))\right\}\mathfrak{c}_\vee\left(\frac{\mathsf{H}_{T,\ell}(x)}{\delta_\varepsilon}\right)\chi_{\text{Đ}_\ell}(x)$$

for all $x \in \mathbf{S}$. We use (d.iv) of Lemma 5.4 to bound the second term on the right of (31).

We now appeal to Lemma 5.10. For $\varepsilon \in (0,1)$, define

$$\Upsilon^\varepsilon_1(x) \stackrel{\text{def}}{=} \sum_{\ell\in\Lambda}\left\{\dot f_\ell^{\delta_\varepsilon}(\mathsf{H}_{T,\ell}(x))\Phi^{\mathfrak{z},\delta_\varepsilon}_{\boldsymbol{\beta}}(x) + \frac{1}{2}\ddot f_\ell^{\delta_\varepsilon}(\mathsf{H}_{T,\ell}(x))\Phi^{\mathfrak{z},\delta_\varepsilon}_{\boldsymbol{\sigma}}(x)\right\}\chi_{\text{Đ}_\ell}(x)$$

for all $x \in \mathbf{S}$. Combining Lemmas 5.4 and 5.10, we have that for some $K_1 > 0$,

(32)
$$\begin{aligned}&\left|\text{Ч}\Upsilon^\varepsilon_1 - \left\{\mathscr{L}(f_\ell^{\delta_\varepsilon}\circ\mathsf{H}_{T,\ell}) - (\mathscr{L}^\dagger_\ell f_\ell^{\delta_\varepsilon})\circ\mathsf{H}_{T,\ell}\right\}\right| \le K_1\delta_\varepsilon\mathfrak{l}(|\mathsf{H}_{T,\ell}|),\\ &|\Upsilon^\varepsilon_1| \le \frac{K_1\mathfrak{l}(\varepsilon)}{\delta_\varepsilon}, \quad \|D\Upsilon^\varepsilon_1\| \le \frac{K_1}{\delta_\varepsilon^3|\mathsf{H}_{T,\ell}|^2}, \quad \|D^2\Upsilon^\varepsilon_1\| \le \frac{K_1}{\delta_\varepsilon^5|\mathsf{H}_{T,\ell}|^4}\end{aligned}$$

on Đ_ℓ for all $\ell \in \Lambda$ and all $\varepsilon \in (0,1)$. For all $\varepsilon \in (0,1)$, define now

$$\Upsilon^\varepsilon_2(x) \stackrel{\text{def}}{=} \varepsilon^2\sum_{\ell\in\Lambda}\Upsilon^\varepsilon_1(x)\mathfrak{c}_\vee\left(\frac{\mathsf{H}_{T,\ell}(x)}{\delta_\varepsilon}\right)\chi_{\text{Đ}_\ell}(x)$$

for all $x \in \mathbf{S}$. Thus for all $x \in \mathbf{S}$ and $\varepsilon \in (0,1)$,

$$(\mathscr{L}^\varepsilon\Upsilon^\varepsilon_2)(x) = \text{I}^\varepsilon(x) + \sum_{i=1}^5 \text{E}^\varepsilon_{\ell,i}(x)\chi_{\text{Đ}_\ell}(x),$$

where for all $\ell \in \Lambda$ and $x \in Đ_\ell$,

$$\begin{aligned}
\mathrm{E}^\varepsilon_{\ell,1}(x) &= \sum_{\ell \in \Lambda} \{(\text{Ч}\Upsilon^\varepsilon_1)(x) \\
&\qquad - \left\{(\mathscr{L}(f^{\delta_\varepsilon}_\ell \circ \mathsf{H}_{T,\ell}))(x) - (\mathscr{L}^\dagger_\ell f^{\delta_\varepsilon}_\ell)(\mathsf{H}_{T,\ell}(x))\right\}\right\} \mathfrak{c}_\vee \left(\frac{\mathsf{H}_{T,\ell}(x)}{\delta_\varepsilon}\right), \\
\mathrm{E}^\varepsilon_{\ell,2}(x) &= -\frac{\varepsilon^2}{\delta_\varepsilon} \sum_{\ell \in \Lambda} \Upsilon^\varepsilon_1(x) \dot{\mathfrak{c}}_\wedge \left(\frac{\mathsf{H}_{T,\ell}(x)}{\delta_\varepsilon}\right) \boldsymbol{\beta}(x), \\
\mathrm{E}^\varepsilon_{\ell,3}(x) &= -\frac{\varepsilon^2}{2\delta^2_\varepsilon} \sum_{\ell \in \Lambda} \Upsilon^\varepsilon_1(x) \ddot{\mathfrak{c}}_\wedge \left(\frac{\mathsf{H}_{T,\ell}(x)}{\delta_\varepsilon}\right) \boldsymbol{\sigma}(x), \\
\mathrm{E}^\varepsilon_{\ell,4}(x) &= \varepsilon^2 \sum_{\ell \in \Lambda} (\mathscr{L}\Upsilon^\varepsilon_1)(x) \mathfrak{c}_\vee \left(\frac{\mathsf{H}_{T,\ell}(x)}{\delta_\varepsilon}\right), \\
\mathrm{E}^\varepsilon_{\ell,5}(x) &= -\frac{\varepsilon^2}{\delta_\varepsilon} \sum_{\ell \in \Lambda} \langle d\Upsilon^\varepsilon_1, d\mathsf{H}_{T,\ell}\rangle (x) \dot{\mathfrak{c}}_\wedge \left(\frac{\mathsf{H}_{T,\ell}(x)}{\delta_\varepsilon}\right).
\end{aligned}$$

We now return to (32). We see that there is a $K_2 > 0$ such that

$$\begin{gathered}
|\mathrm{E}^\varepsilon_{\ell,1}(x)| \le K_2 \delta_\varepsilon \mathfrak{l}(\varepsilon), \qquad |\mathrm{E}^\varepsilon_{\ell,2}(x)| \le K_2 \frac{\varepsilon^2 \mathfrak{l}(\varepsilon)}{\delta^2_\varepsilon}, \qquad |\mathrm{E}^\varepsilon_{\ell,3}(x)| \le K_2 \frac{\varepsilon^2 \mathfrak{l}(\varepsilon)}{\delta^3_\varepsilon}, \\
|\mathrm{E}^\varepsilon_{\ell,4}(x)| \le K_2 \frac{\varepsilon^2}{\delta^9_\varepsilon}, \qquad |\mathrm{E}^\varepsilon_{\ell,5}(x)| \le K_2 \frac{\varepsilon^2}{\delta^6_\varepsilon}, \qquad \text{and} \qquad |\Upsilon^\varepsilon_2(x)| \le K_2 \frac{\varepsilon^2 \mathfrak{l}(\varepsilon)}{\delta_\varepsilon}
\end{gathered}$$

for all $x \in Đ_\ell$, $\ell \in \Lambda$, and $\varepsilon \in (0,1)$. Thus there is a $K_3 > 0$ such that for all $\varepsilon \in (0,1)$,

$$\begin{aligned}
&\mathbb{E}^\varepsilon \left[\left| \int_{s=0}^{t \wedge \mathfrak{e}} \mathrm{I}^\varepsilon(X_s) ds \right| \right] \\
&\qquad \le K_3(1+t) \left\{ \frac{\varepsilon^2 \mathfrak{l}(\varepsilon)}{\delta_\varepsilon} + \delta_\varepsilon \mathfrak{l}(\varepsilon) + \frac{\varepsilon^2 \mathfrak{l}(\varepsilon)}{\delta^2_\varepsilon} + \frac{\varepsilon^2 \mathfrak{l}(\varepsilon)}{\delta^3_\varepsilon} + \frac{\varepsilon^2}{\delta^9_\varepsilon} + \frac{\varepsilon^2}{\delta^6_\varepsilon} \right\} \\
&\qquad\qquad \le 6K_3(1+t) \left\{ \delta_\varepsilon + \frac{\varepsilon^2}{\delta^9_\varepsilon} \right\} \mathfrak{l}(\varepsilon).
\end{aligned}$$

The requirement that $\varkappa < 2/9$ allows us to complete the proof. □

We finally collect things together.

Proof of Theorem 2.10. Fix $(f,g) \in \mathscr{A}^\dagger$, $0 \le r_1 < r_2 \cdots < r_n \le s < t$ and $\{\varphi_j;\, j = 1, 2 \dots n\} \subset C_b(\mathbb{M})$ such that $\|\varphi_j\|_{C(\mathbb{M})} \le 1$ for all $j \in \{1, 2, \dots n\}$.

Then

$$\mathbb{E}^\varepsilon\left[\left\{f(X^M_t)-f(X^M_s)-\int_s^t g(X^M_u)du\right\}\left(\prod_{j=1}^n \varphi_j(X^M_{r_j})\right)^+\right]$$
$$\geq \mathbb{E}^\varepsilon\left[\int_{u=s\wedge\mathfrak{e}}^{t\wedge\mathfrak{e}}\{(\mathscr{L}^\varepsilon F^\varepsilon)(X_u)-g([X_u])\}\,du\left(\prod_{j=1}^n \varphi_j(X^M_{r_j})\right)^+\right]$$
$$-2\|F^\varepsilon - f\circ\mathfrak{m}\|_{C(\mathbb{M})}$$
$$\geq \mathbb{E}^\varepsilon\left[\left\{\int_{u=s\wedge\mathfrak{e}}^{t\wedge\mathfrak{e}}\{(\mathscr{L}^\varepsilon F^\varepsilon)(X_u)-g([X_u])\}\,du\right\}^-\left(\prod_{j=1}^n \varphi_j(X^M_{r_j})\right)^+\right]$$
$$-2\|F^\varepsilon - f\circ\mathfrak{m}\|_{C(\mathbb{M})}$$
$$\geq \mathbb{E}^\varepsilon\left[\left\{\int_{u=s\wedge\mathfrak{e}}^{t\wedge\mathfrak{e}}\{(\mathscr{L}^\varepsilon F^\varepsilon)(X_u)-g([X_u])\}\,du\right\}^-\right]-2\|F^\varepsilon - f\circ\mathfrak{m}\|_{C(\mathbb{M})}.$$

Let $N\to\infty$ and use Lemmas 5.5, 5.7, 5.9, and 5.11 and the second claim of (28). We get that

(33)
$$\varliminf_{N\to\infty}\mathbb{E}^{\varepsilon_N}\left[\left\{f(X^M_t)-f(X^M_s)-\int_s^t g(X^M_u)du\right\}\left(\prod_{j=1}^n \varphi_j(X^M_{r_j})\right)^+\right]\geq 0.$$

Since $\mathscr{A}^\dagger$ is a vector space, $(-f,-g)\in\mathscr{A}^\dagger$, so the fact that (33) holds for all $(f,g)\in\mathscr{A}^\dagger$ implies that

$$\varlimsup_{N\to\infty}\mathbb{E}^{\varepsilon_N}\left[\left\{f(X^M_t)-f(X^M_s)-\int_s^t g(X^M_u)du\right\}\left(\prod_{j=1}^n \varphi_j(X^M_{r_j})\right)^+\right]\leq 0$$

for all $(f,g)\in\mathscr{A}^\dagger$. The combination of this and (33) implies that

$$\lim_{N\to\infty}\mathbb{E}^{\varepsilon_N}\left[\left\{f(X^M_t)-f(X^M_s)-\int_s^t g(X^M_u)du\right\}\left(\prod_{j=1}^n \varphi_j(X^M_{r_j})\right)^+\right]= 0.$$

This clearly implies the stated claim. □

5.1. Proof of Lemma 5.4. The proof heavily uses the arguments of Lemmas A.6, A.7, and A.8 of [Sow03].

Proof of Lemma 5.4. The proof is similar to that of Lemma 7.6 of [Sow03]. Fix $\ell\in\Lambda$. Define

$$\tilde{\boldsymbol{\beta}}_{M,\ell}(h)=\int_{\substack{z\in\text{Đ}_\ell\\ \mathsf{H}_{T,\ell}(z)=h}}\frac{\boldsymbol{\beta}(z)}{\|\bar{\nabla}\mathsf{H}_{T,\ell}(z)\|}\mathscr{H}^1(dz),$$

$$\tilde{\boldsymbol{\sigma}}_{M,\ell}(h) = \int_{\substack{z\in \mathsf{D}_\ell\\ \mathsf{H}_{T,\ell}(z)=h}} \frac{\boldsymbol{\sigma}(z)}{\|\bar\nabla \mathsf{H}_{T,\ell}(z)\|}\mathscr{H}^1(dz),$$
$$\tilde{\mathcal{T}}_\ell(h) = \int_{\substack{z\in \mathsf{D}_\ell\\ \mathsf{H}_{T,\ell}(z)=h}} \frac{1}{\|\bar\nabla \mathsf{H}_{T,\ell}(z)\|}\mathscr{H}^1(dz)$$

for all $h\in\mathcal{I}_\ell$; then $\boldsymbol{\beta}_{M,\ell}(h) = \tilde{\boldsymbol{\beta}}_{M,\ell}(h)/\tilde{\mathcal{T}}_\ell(h)$ and $\boldsymbol{\sigma}_{M,\ell}(h) = \tilde{\boldsymbol{\sigma}}_{M,\ell}(h)/\tilde{\mathcal{T}}_\ell(h)$ for all $h\in\mathcal{I}_\ell$.

Let's next get some bounds on $\tilde{\boldsymbol{\sigma}}_{M,\ell}$, $\tilde{\boldsymbol{\beta}}_{M,\ell}$, $\tilde{\mathcal{T}}_\ell$, and their first two derivatives. Lemma 5.1 of [Sow03] gives bounds on $\tilde{\mathcal{T}}_\ell$ from above and below and on the size of the first two derivatives of $\tilde{\mathcal{T}}_\ell$. Note that $\boldsymbol{\sigma}/\|\bar\nabla \mathsf{H}_{T,\ell}\|$ is bounded and positive on $\mathbb{T}\setminus\mathfrak{X}$. Arguing as in the proofs of Lemmas A.8 and A.9 in [Sow03], we can get bounds on $\tilde{\boldsymbol{\sigma}}_{M,\ell}$ from above and below. Clearly $|\tilde{\boldsymbol{\beta}}_{M,\ell}(h)| \le \|\boldsymbol{\beta}\|_{C(\mathbb{T})}\tilde{\mathcal{T}}_\ell(h)$ for all $h\in\mathcal{I}_\ell$, giving us a bound on the size of $\tilde{\boldsymbol{\beta}}_{M,\ell}$. Directly using Lemma A.7 of [Sow03], we can bound the size of the first two derivatives of $\tilde{\boldsymbol{\beta}}_{M,\ell}$ and $\tilde{\boldsymbol{\sigma}}_{M,\ell}$. Collecting all of this together, we can find a $K_1>0$ such that

$$\frac{1}{K_1}\le \tilde{\boldsymbol{\sigma}}_{M,\ell}(h)\le K_1, \quad \left|\frac{d\tilde{\boldsymbol{\sigma}}_{M,\ell}}{dh}(h)\right| \le K_1\mathfrak{l}(h), \quad \left|\frac{d^2\tilde{\boldsymbol{\sigma}}_{M,\ell}}{dh}(h)\right| \le \frac{K_1}{|h|},$$
$$|\tilde{\boldsymbol{\beta}}_{M,\ell}(h)|\le K_1\mathfrak{l}(h), \quad \left|\frac{d\tilde{\boldsymbol{\beta}}_{M,\ell}}{dh}(h)\right| \le \frac{K_1}{|h|}, \quad \left|\frac{d^2\tilde{\boldsymbol{\beta}}_{M,\ell}}{dh}(h)\right| \le \frac{K_1}{|h|^2},$$
$$\frac{\mathfrak{l}(h)}{K_1}\le \tilde{\mathcal{T}}_\ell(h)\le K_1\mathfrak{l}(h), \quad \left|\frac{d\tilde{\mathcal{T}}_\ell}{dh}(h)\right| \le \frac{K_1}{|h|}, \quad \left|\frac{d^2\tilde{\mathcal{T}}_\ell}{dh}(h)\right| \le \frac{K_1}{|h|^2}$$

for all $h\in\mathcal{I}_\ell$.

Combining things and using Lemma A.8 of [Sow03], we can find a $K_2>0$ such that

$$\frac{1}{K_2\mathfrak{l}(h)}\le \boldsymbol{\sigma}_{M,\ell}(h)\le \frac{K_2}{\mathfrak{l}(h)}, \quad \left|\frac{d\boldsymbol{\sigma}_{M,\ell}}{dh}(h)\right| \le \frac{K_2}{\mathfrak{l}^2(h)|h|},$$
$$\left|\frac{d^2\boldsymbol{\sigma}_{M,\ell}}{dh^2}(h)\right| \le \frac{K_2}{\mathfrak{l}^2(h)|h|^2}$$

for all $h\in\mathcal{I}_\ell$. By calculating as in the proof of Lemma A.8, we can find a constant $K_3>0$ such that

$$|\boldsymbol{\beta}_{M,\ell}(h)|\le K_3, \qquad \left|\frac{d\boldsymbol{\beta}_{M,\ell}}{dh}(h)\right| \le K_3\left\{\frac{1}{\mathfrak{l}(h)|h|}+\frac{\mathfrak{l}(h)}{\mathfrak{l}^2(h)|h|}\right\},$$
$$\left|\frac{d^2\boldsymbol{\beta}_{M,\ell}}{dh^2}(h)\right| \le K_3\left\{\frac{1}{\mathfrak{l}(h)|h|^2}+\frac{1}{\mathfrak{l}^2(h)|h|^2}+\frac{\mathfrak{l}(h)}{\mathfrak{l}^3(h)|h|^2}+\frac{\mathfrak{l}(h)}{\mathfrak{l}^2(h)|h|^2}\right\}$$

for all $h \in \mathcal{I}_\ell$; thus there is a $K_4 > 0$ such that

$$|\boldsymbol{\beta}_{M,\ell}(h)| \le K_4, \quad \left|\frac{d\boldsymbol{\beta}_{M,\ell}}{dh}(h)\right| \le \frac{K_4}{\mathfrak{l}(h)|h|}, \quad \text{and} \quad \left|\frac{d^2\boldsymbol{\beta}_{M,\ell}}{dh^2}(h)\right| \le \frac{K_4}{\mathfrak{l}(h)|h|^2}.$$

We finally define f^δ_ℓ as in the proof of Lemma 7.6 of [Sow03]. We can then find a constant $K_5 > 0$ such that

$$|f^\delta_\ell(h)| \le K_5, \qquad |f^{\delta,(1)}_\ell(h)| \le K_5, \qquad \text{and} \qquad |f^{\delta,(2)}_\ell(h)| \le K_5 \mathfrak{l}(h) \tag{34}$$

for all $h \in \mathcal{I}_\ell$. There is then a constant $K_6 > 0$ such that

$$|f^{\delta,(3)}_\ell(h)| \le K_6 \mathfrak{l}(h) \left\{ \frac{1}{\delta} + \frac{1}{\mathfrak{l}(h)|h|} + \left\{ 1 + \frac{1}{\mathfrak{l}^2(h)|h|} \right\} \mathfrak{l}(h) \right\},$$

and thus a constant $K_7 > 0$ such that

$$|f^{\delta,(3)}_\ell(h)| \le \frac{K_7}{\delta|h|} \tag{35}$$

for all $h \in \mathcal{I}_\ell$. From this, we can find a $K_8 > 0$ such that

$$\begin{aligned}|f^{\delta,(4)}_\ell(h)| \le K_8 \mathfrak{l}(h) \left\{ \frac{1}{\delta^2} + \frac{1}{\mathfrak{l}(h)|h|^2} + \left\{ \frac{1}{\mathfrak{l}(h)|h|} + \frac{1}{\mathfrak{l}^2(h)|h|^2} \right\} \mathfrak{l}(h) \right. \\ \left. + \left\{ 1 + \frac{1}{\mathfrak{l}^2(h)|h|} \right\} \frac{1}{\delta|h|} \right\},\end{aligned}$$

and hence there is a $K_9 > 0$ such that

$$|f^{\delta,(4)}_\ell(h)| \le \frac{K_9}{\delta^2|h|^2}$$

for all $h \in \mathcal{I}_\ell$. Combine this, (34) and (35) to get (30). □

6. Relaxation of the Hamiltonian

We now construct a sequence of approximate Hamiltonians on $\mathbb{R}^2$. The salient features of these will be that they generate a flow on $\mathbb{T}$ which (i) agrees with Ч on the $\mathsf{Đ}_\ell$'s, for which (ii) all of the $\mathfrak{x}_\ell$'s are on the same heteroclinic cycle, and (iii) the flow is periodic on E except on this heteroclinic cycle.

First, recall Lemma 3.3. For each $\ell \in \Lambda$, let $\mathcal{D}^e_\ell$ be the connected component of $\mathfrak{t}^{-1}(\mathcal{D}_\ell)$ which contains $\mathfrak{x}^e_\ell$. Then $\mathfrak{t}\big|_{\mathcal{D}^e_\ell}$ is a diffeomorphism; we let $\check{\mathfrak{t}}_\ell$ be its inverse. For each $\ell \in \Lambda$, we can find an open subset $\mathcal{D}'_\ell$ of $\mathbb{T}$ such that $\mathsf{D}_\ell \subset\subset \mathcal{D}'_\ell \subset\subset \mathcal{D}_\ell$. Define $\mathsf{D}^e_\ell \stackrel{\text{def}}{=} \check{\mathfrak{t}}(\mathsf{D}_\ell)$ and $\mathcal{D}'^{,e}_\ell \stackrel{\text{def}}{=} \check{\mathfrak{t}}(\mathcal{D}'_\ell)$ for all $\ell \in \Lambda$. For each $\ell \in \Lambda$, let $\varpi_\ell \in C^\infty(\mathbb{R}^2; [0,1])$ be such that $\varpi_\ell = 1$ on $\mathcal{D}'^{,e}_\ell$ and $\varpi_\ell = 0$ on $\mathbb{R}^2 \setminus \mathcal{D}^e_\ell$.

Next, fix $\ell \in \Lambda$. Since ϱ is irrational, there are integers $J^{(1)}_{N,\ell}$ and $J^{(2)}_{N,\ell}$ such that

$$\left| J^{(1)}_{N,\ell}\varrho + J^{(2)}_{N,\ell} - \frac{\mathsf{H}(\mathfrak{x}^e_\ell)}{\omega_2} \right| \le |\boldsymbol{\nu}_N|^2.$$

Define now $\mathbf{J}_{N,\ell} \stackrel{\text{def}}{=} \left(J^{(1)}_{N,\ell}, J^{(2)}_{N,\ell}\right)$ and $\text{H}_{N,\ell} \stackrel{\text{def}}{=} \mathsf{H}\left(\mathfrak{x}^e_\ell - \mathbf{J}_{N,\ell}\right)$. We note that thus

$$\text{H}_{N,\ell} = \mathsf{H}(\mathfrak{x}^e_\ell) - J^{(1)}_{N,\ell}\omega_1 - J^{(2)}_{N,\ell}\omega_2 = \omega_2\left\{\frac{\mathsf{H}(\mathfrak{x}^e_\ell)}{\omega_2} - J^{(1)}_{N,\ell}\varrho - J^{(2)}_{N,\ell}\right\}$$

so

$$|\text{H}_{N,\ell}| \le |\omega_2||\boldsymbol{\nu}_N|^2. \tag{36}$$

Defining $\mathbf{e}_1 \stackrel{\text{def}}{=} (1,0) \in \mathbb{R}^2$, set

$$\hat{\mathsf{H}}_N(x) \stackrel{\text{def}}{=} \omega_2\left\{-\langle x, \mathbf{e}_1\rangle_{\mathbb{R}^2} + \sum_{\substack{K\in\mathbb{Z}^2\\ \ell\in\Lambda}} \varpi_\ell(x-K)\langle x-K-\mathbf{J}_{N,\ell}, \mathbf{e}_1\rangle_{\mathbb{R}^2}\right\} - \sum_{\substack{K\in\mathbb{Z}^2\\ \ell\in\Lambda}} \varpi_\ell(x-K)\frac{\text{H}_{N,\ell}}{\boldsymbol{\nu}_N}, \qquad x \in \mathbb{R}^2.$$

We then define a perturbed Hamiltonian and a perturbed frequency vector; set $\mathsf{H}_N(x) \stackrel{\text{def}}{=} \mathsf{H}(x) + \boldsymbol{\nu}_N\hat{\mathsf{H}}_N(x)$ for all $x \in \mathbb{R}^2$ and set $\boldsymbol{\omega}_N \stackrel{\text{def}}{=} \boldsymbol{\omega} - \omega_2\mathbf{e}_1\boldsymbol{\nu}_N = (\omega_2\varrho_N, \omega_2)$.

LEMMA 6.1. *Fix a positive integer N. For any $x \in \mathbb{R}^2$ and $K \in \mathbb{Z}^2$, $\mathsf{H}_N(x+K) = \mathsf{H}_N(x) + \langle\boldsymbol{\omega}_N, K\rangle_{\mathbb{R}^2}$. For any $\ell \in \Lambda$ and any $K \in \mathbb{Z}^2$, $\mathsf{H}_N \equiv \mathsf{H} - H_{N,\ell} - \omega_2\boldsymbol{\nu}_N\langle\mathbf{e}_1, \mathbf{J}_{N,\ell}+K\rangle_{\mathbb{R}^2}$ on $\mathsf{D}^{e,\varsigma}_\ell + K$. In particular, $\mathsf{H}_N(\mathfrak{x}^e_\ell - \mathbf{J}_{N,\ell}) = 0$.*

The proof will be given at the end of the section.

Analogously to (1), we now define the vector fields $\mathsf{Ч}_{e,N} \stackrel{\text{def}}{=} \bar{\nabla}_e\mathsf{H}_N$ and $\hat{\mathsf{Ч}}_{e,N} \stackrel{\text{def}}{=} \bar{\nabla}_e\hat{\mathsf{H}}_N$, and (similarly to (3)) we then define vector fields $\mathsf{Ч}_N$ and $\hat{\mathsf{Ч}}_N$ on $\mathbb{T}$ by requiring that $(\mathsf{Ч}_N\varphi)(\mathfrak{t}(x)) = (\mathsf{Ч}_{e,N}(\varphi\circ\mathfrak{t}))(x)$ and $(\hat{\mathsf{Ч}}_N\varphi)(\mathfrak{t}(x)) = (\hat{\mathsf{Ч}}_{e,N}(\varphi\circ\mathfrak{t}))(x)$ for all $\varphi \in C^\infty(\mathbb{T})$ and $x \in \mathbb{R}^2$. Then

$$\mathsf{Ч}_N = \mathsf{Ч} + \boldsymbol{\nu}_N\hat{\mathsf{Ч}}_N \tag{37}$$

for all $N \in \mathbb{N}$.

LEMMA 6.2. *For each $\ell \in \Lambda$ and $N \in \mathbb{N}$, $\mathsf{Ч}_N = \mathsf{Ч}$ on $\bigcup_{\ell\in\Lambda}\mathsf{D}_\ell$. Furthermore, $\sup_{N\in\mathbb{N},x\in\mathbb{T}}\|\hat{\mathsf{Ч}}_N(x)\| < \infty$. Finally, for N large enough, $\{x \in \mathbf{S} : \mathsf{Ч}_N(x) = 0\} = \{x \in \mathbf{S} : \mathsf{Ч}(x) = 0\}$.*

The proof will be given at the end of the section.

Let $\mathfrak{z}^N$ be the flow of diffeomorphisms of $\mathbb{T}$ defined by

$$\begin{aligned} \dot{\mathfrak{z}}^N_t(x) &\stackrel{\text{def}}{=} \mathsf{Ч}_N(\mathfrak{z}^N_t(x)) \quad t \ge 0 \\ \mathfrak{z}^N_0(x) &= x. \end{aligned} \qquad x \in \mathbb{T}$$

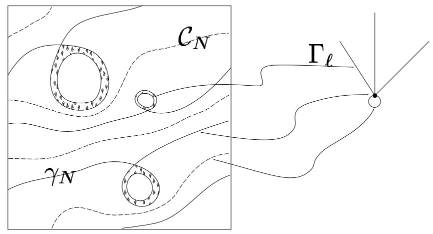


FIGURE 4. γ_N and $\mathcal{C}_N$

By Lemma 6.2, $\mathfrak{z}^N_t = \mathfrak{z}_t$ on $\bigcup_{\ell\in\Lambda} \mathsf{D}_\ell$ for all $t \in \mathbb{R}$. Let now $\sim_N$ denote chain equivalence with respect to $\mathfrak{z}^N$, and let $[x]_N \overset{\text{def}}{=} \{x' \in \mathbb{T} : x' \sim_N x\}$ for all $x \in \mathbb{T}$. Since $\mathfrak{z}^N = \mathfrak{z}$ in the $\mathsf{Đ}_\ell$'s and $\mathfrak{z}$ is periodic in $\mathsf{Đ}_\ell$, all points in all of the $\mathsf{Đ}_\ell$'s are chain-recurrent and $\{[x]_N : x \in \mathsf{Đ}_\ell\} \simeq (0, \hbar_\ell]$ for $\ell \in \Lambda_P$ and $\{[x]_N : x \in \mathsf{Đ}_\ell\} \simeq [-\hbar_\ell, 0)$ for $\ell \in \Lambda_W$. We next claim that $\{[x]_N; x \in \mathsf{E}\}$ is a *circle*. Define

$$\boxed{\boxed{\mathsf{r}_N \overset{\text{def}}{=} \frac{\omega_2}{\mathsf{a}^{(d)}_N}}} \tag{38}$$

for all $N \in \mathbb{N}$. For all $K = (k_1, k_2) \in \mathbb{Z}^2$,

$$\begin{aligned}\mathsf{H}_N(\mathfrak{x}^e_\ell + K) &= \mathsf{H}_N(\mathfrak{x}^e_\ell - \mathbf{J}_{N,\ell}) + \omega_2 \varrho_N (J^{(1)}_{N,\ell} + k_1) + \omega_2 (J^{(2)}_{N,\ell} + k_2) \\ &= \mathsf{r}_N \left\{ \mathsf{a}^{(n)}_N (J^{(1)}_{N,\ell} + k_1) + \mathsf{a}^{(d)}_N (J^{(2)}_{N,\ell} + k_2) \right\};\end{aligned}$$

hence

$$\mathsf{H}_N(\mathfrak{x}^e_\ell + K) \in \mathsf{r}_N \mathbb{Z}. \tag{39}$$

For each $N \in \mathbb{N}$, next define the codimension-one sets

$$\begin{aligned}\gamma_N &\overset{\text{def}}{=} \mathfrak{t}\left\{x \in \mathfrak{t}^{-1}(\overline{\mathsf{E}}) : \mathsf{H}_N(x) \in \mathbb{Z}\mathsf{r}_N\right\} \\ \mathcal{C}_N &\overset{\text{def}}{=} \mathfrak{t}\left\{x \in \mathfrak{t}^{-1}(\mathsf{E}) : \mathsf{H}_N(x) \in \left(\mathbb{Z} + \frac{1}{2}\right)\mathsf{r}_N\right\};\end{aligned}$$

see Figure 4.

LEMMA 6.3. *For large enough $N \in \mathbb{N}$, $\gamma_N \cap (\mathsf{E} \cup \mathfrak{X})$ and γ_N are path-connected and all orbits of $\mathfrak{z}^N$ in $\mathsf{E} \setminus \gamma_N$ are periodic.*

The proof will be given at the end of the section. Thus γ_N is the unique heteroclinic cycle of $\mathfrak{z}^N$ in $\mathbf{S}$, and furthermore all points in E are chain-recurrent under $\mathfrak{z}^N$.

We next claim that there is a local Hamiltonian in $\mathbf{S} \setminus \mathcal{C}_N$.

LEMMA 6.4. *For each $N \in \mathbb{N}$, there is an $\mathsf{H}_N^{\mathrm{loc}} \in B(\mathbf{S})$ such that $\mathsf{H}_N^{\mathrm{loc}}$ is C^∞ on $\mathbf{S}^\circ \setminus \mathcal{C}_N$,*

$$\begin{aligned} \mathsf{H}_N^{\mathrm{loc}}(\mathfrak{t}(x)) &= \mathsf{H}_N(x) - \left\lfloor \frac{\mathsf{H}_N(x)}{\mathsf{r}_N} + \frac{1}{2} \right\rfloor \mathsf{r}_N, \qquad x \in \mathfrak{t}^{-1}(\overline{\mathsf{E}}), \\ \mathsf{H}_N^{\mathrm{loc}}(x) &= \mathsf{H}_{T,\ell}(x), \qquad x \in Đ_\ell,\ \ell \in \Lambda. \end{aligned} \tag{40}$$

This tells us that $\{[x]_N;\ x \in \mathsf{E}\}$ is the circle of circumference r_N. Since γ_N contains the $\partial Đ_\ell$'s, which are limit points of the $Đ_\ell$'s, we have that $\{[x]_N;\ x \in \mathbf{S}\}$ is a *whiskered circle*, where one whisker is attached for each trap. Define $\mathcal{W} \overset{\mathrm{def}}{=} (-1/2, 1/2)$; we thus have that $\mathsf{H}_N^{\mathrm{loc}} : \mathsf{E} \setminus \mathcal{C}_N = \mathsf{r}_N \mathcal{W}$.

REMARK 6.5. By removing $\mathcal{C}_N$, we have approximated E by a long and thin ribbon. More precisely, the width of $\mathsf{E} \setminus \mathcal{C}_N$, as parametrized by $\mathsf{H}_N^{\mathrm{loc}}$, is r_N (the width of $\mathsf{r}_N\mathcal{W}$). Since the area of $\mathsf{E} \setminus \mathcal{C}_N$ is equal to that of E (we have removed only the one-dimensional manifold $\mathcal{C}_N$), the "length" of the ribbon should be of order $1/\mathsf{r}_N$. This can be made more precise by using the change-of-variables formula; the proof of Lemma 7.1 gives a related argument.

We note that $\mathsf{H}_N^{\mathrm{loc}}$ has a very nice form near the $\mathfrak{x}_\ell$'s.

LEMMA 6.6. *For each $N \in \mathbb{N}$,*

$$\mathsf{H}_N^{\mathrm{loc}}(x) = \tilde{\mathsf{H}}(\phi_\ell(x)) - \left\lfloor \frac{\tilde{\mathsf{H}}(\phi_\ell(x))}{\mathsf{r}_N} + \frac{1}{2} \right\rfloor \mathsf{r}_N$$

if $x \in \mathcal{U}_\ell \cap \mathsf{E}$ and $\mathsf{H}_N^{\mathrm{loc}}(x) = \tilde{\mathsf{H}}(\phi_\ell(x))$ if $x \in \mathcal{U}_\ell \cap Đ_\ell$.

We will give the proof at the end of the section.

6.1. Omitted proofs.

Proof of Lemma 6.1. The function $\hat{\mathsf{H}}_N$ consists of a linear part and a periodic part. Via this decomposition, we have that $\hat{\mathsf{H}}_N(x+K) = \hat{\mathsf{H}}_N(x) - \omega_2 \langle K, \mathbf{e}_1\rangle_{\mathbb{R}^2}$ for all $x \in \mathbb{R}^2$ and $K \in \mathbb{Z}^2$. Thus $\mathsf{H}_N(x+K) - \mathsf{H}_N(x) = \langle \boldsymbol{\omega}, K\rangle_{\mathbb{R}^2} - \boldsymbol{\nu}_N \omega_2 \langle \mathbf{e}_1, K\rangle_{\mathbb{R}^2} = \langle \boldsymbol{\omega}_N, K\rangle_{\mathbb{R}^2}$ for all $x \in \mathbb{R}^2$ and $K \in \mathbb{Z}^2$, giving us the first claim.

If $x \in \mathsf{D}_\ell^{e,\varsigma}$, then

$$\hat{\mathsf{H}}_N(x) = \omega_2 \left\{ -\langle x, \mathbf{e}_1\rangle_{\mathbb{R}^2} + \langle x - \mathbf{J}_{N,\ell}, \mathbf{e}_1\rangle_{\mathbb{R}^2} \right\} - \frac{\mathrm{H}_{N,\ell}}{\boldsymbol{\nu}_N} = -\omega_2 \langle \mathbf{J}_{N,\ell}, \mathbf{e}_1\rangle_{\mathbb{R}^2} - \frac{\mathrm{H}_{N,\ell}}{\boldsymbol{\nu}_N},$$

so for all $K \in \mathbb{Z}^2$,

$$\begin{aligned} \mathsf{H}_N(x+K) - \mathsf{H}(x+K) &= \boldsymbol{\nu}_N \hat{\mathsf{H}}_N(x+K) \\ &= \boldsymbol{\nu}_N \left\{ -\omega_2 \langle \mathbf{J}_{N,\ell}, \mathbf{e}_1\rangle_{\mathbb{R}^2} - \frac{\mathrm{H}_{N,\ell}}{\boldsymbol{\nu}_N} - \omega_2 \langle K, \mathbf{e}_1\rangle_{\mathbb{R}^2} \right\} \end{aligned}$$

$$= -\boldsymbol{\nu}_N \omega_2 \left\langle \mathbf{J}_{N,\ell} + K, \mathbf{e}_1 \right\rangle_{\mathbb{R}^2} - \textsc{h}_{N,\ell}.$$

This gives the second claim. We then calculate that $\mathsf{H}_N(\mathfrak{x}_\ell^e - \mathbf{J}_{N,\ell}) = \mathsf{H}(\mathfrak{x}_\ell^e - \mathbf{J}_{N,\ell}) - \textsc{h}_{N,\ell} = 0$, which gives us the last claim. □

Proof of Lemma 6.2. The first claim follows from Lemma 6.1. The second claim is easy. Defining

$$\upsilon \stackrel{\text{def}}{=} \inf\left\{ \|\nabla_e \mathsf{H}(z)\|_e : z \notin \bigcup_{\ell\in\Lambda} \mathsf{D}_\ell^{e,\varsigma} + \mathbb{Z}^2 \right\},$$
$$K \stackrel{\text{def}}{=} \sup\left\{ \|\nabla_e \hat{\mathsf{H}}_N(z)\|_e : z \in \mathbb{R}^2,\ N \in \mathbb{N} \right\},$$

we see that $\upsilon > 0$ and $K < \infty$. If $N \in \mathbb{N}$ large enough that $|\boldsymbol{\nu}_N| K < \upsilon$, then $\{x \in \mathbf{S} : \text{Ч}_N(x) = 0\} \subset \bigcup_{\ell\in\Lambda} \mathsf{D}_\ell^{\varsigma}$. Lemma 6.1 implies that $\mathsf{H}_N - \mathsf{H}$ is locally constant on $\bigcup_{\ell\in\Lambda} \mathsf{D}_\ell^{e,\varsigma} + \mathbb{Z}^2$, so the final claim follows. □

Let's now use a *transversal*. Arnol'd [Arn91] proves that there is a C^∞ map $\zeta : \mathbb{R} \to \mathsf{t}^{-1}(\mathsf{E})$ such that $d\mathsf{H}(\dot{\zeta}(t)) = 1$ and $\zeta(t+\omega_2) = \zeta(t) + (0,1)$ for all $t \in \mathbb{R}$. Define $\mathscr{T}^e \stackrel{\text{def}}{=} \zeta(\mathbb{R})$.

Proof of Lemma 6.3. We start by defining some curves in $\mathsf{t}^{-1}(\mathsf{E} \cup \mathfrak{X})$. For each $x \in \mathsf{t}^{-1}(\mathsf{E} \cup \mathfrak{X})$, let $\{\text{\l}_t^N(x);\, t \in \mathbb{R}\}$ be the unique element of $C(\mathbb{R}; \mathsf{t}^{-1}(\mathsf{E} \cup \mathfrak{X}))$ such that

(i) $\text{\l}_0^N(x) = x$,
(ii) if $\text{\l}_t^N(x) \in \mathsf{t}^{-1}(\mathsf{E})$, then $\dot{\text{\l}}_t^N(x)$ exists and $\dot{\text{\l}}_t^N(x) = \frac{\bar{\nabla}_e \mathsf{H}_N}{\|\bar{\nabla}_e \mathsf{H}_N\|_e}(\text{\l}_t^N(x))$,
(iii) $\{t \in \mathbb{R} : \text{\l}_t^N(x) \in \mathsf{t}^{-1}(\mathfrak{X})\}$ is discrete.

It is easy to see that such a curve is uniquely defined. Indeed, the vector field $\bar{\nabla}_e \mathsf{H}_N / \|\bar{\nabla}_e \mathsf{H}_N\|_e$ is well-defined on all of $\mathsf{t}^{-1}(\bar{\mathsf{E}} \setminus \mathfrak{X})$. Clearly $\mathsf{t}^{-1}(\partial\mathsf{E} \setminus \mathfrak{X})$ is invariant under this vector field, so $\text{\l}_t^N(x)$ is well-defined up to the time when it reaches a point in $\mathsf{t}^{-1}(\mathfrak{X})$ (this time may be ∞). Requirement (iii) means that if we hit a point in $\mathfrak{X}$, we must immediately go back into E. Looking at local coordinates and using the fact that $\bar{\nabla}_e \mathsf{H}_N$ is hyperbolic at points of $\mathsf{t}^{-1}(\mathfrak{X})$, we see that the trajectory of $\text{\l}^N$ must leave $\mathfrak{x}$ along the unstable manifold of Ч_N, and the requirement that it remain in $\mathsf{t}^{-1}(\mathsf{E} \cup \mathfrak{X})$ (as opposed to going into one of the $\mathsf{t}^{-1}(\partial \mathsf{D}_\ell)$'s) specifies the direction along the unstable manifold.

Fix now $\ell \in \Lambda$. We claim that

$$\gamma_N \cap (\mathsf{E} \cup \mathfrak{X}) = \mathsf{t}(\text{\l}_{\mathbb{R}}^N(\mathfrak{x}_\ell^e)). \tag{41}$$

Note that by (39), $\mathsf{H}_N(\mathfrak{x}_\ell^e) = k_\circ \mathsf{r}_N$ for some $k_\circ \in \mathbb{Z}$.

To show the easy inclusion in (41), observe that for any $t \in \mathbb{R}$, $\mathsf{H}_N(\text{\l}_t^N(\mathfrak{x}_\ell^e)) = \mathsf{H}_N(\mathfrak{x}_\ell^e) = k_\circ \mathsf{r}_N \in \mathsf{r}_N\mathbb{Z}$; thus $\mathsf{t}(\text{\l}_{\mathbb{R}}^N(\mathfrak{x}_\ell^e)) \subset \gamma_N \cap (\mathsf{E} \cup \mathfrak{X})$. To see the other direction, fix $x \in \mathsf{t}^{-1}(\mathsf{E} \cup \mathfrak{X})$ such that $\mathsf{H}_N(x) = k\mathsf{r}_N$ for some $k \in \mathbb{Z}$. Since $\mathsf{a}_N^{(n)}$

and $\mathsf{a}_N^{(d)}$ are relatively prime, there is a $(j_1, j_2) \in \mathbb{Z}^2$ such that $\mathsf{a}_N^{(n)} j_1 + \mathsf{a}_N^{(d)} j_2 = k_\circ - k$. Define $\hat{x} \stackrel{\text{def}}{=} x + (j_1, j_2)$; clearly $\mathfrak{t}(\hat{x}) = \mathfrak{t}(x)$. Define also the two times $\tau_\circ \stackrel{\text{def}}{=} \inf\{t \in \mathbb{R} : \mathfrak{l}_t^N(\mathfrak{x}_\ell^e) \in \mathscr{T}^e\}$ and $\hat{\tau} \stackrel{\text{def}}{=} \inf\{t \in \mathbb{R} : \mathfrak{l}_t^N(\hat{x}) \in \mathscr{T}^e\}$; by [Arn91] and perturbative arguments, we know that if $N \in \mathbb{N}$ is large enough, $\tau_\circ$ and $\hat{\tau}$ are finite (independently of the x and ℓ we chose). We note that $\mathsf{H}_N(\mathfrak{l}_{\tau_\circ}^N(\mathfrak{x}_\ell^e)) = \mathsf{H}_N(\mathfrak{x}_\ell^e) = k_\circ \mathsf{r}_N$ and that

$$\begin{aligned}\mathsf{H}_N(\mathfrak{l}_{\hat{\tau}}^N(\hat{x})) &= \mathsf{H}_N(\hat{x}) = \mathsf{H}_N(x) + \langle \boldsymbol{\omega}_N, (j_1, j_2)\rangle_{\mathbb{R}^2} = k\mathsf{r}_N + \omega_2\{\varrho_N j_1 + j_2\} \\ &= \mathsf{r}_N\{k + \mathsf{a}_N^{(n)} j_1 + \mathsf{a}_N^{(d)} j_2\} = k_\circ \mathsf{r}_N.\end{aligned}$$

Since $\{x \in \mathscr{T}^e : \mathsf{H}_N(x) = k_\circ \mathsf{r}_N\}$ is a single point, which we shall denote by x^*, we see that $\mathfrak{l}_{\tau_\circ}^N(\mathfrak{x}_\ell^e) = x^* = \mathfrak{l}_{\hat{\tau}}^N(\hat{x})$. Since the flow $\mathfrak{l}^N$ is unique, $\mathfrak{t}(\mathfrak{l}_{-\hat{\tau}+\tau_\circ}^N(\mathfrak{x}_\ell^e)) = \mathfrak{t}(\mathfrak{l}_{-\hat{\tau}}(x^*)) = \mathfrak{t}(\hat{x}) = \mathfrak{t}(x)$; i.e., $\mathfrak{t}(x) \in \mathfrak{t}(\mathfrak{l}_{\mathbb{R}}^N(\mathfrak{x}_\ell^e))$, finishing the proof of (41).

Since $t \mapsto \mathfrak{l}_t^N(\mathfrak{x}_\ell^e)$ is a piecewise-smooth continuous map, (41) implies that $\gamma_N \cap (\mathsf{E} \cup \mathfrak{X})$ is arcwise connected. Since each $\partial \mathsf{D}_\ell$ is a homoclinic orbit of Ч, the $\partial \mathsf{D}_\ell$'s are arcwise connected. Since each one also intersects $\mathsf{E} \cup \mathfrak{X}$, we can conclude that $\gamma_N = (\gamma_N \cap (\mathsf{E} \cup \mathfrak{X})) \cup \bigcup_{\ell \in \Lambda} \partial \mathsf{D}_\ell$ is arcwise connected.

The periodicity of orbits on $\mathsf{E} \setminus \gamma_N$ follows from Lemma 6.2 and arguments as in the proof of Lemma 2.5 of [Sow05]. □

We next want to prove Lemmas 6.4 and 6.6. First, we need a natural, but technical, result.

LEMMA 6.7. *Suppose that $\mathcal{O} \subset \mathbb{R}^2$ is open and intersects $\mathfrak{t}^{-1}(\partial \mathsf{D}_\ell)$. Suppose further that both $\mathcal{O}$ and $\mathcal{O} \cap \mathfrak{t}^{-1}(\partial \mathsf{D}_\ell)$ are arcwise connected. For each $N \in \mathbb{N}$, there is a $k_N \in \mathbb{Z}$ such that $\mathsf{H}_{T,\ell}(\mathfrak{t}(x)) - \mathsf{H}_N(x) = k_N \mathsf{r}_N$ for all $x \in \mathcal{O} \cap \mathfrak{t}^{-1}(\mathsf{D}_\ell)$. In particular, $\mathsf{H}_N(x) = -k_N \mathsf{r}_N$ for all $x \in \mathcal{O} \cap \mathfrak{t}^{-1}(\partial \mathsf{D}_\ell)$.*

Proof. We first claim that H_N is constant on $\mathcal{O} \cap \mathfrak{t}^{-1}(\partial \mathsf{D}_\ell)$. Let $\{\gamma_t; t \in [0,1]\}$ be a piecewise differentiable path in $\mathcal{O} \cap \mathfrak{t}^{-1}(\partial \mathsf{D}_\ell)$. For all $t \in (0,1)$ for which $\dot{\gamma}_t$ exists, $T\mathfrak{t}\dot{\gamma}_t \in \operatorname{Span} Ч(\mathfrak{t}(\gamma_t))$, so

$$\begin{aligned}0 = \omega(T\mathfrak{t}\dot{\gamma}_t, Ч(\mathfrak{t}(\gamma_t))) &= \omega(T\mathfrak{t}\dot{\gamma}_t, T\mathfrak{t}\bar{\nabla}_e \mathsf{H}(\gamma_t)) \\ &= \omega_e(\dot{\gamma}_t, \bar{\nabla}_e \mathsf{H}(\gamma_t)) = \omega_e(\dot{\gamma}_t, \bar{\nabla}_e \mathsf{H}_N(\gamma_t)) = -d\mathsf{H}_N(\dot{\gamma}_t).\end{aligned}$$

Thus $t \mapsto \mathsf{H}_N(\gamma_t)$ is constant on $[0,1]$. Since (by assumption) $\mathcal{O} \cap \mathfrak{t}^{-1}(\partial \mathsf{D}_\ell)$ is arcwise connected, H_N must indeed be constant on $\mathcal{O} \cap \mathfrak{t}^{-1}(\partial \mathsf{D}_\ell)$.

Fix next $N \in \mathbb{N}$. We next claim that in fact there is a $k_N \in \mathbb{Z}$ such that $\mathsf{H}_N \equiv -k_N \mathsf{r}_N$ on $\mathcal{O} \cap \mathfrak{t}^{-1}(\partial \mathsf{D}_\ell)$. By assumption, there is an $x \in \mathcal{O} \cap \mathfrak{t}^{-1}(\partial \mathsf{D}_\ell)$. Let $\{\mathsf{z}_t; 0 \le t < \infty\}$ be a solution of $\dot{\mathsf{z}}_t = \bar{\nabla}_e \mathsf{H}_N(\mathsf{z}_t)$ such that $\mathsf{z}_0 = x$. Then $\mathfrak{t}(\mathsf{z}_t) = \mathfrak{z}(\mathfrak{t}(x))$ for all $t \ge 0$. Since $\lim_{t\to\infty} \mathfrak{z}_t(x) = \mathfrak{x}_\ell$, $\mathsf{z}_\infty \stackrel{\text{def}}{=} \lim_{t\to\infty} \mathsf{z}_t$ exists and is in $\mathfrak{t}^{-1}\mathfrak{X}$. Keeping (39) in mind, we thus have that $\mathsf{H}_N(x) = \lim_{t\to\infty} \mathsf{H}_N(\mathsf{z}_t) = \mathsf{H}_N(\mathsf{z}_\infty) = -k_N \mathsf{r}_N$ for some $k_N \in \mathbb{Z}$. Since H_N is, by our above arguments, constant on $\mathcal{O} \cap \mathsf{t}^{-1}(\partial \mathsf{D}_\ell)$, we must have that $\mathsf{H}_N(x) = -k_N \mathsf{r}_N$ for all $x \in \mathcal{O} \cap \mathsf{t}^{-1}(\partial \mathsf{D}_\ell)$.

Next, we prove that $\mathsf{H}_{T,\ell} \circ \mathsf{t} - \mathsf{H}_N$ is constant on $\mathcal{O} \cap \mathsf{t}^{-1}(\mathsf{D}_\ell)$. Fix $x \in \mathcal{O} \cap \mathsf{t}^{-1}(\mathsf{D}_\ell)$. Since $\mathcal{O}$ is arcwise connected and intersects $\mathsf{t}^{-1}(\partial \mathsf{D}_\ell)$, there is a piecewise differentiable path $\{\gamma_t;\, t \in [0,1]\}$ such that $\gamma_0 = x$ and $\gamma_1 \in \mathcal{O} \cap \mathsf{t}^{-1}(\partial \mathsf{D}_\ell)$. Let $\tau \overset{\text{def}}{=} \inf\{t \in [0,1] : \gamma_t \in \mathcal{O} \setminus \mathsf{t}^{-1}(\mathsf{D}_\ell)\}$. Then

$$\begin{aligned}
&\mathsf{H}_{T,\ell}(\mathsf{t}(\gamma_\tau)) - \mathsf{H}_{T,\ell}(\mathsf{t}(x)) \\
&\quad = \int_0^\tau \omega(\bar{\nabla}\mathsf{H}_{T,\ell}(\mathsf{t}(\gamma_s)), T\mathsf{t}\dot{\gamma}_s)ds = \int_0^\tau \omega(\text{Ч}(\mathsf{t}(\gamma_s)), T\mathsf{t}\dot{\gamma}_s)ds \\
&\quad = \int_0^\tau \omega(T\mathsf{t}\bar{\nabla}_e\mathsf{H}(\gamma_s), T\mathsf{t}\dot{\gamma}_s)ds = \int_0^\tau \omega_e(\bar{\nabla}_e\mathsf{H}(\gamma_s), \dot{\gamma}_s)ds \\
&\quad = \int_0^\tau \omega_e(\bar{\nabla}_e\mathsf{H}_N(\gamma_s), \dot{\gamma}_s)ds = \int_0^\tau d\mathsf{H}_N(\dot{\gamma}_s)ds = \mathsf{H}_N(\gamma_\tau) - \mathsf{H}_N(x).
\end{aligned}$$

Since $\mathsf{H}_{T,\ell}(\mathsf{t}(\gamma_\tau)) = 0$, $\mathsf{H}_{T,\ell}(\mathsf{t}(x)) = \mathsf{H}_N(x) - \mathsf{H}_N(\gamma_\tau) = \mathsf{H}_N(x) + k_N\mathsf{r}_N$. This implies the first stated claim. The second stated claim follows from the first, by taking x in $\mathcal{O} \cap \mathsf{t}^{-1}(\partial \mathsf{D}_\ell)$. □

Proof of Lemma 6.4. We first show that the right-hand side of (40) is well-defined on $\bar{\mathsf{E}}$. Fix $x \in \mathsf{t}^{-1}(\bar{\mathsf{E}})$ and $K = (k_1, k_2)$ in $\mathbb{Z}^2$. Then $\mathsf{H}_N(x+K) = \mathsf{H}_N(x) + \mathsf{r}_N\left\{\mathsf{a}_N^{(n)}k_1 + \mathsf{a}_N^{(d)}k_2\right\}$. Since $\mathsf{a}_N^{(n)}$ and $\mathsf{a}_N^{(d)}$ are integers, $\mathsf{H}_N(x+K) - \mathsf{H}_N(x) \in \mathsf{r}_N\mathbb{Z}$, so

$$\mathsf{H}_N(x+K) - \left\lfloor \frac{\mathsf{H}_N(x+K)}{\mathsf{r}_N} + \frac{1}{2} \right\rfloor \mathsf{r}_N = \mathsf{H}_N(x) - \left\lfloor \frac{\mathsf{H}_N(x)}{\mathsf{r}_N} + \frac{1}{2} \right\rfloor \mathsf{r}_N.$$

Thus we can *define* $\mathsf{H}_N^{\text{loc}} : \mathbf{S} \to \mathbb{R}$ by setting

$$\mathsf{H}_N^{\text{loc}}(\mathsf{t}(x)) \overset{\text{def}}{=} \mathsf{H}_N(x) - \left\lfloor \frac{\mathsf{H}_N(x)}{\mathsf{r}_N} + \frac{1}{2} \right\rfloor \mathsf{r}_N$$

if $x \in \mathsf{t}^{-1}(\bar{\mathsf{E}})$ and $\mathsf{H}_N^{\text{loc}}(\mathsf{t}(x)) \overset{\text{def}}{=} \mathsf{H}_{T,\ell}(\mathsf{t}(x))$ if $\mathsf{t}(x) \in \text{Đ}_\ell$ and $\ell \in \Lambda$.

To finish the proof, we need to show that $\mathsf{H}_N^{\text{loc}}$ thus defined is smooth on $\mathbf{S}^\circ \setminus \mathcal{C}_N$. It is not difficult to see that it is smooth on $(\mathbf{S}^\circ \setminus \mathcal{C}_N) \setminus \bigcup_{\ell \in \Lambda} \partial \text{Đ}_\ell$. To proceed, fix now $\ell \in \Lambda$ and $x^* \in \partial \mathsf{D}_\ell$.

By constructing local coordinates (use the Morse lemma at $\mathfrak{x}_\ell$; otherwise use standard coordinates as in [Boo86, Theorem 3.14]), we can find an open neighborhood $\mathcal{O}$ of x^* such that t is evenly covered over $\mathcal{O}$, and both $\mathcal{O}$ and $\mathcal{O} \cap \partial\text{Đ}_\ell$ are arcwise connected. Let $\mathcal{O}^e$ be a connected component of $\mathsf{t}^{-1}(\mathcal{O})$, and let $\check{\mathsf{t}}$ be the inverse of $\mathsf{t}\big|_{\mathcal{O}^e}$. Then $\mathcal{O}^e = \check{\mathsf{t}}(\mathcal{O})$ and $\mathcal{O}^e \cap \mathsf{t}^{-1}(\mathsf{D}_\ell) = \check{\mathsf{t}}(\mathcal{O} \cap \mathsf{D}_\ell)$ are arcwise connected. From Lemma 6.7, we thus have that $\mathsf{H}_{T,\ell}(\mathsf{t}(x)) = \mathsf{H}_N(x) - \mathsf{H}_N(\check{\mathsf{t}}(x^*))$ for all $x \in \check{\mathsf{t}}(\mathcal{O} \cap \mathsf{D}_\ell)$ and $\mathsf{H}_N(\check{\mathsf{t}}(x^*)) \in \mathsf{r}_N\mathbb{Z}$. Define $\mathcal{O}' \overset{\text{def}}{=} \{x \in \mathcal{O} : |\mathsf{H}_N(\check{\mathsf{t}}(x)) - \mathsf{H}_N(\check{\mathsf{t}}(x^*))| < \mathsf{r}_N/2\}$. It is fairly easy to see that in fact $\mathsf{H}_N^{\text{loc}}(x) = \mathsf{H}_N(\check{\mathsf{t}}(x)) - \mathsf{H}_N(\check{\mathsf{t}}(x^*))$ for all $x \in \mathcal{O}'$. Since $\check{\mathsf{t}}$ is a diffeomorphism, $\mathsf{H}_N^{\rm loc}$ is indeed smooth at each point in $\partial\mathsf{D}_\ell$, completing the proof. □

Proof of Lemma 6.6. Let $\mathcal{O}$ be the connected component of $\mathfrak{t}^{-1}(\mathcal{U}_\ell\cap Đ_\ell^c)$ which contains $\mathfrak{x}_\ell^e-\mathbf{J}_{N,\ell}$. We have that $\mathsf{H}-\mathsf{H}_N$ and $\mathsf{H}-\tilde{\mathsf{H}}\circ\phi_\ell\circ\mathfrak{t}$ are both constant on $\mathcal{O}$; thus $\mathsf{H}_N-\tilde{\mathsf{H}}\circ\phi_\ell\circ\mathfrak{t}$ is constant on $\mathcal{O}$. Hence for $x\in\mathcal{O}$

$$\begin{aligned}(42)\qquad \mathsf{H}_N(x)-\tilde{\mathsf{H}}(\phi_\ell(\mathfrak{t}(x)))&=\mathsf{H}_N(\mathfrak{x}_\ell^e-\mathbf{J}_{N,\ell})-\tilde{\mathsf{H}}(\phi_\ell(\mathfrak{t}(\mathfrak{x}_\ell^e-\mathbf{J}_{N,\ell})))\\&=-\tilde{\mathsf{H}}(\phi_\ell(\mathfrak{x}_\ell))=-\tilde{\mathsf{H}}(\mathbf{0}_e)=0.\end{aligned}$$

Thus, for $x\in\mathcal{O}\cap\mathfrak{t}^{-1}(\overline{\mathsf{E}})$,

$$\mathsf{H}_N^{\rm loc}(\mathfrak{t}(x))=\tilde{\mathsf{H}}(\phi_\ell(\mathfrak{t}(x))-\left\lfloor\frac{\tilde{\mathsf{H}}(\phi_\ell(\mathfrak{t}(x)))}{\mathsf{r}_N}+\frac{1}{2}\right\rfloor\mathsf{r}_N;$$

while for $x\in\mathcal{O}\cap\mathfrak{t}^{-1}(Đ_\ell)$, the combination of (42) and Lemma 6.7 implies that

$$\mathsf{H}_N^{\rm loc}(\mathfrak{t}(x))=\mathsf{H}_N(x)-\mathsf{H}_N(\mathfrak{x}_\ell^e-\mathbf{J}_{N,\ell})=\mathsf{H}(x)=\tilde{\mathsf{H}}(\phi_\ell(\mathfrak{t}(x))).$$

Since $\mathfrak{t}\big|_{\mathcal{O}}$ is a diffeomorphism with range $\mathcal{U}_\ell\cap Đ_\ell^c$, the stated follows. □

7. Dominant analysis

We now want to use the machinery of [Sow05] to "glue" at γ_N. We have two questions to answer. First of all, if we start on γ_N, what is the relative likelihood of going into each of the $Đ_\ell$'s, versus the relative likelihood of going back to E? Secondly, if we start in E, how long does it (on average) take to get to γ_N, where we are back to the first question. The first question is one of *glueing*, and the second question should involve some sort of *Poisson* equation (recall that one usually studies occupation times by solving Poisson equations). The combination of the likelihood of going back to E and then returning to γ_N at a later time should give the stickiness coefficient of (c.ii), and the relative likelihood of going into the different $Đ_\ell$'s should give (c.iii). Lemma 7.7 is the result which brings all of this into focus.

The long-term behavior of $\mathfrak{z}$ in E should (and will) be important in our calculations. For $\varphi\in C(\mathsf{E})$, define

$$(\mathcal{A}\varphi)([\mathsf{E}])\overset{\rm def}{=}\frac{1}{\mathscr{H}^2(\mathsf{E})}\int_{z\in\mathsf{E}}\varphi(z)\mathscr{H}^2(dz);$$

then for any $x\in\mathsf{E}$,

$$\lim_{T\to\infty}\frac{1}{T}\int_{s=0}^T\varphi(\mathfrak{z}_s(x))ds=(\mathcal{A}\varphi)([\mathsf{E}]);$$

this is an extension of (9).

Let's first focus on γ_N. Recall that [Sow05] gives us a solvability condition for correcting for a *smoothness defect* in certain types of test functions near a homoclinic orbit. Let's write down the function we wish to correct. Similarly to (18), define $\boldsymbol{\sigma}_N$ and $\boldsymbol{\beta}_N$ in $C^\infty(\mathbb{T})$ by requiring that $\boldsymbol{\sigma}_N(\mathfrak{t}(x)) = \langle d\mathsf{H}_N, d\mathsf{H}_N\rangle(x)$ and $\boldsymbol{\beta}_N(\mathfrak{t}(x)) = (\mathscr{L}\mathsf{H}_N)(x)$ for all $x \in \mathbb{R}^2$. Recall the $\textsc{g}_\ell$'s of (19), and define

$$\tilde{\textsc{g}}_N \stackrel{\text{def}}{=} \int_{z\in\gamma_N\cap\mathsf{E}} \frac{\boldsymbol{\sigma}_N(z)}{\|\mathsf{Ч}_N(z)\|}\mathscr{H}^1(dz).$$

We can calculate the asymptotics of $\tilde{\textsc{g}}_N$. Define

$$\daleth \stackrel{\text{def}}{=} (\mathcal{A}\boldsymbol{\sigma})([\mathsf{E}])\mathscr{H}^2(\mathsf{E}). \tag{43}$$

LEMMA 7.1. *We have that* $\lim_{N\to\infty} \mathsf{r}_N\tilde{\textsc{g}}_N = \daleth$.

Proof. It is sufficient to calculate locally. Since $\boldsymbol{\sigma}_N/\|\mathsf{Ч}_N\|$ is bounded, we can excise neighborhoods near the critical points of $\mathsf{Ч}_N$ in $\overline{\mathsf{E}}$ (i.e., points in $\mathfrak{X}$). Similar calculations allow us to excise neighborhoods of $\partial\text{Đ}_\ell$. Let $\mathcal{O}$ be an open subset of E and let $\mathcal{O}^e$ be an open subset of $\mathbb{R}^2$ such that $\mathfrak{t}(\mathcal{O}^e) = \mathcal{O}$, $\mathfrak{t}\big|_{\mathcal{O}^e}$ is a diffeomorphism, and $\overline{\mathcal{O}^e}$ is a compact set which does not contain any critical points of H_N. Then by continuity and change-of-variables formula [EG92, Section 3.4.3],

$$\begin{aligned}
&\lim_{N\to\infty} \mathsf{r}_N \int_{z\in\gamma_N\cap\mathsf{E}\cap\mathcal{O}} \frac{\boldsymbol{\sigma}_N(z)}{\|\mathsf{Ч}_N(z)\|}\mathscr{H}^1(dz)\\
&\qquad = \lim_{N\to\infty} \mathsf{r}_N \int_{\substack{z\in\mathfrak{t}^{-1}(\mathsf{E})\cap\mathcal{O}^e\\ \mathsf{H}_N(z)\in\mathsf{r}_N\mathbb{Z}}} \frac{\boldsymbol{\sigma}_N(z)}{\|\nabla_e\mathsf{H}_N(z)\|_e}\mathscr{H}^1_e(dz)\\
&\qquad\qquad = \int_{z\in\mathfrak{t}^{-1}(\mathsf{E})\cap\mathcal{O}^e} \boldsymbol{\sigma}(z)\mathscr{H}^2_e(dz) = \int_{z\in\mathsf{E}\cap\mathcal{O}} \boldsymbol{\sigma}(z)\mathscr{H}^2(dz).
\end{aligned}$$

This leads to the claimed result. □

Now define

$$\digamma_N(x) \stackrel{\text{def}}{=} \begin{cases} \frac{\mathsf{r}_N}{\daleth}\left\{\sum_{\ell\in\Lambda_W} \textsc{g}_\ell \dot{f}_\ell(0)\right\}\mathsf{H}^{\text{loc}}_N(x) & \text{if } x\in\mathsf{E} \text{ and } \mathsf{H}^{\text{loc}}_N(x) > 0\\ \frac{\mathsf{r}_N}{\daleth}\left\{\sum_{\ell\in\Lambda_P} \textsc{g}_\ell \dot{f}_\ell(0)\right\}\mathsf{H}^{\text{loc}}_N(x) & \text{if } x\in\mathsf{E} \text{ and } \mathsf{H}^{\text{loc}}_N(x) < 0\\ \dot{f}_\ell(0)\mathsf{H}^{\text{loc}}_N(x) & \text{if } x\in\text{Đ}_\ell\\ 0 & \text{if } x\in\mathbf{S} \text{ and } \mathsf{H}^{\text{loc}}_N(x) = 0. \end{cases}$$

This captures, to first order, the behavior of $f\circ\mathfrak{m}$ on the Đ_ℓ's near $\partial\mathsf{E}$. We claim that we can find a small corrector function which compensates for the loss of smoothness of $\digamma_N$ across γ_N. To see this, we first decompose $\mathbf{S}\setminus\gamma_N$ into connected components (see [Sow05]). Note that

$$\{x\in\mathbf{S} : \mathsf{H}^{\text{loc}}_N(x) > 0\} = \{x\in\mathsf{E} : \mathsf{H}^{\text{loc}}_N(x) > 0\} \cup \bigcup_{\ell\in\Lambda_P} \text{Đ}_\ell,$$

$$\{x\in\mathbf{S} : \mathsf{H}^{\text{loc}}_N(x) < 0\} = \{x\in\mathsf{E} : \mathsf{H}^{\text{loc}}_N(x) < 0\} \cup \bigcup_{\ell\in\Lambda_W} \text{Đ}_\ell,$$

both of these being disjoint unions. We then have that

$$\gamma_N \cap \partial\{x \in \mathbf{S} : \mathsf{H}_N^{\rm loc}(x) > 0\} = \left\{(\gamma_N \cap \mathsf{E}) \cup \bigcup_{\ell\in\Lambda_W} \partial \text{Đ}_\ell\right\} \cup \bigcup_{\ell\in\Lambda_P} \partial\text{Đ}_\ell,$$

$$\gamma_N \cap \partial\{x \in \mathbf{S} : \mathsf{H}_N^{\rm loc}(x) < 0\} = \left\{(\gamma_N \cap \mathsf{E}) \cup \bigcup_{\ell\in\Lambda_P} \partial \text{Đ}_\ell\right\} \cup \bigcup_{\ell\in\Lambda_W} \partial\text{Đ}_\ell.$$

We thus have that

$$\begin{aligned}\sum_{\ell\in\Lambda_P} \mathrm{G}_\ell \dot{f}_\ell(0) + \left\{\tilde{\mathrm{G}}_N + \sum_{\ell\in\Lambda_W}\mathrm{G}_\ell\right\}\left\{\frac{\mathsf{r}_N}{\daleth}\sum_{\ell\in\Lambda_W}\mathrm{G}_\ell\dot{f}_\ell(0)\right\}&\\ \approx \sum_{\ell\in\Lambda_P}\mathrm{G}_\ell\dot{f}_\ell(0) + \sum_{\ell\in\Lambda_W}\mathrm{G}_\ell\dot{f}_\ell(0)&\\ \approx \sum_{\ell\in\Lambda_W}\mathrm{G}_\ell\dot{f}_\ell(0) + \left\{\tilde{\mathrm{G}}_N + \sum_{\ell\in\Lambda_P}\mathrm{G}_\ell\right\}\left\{\frac{\mathsf{r}_N}{\daleth}\sum_{\ell\in\Lambda_P}\mathrm{G}_\ell\dot{f}_\ell(0)\right\}&.\end{aligned}$$

This should allow us to glue as in [Sow05]. To make this precise, define

$$\mho_N^+ \overset{\rm def}{=} \frac{\sum_{\ell\in\Lambda_W}\mathrm{G}_\ell\dot{f}_\ell(0)}{\sum_{\ell\in\Lambda_P}\mathrm{G}_\ell}\left\{1 - \frac{\mathsf{r}_N\tilde{\mathrm{G}}_N}{\gimel} - \frac{\mathsf{r}_N}{\gimel}\sum_{\ell\in\Lambda_W}\mathrm{G}_\ell\right\},$$

$$\mho_N^- \overset{\rm def}{=} \frac{\sum_{\ell\in\Lambda_P}\mathrm{G}_\ell\dot{f}_\ell(0)}{\sum_{\ell\in\Lambda_W}\mathrm{G}_\ell}\left\{1 - \frac{\mathsf{r}_N\tilde{\mathrm{G}}_N}{\gimel} - \frac{\mathsf{r}_N}{\gimel}\sum_{\ell\in\Lambda_P}\mathrm{G}_\ell\right\},$$

$$\hat{F}_N(x) \overset{\rm def}{=} \begin{cases} \frac{\mathsf{r}_N}{\daleth}\left\{\sum_{\ell\in\Lambda_W}\mathrm{G}_\ell\dot{f}_\ell(0)\right\}\mathsf{H}_N^{\rm loc}(x) & \text{if } x\in\mathsf{E} \text{ and } \mathsf{H}_N^{\rm loc}(x) > 0\\ \frac{\mathsf{r}_N}{\daleth}\left\{\sum_{\ell\in\Lambda_P}\mathrm{G}_\ell\dot{f}_\ell(0)\right\}\mathsf{H}_N^{\rm loc}(x) & \text{if } x\in\mathsf{E} \text{ and } \mathsf{H}_N^{\rm loc}(x) < 0\\ \{\dot{f}_\ell(0) + \mho_N^+\}\mathsf{H}_N^{\rm loc}(x) & \text{if } x\in\text{Đ}_\ell \text{ and } \ell\in\Lambda_P\\ \{\dot{f}_\ell(0) + \mho_N^-\}\mathsf{H}_N^{\rm loc}(x) & \text{if } x\in\text{Đ}_\ell \text{ and } \ell\in\Lambda_W\\ 0 & \text{if } x\in\mathbf{S} \text{ and } \mathsf{H}_N^{\rm loc}(x) = 0.\end{cases}$$

Then we exactly have that

$$\begin{aligned}\sum_{\ell\in\Lambda_P}\mathrm{G}_\ell\{\dot{f}_\ell(0)+\mho_N^+\} + \left\{\tilde{\mathrm{G}}_N + \sum_{\ell\in\Lambda_W}\mathrm{G}_\ell\right\}\left\{\frac{\mathsf{r}_N}{\daleth}\sum_{\ell\in\Lambda_W}\mathrm{G}_\ell\dot{f}_\ell(0)\right\}&\\ = \sum_{\ell\in\Lambda_P}\mathrm{G}_\ell\dot{f}_\ell(0) + \sum_{\ell\in\Lambda_W}\mathrm{G}_\ell\dot{f}_\ell(0)&\\ = \sum_{\ell\in\Lambda_W}\mathrm{G}_\ell\{\dot{f}_\ell(0)+\mho_N^-\} + \left\{\tilde{\mathrm{G}}_N + \sum_{\ell\in\Lambda_P}\mathrm{G}_\ell\right\}\left\{\frac{\mathsf{r}_N}{\daleth}\sum_{\ell\in\Lambda_P}\mathrm{G}_\ell\dot{f}_\ell(0)\right\}&.\end{aligned}$$

By Lemma 7.1 and the fact that $\lim_{N\to\infty} \mathsf{r}_N = 0$, we have that

$$\lim_{N\to\infty} |\mho_N^+| = \lim_{N\to\infty} |\mho_N^+| = 0. \tag{44}$$

Proposition 7.2 (Glueing). *There are constants $K_{7.2} \in (1,\infty)$ and $\bar{\delta}_{7.2} \in (0,1)$ such that for each $N \in \mathbb{N}$ and $\delta \in (0,\bar{\delta}_{7.2})$ and ε in $(0,1)$ such that $\varepsilon < \sqrt{\delta}/K_{7.2}$ and $\varepsilon < \mathsf{r}_N^{3/2}/4$, there is a function $\Psi_D^{\delta,\varepsilon,N}$ such that $\Psi_D^{\delta,\varepsilon,N} + \hat{\digamma}_N \in C^2(\mathbf{S} \setminus \mathcal{C}_N)$, and such that $|\Psi_D^{\delta,\varepsilon,N}(x)| \le K_{7.2}\varepsilon\mathfrak{l}(\varepsilon)/\mathsf{r}_N^{3/4}$*

$$\begin{aligned}
&(\mathscr{L}^\varepsilon \Psi_D^{\delta,\varepsilon,N})(x) \\
&\quad \ge -K_{7.2}\left\{\frac{|\mu_{N,\varepsilon}|}{\mathsf{r}_N^{5/4}} + \frac{1}{\mathsf{r}_N^{3/4}\sqrt{\delta}}\right\}\mathfrak{l}(\varepsilon)\exp\left[-\frac{1}{K_{7.2}}\left|\frac{\mathsf{H}_N^{\mathrm{loc}}(x)\sqrt{\mathsf{r}_N}}{\varepsilon}\right|\right]\boldsymbol{\sigma}_N(x) \\
&\qquad - \frac{K}{\mathsf{r}_N^{1/4}\sqrt{\delta}}\exp\left[-\frac{1}{K_{7.2}}\left|\frac{\mathsf{H}_N^{\mathrm{loc}}(x)\sqrt{\mathsf{r}_N}}{\varepsilon}\right|\right]\sqrt{\boldsymbol{\sigma}_N(x)} \\
&\qquad - \frac{K_{7.2}}{\varepsilon \mathsf{r}_N^{3/4}}\exp\left[-\left|\frac{\mathsf{H}_N^{\mathrm{loc}}(x)}{\varepsilon\sqrt{\delta}}\right|\right]\boldsymbol{\sigma}_N(x) - \frac{K_{7.2}\varepsilon\mathfrak{l}(\varepsilon)}{\mathsf{r}_N^{3/4}\sqrt{\delta}} \\
&\qquad - \frac{K_{7.2}}{\varepsilon \mathsf{r}_N^{3/4}}\exp\left[-\frac{1}{K_{7.2}}\frac{\sqrt{\delta}}{\varepsilon}\right] - \frac{K_{7.2}}{\varepsilon^{7/3}\sqrt{\delta}}\exp\left[-\frac{1}{K_{7.2}}\left(\frac{\mathsf{r}_N^{3/2}}{\varepsilon}\right)^{1/2}\right]
\end{aligned}$$

for all $x \in \mathbf{S} \setminus (\mathcal{C}_N \cup \gamma_N)$.

We will prove this in Section 10. The unspecified parameter δ comes from the proof of Theorem 3.8 of [Sow05]. Using the calculations of the next section, we will show that the terms which bound $\mathscr{L}^\varepsilon \Psi_D^{\delta,\varepsilon,N}$ from below are in fact small. We will then optimize over δ in Lemma 8.8. We will also later correct for the difference between $\digamma_N$ and $\hat{\digamma}_N$.

Remark 7.3. We would like the "ribbon" of Remark 6.5 to be asymptotically wider than the boundary layer of the glueing corrector. This means that the boundary layer does not interfere with itself across the ribbon. The lower bound on $\mathscr{L}^\varepsilon \Psi_D^{\delta,\varepsilon,N}$ tells us that the glueing corrector has a boundary layer of size $\mathsf{H}_N^{\mathrm{loc}} = O\left(\frac{\varepsilon}{\sqrt{\mathsf{r}_N}}\right)$. Thus we want that $\mathsf{r}_N \gg \frac{\varepsilon}{\sqrt{\mathsf{r}_N}}$, or rather that $\mathsf{r}_N \gg \varepsilon^{2/3}$. The requirement that $\varepsilon < \mathsf{r}_N^{3/2}/4$ is a reflection of this.

Let's now look inside E, i.e., at the issue of the Poisson equation. Here we want to solve the PDE $\mathscr{L}^\varepsilon u \approx g([\mathsf{E}])$ such that $\hat{\digamma}_N$ captures the nonsmooth behavior of u near γ_N. Since γ_N and $\hat{\digamma}_N$ are both given in terms of $\mathsf{H}_N^{\mathrm{loc}}$, let's look at the effect of $\mathscr{L}^\varepsilon$ on functions of $\mathsf{H}_N^{\mathrm{loc}}$. For convenience, define

$\xi_N \in C_p^\infty(\mathbb{T})$ by requiring that

$$\xi_N(\mathsf{t}(x)) \stackrel{\text{def}}{=} \frac{1}{\boldsymbol{\nu}_N}(\bar{\nabla}_e \mathsf{H}, \nabla_e \mathsf{H}_N)_e(x) = (\bar{\nabla}_e \mathsf{H}, \nabla_e \hat{\mathsf{H}}_N)_e(x) \tag{45}$$

for all $x \in \mathbb{R}^2$. If $u(x) = U(\mathsf{H}_N^{\text{loc}}(x)/\mathsf{r}_N)$ (this scaling turns $\mathsf{E} \setminus \mathcal{C}_N$ into a reference strip of *unit* width) on some open subset $\mathcal{O}$ of $\mathsf{E} \setminus \mathcal{C}_N$, where $U \in C^2(\mathbb{R})$, then

$$(\mathscr{L}^\varepsilon u)(x) = \frac{\boldsymbol{\nu}_N \xi_N(x)}{\varepsilon^2 \mathsf{r}_N} \dot{U}\left(\frac{\mathsf{H}_N^{\text{loc}}(x)}{\mathsf{r}_N}\right) + \frac{\boldsymbol{\sigma}_N(x)}{2\mathsf{r}_N^2} \ddot{U}\left(\frac{\mathsf{H}_N^{\text{loc}}(x)}{\mathsf{r}_N}\right) + \frac{\boldsymbol{\beta}_N(x)}{\mathsf{r}_N} \dot{U}\left(\frac{\mathsf{H}_N^{\text{loc}}(x)}{\mathsf{r}_N}\right) \tag{46}$$

for all $x \in \mathcal{O}$. The first two terms are the dominant ones.

The theory of *averaging* tells us that we should replace the coefficients in (46) by *constants*. The operator $\mathscr{L}^\varepsilon$ generates a drift of size $1/\varepsilon^2$ in the direction of Ч; keeping in mind that we are using $Ч_N$ as an approximation of Ч, we have a drift along γ_N of size $1/\varepsilon^2$. Comparing this to the drift and diffusion in the $\nabla \mathsf{H}_N^{\text{loc}}$-direction, we should have a separation of scales, and be able to replace ξ_N and $\boldsymbol{\sigma}_N$ by their averages over the orbits of $\mathfrak{z}^N$ (which is an approximation of the average with respect to $\mathfrak{z}$).

Lemma 7.4. *There is a function $\xi^* \in C^\infty(\mathbb{T})$ and a $K > 0$ such that*
(e.i) $\xi^* \equiv 0$ *in each* $\mathcal{D}_\ell^\varsigma$,
(e.ii) $|\xi_N(x) - \xi^*(x)| \le K|\boldsymbol{\nu}_N|$ *for all* $x \in \mathsf{E}$,
(e.iii) $(\mathcal{A}\xi^*)([\mathsf{E}]) = \frac{\omega_2}{\mathscr{H}^2(\mathsf{E})}$.

Proof. Define

$$\begin{aligned}\xi^{e,*}(x) &\stackrel{\text{def}}{=} \omega_2 \frac{\partial \mathsf{H}}{\partial x_2}(x)\left\{-1 + \sum_{\substack{K\in\mathbb{Z}^2\\ \ell\in\Lambda}} \varpi_\ell(x-K)\right\} \\ &\quad + \omega_2 \left\{\frac{\partial \mathsf{H}}{\partial x_2}(x) \sum_{\substack{K\in\mathbb{Z}^2\\ \ell\in\Lambda}} \frac{\partial \varpi_\ell}{\partial x_1}(x-K)\langle x - K - \mathbf{J}_{N,\ell}, \mathbf{e}_1\rangle_{\mathbb{R}^2}\right. \\ &\quad\quad \left. -\frac{\partial \mathsf{H}}{\partial x_1}(x) \sum_{\substack{K\in\mathbb{Z}^2\\ \ell\in\Lambda}} \frac{\partial \varpi_\ell}{\partial x_2}(x-K)\langle x - K - \mathbf{J}_{N,\ell}, \mathbf{e}_1\rangle_{\mathbb{R}^2}\right\}\end{aligned}$$

for all $x = (x_1, x_2) \in \mathbb{R}^2$ (i.e., we have neglected the contributions involving $\text{H}_{N,\ell}$). From this we clearly see that $\xi^{e,*} \in C_p^\infty(\mathbb{R}^2)$ and we then define ξ^* by setting $\xi^*(\mathsf{t}(x)) = \xi^{e,*}(x)$ for all $x \in \mathbb{R}^2$. It is clear that $\xi^* \equiv 0$ on $\bigcup_{\ell\in\Lambda} \mathsf{D}_\ell^\varsigma$ since for each $\ell \in \Lambda$, $\varpi_\ell = 1$ on $\mathsf{D}_\ell^\varsigma$ and more generally ϖ_ℓ is locally constant on $\bigcup_{\ell\in\Lambda} \mathsf{D}_\ell^\varsigma$. The stated bound on $\xi_N - \xi^*$ is also easy to see (recall (36)). Next, set

$$G_\ell(x) \stackrel{\text{def}}{=} \sum_{K\in\mathbb{Z}^2} \varpi_\ell(x-K)\,\langle x - K - \mathbf{J}_{N,\ell}, \mathbf{e}_1\rangle_{\mathbb{R}^2}$$

for $x = (x_1, x_2) \in \mathbb{R}^2$. Then $G_\ell \in C_p^\infty(\mathbb{R}^2)$ and

$$\xi^{e,*}(x) = -\omega_2 \frac{\partial \mathsf{H}}{\partial x_2}(x) + \omega_2 \sum_{\ell\in\Lambda} (\bar{\nabla}_e \mathsf{H}, \nabla_e G_\ell)_e(x)$$

for all $x = (x_1, x_2) \in \mathbb{R}^2$. We compute that

$$\begin{aligned}
&\int_{(x_1,x_2)\in\mathsf{t}^{-1}(\mathsf{E})\cap[0,1)^2} \frac{\partial \mathsf{H}}{\partial x_2}(x_1,x_2)dx_1dx_2 \\
&\qquad = \int_{(x_1,x_2)\in[0,1)^2} \frac{\partial \mathsf{H}}{\partial x_2}(x_1,x_2)dx_1dx_2 \\
&\qquad\qquad - \sum_{\ell\in\Lambda} \int_{(x_1,x_2)\in\mathsf{t}^{-1}(\mathsf{D}_\ell)\cap[0,1)^2} \frac{\partial \mathsf{H}}{\partial x_2}(x_1,x_2)dx_1dx_2 \\
&\qquad = \int_{x_1=0}^1 \{\mathsf{H}(x_1,1) - \mathsf{H}(x_1,0)\}\,dx_1 - \sum_{\ell\in\Lambda} \int_{z\in\mathsf{D}_\ell^e} \frac{\partial \mathsf{H}}{\partial x_2}(x_1,x_2)dx_1dx_2 = \omega_2.
\end{aligned}$$

The integrals in D_ℓ^e are zero by the divergence theorem since H is constant on $\partial\mathsf{D}_\ell$.

Next, note that $\operatorname{supp} G_\ell \subset \mathcal{D}_\ell^e + \mathbb{Z}^2$ and that $(\bar{\nabla}_e \mathsf{H}, \nabla_e G_\ell)_e(x) = \frac{\partial \mathsf{H}}{\partial x_2}(x)$ for all $x = (x_1, x_2) \in \mathsf{D}_\ell^e + \mathbb{Z}^2$. Thus

$$\begin{aligned}
&\int_{(x_1,x_2)\in\mathsf{t}^{-1}(\mathsf{E})\cap[0,1)^2} (\bar{\nabla}_e \mathsf{H}, \nabla_e G_\ell)_e(x_1,x_2)dx_1dx_2 \\
&\qquad = \int_{(x_1,x_2)\in\mathcal{D}_\ell^e} (\bar{\nabla}_e \mathsf{H}, \nabla_e G_\ell)_e(x_1,x_2)dx_1dx_2 \\
&\qquad\qquad - \int_{(x_1,x_2)\in\mathsf{D}_\ell^e} (\bar{\nabla}_e \mathsf{H}, \nabla_e G_\ell)_e(x_1,x_2)dx_1dx_2 \\
&\qquad = \int_{(x_1,x_2)\in\mathcal{D}_\ell^e} (\bar{\nabla}_e \mathsf{H}, \nabla_e G_\ell)_e(x_1,x_2)dx_1dx_2 \\
&\qquad\qquad - \int_{(x_1,x_2)\in\mathsf{D}_\ell^e} \frac{\partial \mathsf{H}}{\partial x_2}(x_1,x_2)dx_1dx_2.
\end{aligned}$$

Since $\bar{\nabla}_e \mathsf{H}$ has zero divergence and since $G_\ell \equiv 0$ on $\partial\mathcal{D}_\ell^e$, the first term is zero. Since H is constant on $\partial\mathsf{D}_\ell^e$, the second term is also zero. □

Define now

$$\beth \stackrel{\text{def}}{=} \frac{(\mathcal{A}\xi^*)([\mathsf{E}])}{(\mathcal{A}\sigma)([\mathsf{E}])} = \frac{\omega_2}{\daleth} \tag{47}$$

and, for all $N \in \mathbb{N}$ and $\varepsilon \in (0,1)$, $\mu_{N,\varepsilon} \stackrel{\text{def}}{=} \beth\nu_N \mathsf{r}_N/\varepsilon^2$. Combining (15) and (38), we have that

$$|\mu_{N,\varepsilon}| \le \frac{|\beth\omega_2|}{\varepsilon^2 \mathsf{a}^{(d)}_{N+1}(\mathsf{a}^{(d)}_N)^2} \tag{48}$$

for all $\varepsilon \in (0,1)$ and $N \in \mathbb{N}$. After we replace ξ_N by ξ^* and $\boldsymbol{\sigma}_N$ by $\boldsymbol{\sigma}$, then $\mu_{N,\varepsilon}$ is the ratio of the averaged drift to diffusion coefficients in (46). Note that (46) now becomes

$$\begin{aligned}(\mathscr{L}^\varepsilon u)(x) = \frac{1}{\mathsf{r}_N^2}\Bigg\{&\mu_{N,\varepsilon}\frac{\xi_N(x)}{\beth}\dot{U}\left(\frac{\mathsf{H}^{\text{loc}}_N(x)}{\mathsf{r}_N}\right)\\ &+\frac{\boldsymbol{\sigma}_N(x)}{2}\ddot{U}\left(\frac{\mathsf{H}^{\text{loc}}_N(x)}{\mathsf{r}_N}\right) + \mathsf{r}_N\boldsymbol{\beta}_N(x)\dot{U}\left(\frac{\mathsf{H}^{\text{loc}}_N(x)}{\mathsf{r}_N}\right)\Bigg\}.\end{aligned} \tag{49}$$

Then (47) becomes $\left(\mathcal{A}(\xi^*\beth^{-1})\right)([\mathsf{E}]) = (\mathcal{A}\boldsymbol{\sigma})([\mathsf{E}])$. This helps us distinguish between several important cases; viz., when $\mu_{N,\varepsilon}$ is bounded, and when it becomes large.

Let's now get back to our Poisson equation in E. For $N \in \mathbb{N}$ and $\varepsilon \in (0,1)$, define

$$\mathsf{u}^{N,\varepsilon}_P(h) \stackrel{\text{def}}{=} \frac{1}{\mu_{N,\varepsilon}}\left\{h - \frac{1-\exp[-2\mu_{N,\varepsilon}h]}{1-\exp[-2\mu_{N,\varepsilon}]}\right\}, \qquad h \in \mathbb{R}.$$

The importance of $\mathsf{u}^{N,\varepsilon}_P$ is contained in the following

LEMMA 7.5. *For each $\varepsilon \in (0,1)$ and $N \in \mathbb{N}$, $\mathsf{u}^{N,\varepsilon}_P \in C^2([0,1])$ and*

$$\begin{gathered}\mu_{N,\varepsilon}\dot{\mathsf{u}}^{N,\varepsilon}_P(h) + \frac{1}{2}\ddot{\mathsf{u}}^{N,\varepsilon}_P(h) = 1, \qquad h \in (0,1),\\ \mathsf{u}^{N,\varepsilon}_P(1) = \mathsf{u}^{N,\varepsilon}_P(0) = 0,\\ \dot{\mathsf{u}}^{N,\varepsilon}_P(1) - \dot{\mathsf{u}}^{N,\varepsilon}_P(0) = 2.\end{gathered} \tag{50}$$

Furthermore, there is a $K > 0$ such that

$$\left|\mathsf{u}^{N,\varepsilon}_P(h)\right| \le K \qquad \text{and} \qquad \left|\dot{\mathsf{u}}^{N,\varepsilon}_P(h)\right| \le K \tag{51}$$

for all $h \in [0,1]$, $\varepsilon \in (0,1)$, and $N \in \mathbb{N}$.

Proof. The PDE and boundary conditions can be directly checked. Note that

$$\dot{\mathsf{u}}^{N,\varepsilon}_P(h) = \frac{1}{\mu_{N,\varepsilon}} - \frac{2\exp[-2\mu_{N,\varepsilon}h]}{1-\exp[-2\mu_{N,\varepsilon}]}, \qquad h \in (0,1).$$

Note that $\dot{\mathsf{u}}_P^{N,\varepsilon}$ is monotone, so $\sup_{h\in(0,1)}\left|\dot{\mathsf{u}}_P^{N,\varepsilon}(h)\right| \le \left|\dot{\mathsf{u}}_P^{N,\varepsilon}(0)\right| + \left|\dot{\mathsf{u}}_P^{N,\varepsilon}(1)\right|$ for all $\varepsilon \in (0,1)$ and $N \in \mathbb{N}$. To get (51), it is sufficient to consider subsequences $(\varepsilon_k;\, k \in \mathbb{N})$ in $(0,1)$ and $(N_k;\, k \in \mathbb{N})$ in $\mathbb{N}$ such that $\lim_{k\to\infty}\varepsilon_k = 0$ and $\lim_{k\to\infty} N_k = \infty$ and such that $\lim_{k\to\infty}\mu_{N_k,\varepsilon_k}$ exists as an element of $[-\infty,\infty]$. If $\lim_{k\to\infty}\mu_{N_k,\varepsilon_k} \neq 0$, then clearly

$$\varlimsup_{k\to\infty}\left|\dot{\mathsf{u}}_P^{N,\varepsilon}(0)\right| \quad \text{and} \quad \varlimsup_{k\to\infty}\left|\dot{\mathsf{u}}_P^{N,\varepsilon}(1)\right|$$

are finite. Assume next that $\lim_{k\to\infty}\mu_{N_k,\varepsilon_k} = 0$. By taking Taylor expansions of the exponential, we can verify that $\lim_{k\to\infty}\dot{\mathsf{u}}_P^{N,\varepsilon}(0) = -1$ and $\lim_{k\to\infty}\dot{\mathsf{u}}_P^{N,\varepsilon}(0) = 1$. The claimed bound on $\dot{\mathsf{u}}_P^{N,\varepsilon}$ follows. Since $\mathsf{u}_P^{N,\varepsilon}(0) = 0$, the claimed bound on $\mathsf{u}_P^{N,\varepsilon}$ follows from the claimed bound on $\dot{\mathsf{u}}_P^{N,\varepsilon}$. □

Let's now define two constants; set

$$\hat{\mathsf{u}}_+^{N,\varepsilon} \overset{\text{def}}{=} \left\{\frac{g([\mathsf{E}])\dot{\mathsf{u}}_P^{N,\varepsilon}(1)}{(\mathcal{A}\sigma)([\mathsf{E}])} - \frac{1}{\daleth}\sum_{\ell\in\Lambda_P} \mathrm{G}_\ell \dot{f}_\ell(0)\right\}\mathsf{r}_N + \mho_N^+,$$
$$\hat{\mathsf{u}}_-^{N,\varepsilon} \overset{\text{def}}{=} \left\{\frac{g([\mathsf{E}])\dot{\mathsf{u}}_P^{N,\varepsilon}(0)}{(\mathcal{A}\sigma)([\mathsf{E}])} - \frac{1}{\daleth}\sum_{\ell\in\Lambda_W} \mathrm{G}_\ell \dot{f}_\ell(0)\right\}\mathsf{r}_N + \mho_N^-$$

for all $\varepsilon \in (0,1)$ and $N \in \mathbb{N}$. In light of (44) and the bound on $\dot{\mathsf{u}}_P^{N,\varepsilon}$ in Lemma 7.5, we have that

$$\lim_{\substack{N\to\infty\\ \varepsilon\to 0}}\left|\hat{\mathsf{u}}_+^{N,\varepsilon}\right| = \lim_{\substack{N\to\infty\\ \varepsilon\to 0}}\left|\hat{\mathsf{u}}_-^{N,\varepsilon}\right| = 0. \tag{52}$$

Finally, define

$$\begin{aligned}\mathsf{U}_P^{N,\varepsilon}(x) \overset{\text{def}}{=} {}& \frac{g([\mathsf{E}])}{(\mathcal{A}\sigma)([\mathsf{E}])}\mathsf{r}_N^2 \mathsf{u}_P^{N,\varepsilon}\left(\iota\left(\frac{\mathsf{H}_N^{\text{loc}}(x)}{\mathsf{r}_N}\right)\right)\chi_{\mathsf{E}}(x)\\ &+ \sum_{\ell\in\Lambda_P}\hat{\mathsf{u}}_+^{N,\varepsilon}\mathsf{H}_{T,\ell}(x)\chi_{\text{Đ}_\ell}(x) + \sum_{\ell\in\Lambda_W}\hat{\mathsf{u}}_-^{N,\varepsilon}\mathsf{H}_{T,\ell}(x)\chi_{\text{Đ}_\ell}(x)\end{aligned}$$

for all $x \in \mathbf{S}$, $\varepsilon \in (0,1)$, and $N \in \mathbb{N}$ (where ι is as was defined in (13)).

LEMMA 7.6. *For each* $\varepsilon \in (0,1)$ *and* $N \in \mathbb{N}$, $\mathsf{U}_P^{N,\varepsilon} \in C^\infty(\mathbf{S}\setminus\gamma_N)\cap C(\mathbf{S})$.

Proof. Clearly $\mathsf{U}_P^{N,\varepsilon}$ is C^∞ on $\bigcup_{\ell\in\Lambda}\text{Đ}_\ell$. Lemma 6.4 ensures that $\mathsf{H}_N^{\text{loc}}$ is smooth on $\mathsf{E}\setminus\mathcal{C}_N$. Thus $\iota(\mathsf{H}_N^{\text{loc}}/\mathsf{r}_N)$ is smooth on all of $\mathsf{E}\setminus\mathcal{C}_N$ except possibly the set

$$\left\{x\in\mathsf{E} : \frac{\mathsf{H}_N^{\text{loc}}(x)}{\mathsf{r}_N}\in\mathbb{Z}\right\} = \{x\in\mathsf{E} : \mathsf{H}_N^{\text{loc}}(x) = 0\} = \gamma_N.$$

We proceed by checking that $\mathsf{U}_P^{N,\varepsilon}$ is continuous at γ_N. Since

$$\lim_{\substack{x\to\gamma_N\\ x\in\mathsf{E}}} (\mathsf{H}_N^{\mathrm{loc}}(x)/\mathsf{r}_N) = 0,$$

$$\mathsf{u}_P^{N,\varepsilon}(\iota(0^+)) = \mathsf{u}_P^{N,\varepsilon}(0) = 0 = \mathsf{u}_P^{N,\varepsilon}(1) = \mathsf{u}_P^{N,\varepsilon}(\iota(0^-)),$$

$\mathsf{U}_P^{N,\varepsilon}$ is continuous at $\gamma_N \cap \mathsf{E}$. Similarly, since the common value of $\mathsf{u}_P^{N,\varepsilon}(\iota(0^+))$ and $\mathsf{u}_P^{N,\varepsilon}(\iota(0^-))$ is zero, $\mathsf{U}_P^{N,\varepsilon}$ is continuous at the $\partial\mathsf{D}_\ell$'s.

Thus far, we have proved that $\mathsf{U}_P^{N,\varepsilon}$ is continuous on $\mathbf{S}\setminus\mathcal{C}_N$ and that it is smooth on $\mathbf{S}\setminus(\mathcal{C}_N\cup\gamma_N)$. It remains to show that $\mathsf{U}_P^{N,\varepsilon}$ is smooth on $\mathcal{C}_N$. For any $x \in \mathsf{t}^{-1}(\mathsf{E})$,

$$\iota\left(\frac{\mathsf{H}_N^{\mathrm{loc}}(\mathsf{t}(x))}{\mathsf{r}_N}\right) = \iota\left(\frac{\mathsf{H}_N(x)}{\mathsf{r}_N} - \left\lfloor \frac{\mathsf{H}_N(x)}{\mathsf{r}_N} + \frac{1}{2}\right\rfloor\right) = \iota\left(\frac{\mathsf{H}_N(x)}{\mathsf{r}_N}\right)$$

(since $\iota(z+k)=\iota(z)$ for all $z\in\mathbb{R}$ and $k\in\mathbb{Z}$). Since ι is smooth at all points of $\mathbb{Z}+\frac{1}{2}$, $\iota(\mathsf{H}_N^{\mathrm{loc}}/\mathsf{r}_N)$ is smooth at all points of $\mathcal{C}_N$. Since $\iota\left(\mathbb{Z}+\frac{1}{2}\right)=\frac{1}{2}$ and $\mathsf{U}_P^{N,\varepsilon}$ is smooth at $\frac{1}{2}$, the claimed result follows. □

We also have

LEMMA 7.7. *For each $\varepsilon\in(0,1)$ and $N\in\mathbb{N}$, $\mathsf{U}_P^{N,\varepsilon} - \hat{F}_N + f_{\mathrm{outer}}$ is C^1 at γ_N.*

Proof. Clearly $\mathsf{U}_P^{N,\varepsilon} - \hat{F}_N + f_{\mathrm{outer}}$ is continuous at γ_N since $\mathsf{U}_P^{N,\varepsilon}$, $\hat{F}_N$, and f_{outer} are all continuous at γ_N.

Consider next differentiability at points of $\gamma_N\cap\mathsf{E}$. We need to show that

$$\frac{g([\mathsf{E}])}{(\mathcal{A}\sigma)([\mathsf{E}])}\mathsf{r}_N \dot{u}_P^{N,\varepsilon}(1) - \frac{\mathsf{r}_N}{\daleth}\sum_{\ell\in\Lambda_P} \mathrm{G}_\ell \dot{f}_\ell(0) = \frac{g([\mathsf{E}])}{(\mathcal{A}\sigma)([\mathsf{E}])}\mathsf{r}_N \dot{u}_P^{N,\varepsilon}(0) - \frac{\mathsf{r}_N}{\daleth}\sum_{\ell\in\Lambda_W} \mathrm{G}_\ell \dot{f}_\ell(0);$$

the first term is the transversal derivative of $\mathsf{U}_P^{N,\varepsilon} - \hat{F}_N + f_{\mathrm{outer}}$ when we approach $\gamma_N\cap\mathsf{E}$ from the direction where $\mathsf{H}_N^{\mathrm{loc}}<0$, and the second term is the transversal derivative of $\mathsf{U}_P^{N,\varepsilon} - \hat{F}_N + f_{\mathrm{outer}}$ when we approach $\gamma_N\cap\mathsf{E}$ from the direction where $\mathsf{H}_N^{\mathrm{loc}}>0$. In light of (43), this is equivalent to showing that

$$g([\mathsf{E}])\left\{\dot{u}_P^{N,\varepsilon}(1) - \dot{u}_P^{N,\varepsilon}(0)\right\} = \frac{1}{\mathscr{H}^2(\mathsf{E})}\left\{\sum_{\ell\in\Lambda_P}\mathrm{G}_\ell\dot{f}_\ell(0) - \sum_{\ell\in\Lambda_W}\mathrm{G}_\ell\dot{f}_\ell(0)\right\},$$

and substituting the last line of (50), this is in turn equivalent to showing that

$$2g([\mathsf{E}])\mathscr{H}^2(\mathsf{E}) = \sum_{\ell\in\Lambda_P}\mathrm{G}_\ell\dot{f}_\ell(0) - \sum_{\ell\in\Lambda_W}\mathrm{G}_\ell\dot{f}_\ell(0).$$

This is exactly the glueing condition (11).

Next, consider $\partial Đ_\ell$. Here we use the facts that

$$\frac{g([\mathsf{E}])}{(\mathcal{A}\sigma)([\mathsf{E}])}\mathsf{r}_N\dot{\mathsf{u}}_P^{N,\varepsilon}(1) - \frac{\mathsf{r}_N}{\daleth}\sum_{\ell\in\Lambda_P} \mathrm{G}_\ell \dot{f}_\ell(0) = \hat{\mathsf{u}}_+^{N,\varepsilon} - \{\dot{f}_\ell(0) + \mho_N^+\} + \dot{f}_\ell(0),$$
$$\frac{g([\mathsf{E}])}{(\mathcal{A}\sigma)([\mathsf{E}])}\mathsf{r}_N\dot{\mathsf{u}}_P^{N,\varepsilon}(0) - \frac{\mathsf{r}_N}{\daleth}\sum_{\ell\in\Lambda_W} \mathrm{G}_\ell \dot{f}_\ell(0) = \hat{\mathsf{u}}_-^{N,\varepsilon} - \{\dot{f}_\ell(0) + \mho_N^-\} + \dot{f}_\ell(0);$$

we use the first equality if $\ell \in \Lambda_P$, and the second if $\ell \in \Lambda_W$. □

Since

$$\mathsf{U}_P^{N,\varepsilon} + \Psi_D^{\delta,\varepsilon,N} + f_{\text{outer}} = \{\mathsf{U}_P^{N,\varepsilon} - \hat{F}_N + f_{\text{outer}}\} + \{\Psi_D^{\delta,\varepsilon,N} + \hat{F}_N\},$$

this lemma means that $\mathsf{U}_P^{N,\varepsilon} + \Psi_D^{\delta,\varepsilon,N} + f_{\text{outer}}$ is C^1 at γ_N. In the proof of Proposition 5.2 (at the end of Section 9), we will use this to help construct Ψ_A^ε of Proposition 5.2; see (84).

We next want to study $\mathscr{L}^\varepsilon \mathsf{U}_P^{N,\varepsilon}$. Define

$$\mathsf{v}_P^{N,\varepsilon}(h) \stackrel{\text{def}}{=} \frac{\exp[-2\mu_{N,\varepsilon}h]}{1-\exp[-2\mu_{N,\varepsilon}]}, \qquad h \in [0,1],$$
$$\mathsf{V}_P^{N,\varepsilon}(x) \stackrel{\text{def}}{=} \mathsf{v}_P^{N,\varepsilon}\left(\iota\left(\frac{\mathsf{H}_N^{\text{loc}}(x)}{\mathsf{r}_N}\right)\right), \qquad x \in \mathsf{E}.$$

By (49), we have

$$(\mathscr{L}^\varepsilon \mathsf{U}_P^{N,\varepsilon})(x) = g([\mathsf{E}])\chi_{\mathsf{E}}(x) + g([\mathsf{E}])\chi_{\mathsf{E}}(x)\sum_{i=1}^4 \mathscr{E}_i^{N,\varepsilon}(x)$$
$$+ \sum_{\ell\in\Lambda_P} \hat{\mathsf{u}}_+^{N,\varepsilon}\boldsymbol{\beta}(x)\chi_{Đ_\ell}(x) + \sum_{\ell\in\Lambda_W} \hat{\mathsf{u}}_-^{N,\varepsilon}\boldsymbol{\beta}(x)\chi_{Đ_\ell}(x),$$

where

(53)

$$
\begin{aligned}
\mathscr{E}^{N,\varepsilon}_{b,1}(x) &\overset{\text{def}}{=} \frac{1}{(\mathcal{A}\xi^*)([\mathsf{E}])}\left\{\xi^*(x)-(\mathcal{A}\xi^*)([\mathsf{E}])\right\} \qquad \text{(Lemma 9.4)},\\
\mathscr{E}^{N,\varepsilon}_{b,2}(x) &\overset{\text{def}}{=} \frac{1}{(\mathcal{A}\xi^*)([\mathsf{E}])}\left\{\xi_N(x)-\xi^*(x)\right\} \qquad \text{(Lemma 7.8)},\\
\mathscr{E}^{N,\varepsilon}_{b,3}(x) &\overset{\text{def}}{=} \left\{\frac{\boldsymbol{\sigma}(x)}{(\mathcal{A}\boldsymbol{\sigma})([\mathsf{E}])}-\frac{\xi^*(x)}{(\mathcal{A}\xi^*)([\mathsf{E}])}\right\}(2\mu_{N,\varepsilon})\mathsf{V}^{N,\varepsilon}_P(x) \qquad \text{(Lemma 9.3)},\\
\mathscr{E}^{N,\varepsilon}_{b,4}(x) &\overset{\text{def}}{=} \frac{1}{(\mathcal{A}\boldsymbol{\sigma})([\mathsf{E}])}\{\boldsymbol{\sigma}_N(x)-\boldsymbol{\sigma}(x)\}(2\mu_{N,\varepsilon})\mathsf{V}^{N,\varepsilon}_P(x) \qquad \text{(Lemma 9.2)},\\
\mathscr{E}^{N,\varepsilon}_{b,5}(x) &\overset{\text{def}}{=} -\frac{1}{(\mathcal{A}\xi^*)([\mathsf{E}])}\{\xi_N(x)-\xi^*(x)\}(2\mu_{N,\varepsilon})\mathsf{V}^{N,\varepsilon}_P(x) \qquad \text{(Lemma 9.2)},\\
\mathscr{E}^{N,\varepsilon}_{b,6}(x) &\overset{\text{def}}{=} \frac{\boldsymbol{\beta}_N(x)}{(\mathcal{A}\boldsymbol{\sigma})([\mathsf{E}])}\mathsf{r}_N\dot{\mathsf{u}}^{N,\varepsilon}_P\left(\iota\left(\frac{\mathsf{H}^{\text{loc}}_N(x)}{\mathsf{r}_N}\right)\right) \qquad \text{(Lemma 7.9)}.
\end{aligned}
$$

We want to show that the effect of the $\mathscr{E}^{N,\varepsilon}_{b,i}$'s is negligible. This is always true for two of them.

Lemma 7.8. *There is a $K>0$ such that* $\sup_{x\in\mathsf{E}}|\mathscr{E}^{N,\varepsilon}_{b,2}(x)|\le K|\boldsymbol{\nu}_N|$ *for all $\varepsilon\in(0,1)$ and $N\in\mathbb{N}$.*

Proof. Use (e.ii) of Lemma 7.4. □

Lemma 7.9. *There is a constant $K>0$ such that* $\sup_{x\in\mathsf{E}}|\mathscr{E}^{N,\varepsilon}_{b,6}(x)|\le K\mathsf{r}_N$ *for all $\varepsilon\in(0,1)$ and $N\in\mathbb{N}$.*

Proof. Since $\lim_{N\to\infty}\mathsf{r}_N=0$, the desired result follows from Lemma 7.5. □

The remainder of the error terms will take some work to bound.

8. Residence time and averaging

Not surprisingly, the $\mathscr{E}^{N,\varepsilon}_{b,i}$ for $i\in\{1,3,4,5\}$ are small due to the interplay of (i) a bound on residence time near γ_N and (ii) averaging. In this section, we develop some relevant technical estimates. Our analysis of the error terms of (53) will be completed in the next section.

For each $N\in\mathbb{N}$, define the set $\mathcal{N}_N\overset{\text{def}}{=}\mathbf{S}\setminus(\mathcal{C}_N\cup\gamma_N)$; on $\mathcal{N}_N$, $\mathfrak{z}^N$ is periodic and $\mathsf{H}^{\text{loc}}_N$ is smooth.

The calculations in E will be fairly complicated, so to get warmed up, we will start with the proof of Lemma 5.6. Lemma 5.6 is essentially a *diffusive* bound; it shows that as long as X is in one of the $\mathsf{Đ}_\ell$'s, diffusivity prevents it from spending too much time near $\partial\mathsf{E}$.

Proof of Lemma 5.6. Define

$$\Upsilon_1^\delta(h) \overset{\text{def}}{=} 2\int_{r=0}^h (h-r)\mathfrak{c}_\wedge(r)dr, \qquad h\in\mathbb{R},$$

$$\Upsilon_2^\delta(x) \overset{\text{def}}{=} \sum_{\ell\in\Lambda}\delta^2\Upsilon_1^\delta\left(\frac{\mathsf{H}_{T,\ell}(x)}{\delta}\right)\chi_{\text{Đ}_\ell}(x), \qquad x\in\mathbf{S}.$$

Then $\Upsilon_1^\delta(0)=\dot\Upsilon_1^\delta(0)=0$; thus $\Upsilon_2^\delta\in C^1(\mathbb{T})$. Furthermore, there is a $K>0$ such that $|\dot\Upsilon_1^\delta(h)|\le K$ and $|\Upsilon_1^\delta(h)|\le K|h|$ for all $h\in\mathbb{R}$. We can also calculate that

$$(\mathscr{L}^\varepsilon\Upsilon_2^\delta)(x) = \sum_{\ell\in\Lambda}\mathfrak{c}_\wedge\left(\frac{\mathsf{H}_{T,\ell}(x)}{\delta}\right)\boldsymbol{\sigma}(x)\chi_{\text{Đ}_\ell}(x)+\delta\sum_{\ell\in\Lambda}\dot\Upsilon_1^\delta\left(\frac{\mathsf{H}_{T,\ell}(x)}{\delta}\right)\boldsymbol{\beta}(x)\chi_{\text{Đ}_\ell}(x)$$

for all ε and δ in $(0,1)$ and all $x\in\mathbf{S}\setminus\partial\mathsf{E}$. This fairly easily leads to the first stated claim (one must use a smoothing argument as in [Sow02, Lemma 6.7] to approximate Υ_2^δ by elements of $C^2(\mathbb{T})$ before applying the martingale problem). To get the second stated claim, we use Young's inequality to see that

$$\sqrt{\boldsymbol{\sigma}(x)} \le \frac{\boldsymbol{\sigma}(x)}{2\sqrt{\delta}}+\frac{\sqrt{\delta}}{2}$$

for all $x\in\mathbf{S}$ and $\delta\in(0,1)$. □

Let's next construct a similar diffusive bound to control the amount of time that X spends near γ_N. This bound does not restrict X to lie in one of the Đ_ℓ's, so we must contend with the fact that $\mathsf{H}_N^{\text{loc}}(X)$ has a drift of size $\mu_{N,\varepsilon}/\mathsf{r}_N^2$ in E (see (49)), which may be large. Note that $|\mathsf{r}_N|\le|\omega_2|$.

LEMMA 8.1. *There is a constant $K>0$ such that*

$$\mathbb{E}^\varepsilon\left[\int_{u=0}^{t\wedge\mathfrak{e}}\eta\left(\frac{\mathsf{H}_N^{\text{loc}}(X_u)}{\delta\mathsf{r}_N}\right)\boldsymbol{\sigma}_N(X_u)du\right]\le K(1+t)\delta\left\{1+|\mu_{N,\varepsilon}|\right\}\int_{\mathbb{R}}|\eta(z)|dz$$

for all δ and ε in $(0,1)$, all $N\in\mathbb{N}$, all $t>0$, and all $\eta\in C(\mathbb{R})\cap L^1(\mathbb{R})$ such that $\eta\ge 0$.

Proof. The proof is similar to that of Lemma 5.6. Fix δ, ε, N, t, and η as required. Set

$$\bar\eta_\delta \overset{\text{def}}{=} \int_{z=-1/(2\delta)}^{1/(2\delta)}\eta(z)dz,$$

$$\Upsilon_1^\delta(h) \overset{\text{def}}{=} \eta\left(\frac{h}{\delta}\right)-\delta\bar\eta_\delta, \qquad h\in\mathbb{R},$$

$$\Upsilon_2^\delta(h) \overset{\text{def}}{=} 2\int_{s=-1/2}^{h} (h-s)\Upsilon_1^\delta(s)ds$$
$$- 2(h+1/2)\int_{s\in\mathcal{W}} (1/2-s)\Upsilon_1^\delta(s)ds, \qquad h \in \mathbb{R},$$
$$\Upsilon_3^{\delta,N}(x) \overset{\text{def}}{=} \mathsf{r}_N^2 \Upsilon_2^\delta\left(\frac{\mathsf{H}_N^{\text{loc}}(x)}{\mathsf{r}_N}\right), \qquad x \in \mathbf{S}.$$

Then $\Upsilon_2^\delta \in C^2(\mathbb{R})$ and

$$\frac{1}{2}\ddot{\Upsilon}_2^\delta(h) = \eta\left(\frac{h}{\delta}\right) - \delta\bar{\eta}_\delta, \qquad h \in \mathbb{R},$$
$$\dot{\Upsilon}_2^\delta(h) = 2\int_{s=-1/2}^{h} \Upsilon_1^\delta(r)dr - 2\int_{r\in\mathcal{W}} \Upsilon_2^\delta(r)(1/2-r)dr, \qquad h \in \mathbb{R},$$
$$\Upsilon_2^\delta\left(-\frac{1}{2}\right) = \Upsilon_2^\delta\left(\frac{1}{2}\right) = 0 \quad \text{and} \quad \dot{\Upsilon}_2^\delta\left(-\frac{1}{2}\right) = \dot{\Upsilon}_2^\delta\left(\frac{1}{2}\right).$$

Thus $\Upsilon_3^{\delta,N} \in C^1(\mathbf{S}) \cap C^2(\mathcal{N}_N)$ and

$$(\mathscr{L}^\varepsilon \Upsilon_3^{\delta,N})(x) = \eta\left(\frac{\mathsf{H}_N^{\text{loc}}(x)}{\delta \mathsf{r}_N}\right)\boldsymbol{\sigma}_N(x) + \mathrm{E}^{\delta,\varepsilon,N}(x)$$

for all $x \in \mathcal{N}_N$, where

$$\mathrm{E}^{\delta,\varepsilon,N}(x) = -\delta\bar{\eta}_\delta\boldsymbol{\sigma}_N(x) + \dot{\Upsilon}_2^\delta\left(\frac{\mathsf{H}_N^{\text{loc}}(x)}{\mathsf{r}_N}\right)\left\{\mu_{N,\varepsilon}\frac{\xi_N(x)}{\beth} + \mathsf{r}_N\boldsymbol{\beta}_N(x)\right\}$$

for all $x \in \mathsf{E}$. Note that $|\dot{\Upsilon}_2^\delta(h)| \le 8\delta\|\eta\|_{L^1(\mathbb{R})}$ and $|\Upsilon_2^\delta(h)| \le 8\delta\|\eta\|_{L^1(\mathbb{R})}(|h| + 1/2)$ for all $h \in \mathbb{R}$. Set

$$K \overset{\text{def}}{=} \sup\left\{\boldsymbol{\sigma}_N(x), \frac{|\xi_N(x)|}{|\beth|}, |\boldsymbol{\beta}_N(x)|, x \in \mathbf{S}, N \in \mathbb{N}\right\}.$$

Then

$$|\mathrm{E}^{\delta,\varepsilon,N}(x)| \le K\delta\left\{1 + 8|\mu_{N,\varepsilon}| + 8\mathsf{r}_N\right\}\|\eta\|_{L^1(\mathbb{R})},$$
$$|\Upsilon_3^{\delta,N}(x)| \le 8\delta\left(\left|\frac{\hbar}{\mathsf{r}_N}\right| + \frac{\mathsf{r}_N}{2\mathsf{r}_N} + \frac{1}{2}\right)\mathsf{r}_N^2\|\eta\|_{L^1(\mathbb{R})} \le 8\delta\left(|\hbar| + |\omega_2|\right)\mathsf{r}_N\|\eta\|_{L^1(\mathbb{R})};$$

recall the assumption that $\mathsf{r}_N < 1$. Combine estimates to get the desired result. □

Define now $\mathfrak{E}(z) \overset{\text{def}}{=} \exp\left[-\sqrt{z^2+1}\right]$ for all $z \in \mathbb{R}$ and note that

$$K_{(54)} \overset{\text{def}}{=} \sup_{z\in\mathbb{R}} \frac{\mathfrak{c}_\wedge(z)}{\mathfrak{E}(z)} \tag{54}$$

is finite. Also note that since $1 + z^2 \le (1+|z|)^2$ for all $z \in \mathbb{R}$, $|z| \le \sqrt{1+z^2} \le 1 + |z|$ for all $z \in \mathbb{R}$, so

$$e^{-1}\exp\left[-|z|\right] \le \mathfrak{E}(z) \le \exp\left[-|z|\right] \tag{55}$$

for all $z \in \mathbb{R}$. We then have

Lemma 8.2 (Diffusive Bound). *There is a constant $K_{8.2} > 0$ such that*

$$\mathbb{E}^\varepsilon\left[\int_{u=0}^{t\wedge\mathfrak{e}} \mathfrak{E}\left(\frac{\mathsf{H}_N^{\text{loc}}(X_u)}{\delta \mathsf{r}_N}\right)\boldsymbol{\sigma}_N(X_u)du\right] \le K_{8.2}(1+t)\delta\left\{1+|\mu_{N,\varepsilon}|\right\},$$
$$\mathbb{E}^\varepsilon\left[\int_{u=0}^{t\wedge\mathfrak{e}} \mathfrak{E}\left(\frac{\mathsf{H}_N^{\text{loc}}(X_u)}{\delta \mathsf{r}_N}\right)\sqrt{\boldsymbol{\sigma}_N(X_u)}du\right] \le K_{8.2}(1+t)\sqrt{\delta\left\{1+|\mu_{N,\varepsilon}|\right\}}$$

for all δ and ε in $(0,1)$, all $N \in \mathbb{N}$, and all $t > 0$.

Proof. The first statement directly follows from Lemma 8.1 by using $\eta \overset{\text{def}}{=} \mathfrak{E}$. To get the second claimed result, we use Young's inequality to see that

$$\sqrt{\boldsymbol{\sigma}_N(x)} \le \frac{\sqrt{\delta\{1+|\mu_{N,\varepsilon}|\}}}{2} + \frac{1}{2\sqrt{\delta\{1+|\mu_{N,\varepsilon}|\}}}\boldsymbol{\sigma}_N(x)$$

for all $x \in \mathbf{S}$. Using the first claim, we can now get the second. □

We now can prove our basic residence time result in E. As a preliminary, we first state a somewhat technical estimate near the $\mathfrak{x}_\ell$'s. This estimate is like Lemma 5.3 of [Sow05].

Lemma 8.3. *There is a $\mathfrak{h} \in C^\infty(\mathbb{T})$, a collection $\{Ƀ^\varepsilon; \varepsilon \in (0,1)\}$ of elements of $C^\infty(\mathbb{T})$, and a constant $K_{8.3} > 0$ such that*

(i) $0 \le \mathfrak{h} \le 1$, *and $\mathfrak{h} = 1$ in a neighborhood of each of the $\mathfrak{x}_\ell$'s,*
(ii) $\operatorname{supp} Ƀ \subset \bigcup_{\ell\in\Lambda}\mathcal{U}_\ell$,

and such that $\mathfrak{h}(x) \le (\mathscr{L}^\varepsilon Ƀ^\varepsilon)(x) + K_{8.3}\{\mathfrak{l}(\varepsilon)\boldsymbol{\sigma}(x)+\varepsilon\}$, $|Ƀ^\varepsilon(x)| \le K_{8.3}\varepsilon^2\mathfrak{l}(\varepsilon)$, and $\sqrt{\langle dɃ^\varepsilon, dɃ^\varepsilon\rangle(x)} \le K_{8.3}\varepsilon$ for all $\varepsilon \in (0,1)$ and $x \in \mathbb{T}$.

We delay the proof of this result until the end of the section. Observe that

$$v_{(56)} \overset{\text{def}}{=} \inf\{\boldsymbol{\sigma}_N(x) + \mathfrak{h}(x) : x \in \mathbf{S},\ N \in \mathbb{N}\} \tag{56}$$

is positive; thus

(57)
$$1 \le \frac{1}{v_{(56)}}\{\boldsymbol{\sigma}_N(x)+\mathfrak{h}(x)\} \le \frac{1+K_{8.3}}{v_{(56)}}\mathfrak{l}(\varepsilon)\boldsymbol{\sigma}_N(x) + \frac{K_{8.3}}{v_{(56)}}\varepsilon + \frac{1}{v_{(56)}}(\mathscr{L}^\varepsilon Ƀ^\varepsilon)(x)$$

for all $x \in \mathbf{S}$ and $N \in \mathbb{N}$.

Proposition 8.4. *There is a $K > 0$ such that for all $t \ge 0$,*

$$\mathbb{E}^\varepsilon\left[\int_{u=0}^{t\wedge\mathfrak{e}} \mathfrak{c}_\wedge\left(\frac{\mathsf{H}_N^{\text{loc}}(X_u)}{\delta_1 \mathsf{r}_N}\right)du\right] \le K(1+t)\left\{\delta_1(1+|\mu_{N,\varepsilon}|) + \frac{\varepsilon\sqrt{1+|\mu_{N,\varepsilon}|}}{\mathsf{r}_N\sqrt{\delta_1}}\right\}\mathfrak{l}(\varepsilon) \tag{58}$$

for all $\delta_1 \in (0,1/4)$ *and* $\varepsilon \in (0,1)$ *and* $N \in \mathbb{N}$ *and such that*

$$\sum_{\ell\in\Lambda} \mathbb{E}^\varepsilon\left[\int_{u=0}^{t\wedge\mathfrak{e}} \mathfrak{c}_\wedge\left(\frac{\mathsf{H}_N^{\mathrm{loc}}(X_u)}{\delta_2}\right)\chi_{Ð_\ell}(X_u)du\right] \le K(1+t)\left\{\delta_2 + \frac{\varepsilon}{\sqrt{\delta_2}} + \left(\frac{\varepsilon\{1+|\mu_{N,\varepsilon}|\}}{r_N}\right)^{2/3}\right\}\mathfrak{l}(\varepsilon) \tag{59}$$

for all δ_2 *and* $\varepsilon \in (0,1)$ *and* $N \in \mathbb{N}$ *such that* $\delta_2 > \varepsilon^2$ *and* $\varepsilon < \mathsf{r}_N/8$.

Proof. Fix δ_d, δ_e, and ε in $(0,1)$, $N \in \mathbb{N}$, and $t > 0$. Define

$$\mathsf{c}^{\delta_d,\delta_e,N}(x) \stackrel{\mathrm{def}}{=} \mathfrak{c}_\wedge\left(\frac{\mathsf{H}_N^{\mathrm{loc}}(x)}{\delta_d}\right)\chi_{\mathbf{S}\setminus\mathsf{E}}(x) + \mathfrak{c}_\wedge\left(\frac{\mathsf{H}_N^{\mathrm{loc}}(x)}{\delta_e \mathsf{r}_N}\right)\chi_{\mathsf{E}}(x)$$

for all $x \in \mathsf{E}$ (remember that according to Lemma 6.4, $\mathsf{H}_N^{\mathrm{loc}}$ agrees with the $\mathsf{H}_{T,\ell}$'s on the $Ð_\ell$'s). By (57), we have that for all $x \in \mathbf{S}$,

$$\mathsf{c}^{\delta_d,\delta_e,N}(x) \le \frac{K_{8.3}}{\upsilon_{(56)}}\varepsilon + \frac{1+K_{8.3}}{\upsilon_{(56)}}\{\mathrm{E}_1^{\delta_d,\varepsilon,N}(x) + \mathrm{E}_3^{\delta_d,\varepsilon,N}(x)\} + \frac{1}{\upsilon_{(56)}}\mathrm{I}^{\delta_d,\delta_e,\varepsilon,N}(x), \tag{60}$$

where

$$\mathrm{E}_1^{\delta_d,\varepsilon,N}(x) = \mathfrak{l}(\varepsilon)\mathfrak{c}_\wedge\left(\frac{\mathsf{H}_N^{\mathrm{loc}}(x)}{\delta_d}\right)\boldsymbol{\sigma}(x)\chi_{\mathbf{S}\setminus\mathsf{E}}(x),$$
$$\mathrm{E}_2^{\delta_e,\varepsilon,N}(x) = \mathfrak{l}(\varepsilon)\mathfrak{c}_\wedge\left(\frac{\mathsf{H}_N^{\mathrm{loc}}(x)}{\delta_e \mathsf{r}_N}\right)\boldsymbol{\sigma}_N(x)\chi_{\mathsf{E}}(x),$$
$$\mathrm{I}^{\delta_d,\delta_e,\varepsilon,N}(x) = (\mathscr{L}^\varepsilon Ъ^\varepsilon)(x)\mathsf{c}^{\delta_d,\delta_e,N}(x).$$

From Lemmas 5.6 and 8.2, we have that

$$\begin{aligned}\mathbb{E}^\varepsilon\left[\int_{u=0}^{t\wedge\mathfrak{e}} \mathrm{E}_1^{\delta_d,\varepsilon,N}(X_u)du\right] &\le K_{5.6}(1+t)\delta_d\mathfrak{l}(\varepsilon),\\ \mathbb{E}^\varepsilon\left[\int_{u=0}^{t\wedge\mathfrak{e}} \mathrm{E}_2^{\delta_e,\varepsilon,N}(X_u)du\right] &\le K_{(54)}K_{8.2}(1+t)\delta_e\mathfrak{l}(\varepsilon)\{1+|\mu_{N,\varepsilon}|\}.\end{aligned} \tag{61}$$

Define next $\Upsilon^{\delta_d,\delta_e,\varepsilon,N}(x) \stackrel{\mathrm{def}}{=} Ъ^\varepsilon(x)\mathsf{c}^{\delta_d,\delta_e,N}(x)$ for all $x \in \mathbf{S}$. If $\delta_e \in (0,1/4)$, then $\Upsilon^{\delta_d,\delta_\varepsilon,\varepsilon,N}$ is smooth (in particular, it is smooth at $\mathcal{C}_N$). Noting also that $Ч_N = Ч$ on the support of $Ъ^\varepsilon$, we have that

$$(\mathscr{L}^\varepsilon\Upsilon^{\delta_d,\delta_e,\varepsilon,N})(x) = \mathrm{I}^{\delta_d,\delta_e,\varepsilon,N} + \mathrm{E}_3^{\delta_d,\varepsilon,N}(x)\chi_{\mathbf{S}\setminus\mathsf{E}}(x) + \mathrm{E}_4^{\delta_e,\varepsilon,N}(x)\chi_{\mathsf{E}}(x)$$

for all $x \in \mathbf{S}$, where

$$\mathrm{E}_3^{\delta_d,\varepsilon,N}(x) = Ъ^\varepsilon(x)\left\{\dot{\mathfrak{c}}_\wedge\left(\frac{\mathsf{H}_N^{\mathrm{loc}}(x)}{\delta_d}\right)\frac{\boldsymbol{\beta}(x)}{\delta_d} + \ddot{\mathfrak{c}}_\wedge\left(\frac{\mathsf{H}_N^{\mathrm{loc}}(x)}{\delta_d}\right)\frac{\boldsymbol{\sigma}(x)}{2\delta_d^2}\right\} + \frac{1}{\delta_d}\left\langle dЪ^\varepsilon, d\mathsf{H}_N^{\mathrm{loc}}\right\rangle(x)\dot{\mathfrak{c}}_\wedge\left(\frac{\mathsf{H}_N^{\mathrm{loc}}(x)}{\delta_d}\right),$$

$$\mathrm{E}_4^{\delta_e,\varepsilon,N}(x) = \mathfrak{P}^\varepsilon(x)\left\{\dot{\mathfrak{c}}_\wedge\left(\frac{\mathsf{H}_N^{\mathrm{loc}}(x)}{\delta_e \mathsf{r}_N}\right)\frac{\boldsymbol{\beta}_N(x)}{\mathsf{r}_N\delta_e} + \ddot{\mathfrak{c}}_\wedge\left(\frac{\mathsf{H}_N^{\mathrm{loc}}(x)}{\delta_e \mathsf{r}_N}\right)\frac{\boldsymbol{\sigma}_N(x)}{2\delta_e^2\mathsf{r}_N^2}\right\} + \frac{1}{\delta_e\mathsf{r}_N}\left\langle d\mathfrak{P}^\varepsilon, d\mathsf{H}_N^{\mathrm{loc}}\right\rangle(x)\dot{\mathfrak{c}}_\wedge\left(\frac{\mathsf{H}_N^{\mathrm{loc}}(x)}{\delta_e \mathsf{r}_N}\right)$$

for all $x \in \mathbb{T}$. It is easy to see that there is a $K_1 > 0$ such that $|\dot{\mathfrak{c}}_\wedge(z)| \le K_1\mathfrak{E}(z)$ and $|\ddot{\mathfrak{c}}_\wedge(z)| \le K_1\mathfrak{E}(z)$ for all $z \in \mathbb{R}$; also recall (22). Thus, there is a $K_2 > 0$ such that

$$\begin{aligned}\left|\mathrm{E}_3^{\delta_d,\varepsilon,N}(x)\right| &\le \frac{K_2\varepsilon^2\mathfrak{l}(\varepsilon)}{\delta_d} + \frac{K_2\varepsilon^2\mathfrak{l}(\varepsilon)}{\delta_d^2}\mathfrak{c}_\wedge\left(\frac{\mathsf{H}_N^{\mathrm{loc}}(x)}{2\delta_d}\right)\boldsymbol{\sigma}(x) \\ &\quad + \frac{K_2\varepsilon}{\delta_d}\mathfrak{c}_\wedge\left(\frac{\mathsf{H}_N^{\mathrm{loc}}(x)}{2\delta_d}\right)\sqrt{\boldsymbol{\sigma}(x)}, \\ \left|\mathrm{E}_4^{\delta_e,\varepsilon,N}(x)\right| &\le \frac{K_2\varepsilon^2\mathfrak{l}(\varepsilon)}{\delta_e\mathsf{r}_N} + \frac{K_2\varepsilon^2\mathfrak{l}(\varepsilon)}{\delta_e^2\mathsf{r}_N^2}\mathfrak{E}\left(\frac{\mathsf{H}_N^{\mathrm{loc}}(x)}{\delta_e\mathsf{r}_N}\right)\boldsymbol{\sigma}_N(x) \\ &\quad + \frac{K_2\varepsilon}{\delta_e\mathsf{r}_N}\mathfrak{E}\left(\frac{\mathsf{H}_N^{\mathrm{loc}}(x)}{\delta_e\mathsf{r}_N}\right)\sqrt{\boldsymbol{\sigma}_N(x)}\end{aligned}$$

and $|\Upsilon^{\delta_d,\delta_e,\varepsilon,N}(x)| \le K_2\varepsilon^2\mathfrak{l}(\varepsilon)$ for all δ_d and ε in $(0,1)$, $\delta_e \in (0,1/4)$, all $N \in \mathbb{N}$, and all $x \in \mathbf{S}$. Combining and using Lemmas 5.6 and 8.2, we can find a $K_3 > 0$ such that

(62)
$$\mathbb{E}^\varepsilon\left[\int_{u=0}^{t\wedge\mathfrak{e}}\mathrm{I}^{\delta_d,\delta_e,\varepsilon,N}(X_u)du\right] \le K_3(1+t)\left\{\varepsilon^2\mathfrak{l}(\varepsilon) + \frac{\varepsilon^2\mathfrak{l}(\varepsilon)}{\delta_d} + \frac{\varepsilon^2\mathfrak{l}(\varepsilon)\delta_d}{\delta_d^2} + \frac{\varepsilon\sqrt{\delta_d}}{\delta_d}, + \frac{\varepsilon^2\mathfrak{l}(\varepsilon)}{\delta_e\mathsf{r}_N} + \frac{\varepsilon^2\mathfrak{l}(\varepsilon)\delta_e\{1+|\mu_{N,\varepsilon}|\}}{\delta_e^2\mathsf{r}_N^2} + \frac{\varepsilon\sqrt{\delta_e\{1+|\mu_{N,\varepsilon}|\}}}{\delta_e\mathsf{r}_N}\right\}$$

for all δ_d and ε in $(0,1)$, all $\delta_e \in (0,1/4)$, all $N \in \mathbb{N}$, and all $t > 0$. Combining (60), (61), and (62), we have that there is a $K_4 > 0$ such that

$$\mathbb{E}^\varepsilon\left[\int_{u=0}^{t\wedge\mathfrak{e}}\mathsf{c}^{\delta_d,\delta_e,N}(X_u)du\right] \le K_4(1+t)\left\{\varepsilon + \varepsilon^2 + \delta_d + \delta_e\{1+|\mu_{N,\varepsilon}|\} + \frac{\varepsilon^2}{\delta_d} + \frac{\varepsilon}{\sqrt{\delta_d}} + \frac{\varepsilon^2}{\delta_e\mathsf{r}_N} + \frac{\varepsilon^2\{1+|\mu_{N,\varepsilon}|\}}{\delta_e\mathsf{r}_N^2} + \frac{\varepsilon\sqrt{1+|\mu_{N,\varepsilon}|}}{\sqrt{\delta_e}\mathsf{r}_N}\right\}\mathfrak{l}(\varepsilon) \tag{63}$$

for all δ_d and ε in $(0,1)$, all $\delta_e \in (0,1/4)$, all $N \in \mathbb{N}$, and all $t > 0$.

Consider now the quantity in braces in (63). Since ε and δ_d are in $(0,1)$, $\varepsilon^2 < \varepsilon < \varepsilon/\sqrt{\delta_d}$. Since $(1+|\mu_{N,\varepsilon}|)/\mathsf{r}_N > 1/\mathsf{r}_N > 1/|\omega_2|$, the seventh term is effectively smaller than the eighth. Since we are interested in showing that residence time is small, we should also restrict our interest to δ_d and δ_e such that

$$\delta_d > \varepsilon^2 \qquad \text{and} \qquad \delta_e > \left(\frac{\varepsilon}{\mathsf{r}_N}\right)^2 \{1 + |\mu_{N,\varepsilon}|\}. \tag{64}$$

When this is so, the fifth term in braces in (63) is smaller than the sixth term and the eighth term is smaller than the ninth. Thus, for all δ_d and ε in $(0,1)$, and all $\delta_e \in (0, 1/4)$, all $N \in \mathbb{N}$, and all $t > 0$ such that (64) holds, we have that

$$\begin{aligned} &\mathbb{E}^\varepsilon\left[\int_{u=0}^{t\wedge\mathfrak{e}} \mathsf{c}^{\delta_d,\delta_e,N}(X_u)du\right] \\ &\qquad \le K_4(1+t)\left\{\delta_d + \delta_e\{1+|\mu_{N,\varepsilon}|\} + \frac{\varepsilon}{\sqrt{\delta_d}} + \frac{\varepsilon\sqrt{1+|\mu_{N,\varepsilon}|}}{\sqrt{\delta_e}\mathsf{r}_N}\right\}\mathfrak{l}(\varepsilon). \end{aligned} \tag{65}$$

We first prove (58). If $\delta_1 \le (\varepsilon/\mathsf{r}_N)^2\{1+|\mu_{N,\varepsilon}|\}$, then

$$\mathfrak{c}_\wedge\left(\frac{\mathsf{H}_N^{\mathrm{loc}}}{\delta_1\mathsf{r}_N}\right) \le 1 \le \frac{\varepsilon\sqrt{1+|\mu_{N,\varepsilon}|}}{\mathsf{r}_N\sqrt{\delta_1}} \le \delta_1(1+|\mu_{N,\varepsilon}|) + \frac{\varepsilon\sqrt{1+|\mu_{N,\varepsilon}|}}{\mathsf{r}_N\sqrt{\delta_1}}.$$

If $\delta_1 < (\varepsilon/\mathsf{r}_N)^2\{1+|\mu_{N,\varepsilon}|\}$, we use (65) with $\delta_e = \delta_1$ and $\delta_d = \delta_1\mathsf{r}_N$. Note that when $\delta_1 < (\varepsilon/\mathsf{r}_N)^2\{1+|\mu_{N,\varepsilon}|\}$,

$$\begin{aligned} \delta_d = \delta_1\mathsf{r}_N > \frac{\varepsilon^2}{\mathsf{r}_N}\{1+|\mu_{N,\varepsilon}|\} \ge \varepsilon^2 \qquad &(\text{first requirement of (64)}), \\ \delta_d = \delta_1\mathsf{r}_N < \delta_1|\omega_2| < \delta_1\{1+|\mu_{N,\varepsilon}|\}|\omega_2| \qquad &(\text{use in (65)}), \\ \frac{\varepsilon}{\sqrt{\delta_d}} \le |\omega_2|^{1/2}\frac{\varepsilon\sqrt{1+|\mu_{N,\varepsilon}|}}{\sqrt{\delta_1\mathsf{r}_N}} \qquad &(\text{use in (65)}). \end{aligned}$$

We can now get (58) from (65).

We next prove (59). If $1+|\mu_{N,\varepsilon}| \ge \mathsf{r}_N/\varepsilon$, then

$$\mathfrak{c}_\wedge\left(\frac{\mathsf{H}_N^{\mathrm{loc}}}{\delta_2}\right) \le 1 \le \left(\frac{\varepsilon}{\mathsf{r}_N}\{1+|\mu_{N,\varepsilon}|\}\right)^{2/3} \le \delta_2 + \frac{\varepsilon}{\sqrt{\delta_2}} + \left(\frac{\varepsilon}{\mathsf{r}_N}\{1+|\mu_{N,\varepsilon}|\}\right)^{2/3}.$$

If $1+|\mu_{N,\varepsilon}| < \mathsf{r}_N/\varepsilon$, we use (65) with $\delta_d = \delta_2$ and set

$$\delta_e = \left(\frac{\varepsilon}{\mathsf{r}_N}\right)^{2/3}\frac{1}{\{1+|\mu_{N,\varepsilon}|\}^{1/3}}.$$

When $1+|\mu_{N,\varepsilon}| < \mathsf{r}_N/\varepsilon$ and $\varepsilon < \mathsf{r}_N/8$,

$$\begin{aligned} \delta_e\{1+|\mu_{N,\varepsilon}|\} + \frac{\varepsilon\sqrt{1+|\mu_{N,\varepsilon}|}}{\mathsf{r}_N\sqrt{\delta_e}} = 2\left(\frac{\varepsilon}{\mathsf{r}_N}\{1+|\mu_{N,\varepsilon}|\}\right)^{2/3} \qquad &(\text{use in (65)}), \\ \delta_e \le \left(\frac{\varepsilon}{\mathsf{r}_N}\right)^{2/3} < \left(\frac{1}{8}\right)^{2/3} = \frac{1}{4}, \end{aligned}$$

$$\delta_e = \left(\frac{\varepsilon}{\mathsf{r}_N}\right)^2 \{1+|\mu_{N,\varepsilon}|\} \left(\frac{\mathsf{r}_N/\varepsilon}{1+|\mu_{N,\varepsilon}|}\right)^{4/3} \geq \left(\frac{\varepsilon}{\mathsf{r}_N}\right)^2 \{1+|\mu_{N,\varepsilon}|\}$$
(second requirement of (64)).

We can now get (59) from (65). □

We finally give

Proof of Proposition 5.8. We use Lemma 8.4. First, note that if $\delta > \varepsilon^{2/3}$, then $\delta > \varepsilon^2$ and $\varepsilon/\sqrt{\delta} = \delta(\varepsilon/\delta^{3/2}) \leq \delta$. From (38) and (48), we have that there is a constant $K > 0$ such that

$$\frac{\varepsilon}{\mathsf{r}_N}\{1+|\mu_{N,\varepsilon}|\} \leq K\left\{\varepsilon \mathsf{a}_N^{(d)} + \frac{1}{\varepsilon \mathsf{a}_{N+1}^{(d)} \mathsf{a}_N^{(d)}}\right\}$$

for all $\varepsilon \in (0,1)$ and $N \in \mathbb{N}$. □

While this result gives us some control over the amount of time spent near γ_N, it is weaker than we need, particularly when $|\mu_{N,\varepsilon}|$ becomes large, and when we bound $\mathscr{E}_{b,3}^{N,\varepsilon}$. To extend the bound on residence time near γ_N, we average ξ^* near γ_N. Note that we want to carry out this average along the integral curves of $\mathfrak{z}^N$ rather than $\mathfrak{z}$, since the orbits of $\mathfrak{z}^N$ are periodic (while those of $\mathfrak{z}$ are not). Hopefully, averages over integral curves of $\mathfrak{z}^N$ are sufficiently close to those of $\mathfrak{z}$. Since the orbit time of Ψ_N tends to infinity near γ_N, we also need to average with respect to a speeded-up version of $\mathfrak{z}$ (whose orbit time near γ_N stays bounded). Note that because $\mathfrak{z}$ is ambiguous at γ_N, we cannot average exactly *on* γ_N. For each $N \in \mathbb{N}$, define the vector field

$$\bar{\Psi}_N(x) \stackrel{\text{def}}{=} \frac{\Psi_N(x)}{\boldsymbol{\sigma}(x)}, \qquad x \in \overline{\mathsf{E}} \setminus \mathfrak{X}$$

and let $\mathbf{p}^N$ be the flow of diffeomorphisms of E generated by $\bar{\Psi}_N$; i.e., by

$$\begin{aligned} \dot{\mathbf{p}}_t^N(x) &= \bar{\Psi}_N(\mathbf{p}_t^N(x)), \qquad t \in \mathbb{R} \\ \mathbf{p}_0^N(x) &= x \end{aligned} \qquad x \in \mathsf{E}$$

(since E is invariant under $\mathfrak{z}^N$, $\mathbf{p}^N$ is well-defined on E).

LEMMA 8.5. *There is a $\lambda_{8.5} \in (0,1)$ such that for each $\lambda \in (0, \lambda_{8.5})$, there is a constant $K_\lambda^{8.5}$, an $N_\lambda^{8.5} \in \mathbb{N}$, a collection $\{\check{\Phi}_N^{\mathbf{p};\lambda}; N \in \mathbb{N}, N \geq N_\lambda^{8.5}\}$ of functions, and a collection $\{\mathscr{E}^{8.5}(\lambda, N); N \in \mathbb{N}, N \geq N_\lambda^{8.5}\}$ of numbers such that such that for each integer $N \geq N_\lambda^{8.5}$, $\check{\Phi}_N^{\mathbf{p};\lambda} \in C^2(\mathcal{N}_N)$ and $|\check{\Phi}_N^{\mathbf{p};\lambda}(x)| \leq K_\lambda^{8.5}$, $\|D\check{\Phi}_N^{\mathbf{p};\lambda}(x)\| \leq K_\lambda^{8.5}$, and $\|D^2\check{\Phi}_N^{\mathbf{p};\lambda}(x)\| \leq K_\lambda^{8.5}$ for all $x \in \mathcal{N}_N$ and such that*

$$(66) \qquad \left|(\Psi\check{\Phi}_N^{\mathbf{p};\lambda})(x) - \{\xi^*(x) - \mathfrak{I}\boldsymbol{\sigma}(x)\}\right| \leq \mathscr{E}^{8.5}(\lambda, N)\boldsymbol{\sigma}(x) + K_\lambda^{8.5}\mathsf{r}_N$$

for all $\lambda \in (0, \lambda_{8.5})$*, all integers* $N \geq N^{8.5}_\lambda$*, and all* $x \in \mathcal{N}_N$*, and such that* $\lim_{\lambda \to 0} \overline{\lim}_{N \to \infty} \mathscr{E}^{8.5}(\lambda, N) = 0$.

Before proceeding, define

$$\bar{\xi}^*(x) \stackrel{\text{def}}{=} \frac{\xi^*(x)}{\boldsymbol{\sigma}(x)}, \qquad x \in \overline{\mathsf{E}} \setminus \mathfrak{X}; \tag{67}$$

we will use $\bar{\xi}^*$ in the proof of Lemma 8.5; we will also need it below. Note that $\bar{\xi}^*$ is bounded. Also, note that there is a $K_{(68)} > 0$ such that

$$\boldsymbol{\sigma} \leq K_{(68)} \boldsymbol{\sigma}_N \tag{68}$$

on $\mathbf{S}$ for all $N \in \mathbb{N}$.

We now have

Lemma 8.6. *There is a* $K > 0$ *and an* $N^{8.6} \in \mathbb{N}$ *such that for all* $t \geq 0$*,*

$$\begin{aligned}\mathbb{E}^\varepsilon &\left[\int_{u=0}^{t \wedge \mathfrak{e}} \mathfrak{E}\left(\frac{\mathsf{H}^{\text{loc}}_N(X_u)|\mu_{N,\varepsilon}|}{r_N}\right) \boldsymbol{\sigma}_N(X_u) \chi_{\mathsf{E}}(X_u) du\right] \\ &\qquad \leq K(1+t)\left\{\frac{1}{|\mu_{N,\varepsilon}|+1} + r_N\{1+|\mu_{N,\varepsilon}|\} + \frac{\varepsilon^2\{1+|\mu_{N,\varepsilon}|\}^4}{r_N^2}\right\}\end{aligned}$$

for all $\varepsilon \in (0,1)$ *and all integers* N *such that* $N \geq N^{8.6}$.

Proof. Fix $\varepsilon \in (0,1)$ and $N \in \mathbb{N}$. The relevance of this bound is clearly when $|\mu_{N,\varepsilon}|$ is large; i.e., when $1/|\mu_{N,\varepsilon}|$ is small. As we shall see, it will be convenient to focus on the case that $|\mu_{N,\varepsilon}| > 2$. If $|\mu_{N,\varepsilon}| \leq 2$, then $1 \leq 3/(1+|\mu_{N,\varepsilon}|)$, so

$$\begin{aligned}\mathfrak{E}\left(\frac{\mathsf{H}^{\text{loc}}_N|\mu_{N,\varepsilon}|}{\mathsf{r}_N}\right)\boldsymbol{\sigma}_N &\leq \sup_{N \in \mathbb{N}} \|\boldsymbol{\sigma}_N\|_{C(\mathbb{T})} \leq \sup_{N \in \mathbb{N}} \|\boldsymbol{\sigma}_N\|_{C(\mathbb{T})} \frac{3}{1+|\mu_{N,\varepsilon}|} \\ &\leq 3 \sup_{N \in \mathbb{N}} \|\boldsymbol{\sigma}_N\|_{C(\mathbb{T})} \left\{\frac{1}{1+|\mu_{N,\varepsilon}|} + \frac{\varepsilon^2 \mu^4_{N,\varepsilon}}{\mathsf{r}_N^2}\right\}.\end{aligned}$$

Assume now that $|\mu_{N,\varepsilon}| > 2$. For convenience, define

$$\delta_{N,\varepsilon} \stackrel{\text{def}}{=} \frac{1}{|\mu_{N,\varepsilon}|} \qquad \text{and} \qquad \mathsf{C}_{N,\varepsilon} \stackrel{\text{def}}{=} \int_{-1/(2\delta_{N,\varepsilon})}^{1/(2\delta_{N,\varepsilon})} \mathfrak{E}(s) ds;$$

then $0 \leq \mathsf{C}_{N,\varepsilon} \leq \int_{s \in \mathbb{R}} \mathfrak{E}(s) ds < \infty$.

We first solve an averaged PDE. Set

$$\begin{aligned}\Upsilon^{\varepsilon,N}_1(h) &\stackrel{\text{def}}{=} 2\int_{s=-1/2}^{h} \mathfrak{E}\left(\frac{s}{\delta_{N,\varepsilon}}\right) \exp\left[-|\mu_{N,\varepsilon}|(h-s)\right] ds \\ &\quad + 2\frac{\exp\left[-|\mu_{N,\varepsilon}|\left(h+\frac{1}{2}\right)\right]}{1-\exp\left[-|\mu_{N,\varepsilon}|\right]} \int_{s \in \mathcal{W}} \mathfrak{E}\left(\frac{s}{\delta_{N,\varepsilon}}\right) \exp\left[-|\mu_{N,\varepsilon}|\left(\frac{1}{2}-s\right)\right] ds,\end{aligned}$$

$$\Upsilon_2^{\varepsilon,N}(h) \stackrel{\text{def}}{=} \int_{s=-1/2}^{h} \Upsilon_1^{\varepsilon,N}(s)ds - \frac{2\delta_{N,\varepsilon}\mathsf{C}_{N,\varepsilon}}{|\mu_{N,\varepsilon}|}\left(h+\frac{1}{2}\right),$$

$$\Upsilon_3^{\varepsilon,N}(h) \stackrel{\text{def}}{=} \Upsilon_2^{\varepsilon,N}(\mathfrak{s}(\mu_{N,\varepsilon})h)$$

for all $h \in \mathcal{W}$, and define

$$\Upsilon_4^{\varepsilon,N}(x) \stackrel{\text{def}}{=} \mathsf{r}_N^2 \Upsilon_3^{\varepsilon,N}\left(\frac{\mathsf{H}_N^{\text{loc}}(x)}{\mathsf{r}_N}\right)\chi_{\mathsf{E}}(x) + \sum_{\ell\in\Lambda}\left\{\mathsf{r}_N^2\Upsilon_3^{\varepsilon,N}(0) + \mathsf{r}_N\dot{\Upsilon}_3^{\varepsilon,N}(0)\mathsf{H}_{T,\ell}(x)\right\}\chi_{\text{Ð}_\ell}(x)$$

for all $x \in \mathbf{S}$. Then $\Upsilon_3^{\varepsilon,N} \in C^\infty(\mathcal{W})$,

$$\frac{\mu_{N,\varepsilon}}{2}\dot{\Upsilon}_3^{\varepsilon,N}(h) + \frac{1}{2}\ddot{\Upsilon}_3^{\varepsilon,N}(h) = \mathfrak{E}\left(\frac{h}{\delta_{N,\varepsilon}}\right) - \delta_{N,\varepsilon}\mathsf{C}_{N,\varepsilon}, \qquad h \in \mathcal{W},$$

$$\Upsilon_3^{\varepsilon,N}(-1/2) = \Upsilon_3^{\varepsilon,N}(1/2) = 0 \qquad \text{and} \qquad \dot{\Upsilon}_3^{\varepsilon,N}(-1/2) = \dot{\Upsilon}_3^{\varepsilon,N}(1/2).$$

The true averaged PDE would have $\mu_{N,\varepsilon}$ in place of $\mu_{N,\varepsilon}/2$; we allow for an averaging error.

Note that

$$\dot{\Upsilon}_3^{\varepsilon,N}(h) = \mathfrak{s}(\mu_{N,\varepsilon})\Upsilon_1^{\varepsilon,N}(\mathfrak{s}(\mu_{N,\varepsilon})h) - \frac{2\delta_{N,\varepsilon}\mathsf{C}_{N,\varepsilon}}{\daleth\mu_{N,\varepsilon}}, \qquad h \in \mathcal{W}.$$

Then for $x \in \mathbf{S} \setminus \gamma_N$,

$$(\mathscr{L}^\varepsilon\Upsilon_4^{\varepsilon,N})(x) = \mathfrak{E}\left(\frac{\mathsf{H}_N^{\text{loc}}(x)}{\delta_{N,\varepsilon}\mathsf{r}_N}\right)\boldsymbol{\sigma}_N(x) + \mathrm{I}_1^{\varepsilon,N}(x) + \sum_{i=1}^{6}\mathrm{E}_i^{\varepsilon,N}(x),$$

where

$$\mathrm{I}_1^{\varepsilon,N}(x) = |\mu_{N,\varepsilon}|\left\{\frac{\xi^*(x)}{\daleth} - \frac{\boldsymbol{\sigma}(x)}{2}\right\}\Upsilon_1^{\varepsilon,N}\left(\mathfrak{s}(\mu_{N,\varepsilon})\frac{\mathsf{H}_N^{\text{loc}}(x)}{\mathsf{r}_N}\right)\chi_{\mathsf{E}}(x),$$

$$\mathrm{E}_1^{\varepsilon,N}(x) = \frac{|\mu_{N,\varepsilon}|}{\daleth}\{\xi_N(x) - \xi^*(x)\}\Upsilon_1^{\varepsilon,N}\left(\mathfrak{s}(\mu_{N,\varepsilon})\frac{\mathsf{H}_N^{\text{loc}}(x)}{\mathsf{r}_N}\right)\chi_{\mathsf{E}}(x),$$

$$\mathrm{E}_2^{\varepsilon,N}(x) = \frac{|\mu_{N,\varepsilon}|}{2}\{\boldsymbol{\sigma}(x) - \boldsymbol{\sigma}_N(x)\}\Upsilon_1^{\varepsilon,N}\left(\mathfrak{s}(\mu_{N,\varepsilon})\frac{\mathsf{H}_N^{\text{loc}}(x)}{\mathsf{r}_N}\right)\chi_{\mathsf{E}}(x),$$

$$\mathrm{E}_3^{\varepsilon,N}(x) = -\frac{2\mathsf{C}_{N,\varepsilon}\delta_{N,\varepsilon}}{\daleth}\left\{\frac{\xi_N(x)}{\daleth} - \frac{\boldsymbol{\sigma}_N(x)}{2}\right\}\chi_{\mathsf{E}}(x),$$

$$\mathrm{E}_4^{\varepsilon,N}(x) = -\mathsf{C}_{N,\varepsilon}\delta_{N,\varepsilon}\boldsymbol{\sigma}_N(x)\chi_{\mathsf{E}}(x),$$

$$\mathrm{E}_5^{\varepsilon,N}(x) = \mathsf{r}_N\boldsymbol{\beta}_N(x)\dot{\Upsilon}_3^{\varepsilon,N}\left(\frac{\mathsf{H}_N^{\text{loc}}(x)}{\mathsf{r}_N}\right)\chi_{\mathsf{E}}(x),$$

$$\mathrm{E}_6^{\varepsilon,N}(x) = \sum_{\ell\in\Lambda}\mathsf{r}_N\boldsymbol{\beta}(x)\dot{\Upsilon}_3^{\varepsilon,N}(0)\chi_{\text{Ð}_\ell}(x).$$

Let's now bound $\dot{\Upsilon}_3^{\varepsilon,N}$. We first note that $\Upsilon_1^{\varepsilon,N}$ is nonnegative and $\mathfrak{E}$ is bounded. Thus, there is a $K_1>0$ such that

$$0 \le \Upsilon_1^{\varepsilon,N}(h) \le \frac{K_1}{|\mu_{N,\varepsilon}|} = K_1\delta_{N,\varepsilon} \tag{69}$$

for all $\varepsilon\in(0,1)$ and $N\in\mathbb{N}$ such that $|\mu_{N,\varepsilon}|>2$ (note that $1-\exp[-2|\mu_{N,\varepsilon}|]\ge 1-e^{-4}$ when $|\mu_{N,\varepsilon}|>2$). We then can find a $K_2>0$ such that $|\dot{\Upsilon}_2^{\varepsilon,N}(h)|\le K_2\delta_{N,\varepsilon}$ and $|\Upsilon_2^{\varepsilon,N}(h)|\le K_2\delta_{N,\varepsilon}$ for all $h\in\mathcal{W}$, $\varepsilon\in(0,1)$, and $N\in\mathbb{N}$ such that $|\mu_{N,\varepsilon}|>2$. Thus, there is a constant $K_3>0$ such that for all $x\in\mathbf{S}$, $\varepsilon\in(0,1)$, and $N\in\mathbb{N}$ such that $|\mu_{N,\varepsilon}|>2$

$$|\mathrm{E}_i^{\varepsilon,N}(x)| \le K_3|\mu_{N,\varepsilon}||\boldsymbol{\nu}_N| = \frac{K_3}{|\gimel|}\frac{\varepsilon^2\mu_{N,\varepsilon}^2}{\mathsf{r}_N}$$

for $i\in\{1,2\}$ and $|\mathrm{E}_i^{\varepsilon,N}(x)|\le K_3\delta_{N,\varepsilon}$ for $i\in\{3,4,5,6\}$. Hence, there is a $K_4>0$ such that

$$\begin{aligned}\mathbb{E}^\varepsilon\left[\int_{u=0}^{t\wedge\mathfrak{e}}\mathfrak{E}\left(\frac{\mathsf{H}_N^{\mathrm{loc}}(X_u)}{\delta_{N,\varepsilon}\mathsf{r}_N}\right)\boldsymbol{\sigma}_N(X_u)\chi_{\mathsf{E}}(X_u)du\right]\\ +\mathbb{E}^\varepsilon\left[\int_{u=0}^{t\wedge\mathfrak{e}}\mathrm{I}_1^{\varepsilon,N}(X_u)\chi_{\mathsf{E}}(X_u)du\right]\le K_4(1+t)\left\{\delta_{N,\varepsilon}+\frac{\varepsilon^2\mu_{N,\varepsilon}^2}{\mathsf{r}_N}\right\}\end{aligned} \tag{70}$$

for all $\varepsilon\in(0,1)$ and $N\in\mathbb{N}$ such that $|\mu_{N,\varepsilon}|>2$.

We now need to bound $\mathrm{I}_1^{\varepsilon,N}$ from below. We should be able to do this by *averaging*. Fix $\lambda^*\in(0,\lambda_{8.5})$ and an integer $N^{8.6}\ge N_{\lambda^*}^{8.5}$ such that $\mathscr{E}^{8.5}(\lambda_*,N)<\frac{|\gimel|}{2}$ for all integers $N\ge N^{8.6}$. Then

$$\left|(\text{Ч}\breve{\Phi}_N^{\mathbf{p},\lambda^*})(x)-\{\xi^*(x)-\gimel\boldsymbol{\sigma}(x)\}\right|\le\frac{|\gimel|}{2}\boldsymbol{\sigma}(x)+K_{\lambda_*}^{8.6}\mathsf{r}_N$$

for all $x\in\mathcal{N}_N$ and all integers $N\ge N^{8.6}$. Then for all integers $N\ge N^{8.6}$ and $x\in\mathcal{N}_N$,

$$\frac{\xi^*(x)}{\gimel}-\boldsymbol{\sigma}(x)-\frac{(\text{Ч}\breve{\Phi}_N^{\mathbf{p},\lambda^*})(x)}{\gimel}\ge-\frac{\boldsymbol{\sigma}(x)}{2}-\frac{K_{\lambda_*}^{8.6}}{|\gimel|}\mathsf{r}_N$$

and hence

$$\frac{\xi^*(x)}{\gimel}-\frac{\boldsymbol{\sigma}(x)}{2}\ge\frac{(\text{Ч}\breve{\Phi}_N^{\mathbf{p},\lambda^*})(x)}{\gimel}-\frac{K_{\lambda_*}^{8.6}}{|\gimel|}\mathsf{r}_N.$$

To carry out the calculations, we need to stay away from γ_N and $\mathcal{C}_N$, since $\breve{\Phi}_N^{\mathbf{p},\lambda^*}$ loses smoothness on these curves. Fix ε and N as required in the claim. Define $\vartheta_{N,\varepsilon}\overset{\mathrm{def}}{=}1/\mu_{N,\varepsilon}^2$. Then (recalling the positivity claim of (69)) we have that $\text{I}_1^{\varepsilon,N} \geq \text{I}_2^{\varepsilon,N} + \text{I}_3^{\varepsilon,N} - K^{8.6}_{\lambda_*}\text{E}_7^{\varepsilon,N}/|\daleth|$ on $\mathbf{S}$, where

$$\begin{aligned}
\text{I}_2^{\varepsilon,N}(x) &\overset{\text{def}}{=} \frac{|\mu_{N,\varepsilon}|}{\daleth}(\text{Ч}\breve{\Phi}^{\mathsf{p},\lambda}_N)(x)\Upsilon_1^{\varepsilon,N}\left(\mathfrak{s}(\mu_{N,\varepsilon})\frac{\mathsf{H}^{\text{loc}}_N(x)}{\mathsf{r}_N}\right)\left\{\mathfrak{c}_\vee\left(\frac{\mathsf{H}^{\text{loc}}_N(x)}{\vartheta_{N,\varepsilon}\mathsf{r}_N}\right)\right.\\
&\qquad \left. -\mathfrak{c}_\wedge\left(\frac{|\mathsf{H}^{\text{loc}}_N(x)|-\mathsf{r}_N/2}{\vartheta_{N,\varepsilon}\mathsf{r}_N}\right)\right\}\chi_{\mathsf{E}}(x),\\
\text{I}_3^{\varepsilon,N}(x) &\overset{\text{def}}{=} |\mu_{N,\varepsilon}|\left\{\frac{\xi^*(x)}{\daleth}-\frac{\boldsymbol{\sigma}(x)}{2}\right\}\Upsilon_1^{\varepsilon,N}\left(\mathfrak{s}(\mu_{N,\varepsilon})\frac{\mathsf{H}^{\text{loc}}_N(x)}{\mathsf{r}_N}\right)\left\{\mathfrak{c}_\wedge\left(\frac{\mathsf{H}^{\text{loc}}_N(x)}{\vartheta_{N,\varepsilon}\mathsf{r}_N}\right)\right.\\
&\qquad \left. +\mathfrak{c}_\wedge\left(\frac{|\mathsf{H}^{\text{loc}}_N(x)|-\mathsf{r}_N/2}{\vartheta_{N,\varepsilon}\mathsf{r}_N}\right)\right\}\chi_{\mathsf{E}}(x),\\
\text{E}_7^{\varepsilon,N} &\overset{\text{def}}{=} \mathsf{r}_N|\mu_{N,\varepsilon}|\Upsilon_1^{\varepsilon,N}\left(\mathfrak{s}(\mu_{N,\varepsilon})\frac{\mathsf{H}^{\text{loc}}_N(x)}{\mathsf{r}_N}\right)\left\{\mathfrak{c}_\vee\left(\frac{\mathsf{H}^{\text{loc}}_N(x)}{\vartheta_{N,\varepsilon}\mathsf{r}_N}\right)\right.\\
&\qquad \left. -\mathfrak{c}_\wedge\left(\frac{|\mathsf{H}^{\text{loc}}_N(x)|-\mathsf{r}_N/2}{\vartheta_{N,\varepsilon}\mathsf{r}_N}\right)\right\}\chi_{\mathsf{E}}(x);
\end{aligned}$$

the requirement that $|\mu_{N,\varepsilon}|>2$ implies that $\text{I}_2^{\varepsilon,N}$ is smooth; in particular, 0 is not in the support of $h\mapsto\mathfrak{c}_\wedge\left(\left|\frac{h-1/2}{\vartheta_{N,\varepsilon}}\right|\right)$. It is easy to see that there is a constant $K_6>0$ such that

$$(71)\qquad |\text{E}_7^{\varepsilon,N}(x)|\leq K_6|\mu_{N,\varepsilon}|\mathsf{r}_N$$

for all $x\in\mathcal{N}_N$. To bound $\text{I}_3^{\varepsilon,N}$, use Lemma 8.1 with $\delta=\vartheta_{N,\varepsilon}$ and $\eta(z)\overset{\text{def}}{=}\mathfrak{c}_\wedge(z)+\mathfrak{c}_\wedge(z-1/(2\vartheta_{N,\varepsilon}))$ for $z\in\mathbb{R}$. Recall (68) and that $\bar{\xi}^*$ of (67) is bounded. Thus there is a constant $K_7>0$ such that

(72)
$$\mathbb{E}^\varepsilon\left[\int_{s=0}^{t\wedge\mathfrak{e}}\text{I}_3^{\varepsilon,N}(X_s)ds\right]\leq K_7(1+t)|\mu_{N,\varepsilon}|\delta_{N,\varepsilon}\vartheta_{N,\varepsilon}(1+|\mu_{N,\varepsilon}|)\leq\frac{3K_7}{2}(1+t)\delta_{N,\varepsilon}$$

(note that $\sup_{z\geq 2}\frac{1+z}{z}=\frac{3}{2}$) for all ε, N, and t as required in the claim.

Finally, we average. Fix ε and N as required in the claim. Define

$$\begin{aligned}
\Upsilon_5^{\varepsilon,N}(h) &\overset{\text{def}}{=} \Upsilon_1^{\varepsilon,N}(\mathfrak{s}(\mu_{N,\varepsilon})h)\left\{\mathfrak{c}_\vee\left(\frac{h}{\vartheta_{N,\varepsilon}}\right)-\mathfrak{c}_\wedge\left(\left|\frac{h-1/2}{\vartheta_{N,\varepsilon}}\right|\right)\right\}, \qquad h\in\mathcal{W},\\
\Upsilon_6^{\varepsilon,N}(x) &\overset{\text{def}}{=} \frac{\varepsilon^2}{\daleth}\breve{\Phi}^{\mathsf{p},\lambda^*}_N(x)|\mu_{N,\varepsilon}|\Upsilon_5^{\varepsilon,N}\left(\frac{\mathsf{H}^{\text{loc}}_N(x)}{\mathsf{r}_N}\right)\chi_{\mathsf{E}}(x), \qquad x\in\mathbf{S}.
\end{aligned}$$

Then

$$(\mathscr{L}^\varepsilon\Upsilon_6^{\varepsilon,N})(x)=\text{I}_2^{\varepsilon,N}(x)+\frac{1}{\daleth}\chi_{\mathsf{E}}(x)\sum_{i=8}^{11}\text{E}_i^{\varepsilon,N}(x)$$

for all $x\in\mathbf{S}$ where

$$\text{E}_8^{\varepsilon,N}(x)=\varepsilon^2|\mu_{N,\varepsilon}|\Upsilon_5^{\varepsilon,N}\left(\frac{\mathsf{H}^{\text{loc}}_N(x)}{\mathsf{r}_N}\right)(\mathscr{L}\breve{\Phi}^{\mathsf{p},\lambda^*}_N)(x),$$

$$\mathrm{E}_9^{\varepsilon,N}(x) = \frac{\varepsilon^2|\mu_{N,\varepsilon}|}{\mathsf{r}_N^2}\dot{\Upsilon}_5^{\varepsilon,N}\left(\frac{\mathsf{H}_N^{\mathrm{loc}}(x)}{\mathsf{r}_N}\right)\left\{\frac{\mu_{N,\varepsilon}}{\daleth}\xi_N(x) + \mathsf{r}_N\boldsymbol{\beta}_N(x)\right\}\breve{\Phi}_N^{\mathbf{p},\lambda^*}(x),$$
$$\mathrm{E}_{10}^{\varepsilon,N}(x) = \frac{\varepsilon^2|\mu_{N,\varepsilon}|}{2\mathsf{r}_N^2}\ddot{\Upsilon}_5^{\varepsilon,N}\left(\frac{\mathsf{H}_N^{\mathrm{loc}}(x)}{\mathsf{r}_N}\right)\breve{\Phi}_N^{\mathbf{p},\lambda^*}(x)\boldsymbol{\sigma}_N(x),$$
$$\mathrm{E}_{11}^{\varepsilon,N}(x) = \frac{\varepsilon^2|\mu_{N,\varepsilon}|}{\mathsf{r}_N}\dot{\Upsilon}_5^{\varepsilon,N}\left(\frac{\mathsf{H}_N^{\mathrm{loc}}(x)}{\mathsf{r}_N}\right)\left\langle d\breve{\Phi}_N^{\mathbf{p},\lambda^*}, d\mathsf{H}_N^{\mathrm{loc}}\right\rangle(x).$$

Note that

$$\dot{\Upsilon}_1^{\varepsilon,N}(h) = 2\mathfrak{E}\left(\frac{h}{\delta_{N,\varepsilon}}\right) - |\mu_{N,\varepsilon}|\Upsilon_1^{\varepsilon,N}(h),$$
$$\ddot{\Upsilon}_1^{\varepsilon,N}(h) = \frac{2}{\delta_{N,\varepsilon}}\dot{\mathfrak{E}}\left(\frac{h}{\delta_{N,\varepsilon}}\right) - |\mu_{N,\varepsilon}|\dot{\Upsilon}_1^{\varepsilon,N}(h)$$
$$= \frac{2}{\delta_{N,\varepsilon}}\dot{\mathfrak{E}}\left(\frac{h}{\delta_{N,\varepsilon}}\right) - 2|\mu_{N,\varepsilon}|\mathfrak{E}\left(\frac{h}{\delta_{N,\varepsilon}}\right) + \mu_{N,\varepsilon}^2\Upsilon_1^{\varepsilon,N}(h)$$

for all $h \in \mathcal{W}$. Use now (69). We can thus find a constant $K_8 > 0$ such that

$$|\dot{\Upsilon}_1^{\varepsilon,N}(h)| \le K_8,$$
$$|\ddot{\Upsilon}_1^{\varepsilon,N}(h)| \le K_8\left(\frac{1}{\delta_{N,\varepsilon}} + |\mu_{N,\varepsilon}| + \frac{\mu_{N,\varepsilon}^2}{|\mu_{N,\varepsilon}|}\right) \le 3K_8|\mu_{N,\varepsilon}|$$

for all $h \in \mathcal{W}$ and ε and N as required. Thus, there is a $K_9 > 0$ such that

$$|\dot{\Upsilon}_5^{\varepsilon,N}(h)| \le K_9\left\{1 + \frac{1}{|\mu_{N,\varepsilon}|\vartheta_{N,\varepsilon}}\right\} \le 2K_9|\mu_{N,\varepsilon}|,$$
$$|\ddot{\Upsilon}_5^{\varepsilon,N}(h)| \le K_9\left\{\frac{1}{|\mu_{N,\varepsilon}|\vartheta_{N,\varepsilon}^2} + \frac{1}{\vartheta_{N,\varepsilon}} + |\mu_{N,\varepsilon}|\right\} \le 3K_9|\mu_{N,\varepsilon}|^3$$

for all $h \in \mathcal{W}$ and ε and N as required. Consequently, there is a $K_{10} > 0$ such that

$$|\mathrm{E}_8^{\varepsilon,N}(x)| \le K_{10}\varepsilon^2, \qquad |\mathrm{E}_9^{\varepsilon,N}(x)| \le K_{10}\varepsilon^2\frac{|\mu_{N,\varepsilon}|^3}{\mathsf{r}_N^2}, \qquad |\mathrm{E}_{10}^{\varepsilon,N}(x)| \le K_{10}\varepsilon^2\frac{\mu_{N,\varepsilon}^4}{\mathsf{r}_N^2},$$
$$|\mathrm{E}_{11}^{\varepsilon,N}(x)| \le K_{10}\varepsilon^2\frac{|\mu_{N,\varepsilon}|^2}{\mathsf{r}_N}, \qquad |\Upsilon_6^{\varepsilon,N}(x)| \le K_{10}\varepsilon^2$$

for all $x \in \mathcal{N}_N$ and ε and N as required. Hence, taking the largest of the numerators and smallest of the denominators, there is a $K_{11} > 0$ such that

$$\left|\mathbb{E}^\varepsilon\left[\int_{s=0}^{t\wedge\mathfrak{e}} \mathrm{I}_2^{\varepsilon,N}(X_s)ds\right]\right| \le K_{11}(1+t)\frac{\varepsilon^2\mu_{N,\varepsilon}^4}{\mathsf{r}_N^2}.$$

for all ε and N as required. Combine this, (70), (71), and (72), and again take the largest of the numerators and smallest of the denominators. Again using the fact that $\sup_{z\ge 2}\frac{1+z}{z} = \frac{3}{2}$, we have that $\delta_{N,\varepsilon} \le \frac{3}{2\{1+|\mu_{N,\varepsilon}|\}}$. □

8.1. Proof of Lemma 8.3. To set up some notation, define $\tilde{\mathbf{x}}_1(x) \stackrel{\text{def}}{=} x_1$ for all $x = (x_1, x_2) \in \mathbb{R}^2$, and define

$$\mathsf{w}(z) = \int_{y=0}^{|z|} \left\{ e^{-y^2/2} \int_{z=0}^{y} e^{z^2/2} dz \right\} dy$$

for all $z \in \mathbb{R}$. The properties of w are given in Lemma 5.2 of [Sow05]. In particular, w is smooth and $\ddot{\mathsf{w}}(z) + z\dot{\mathsf{w}}(z) = 1$ and $z\dot{\mathsf{w}}(z) > 0$ for all $z \in \mathbb{R}$, and there is a constant $K > 0$ such that

$$|\mathsf{w}(z)| \le K\mathfrak{l}(|z|^{-1}), \qquad |\dot{\mathsf{w}}(z)| \le \frac{K}{|z|+1}, \qquad \text{and} \qquad |\ddot{\mathsf{w}}(z)| \le K \tag{73}$$

for all $z \in \mathbb{R}$ (the bound on w and $\dot{\mathsf{w}}$ were given in Lemma 5.2 of [Sow05]; the bound on $\ddot{\mathsf{w}}$ comes from combining the bound on $\dot{\mathsf{w}}$ with the PDE for w).

Proof of Lemma 8.3. Let $\varkappa \in (0,1)$ be such that $x \in \tilde{\mathcal{U}}$ if $\mathbf{n}(x) < 2\varkappa$ ($\mathbf{n}$ was defined in Subsection 3.2). Define $\tilde{\mathfrak{H}}(x) \stackrel{\text{def}}{=} \mathfrak{c}_\wedge \left(\frac{\mathbf{n}(x)}{\varkappa}\right)$ for all $x \in \mathbb{R}^2$. Then $0 \le \tilde{\mathfrak{H}} \le 1$, $\tilde{\mathfrak{H}}(x) = 1$ if $\|x\|_e \le \sqrt{\varkappa}$, and $\operatorname{supp} \tilde{\mathfrak{H}}(x) \subset \tilde{\mathcal{U}}$. Define the constants

$$\begin{aligned} \alpha_1 &\stackrel{\text{def}}{=} \inf\left\{ \sqrt{\frac{2}{\langle d\mathbf{x}_1, d\mathbf{x}_1\rangle_\ell^\sim (x) \tilde{\mathrm{B}}_\ell(x)}} : x \in \tilde{\mathcal{U}},\, \ell \in \Lambda \right\}, \\ \alpha_2 &\stackrel{\text{def}}{=} \sup\left\{ \frac{2}{\alpha_1^2 \langle d\mathbf{x}_1, d\mathbf{x}_1\rangle_\ell^\sim (x)} : x \in \tilde{\mathcal{U}},\, \ell \in \Lambda \right\}; \end{aligned}$$

in light of (2), $\alpha_1 > 0$, so α_2 is well-defined.

Define $\tilde{\mathfrak{b}}^\varepsilon(x) \stackrel{\text{def}}{=} \varepsilon^2 \alpha_2 \mathsf{w}(\alpha_1 \mathbf{x}_1(x)/\varepsilon)\, \tilde{\mathfrak{H}}(x)$ for all $x \in \mathbb{R}^2$ and $\varepsilon \in (0,1)$. Note that $\tilde{\mathfrak{b}}^\varepsilon \in C_c^\infty(\tilde{\mathcal{U}})$. We then compute that $(\tilde{\mathscr{L}}_\ell^\varepsilon \tilde{\mathfrak{b}}^\varepsilon)(x) = A_\ell^\varepsilon(x)\tilde{\mathfrak{H}}(x) + \alpha_2\{\mathrm{E}_{\ell,1}^\varepsilon(x) + \mathrm{E}_{\ell,2}^\varepsilon(x)\}$ for all $\varepsilon \in (0,1)$ and $x \in \tilde{\mathcal{U}}$ and $\ell \in \Lambda$, where

$$\begin{aligned} A_\ell^\varepsilon(x) &\stackrel{\text{def}}{=} \alpha_2 \left\{ \frac{\alpha_1 \mathbf{x}_1(x)}{\tilde{\mathrm{B}}_\ell(x)\varepsilon} \dot{\mathsf{w}}\left(\alpha_1 \frac{\mathbf{x}_1(x)}{\varepsilon}\right) + \frac{\alpha_1^2 \langle d\mathbf{x}_1, d\mathbf{x}_2\rangle_\ell^\sim (x)}{2} \ddot{\mathsf{w}}\left(\alpha_1 \frac{\mathbf{x}_1(x)}{\varepsilon}\right) \right\}, \\ \mathrm{E}_{\ell,1}^\varepsilon(x) &\stackrel{\text{def}}{=} \mathsf{w}\left(\alpha_1 \frac{\mathbf{x}_1(x)}{\varepsilon}\right) \frac{(\bar{\nabla}_e \tilde{\mathsf{H}}, \nabla_e \tilde{\mathfrak{H}})_e(x)}{\tilde{\mathrm{B}}_\ell(x)}, \\ \mathrm{E}_{\ell,2}^\varepsilon(x) &\stackrel{\text{def}}{=} \varepsilon\alpha_1 \dot{\mathsf{w}}\left(\alpha_1 \frac{\mathbf{x}_1(x)}{\varepsilon}\right) \left\{ (\tilde{\mathscr{L}}_\ell \mathbf{x}_1)(x)\tilde{\mathfrak{H}}(x) + \left\langle d\mathbf{x}_1, d\tilde{\mathfrak{H}}\right\rangle_\ell^\sim (x) \right\} \\ &\quad + \varepsilon^2 \mathsf{w}\left(\alpha_1 \frac{\mathbf{x}_1(x)}{\varepsilon}\right) (\tilde{\mathscr{L}}_\ell \tilde{\mathfrak{H}})(x). \end{aligned}$$

From the PDE and bounds of (73), it is fairly easy to see that there is a $K_1 > 0$ such that $A_\ell^\varepsilon(x) \ge 1$ and $|\mathrm{E}_{\ell,2}^\varepsilon| \le K_1\{\varepsilon + \varepsilon^2 \mathfrak{l}(\varepsilon)\}$ for all $\varepsilon \in (0,1)$ and $x \in \tilde{\mathcal{U}}$. To bound $\mathrm{E}_{\ell,1}^\varepsilon$, we first note that $(\bar{\nabla}_e \tilde{\mathsf{H}}, \nabla_e \tilde{\mathfrak{H}})_e(x)$ is nonzero only when

$\sqrt{\varkappa} \le \|x\|_e \le \sqrt{2\varkappa}$. Since $\left\langle d\tilde{\mathsf{H}}, d\tilde{\mathsf{H}}\right\rangle_{\sim}(x) > 0$ for all $x \in \tilde{\mathcal{U}} \setminus \{0\}$; thus

$$\sup_{x \in \tilde{\mathcal{U}} \setminus \{\mathsf{0}_e\}} \frac{\left|(\bar{\nabla}_e \tilde{\mathsf{H}}, \nabla_e \tilde{\mathfrak{H}})_e(x)\right|}{\left\langle d\tilde{\mathsf{H}}, d\tilde{\mathsf{H}}\right\rangle_{\sim}(x)}$$

is finite. Hence there is a $K_2 > 0$ such that $|\mathrm{E}^{\varepsilon}_{\ell,1}(x)| \le K_2 \mathfrak{l}(\varepsilon) \left\langle d\tilde{\mathsf{H}}, d\tilde{\mathsf{H}}\right\rangle_{\sim}(x)$ for all $\varepsilon \in (0,1)$ and $x \in \tilde{\mathcal{U}}$. Combine things together to see that for some constant $K_3 > 0$,

$$\tilde{\mathfrak{H}}(x) \le (\mathscr{L}^{\varepsilon}_{\ell} \tilde{\mathfrak{b}}^{\varepsilon})(x) + K_3 \left\{ \mathfrak{l}(\varepsilon) \left\langle d\tilde{\mathsf{H}}, d\tilde{\mathsf{H}}\right\rangle_{\sim}(x) + \varepsilon \right\}$$

for all $x \in \tilde{\mathcal{U}}$ and $\varepsilon \in (0,1)$.

For all $\ell \in \Lambda$ and $x \in \mathbb{T}$, define now $\mathfrak{H}(x) \stackrel{\text{def}}{=} \sum_{\ell \in \Lambda} \tilde{\mathfrak{H}}(\phi_\ell(x))\chi_{\mathcal{U}_\ell}(x)$ and $\mathfrak{b}^{\varepsilon}(x) \stackrel{\text{def}}{=} \sum_{\ell \in \Lambda} \tilde{\mathfrak{b}}^{\varepsilon}(\phi_\ell(x))\chi_{\mathcal{U}_\ell}(x)$ for all $\varepsilon \in (0,1)$. The bound on $\mathfrak{b}^{\varepsilon}$ and $\sqrt{\langle d\mathfrak{b}^{\varepsilon}, d\mathfrak{b}^{\varepsilon}\rangle}$ are fairly easy to see; use Lemma 5.2 of [Sow05]. □

Finally, let's state the lemma which will allow us to average ξ^* in $\mathcal{N}_N$ away from γ_N. This will be useful in bounding $\mathscr{E}^{N,\varepsilon}_{b,1}$. For each $N \in \mathbb{N}$ and $x \in \mathcal{N}_N$, define

$$\Phi^{\mathfrak{z}_N,\lambda}_{\xi^*}(x) \stackrel{\text{def}}{=} \int_{s=0}^{\infty} e^{-\lambda \mathsf{r}_N s} \xi^*(\mathfrak{z}^N_s(x))ds.$$

Lemma 8.7. *There is a constant $K > 0$ and a sequence $\{\mathscr{E}^{8.7}_n; n \in \mathbb{N}\}$ of positive real numbers such that for $\lambda \in (0,1)$ and $N \in \mathbb{N}$,*

$$\begin{gathered}
\left| \mathsf{Ч}\Phi^{\mathfrak{z}_N,\lambda}_{\xi^*} - \{\xi^* - (\mathcal{A}\xi^*)([\mathsf{E}])\}\right| \\
\le K \left\{ \lambda + \mathsf{r}_N \mathfrak{l}^2(\mathsf{H}^{\mathrm{loc}}_N) + \frac{|\boldsymbol{\nu}_N|}{\lambda^2 \mathsf{r}^2_N |\mathsf{H}^{\mathrm{loc}}_N|} + \mathscr{E}^{8.7}_N \right\}, \\
\left|\Phi^{\mathfrak{z}_N,\lambda}_{\xi^*}\right| \le \frac{K}{\lambda \mathsf{r}_N}, \qquad \left\| D\Phi^{\mathfrak{z}_N,\lambda}_{\xi^*}\right\| \le \frac{K}{\lambda^2 \mathsf{r}^2_N |\mathsf{H}^{\mathrm{loc}}_N|}, \\
\left\| D^2\Phi^{\mathfrak{z}_N,\lambda}_{\xi^*}\right\| \le \frac{K}{\lambda^3 \mathsf{r}^3_N |\mathsf{H}^{\mathrm{loc}}_N|^2}
\end{gathered}$$

on $\mathcal{N}_N$, and such that $\lim_{N\to\infty} \mathscr{E}^{8.7}_N = 0$.

We close this section by using the above results to bound the errors in Proposition 7.2. Define

$$\begin{aligned}
\delta_{N,\varepsilon} &\stackrel{\text{def}}{=} \left(\frac{\varepsilon\{1 + |\mu_{N,\varepsilon}|\}}{\mathsf{r}^{9/4}_N} + \frac{\sqrt{\varepsilon\{1 + |\mu_{N,\varepsilon}|\}}}{\mathsf{r}_N} \right) \cdot \frac{\mathsf{r}^{7/4}_N}{1 + |\mu_{N,\varepsilon}|} \\
&= \frac{\varepsilon}{\mathsf{r}^{1/2}_N} + \frac{\varepsilon^{1/2}\mathsf{r}^{3/4}_N}{\{1 + |\mu_{N,\varepsilon}|\}^{1/2}}
\end{aligned} \tag{74}$$

and then set $\Psi_C^{N,\varepsilon} \stackrel{\text{def}}{=} \Psi_D^{\delta_{N,\varepsilon},\varepsilon,N}$.

LEMMA 8.8. *There is a constant* $K_{8.8} > 1$ *such that for each* $N \in \mathbb{N}$ *and* ε *in* $(0,1)$ *such that* $\varepsilon < r_N^{3/2}/K_{8.8}$*, we have that* $\sup_{x\in\mathbf{S}} |\Psi_C^{\varepsilon,N}(x)| \le K_{8.8}\varepsilon\mathfrak{l}(\varepsilon)/r_N^{3/4}$ *and*

$$\mathbb{E}^\varepsilon\left[\left\{\int_{u=s\wedge\mathfrak{e}}^{t\wedge\mathfrak{e}}(\mathscr{L}^\varepsilon\Psi_C^{\varepsilon,N})(X_u)du\right\}^-\right] \ge -\frac{K_{8.8}(1+t)}{\varepsilon^{17/6}}\exp\left[-\frac{1}{K_{8.8}}\left(\frac{r_N^{3/2}}{\varepsilon}\right)^{1/2}\right] - K_{8.8}(1+t)\left\{\frac{\varepsilon^{1/2}\{1+|\mu_{N,\varepsilon}|\}}{r_N^2}+\frac{\varepsilon^{1/4}\{1+|\mu_{N,\varepsilon}|\}^{3/4}}{r_N^{11/8}} + \frac{\varepsilon\{1+|\mu_{N,\varepsilon}|\}^2}{r_N^{11/4}}\right\}\mathfrak{l}(\varepsilon)$$

for all $0 \le s \le t$*.*

Proof. We can use Lemma 8.2 with δ being either $\varepsilon/\mathsf{r}_N^{3/2}$ or $\varepsilon\sqrt{\delta_{N,\varepsilon}}/\mathsf{r}_N$; to do so, we need that $\varepsilon/\mathsf{r}_N^{3/2} < 1$ and $\varepsilon\sqrt{\delta_{N,\varepsilon}}/\mathsf{r}_N < 1$. We get that there is a constant $K > 0$ such that

$$\begin{aligned}\mathbb{E}^\varepsilon&\left[\left\{\int_{u=s\wedge\mathfrak{e}}^{t\wedge\mathfrak{e}}(\mathscr{L}^\varepsilon\Psi_C^{\varepsilon,N})(X_u)du\right\}^-\right]\\ &\ge -K(1+t)\left\{\frac{\varepsilon\mathfrak{l}(\varepsilon)}{\mathsf{r}_N^{3/4}\sqrt{\delta_{N,\varepsilon}}}+\frac{1}{\varepsilon\mathsf{r}_N^{3/4}}\exp\left[-\frac{1}{K_{7.2}}\frac{\sqrt{\delta_{N,\varepsilon}}}{\varepsilon}\right]\right.\\ &\quad\left.+\frac{1}{\varepsilon^{7/3}\sqrt{\delta_{N,\varepsilon}}}\exp\left[-\frac{1}{K_{7.2}}\left(\frac{\mathsf{r}_N^{3/2}}{\varepsilon}\right)^{1/2}\right]\right\}\\ &\quad - K(1+t)\left\{\frac{|\mu_{N,\varepsilon}|}{\mathsf{r}_N^{5/4}}+\frac{1}{\mathsf{r}_N^{3/4}\sqrt{\delta_{N,\varepsilon}}}\right\}\frac{\varepsilon}{\mathsf{r}_N^{3/2}}\{1+|\mu_{N,\varepsilon}|\}\mathfrak{l}(\varepsilon)\\ &\quad - K(1+t)\frac{1}{\mathsf{r}_N^{1/4}\sqrt{\delta_{N,\varepsilon}}}\sqrt{\frac{\varepsilon}{\mathsf{r}_N^{3/2}}\{1+|\mu_{N,\varepsilon}|\}}\\ &\quad - \frac{K(1+t)}{\varepsilon\mathsf{r}_N^{3/4}}\frac{\varepsilon\sqrt{\delta_{N,\varepsilon}}}{\mathsf{r}_N}\{1+|\mu_{N,\varepsilon}|\}\end{aligned}$$

$$
\begin{aligned}
&\geq -K(1+t)\left\{\frac{1}{\varepsilon \mathsf{r}_N^{3/4}}\exp\left[-\frac{1}{K_{7.2}}\frac{\sqrt{\delta_{N,\varepsilon}}}{\varepsilon}\right] + \frac{1}{\varepsilon \mathsf{r}_N^{7/3}\sqrt{\delta_{N,\varepsilon}}}\exp\left[-\frac{1}{K_{7.2}}\left(\frac{\mathsf{r}_N^{3/2}}{\varepsilon}\right)^{1/2}\right]\right\} \\
&\quad - K(1+t)I_{N,\varepsilon}(\delta_{N,\varepsilon})\mathfrak{l}(\varepsilon) - K(1+t)\frac{\varepsilon\{1+|\mu_{N,\varepsilon}|\}^2}{\mathsf{r}_N^{11/4}}\mathfrak{l}(\varepsilon)
\end{aligned}
$$

for all $\varepsilon \in (0,1)$ and $N \in \mathbb{N}$ such that (combining the requirements of Proposition 7.2 and Lemma 8.2)

$$
\begin{gathered}
\frac{\varepsilon}{\mathsf{r}_N^{3/2}} < 1, \qquad \frac{\varepsilon\sqrt{\delta_{N,\varepsilon}}}{\mathsf{r}_N} < 1, \qquad \delta_{N,\varepsilon} < \bar{\delta}_{7.2}, \\
\varepsilon < \frac{\sqrt{\delta_{N,\varepsilon}}}{K_{7.2}}, \qquad \frac{\varepsilon}{\mathsf{r}_N^{3/2}} < \frac{1}{4},
\end{gathered}
\tag{75}
$$

where for convenience we have defined

$$
I_{N,\varepsilon}(\delta) = \frac{1}{\sqrt{\delta}}\left\{\frac{\varepsilon}{\mathsf{r}_N^{3/4}} + \frac{\varepsilon\{1+|\mu_{N,\varepsilon}|\}}{\mathsf{r}_N^{9/4}} + \frac{\sqrt{\varepsilon\{1+|\mu_{N,\varepsilon}|\}}}{\mathsf{r}_N}\right\} + \sqrt{\delta}\frac{\{1+|\mu_{N,\varepsilon}|\}}{\mathsf{r}_N^{7/4}}
$$

for all δ and ε in $(0,1)$ and $N \in \mathbb{N}$.

Bounding $\delta_{N,\varepsilon}$ from below by the first term after the last equality of (74), we have that

$$
\begin{gathered}
\frac{1}{\varepsilon^{7/3}\sqrt{\delta_{N,\varepsilon}}} \leq \frac{1}{\varepsilon^{7/3}}\sqrt{\frac{\mathsf{r}_N^{1/2}}{\varepsilon}} = \frac{1}{\varepsilon^{17/6}}, \\
\frac{\sqrt{\delta_{N,\varepsilon}}}{\varepsilon} \geq \frac{\varepsilon^{1/2}}{\varepsilon \mathsf{r}_N^{1/4}} = \frac{1}{\varepsilon^{1/2}\mathsf{r}_N^{1/4}} = \left(\frac{\mathsf{r}_N^{3/2}}{\varepsilon}\right)^{1/2}\frac{1}{\mathsf{r}_N} \geq \frac{1}{|\omega_2|}\left(\frac{\mathsf{r}_N^{3/2}}{\varepsilon}\right)^{1/2}
\end{gathered}
$$

for all $\varepsilon \in (0,1)$ and $N \in \mathbb{N}$. Thus,

$$
\begin{aligned}
&\frac{1}{\varepsilon \mathsf{r}_N^{3/4}}\exp\left[-\frac{1}{K_{7.2}}\frac{\sqrt{\delta_{N,\varepsilon}}}{\varepsilon}\right] + \frac{1}{\varepsilon^{7/3}\sqrt{\delta_{N,\varepsilon}}}\exp\left[-\frac{1}{K_{7.2}}\left(\frac{\mathsf{r}_N^{3/2}}{\varepsilon}\right)^{1/2}\right] \\
&\qquad \leq \left\{\frac{1}{\varepsilon(\varepsilon^{2/3})^{3/4}} + \frac{1}{\varepsilon^{17/6}}\right\}\exp\left[-\frac{1}{K_{7.2}(1+|\omega_2|)}\left(\frac{\mathsf{r}_N^{3/2}}{\varepsilon}\right)^{1/2}\right] \\
&\qquad \leq \left\{\frac{1}{\varepsilon^{3/2}} + \frac{1}{\varepsilon^{17/6}}\right\}\exp\left[-\frac{1}{K_{7.2}(1+|\omega_2|)}\left(\frac{\mathsf{r}_N^{3/2}}{\varepsilon}\right)^{1/2}\right]
\end{aligned}
$$

$$\leq \frac{2}{\varepsilon^{17/6}} \exp\left[-\frac{1}{K_{7.2}(1+|\omega_2|)}\left(\frac{\mathsf{r}_N^{3/2}}{\varepsilon}\right)^{1/2}\right]$$

for all $\varepsilon \in (0,1)$ and $N \in \mathbb{N}$.

Since ε and r_N are in $(0,1)$ and $1+|\mu_{N,\varepsilon}| \geq 1$,

$$\frac{\varepsilon}{\mathsf{r}_N^{3/4}} = \frac{\varepsilon\{1+|\mu_{N,\varepsilon}|\}}{\mathsf{r}_N^{9/4}} \frac{\mathsf{r}_N^{6/4}}{1+|\mu_{N,\varepsilon}|} \leq \frac{\varepsilon\{1+|\mu_{N,\varepsilon}|\}}{\mathsf{r}_N^{9/4}}|\omega_2|^{3/2}.$$

Thus we have that

$$\begin{aligned} I_{N,\varepsilon}(\delta_{N,\varepsilon}) &\leq \frac{1+|\omega_2|^{3/2}}{\sqrt{\delta_{N,\varepsilon}}}\left\{\frac{\varepsilon\{1+|\mu_{N,\varepsilon}|\}}{\mathsf{r}_N^{9/4}} + \frac{\sqrt{\varepsilon\{1+|\mu_{N,\varepsilon}|\}}}{\mathsf{r}_N}\right\} \\ &\quad + \sqrt{\delta_{N,\varepsilon}}\frac{\{1+|\mu_{N,\varepsilon}|\}}{\mathsf{r}_N^{7/4}} \\ &\leq \{3+|\omega_2|^{3/2}\}\sqrt{\frac{\varepsilon\{1+|\mu_{N,\varepsilon}|\}}{\mathsf{r}_N^{9/4}} + \frac{\sqrt{\varepsilon\{1+|\mu_{N,\varepsilon}|\}}}{\mathsf{r}_N}}\sqrt{\frac{\{1+|\mu_{N,\varepsilon}|\}}{\mathsf{r}_N^{7/4}}} \\ &= \{3+|\omega_2|^{3/2}\}\left\{\frac{\varepsilon\{1+|\mu_{N,\varepsilon}|\}^2}{\mathsf{r}_N^4} + \frac{\varepsilon^{1/2}\{1+|\mu_{N,\varepsilon}|\}^{3/2}}{\mathsf{r}_N^{11/4}}\right\}^{1/2} \\ &= \{6+2|\omega_2|^{3/2}\}\left\{\frac{\varepsilon^{1/2}\{1+|\mu_{N,\varepsilon}|\}}{\mathsf{r}_N^2} + \frac{\varepsilon^{1/4}\{1+|\mu_{N,\varepsilon}|\}^{3/4}}{\mathsf{r}_N^{11/8}}\right\} \end{aligned}$$

for all $\varepsilon \in (0,1)$ and $N \in \mathbb{N}$; the choice (74) of $\delta_{N,\varepsilon}$ makes both terms on the right of the first inequality to have the same order.

We need to check that there is a $K > 1$ such that (75) holds when $\varepsilon < \mathsf{r}_N^{3/2}/K$. We start by rewriting the constraints of (75) as

$$\varepsilon < \frac{\mathsf{r}_N^{3/2}}{4}, \quad \frac{\varepsilon}{\mathsf{r}_N^{1/2}} + \frac{\varepsilon^{1/2}\mathsf{r}_N^{3/4}}{\{1+|\mu_{N,\varepsilon}|\}^{1/2}} < \frac{\mathsf{r}_N^2}{\varepsilon^2}, \quad \frac{\varepsilon}{\mathsf{r}_N^{1/2}} + \frac{\varepsilon^{1/2}\mathsf{r}_N^{3/4}}{\{1+|\mu_{N,\varepsilon}|\}^{1/2}} < \bar{\delta}_{7.2},$$
$$\varepsilon^2 \leq \frac{1}{K_{7.2}^2}\left\{\frac{\varepsilon}{\mathsf{r}_N^{1/2}} + \frac{\varepsilon^{1/2}\mathsf{r}_N^{3/4}}{\{1+|\mu_{N,\varepsilon}|\}^{1/2}}\right\}.$$

(the last requirement of (75) implies the first). These inequalities hold if

$$\varepsilon < \frac{\mathsf{r}_N^{3/2}}{4}, \quad \frac{\varepsilon}{\mathsf{r}_N^{1/2}} < \frac{\mathsf{r}_N^2}{2\varepsilon^2}, \quad \varepsilon^{1/2}\mathsf{r}_N^{3/4} < \frac{\mathsf{r}_N^2}{2\varepsilon^2}, \quad \frac{\varepsilon}{\mathsf{r}_N^{1/2}} < \frac{\bar{\delta}_{7.2}}{2}, \quad \varepsilon^{1/2}\mathsf{r}_N^{3/4} < \frac{\bar{\delta}_{7.2}}{2},$$
$$\varepsilon^2 < \frac{\varepsilon}{K_{7.2}^2\mathsf{r}_N^{1/2}};$$

i.e., if

$$\varepsilon < \min\left\{\frac{\mathsf{r}_N^{3/2}}{4}, \frac{\mathsf{r}_N^{5/6}}{2^{1/3}}, \frac{\mathsf{r}_N^{1/2}}{2^{2/5}}, \frac{\bar{\delta}_{7.2}\mathsf{r}_N^{1/2}}{2}, \frac{\bar{\delta}_{7.2}^2}{4\mathsf{r}_N^{3/2}}, \frac{1}{K_{7.2}^2\mathsf{r}_N^{1/2}}\right\},$$

which in turn is true if

$$\varepsilon < \frac{\bar{\delta}_{7.2}^2}{(K_{7.2}^2+4)(1+|\omega_2|)^3}\mathsf{r}_N^{3/2}.$$

In verifying this, note that if $-3/2 \le \alpha \le 3/2$,

$$\frac{\mathsf{r}_N^{3/2}}{(1+|\omega_2|)^3} = \mathsf{r}_N^{\alpha}\frac{\mathsf{r}_N^{3/2-\alpha}}{(1+|\omega_2|)^3} \le \mathsf{r}_N^{\alpha}\frac{(1+|\omega_2|)^{3/2-\alpha}}{(1+|\omega_2|)^3} = \mathsf{r}_N^{\alpha}\frac{1}{(1+|\omega_2|)^{3/2+\alpha}} \le \mathsf{r}_N^{\alpha}.$$

□

9. Error estimates in $\mathcal{N}_N$

We now want to continue the bounds of Lemmas 7.8 and 7.9 and bound the remaining $\mathscr{E}_{b,i}^{N,\varepsilon}$'s of (53).

We start with a useful lemma.

LEMMA 9.1. *There is a $K_{9.1} > 0$ such that*

$$\left|(2\mu_{N,\varepsilon})V_P^{N,\varepsilon}(x)\right| \le K_{9.1}\{1+|\mu_{N,\varepsilon}|\}\mathfrak{E}\left(\frac{\mathsf{H}_N^{\text{loc}}(x)\mu_{N,\varepsilon}}{r_N}\right)$$

for all $\varepsilon \in (0,1)$ and $N \in \mathbb{N}$ and all $x \in \mathsf{E}$.

Proof. First fix $h \in \mathcal{W}$. We claim that

$$\left|2\mu_{N,\varepsilon}\mathsf{v}_P^{N,\varepsilon}(\iota(h))\right| \le \frac{|2\mu_{N,\varepsilon}|}{1-\exp[-2|\mu_{N,\varepsilon}|]}\exp[-2|\mu_{N,\varepsilon}h|]. \tag{76}$$

Assume first that $\mu_{N,\varepsilon} > 0$. Then

$$|1-\exp[-2\mu_{N,\varepsilon}]| = 1-\exp[-2\mu_{N,\varepsilon}] = 1-\exp[-2|\mu_{N,\varepsilon}|]. \tag{77}$$

We also calculate that $\iota(h) = h \ge h$ if $0 \le h < 1/2$, and $\iota(h) = h+1 \ge -h$ if $-1/2 < h < 0$, implying that

$$-2\mu_{N,\varepsilon}\iota(h) = -2|\mu_{N,\varepsilon}|\iota(h) \le -2|\mu_{N,\varepsilon}||h| = -2|\mu_{N,\varepsilon}h|.$$

Combining this and (77), we get (76).

Assume next that $\mu_{N,\varepsilon} < 0$. We then rewrite $\mathsf{v}_P^{N,\varepsilon}(\iota(h))$ as

$$\mathsf{v}_P^{N,\varepsilon}(\iota(h)) = \frac{\exp[2\mu_{N,\varepsilon}(1-\iota(h))]}{\exp[2\mu_{N,\varepsilon}]-1}.$$

We then have that $|\exp[-2\mu_{N,\varepsilon}]-1| = 1-\exp[2\mu_{N,\varepsilon}] = 1-\exp[-2|\mu_{N,\varepsilon}|]$. We also have that $1-\iota(h) = 1-h \ge h$ if $0 \le h < 1/2$, and $1-\iota(h) = (1-(h+1)) \ge -h$ if $-1/2 < h < 0$. This implies that

$$2\mu_{N,\varepsilon}(1-\iota(h)) = -2|\mu_{N,\varepsilon}|(1-\iota(h)) \le -2|\mu_{N,\varepsilon}||h| \le -2|\mu_{N,\varepsilon}h|.$$

Thus we again get (76).

We finish the proof by noting that $\sup_{z>0}\{z(1+z)^{-1}(1-e^{-z})^{-1}\}$ is finite and by then using (55). □

We can now bound $\mathscr{E}^{N,\varepsilon}_{b,4}$ and $\mathscr{E}^{N,\varepsilon}_{b,5}$ of (53).

LEMMA 9.2. *There is a* $K>0$ *such that* $\sup_{x\in\mathsf{E}}|\mathscr{E}^{N,\varepsilon}_{b,i}(x)|\le K\varepsilon^2\{1+|\mu_{N,\varepsilon}|\}^2/\mathsf{r}_N$ *for all* $\varepsilon\in(0,1)$, $N\in\mathbb{N}$ *and* $i\in\{4,5\}$.

Proof. We first note that there is a $K_1>0$ such that $|\boldsymbol{\sigma}_N(x)-\boldsymbol{\sigma}(x)|\le K_1|\boldsymbol{\nu}_N|$ and $|\xi_N(x)-\xi^*(x)|\le K_1|\boldsymbol{\nu}_N|$ for all $x\in\mathcal{N}_N$. By combining this with Lemma 9.1, we can find a constant $K>0$ such that

$$\begin{aligned}|\mathscr{E}^{N,\varepsilon}_{b,i}(x)| &\le K|\boldsymbol{\nu}_N|\{1+|\mu_{N,\varepsilon}|\}=\frac{K}{|\daleth|}\frac{\varepsilon^2}{\mathsf{r}_N}|\mu_{N,\varepsilon}|\{1+|\mu_{N,\varepsilon}|\}\\ &\le \frac{K}{|\daleth|}\frac{\varepsilon^2}{\mathsf{r}_N}\{1+|\mu_{N,\varepsilon}|\}^2.\end{aligned}$$ □

We can bound $\mathscr{E}^{N,\varepsilon}_{b,3}$ by combining Lemmas 8.2 and Lemma 8.5.

LEMMA 9.3. *There is a* $K>0$ *such that for all* $0\le s\le t$,

$$\begin{aligned}&\mathbb{E}^{\varepsilon}\left[\left|\int_{u=s\wedge\mathfrak{e}}^{t\wedge\mathfrak{e}}\mathscr{E}^{N,\varepsilon}_{b,3}(X_u)\chi_{\mathsf{E}}(X_u)du\right|\right]\\ &\quad\le K\mathscr{E}^{8.5}(\lambda,N)(1+t)\left\{1+\mathsf{r}_N\{1+|\mu_{N,\varepsilon}|\}^2+\left(\frac{\varepsilon}{\mathsf{r}_N}\right)^2\{1+|\mu_{N,\varepsilon}|\}^5\right\}\\ &\qquad\qquad+KK^{8.5}_\lambda(1+t)\{1+|\mu_{N,\varepsilon}|\}^{7/3}\left(\frac{\varepsilon}{\mathsf{r}_N}\right)^{2/3}\end{aligned}$$

for all $\lambda\in(0,\lambda_{8.5})$ *and all integers* $N\ge N^{8.5}_\lambda$ *such that*

$$1+|\mu_{N,\varepsilon}|\le\frac{\mathsf{r}_N^2}{\varepsilon^2}. \tag{78}$$

Proof. Fix $\varepsilon\in(0,1)$, $N\in\mathbb{N}$, and $x\in\mathsf{E}$. We first write that

$$\mathscr{E}^{N,\varepsilon}_{b,3}(x)=-\frac{1}{(\mathcal{A}\xi^*)([\mathsf{E}])}\left\{\xi^*(x)-\daleth\boldsymbol{\sigma}(x)\right\}(2\mu_{N,\varepsilon})\mathsf{V}^{N,\varepsilon}_P(x).$$

Define now a relaxation parameter $\vartheta_{N,\varepsilon}\overset{\text{def}}{=}\{1+|\mu_{N,\varepsilon}|\}^{1/3}(\varepsilon/\mathsf{r}_N)^{2/3}$; the requirement of (78) is exactly that $\vartheta_{N,\varepsilon}\in(0,1)$. Then

$$\mathscr{E}^{N,\varepsilon}_{b,3}(x)\overset{\text{def}}{=}-\frac{1}{(\mathcal{A}\xi^*)([\mathsf{E}])}\{\imath^{\varepsilon,N}_1(x)+\imath^{\varepsilon,N}_2(x)\},$$

where (recalling (67)),

$$\mathrm{I}_1^{\varepsilon,N}(x) \stackrel{\text{def}}{=} \{\bar{\xi}^*(x) - \daleth\}(2\mu_{N,\varepsilon})\mathsf{V}_P^{N,\varepsilon}(x)\mathfrak{c}_\wedge\left(\frac{\mathsf{H}_N^{\text{loc}}(x)}{\vartheta_{N,\varepsilon}\mathsf{r}_N}\right)\boldsymbol{\sigma}(x),$$

$$\mathrm{I}_2^{\varepsilon,N}(x) \stackrel{\text{def}}{=} \{\xi^*(x) - \daleth\boldsymbol{\sigma}(x)\}(2\mu_{N,\varepsilon})\mathsf{V}_P^{N,\varepsilon}(x)\mathfrak{c}_\vee\left(\frac{\mathsf{H}_N^{\text{loc}}(x)}{\vartheta_{N,\varepsilon}\mathsf{r}_N}\right).$$

We now use (54) and Lemma 9.1. Thus there is a constant $K_1 > 0$ such that

$$|\mathrm{I}_1^{\varepsilon,N}(x)| \le K\{1+|\mu_{N,\varepsilon}|\}\mathfrak{E}\left(\frac{\mathsf{H}_N^{\text{loc}}(x)\mu_{N,\varepsilon}}{\mathsf{r}_N}\right)\mathfrak{c}_\wedge\left(\frac{\mathsf{H}_N^{\text{loc}}(x)}{\vartheta_{N,\varepsilon}\mathsf{r}_N}\right)\boldsymbol{\sigma}_N(x)$$
$$= K\{1+|\mu_{N,\varepsilon}|\}\mathfrak{c}_\wedge\left(\frac{\mathsf{H}_N^{\text{loc}}(x)}{\vartheta_{N,\varepsilon}\mathsf{r}_N}\right)\boldsymbol{\sigma}_N(x)$$

(note that $\|\mathfrak{E}\|_{C(\mathbb{R})} = 1$). Thus by Lemma 8.2, there is a $K > 0$ such that

$$\text{(79)}\quad \mathbb{E}^\varepsilon\left[\int_{s=0}^{t\wedge\mathfrak{e}}\left|\mathrm{I}_1^{\varepsilon,N}(X_s)\right|\chi_{\mathsf{E}}(X_s)ds\right] \le K(1+t)\{1+|\mu_{N,\varepsilon}|\}^2\vartheta_{N,\varepsilon}$$
$$= K(1+t)\{1+|\mu_{N,\varepsilon}|\}^{7/3}\left(\frac{\varepsilon}{\mathsf{r}_N}\right)^{2/3}$$

for all $\varepsilon \in (0,1)$ and $N \in \mathbb{N}$.

We bound $\mathrm{I}_2^{\varepsilon,N}$ by averaging. Fix $\varepsilon \in (0,1)$, $\lambda \in (0,\lambda_{8.5})$, an integer $N \ge N_\lambda^{8.5}$, and $x \in \mathbf{S}$. Define

$$\Upsilon^{\varepsilon,\lambda,N}(x) \stackrel{\text{def}}{=} \varepsilon^2\breve{\Phi}_N^{\mathsf{p},\lambda}(x)(2\mu_{N,\varepsilon})\mathsf{V}_P^{N,\varepsilon}(x)\mathfrak{c}_\vee\left(\frac{\mathsf{H}_N^{\text{loc}}(x)}{\vartheta_{N,\varepsilon}\mathsf{r}_N}\right)\chi_{\mathsf{E}}(x).$$

Then

$$(\mathscr{L}^\varepsilon\Upsilon^{\varepsilon,\lambda,N})(x) = \left\{\mathrm{I}_2^{\varepsilon,N}(x) + \mathrm{I}_3^{\varepsilon,\lambda,N}(x) + \sum_{i=1}^4 \mathrm{E}_i^{\varepsilon,\lambda,N}(x)\right\}\chi_{\mathsf{E}}(x)$$

where

$$\mathrm{I}_3^{\varepsilon,\lambda,N}(x) = \left\{(\text{Ч}\breve{\Phi}_N^{\mathsf{p},\lambda})(x) - \{\xi^*(x) - \daleth\boldsymbol{\sigma}(x)\}\right\}(2\mu_{N,\varepsilon})\mathsf{V}_P^{N,\varepsilon}(x)\mathfrak{c}_\vee\left(\frac{\mathsf{H}_N^{\text{loc}}(x)}{\vartheta_{N,\varepsilon}\mathsf{r}_N}\right),$$

$$\mathrm{E}_1^{\varepsilon,\lambda,N}(x) = \frac{\varepsilon^2}{\mathsf{r}_N^2}\left\{\mu_{N,\varepsilon}\frac{\xi_N(x)}{\daleth} + \mathsf{r}_N\boldsymbol{\beta}_N(x)\right\}\left\{-2\mu_{N,\varepsilon}\mathfrak{c}_\vee\left(\frac{\mathsf{H}_N^{\text{loc}}(x)}{\vartheta_{N,\varepsilon}\mathsf{r}_N}\right)\right.$$
$$\left.+\frac{1}{\vartheta_{N,\varepsilon}}\dot{\mathfrak{c}}_\wedge\left(\frac{\mathsf{H}_N^{\text{loc}}(x)}{\vartheta_{N,\varepsilon}\mathsf{r}_N}\right)\right\}(2\mu_{N,\varepsilon})\mathsf{V}_P^{N,\varepsilon}(x)\breve{\Phi}_N^{\mathsf{p},\lambda}(x),$$

$$\mathrm{E}_2^{\varepsilon,\lambda,N}(x) = \frac{\varepsilon^2}{2\mathsf{r}_N^2}\boldsymbol{\sigma}_N(x)\left\{4\mu_{N,\varepsilon}^2\mathfrak{c}_\vee\left(\frac{\mathsf{H}_N^{\text{loc}}(x)}{\vartheta_{N,\varepsilon}\mathsf{r}_N}\right) + \frac{4\mu_{N,\varepsilon}}{\vartheta_{N,\varepsilon}}\dot{\mathfrak{c}}_\wedge\left(\frac{\mathsf{H}_N^{\text{loc}}(x)}{\vartheta_{N,\varepsilon}\mathsf{r}_N}\right)\right.$$
$$\left.-\frac{1}{\vartheta_{N,\varepsilon}^2}\ddot{\mathfrak{c}}_\wedge\left(\frac{\mathsf{H}_N^{\text{loc}}(x)}{\vartheta_{N,\varepsilon}\mathsf{r}_N}\right)\right\}(2\mu_{N,\varepsilon})\mathsf{V}_P^{N,\varepsilon}(x)\breve{\Phi}_N^{\mathsf{p},\lambda}(x),$$

$$\mathrm{E}_3^{\varepsilon,\lambda,N}(x) = \varepsilon^2(2\mu_{N,\varepsilon})\mathsf{V}_P^{N,\varepsilon}(x)\mathfrak{c}_\vee\left(\frac{\mathsf{H}_N^{\mathrm{loc}}(x)}{\vartheta_{N,\varepsilon}\mathsf{r}_N}\right)(\mathscr{L}\breve{\Phi}_N^{\mathbf{p},\lambda})(x),$$

$$\mathrm{E}_4^{\varepsilon,\lambda,N}(x) = \frac{\varepsilon^2}{\mathsf{r}_N}\left\{-2\mu_{N,\varepsilon}\mathfrak{c}_\vee\left(\frac{\mathsf{H}_N^{\mathrm{loc}}(x)}{\vartheta_{N,\varepsilon}\mathsf{r}_N}\right)\right.$$
$$\left.-\frac{1}{\vartheta_{N,\varepsilon}}\dot{\mathfrak{c}}_\wedge\left(\frac{\mathsf{H}_N^{\mathrm{loc}}(x)}{\vartheta_{N,\varepsilon}\mathsf{r}_N}\right)\right\}(2\mu_{N,\varepsilon})\mathsf{V}_P^{N,\varepsilon}(x)\left\langle d\mathsf{H}_N^{\mathrm{loc}}, d\breve{\Phi}_N^{\mathbf{p},\lambda}\right\rangle(x)$$

for all $x \in \mathsf{E}$. Using Lemma 9.1, we can thus find a constant $K_1 > 0$ such that[3]

$$(80)\quad \begin{aligned} |\mathrm{E}_i^{\varepsilon,\lambda,N}| &\le K_1 K_\lambda^{8.5}\{1+|\mu_{N,\varepsilon}|\}^3 \frac{\varepsilon^2}{\mathsf{r}_N^2\vartheta_{N,\varepsilon}^2} \\ &= K_1 K_\lambda^{8.5}\{1+|\mu_{N,\varepsilon}|\}^{7/3}\left(\frac{\varepsilon}{\mathsf{r}_N}\right)^{2/3}, \qquad i\in\{1,2,3,4\}, \\ |\Upsilon^{\varepsilon,\lambda,N}| &\le K_1 K_\lambda^{8.5}\varepsilon^2\{1+|\mu_{N,\varepsilon}|\} = K_1 K_\lambda^{8.5}\{1+|\mu_{N,\varepsilon}|\}\frac{\varepsilon^2|\omega_2|^2}{\mathsf{r}_N^2\vartheta_{N,\varepsilon}^2} \\ &\le K_1 K_\lambda^{8.5}|\omega_2|^2\{1+|\mu_{N,\varepsilon}|\}^{7/3}\left(\frac{\varepsilon}{\mathsf{r}_N}\right)^{2/3} \end{aligned}$$

on E for all $\varepsilon \in (0,1)$, $\lambda \in (0,\lambda_{8.5})$, and all integers $N \ge N_\lambda^{8.5}$. Combining Lemmas 8.6 and 9.1, we furthermore can find a constant $K_2 > 0$ such that

$$\begin{aligned} &\mathbb{E}^\varepsilon\left[\left|\int_{u=s\wedge\mathfrak{e}}^{t\wedge\mathfrak{e}} \mathrm{I}_3^{\varepsilon,\lambda,N}(X_u)\chi_{\mathsf{E}}(X_u)du\right|\right] \\ &\quad\le K_{9.1}K_{(68)}\mathscr{E}^{8.5}(\lambda,N)\{1+|\mu_{N,\varepsilon}|\} \\ &\qquad\quad\times\mathbb{E}^\varepsilon\left[\int_{u=s\wedge\mathfrak{e}}^{t\wedge\mathfrak{e}}\mathfrak{E}\left(\frac{\mathsf{H}_N^{\mathrm{loc}}(X_u)\mu_{N,\varepsilon}}{\mathsf{r}_N}\right)\boldsymbol{\sigma}_N(X_u)\chi_{\mathsf{E}}(X_u)du\right] \\ &\qquad+K_{9.1}K_{(68)}K_\lambda^{8.5}\mathsf{r}_N\{1+|\mu_{N,\varepsilon}|\} \\ &\quad\le K_2\mathscr{E}^{8.5}(\lambda,N)\left\{1+\mathsf{r}_N\{1+|\mu_{N,\varepsilon}|\}^2+\left(\frac{\varepsilon}{\mathsf{r}_N}\right)^2\{1+|\mu_{N,\varepsilon}|\}^5\right\} \\ &\qquad+K_{9.1}K_{(68)}K_\lambda^{8.5}\mathsf{r}_N\{1+|\mu_{N,\varepsilon}|\} \end{aligned}$$

for all $\varepsilon \in (0,1)$, $\lambda \in (0,\lambda_{8.5})$, and all integers $N \ge N_\lambda^{8.5}$. Combine things to get the stated result. □

We finally bound $\mathscr{E}_{b,1}^{N,\varepsilon}$.

[3]The relaxation parameter $\vartheta_{N,\varepsilon}$ was chosen so that the errors in (79) and the first line of (80) would be of the same order.

LEMMA 9.4. *There is a $K>0$ such that*

$$\mathbb{E}^\varepsilon\left[\left|\int_{u=s\wedge\mathfrak{e}}^{t\wedge\mathfrak{e}}\mathscr{E}_{b,1}^{N,\varepsilon}(X_u)\chi_{\mathsf{E}}(X_u)du\right|\right] \le K(1+t)\left\{\frac{\varepsilon^{1/3}}{r_N^{5/6}}\{1+|\mu_{N,\varepsilon}|\}^{1/2}+\frac{\varepsilon^{2/3}}{r_N^{2/3}}\{1+|\mu_{N,\varepsilon}|\}^{2/3}\right\}\mathfrak{l}(\varepsilon) + K(1+t)\left\{\varepsilon\mathfrak{l}(\varepsilon)+\mathsf{r}_N\mathfrak{l}^2(\varepsilon)+\mathsf{r}_N\mathfrak{l}^2(\mathsf{r}_N)+\mathsf{r}_N\mathfrak{l}^2\left(\frac{1}{1+|\mu_{N,\varepsilon}|}\right)+\mathscr{E}_N^{8.7}\right\}$$

for all $\varepsilon\in(0,1)$ and $N\in\mathbb{N}$ such that

$$\varepsilon<\frac{r_N^{5/2}}{128(1+|\omega_2|)},\qquad \textit{and}\qquad 1+|\mu_{N,\varepsilon}|<\frac{r_N^{15/2}}{\varepsilon^3}. \tag{81}$$

Proof. Fix a relaxation parameter $\vartheta\in(0,1/4)$. For all $\varepsilon\in(0,1)$, $N\in\mathbb{N}$, and $x\in\mathsf{E}$, $\mathscr{E}_{b,1}^{N,\varepsilon}(x)=\text{I}^{\varepsilon,N,\vartheta}(x)+\text{E}_1^{\varepsilon,N,\vartheta}(x)$, where

$$\text{I}^{\varepsilon,N,\vartheta}(x)\overset{\text{def}}{=}\{\xi^*(x)-(\mathcal{A}\xi^*)([\mathsf{E}])\}\mathfrak{c}_\vee\left(\frac{\mathsf{H}_N^{\text{loc}}(x)}{\vartheta\mathsf{r}_N}\right),$$

$$\text{E}_1^{\varepsilon,N,\vartheta}(x)\overset{\text{def}}{=}\{\xi^*(x)-(\mathcal{A}\xi^*)([\mathsf{E}])\}\mathfrak{c}_\wedge\left(\frac{\mathsf{H}_N^{\text{loc}}(x)}{\vartheta\mathsf{r}_N}\right);$$

the requirement that $\vartheta\in(0,1/4)$ ensures that $\mathfrak{c}_\vee\left(\frac{\mathsf{H}_N^{\text{loc}}}{\vartheta\mathsf{r}_N}\right)$ is smooth, particularly at $\mathcal{C}_N$. Using Proposition 8.4 on $\text{E}_1^{\varepsilon,N,\vartheta}$, we have that there is a $K_1>0$ such that

$$\mathbb{E}^\varepsilon\left[\int_{s=0}^{t\wedge\mathfrak{e}}\left|\text{E}_1^{\varepsilon,N,\vartheta}(X_s)\right|\chi_{\mathsf{E}}(X_s)ds\right] \le K_1(1+t)\left\{\vartheta\{1+|\mu_{N,\varepsilon}|\}+\frac{\varepsilon\sqrt{1+|\mu_{N,\varepsilon}|}}{\mathsf{r}_N\sqrt{\vartheta}}+\varepsilon\right\}\mathfrak{l}(\varepsilon)$$

for all $\varepsilon\in(0,1)$ and $N\in\mathbb{N}$ and all $\vartheta\in(0,1/4)$.

To bound $\text{I}^{\varepsilon,N,\vartheta}$, we average. Fix $\varepsilon\in(0,1)$, $\vartheta\in(0,1/4)$, $N\in\mathbb{N}$, and $\lambda\in(0,1)$. For all $x\in\mathbf{S}$, define

$$\Upsilon^{\varepsilon,\lambda,N,\vartheta}\overset{\text{def}}{=}\varepsilon^2\mathfrak{c}_\vee\left(\frac{\mathsf{H}_N^{\text{loc}}(x)}{\vartheta\mathsf{r}_N}\right)\Phi_{\xi^*}^{\mathfrak{z}_N,\lambda}(x)\chi_{\mathsf{E}}(x);$$

then

$$(\mathscr{L}^\varepsilon\Upsilon^{\varepsilon,\lambda,N,\vartheta})(x)=\left\{\text{I}^{\varepsilon,N,\vartheta}(x)+\sum_{i=2}^7\text{E}_i^{\varepsilon,\lambda,N,\vartheta}(x)\right\}\chi_{\mathsf{E}}(x)$$

for all $x\in\mathbf{S}$, where

$$\text{E}_2^{\varepsilon,\lambda,N,\vartheta}(x)\overset{\text{def}}{=}\mathfrak{c}_\vee\left(\frac{\mathsf{H}_N^{\text{loc}}(x)}{\vartheta\mathsf{r}_N}\right)\left\{(\text{Ч}\Phi_{\xi^*}^{\mathfrak{z}_N,\lambda})(x)-\{\xi^*(x)-(\mathcal{A}\xi^*)(x)\}\right\},$$

$$\text{E}_3^{\varepsilon,\lambda,N,\vartheta}(x) \overset{\text{def}}{=} \varepsilon^2 \mathfrak{c}_\vee \left(\frac{\mathsf{H}_N^{\text{loc}}(x)}{\vartheta \mathsf{r}_N}\right) (\mathscr{L}\Phi_{\xi^*}^{\mathfrak{z}_N,\lambda})(x),$$

$$\text{E}_4^{\varepsilon,\lambda,N,\vartheta}(x) \overset{\text{def}}{=} -\frac{\varepsilon^2}{\vartheta \mathsf{r}_N^2}\frac{\xi_N(x)}{\daleth}\dot{\mathfrak{c}}_\wedge \left(\frac{\mathsf{H}_N^{\text{loc}}(x)}{\vartheta \mathsf{r}_N}\right) \Phi_{\xi^*}^{\mathfrak{z}_N,\lambda}(x)\left\{\mu_{N,\varepsilon}\frac{\xi_N(x)}{\daleth} + \mathsf{r}_N \boldsymbol{\beta}_N(x)\right\},$$

$$\text{E}_5^{\varepsilon,\lambda,N,\vartheta}(x) \overset{\text{def}}{=} -\frac{\varepsilon^2}{2\vartheta^2 \mathsf{r}_N^2}\ddot{\mathfrak{c}}_\wedge \left(\frac{\mathsf{H}_N^{\text{loc}}(x)}{\vartheta \mathsf{r}_N}\right) \Phi_{\xi^*}^{\mathfrak{z}_N,\lambda}(x)\boldsymbol{\sigma}_N(x),$$

$$\text{E}_6^{\varepsilon,\lambda,N,\vartheta}(x) \overset{\text{def}}{=} -\frac{\varepsilon^2}{\vartheta \mathsf{r}_N}\dot{\mathfrak{c}}_\wedge \left(\frac{\mathsf{H}_N^{\text{loc}}(x)}{\vartheta \mathsf{r}_N}\right) \left\langle d\mathsf{H}_N^{\text{loc}}, d\Phi_{\xi^*}^{\mathfrak{z}_N,\lambda}\right\rangle (x)$$

for all $x \in \mathsf{E}$. Use now Lemma 8.7; there is then a constant $K_2 > 0$ such that

$$\begin{aligned}
|\text{E}_2^{\varepsilon,\lambda,N,\vartheta}(x)| &\le K_2\left\{\lambda + \mathsf{r}_N \mathfrak{l}^2(\vartheta \mathsf{r}_N) + \frac{|\boldsymbol{\nu}_N|}{\lambda^2 \mathsf{r}_N^3 \vartheta} + \mathscr{E}_N^{8.7}\right\} \\
&= K_2\left\{\lambda + \frac{1}{\daleth}\frac{\varepsilon^2|\mu_{N,\varepsilon}|}{\lambda^2 \mathsf{r}_N^4\vartheta} + \mathsf{r}_N \mathfrak{l}^2(\vartheta \mathsf{r}_N) + \mathscr{E}_N^{8.7}\right\},
\end{aligned}$$

$$|\text{E}_3^{\varepsilon,\lambda,N,\vartheta}(x)| = \frac{K_2}{\lambda^3}\frac{\varepsilon^2}{\mathsf{r}_N^5\vartheta^2}, \qquad |\text{E}_4^{\varepsilon,\lambda,N,\vartheta}(x)| \le \frac{K_2}{\lambda}\frac{\varepsilon^2}{\mathsf{r}_N^3\vartheta}\{1+|\mu_{N,\varepsilon}|\},$$

$$|\text{E}_5^{\varepsilon,\lambda,N,\vartheta}(x)| \le \frac{K_2}{\lambda}\frac{\varepsilon^2}{\mathsf{r}_N^3\vartheta^2}, \qquad |\text{E}_6^{\varepsilon,\lambda,N,\vartheta}(x)| \le \frac{K_2}{\lambda^2}\frac{\varepsilon^2}{\mathsf{r}_N^4\vartheta^2},$$

$$|\Upsilon^{\varepsilon,\lambda,N,\vartheta}(x)| \le \frac{K_2}{\lambda}\frac{\varepsilon^2}{\mathsf{r}_N}$$

for all ε and λ in $(0,1)$, all $\vartheta \in (0,1/4)$, all $N \in \mathbb{N}$, and all $x \in \mathsf{E}$. Thus, there is a constant $K_3 > 0$ such that

$$\begin{aligned}
\mathbb{E}^\varepsilon\left[\left|\int_{u=s\wedge\mathfrak{e}}^{t\wedge\mathfrak{e}} \text{I}^{\varepsilon,N,\vartheta}(X_u)\chi_{\mathsf{E}}(X_u)du\right|\right] &\le K_3(1+t)\left\{\lambda + \frac{\varepsilon^2\{1+|\mu_{N,\varepsilon}|\}}{\lambda^3 \mathsf{r}_N^5\vartheta^2}\right\} \\
&\quad + K_3(1+t)\{\mathsf{r}_N \mathfrak{l}^2(\vartheta \mathsf{r}_N) + \mathscr{E}_N^{8.7}\}
\end{aligned}$$

for all ε and λ in $(0,1)$, all $\vartheta \in (0,1/4)$, and all $N \in \mathbb{N}$. Combining things, we find that there is a constant $K_4 > 0$ such that

$$\begin{aligned}
&\mathbb{E}^\varepsilon\left[\left|\int_{u=s\wedge\mathfrak{e}}^{t\wedge\mathfrak{e}} \mathscr{E}_{b,1}^{N,\varepsilon}(X_u)\chi_{\mathsf{E}}(X_u)du\right|\right] \\
&\qquad\le K_4(1+t)I_{N,\varepsilon}(\lambda,\vartheta)\mathfrak{l}(\varepsilon) \\
&\qquad\quad + K_4(1+t)\{\varepsilon\mathfrak{l}(\varepsilon) + \mathsf{r}_N \mathfrak{l}^2(\vartheta \mathsf{r}_N) + \mathscr{E}_N^{8.7}\}
\end{aligned}$$

for all $\varepsilon \in (0,1)$ and $N \in \mathbb{N}$ and all real λ and ϑ such that

$$0 < \lambda < 1 \qquad \text{and} \qquad 0 < \vartheta < \frac{1}{4}, \tag{82}$$

where we have for convenience defined

$$I_{N,\varepsilon}(\lambda,\vartheta) \stackrel{\text{def}}{=} \lambda + \frac{\varepsilon^2\{1+|\mu_{N,\varepsilon}|\}}{\lambda^3\vartheta^2 \mathsf{r}_N^5} + \vartheta\{1+|\mu_{N,\varepsilon}|\} + \frac{\varepsilon\sqrt{1+|\mu_{N,\varepsilon}|}}{\mathsf{r}_N\vartheta^{1/2}} \tag{83}$$

for all λ and ϑ in $(0,\infty)$.

Fix now $\varepsilon \in (0,1)$ and $N \in \mathbb{N}$. Define

$$\vartheta_{N,\varepsilon} \stackrel{\text{def}}{=} \frac{\left(\frac{\varepsilon\sqrt{1+|\mu_{N,\varepsilon}|}}{\mathsf{r}_N^{5/2}} + \frac{\varepsilon^2\{1+|\mu_{N,\varepsilon}|\}}{\mathsf{r}_N^2}\right)^{1/3}}{\{1+|\mu_{N,\varepsilon}|\}^{2/3}},$$

$$\lambda_{N,\varepsilon} \stackrel{\text{def}}{=} \frac{\varepsilon^{1/2}\{1+|\mu_{N,\varepsilon}|\}^{1/4}}{\mathsf{r}_N^{5/4}\vartheta_{N,\varepsilon}^{1/2}} = \frac{\varepsilon^{1/2}}{\mathsf{r}_N^{5/4}} \frac{\{1+|\mu_{N,\varepsilon}|\}^{7/12}}{\left(\frac{\varepsilon\sqrt{1+|\mu_{N,\varepsilon}|}}{\mathsf{r}_N^{5/2}} + \frac{\varepsilon^2\{1+|\mu_{N,\varepsilon}|\}}{\mathsf{r}_N^2}\right)^{1/6}}.$$

We chose $\lambda_{N,\varepsilon}$ to make both λ-terms in (83) to be of the same order;

$$\begin{aligned} I_{N,\varepsilon}(\lambda_{N,\varepsilon},\vartheta_{N,\varepsilon}) &= 2\frac{\varepsilon^{1/2}\{1+|\mu_{N,\varepsilon}|\}^{1/4}}{\mathsf{r}_N^{5/4}\vartheta_{N,\varepsilon}^{1/2}} + \vartheta_{N,\varepsilon}\{1+|\mu_{N,\varepsilon}|\} + \frac{\varepsilon\sqrt{1+|\mu_{N,\varepsilon}|}}{\mathsf{r}_N\vartheta_{N,\varepsilon}^{1/2}} \\ &\le \frac{3}{\vartheta_{N,\varepsilon}^{1/2}}\sqrt{\frac{\varepsilon\sqrt{1+|\mu_{N,\varepsilon}|}}{\mathsf{r}_N^{5/2}} + \frac{\varepsilon^2\{1+|\mu_{N,\varepsilon}|\}}{\mathsf{r}_N^2}} + \vartheta_{N,\varepsilon}\{1+|\mu_{N,\varepsilon}|\}. \end{aligned}$$

We then chose $\vartheta_{N,\varepsilon}$ to make both terms in the final expression of the same order;

$$\begin{aligned} I_{N,\varepsilon}(\lambda_{N,\varepsilon},\vartheta_{N,\varepsilon}) &\le 4\left(\frac{\varepsilon\sqrt{1+|\mu_{N,\varepsilon}|}}{\mathsf{r}_N^{5/2}} + \frac{\varepsilon^2\{1+|\mu_{N,\varepsilon}|\}}{\mathsf{r}_N^2}\right)^{1/3}\{1+|\mu_{N,\varepsilon}|\}^{1/3} \\ &\le 8\frac{\varepsilon^{1/3}}{\mathsf{r}_N^{5/6}}\{1+|\mu_{N,\varepsilon}|\}^{1/2} + 8\frac{\varepsilon^{2/3}}{\mathsf{r}_N^{2/3}}\{1+|\mu_{N,\varepsilon}|\}^{2/3}. \end{aligned}$$

We now need to show that (81) implies that $\vartheta_{N,\varepsilon}$ and $\lambda_{N,\varepsilon}$ satisfy the admissibility requirements of (82). Clearly $\lambda_{N,\varepsilon}$ and $\vartheta_{N,\varepsilon}$ are positive. We also need that

$$\left(\frac{\varepsilon\sqrt{1+|\mu_{N,\varepsilon}|}}{\mathsf{r}_N^{5/2}} + \frac{\varepsilon^2\{1+|\mu_{N,\varepsilon}|\}}{\mathsf{r}_N^2}\right)^{1/3} < \frac{1}{4}\{1+|\mu_{N,\varepsilon}|\}^{2/3} \qquad \left(\vartheta_{N,\varepsilon} < \frac{1}{4}\right),$$

$$\frac{\varepsilon^{1/2}}{\mathsf{r}_N^{5/4}}\{1+|\mu_{N,\varepsilon}|\}^{7/12} < \left(\frac{\varepsilon\sqrt{1+|\mu_{N,\varepsilon}|}}{\mathsf{r}_N^{5/2}} + \frac{\varepsilon^2\{1+|\mu_{N,\varepsilon}|\}}{\mathsf{r}_N^2}\right)^{1/6} \qquad (\lambda_{N,\varepsilon} < 1).$$

These are equivalent to the requirements that

$$\frac{\varepsilon\sqrt{1+|\mu_{N,\varepsilon}|}}{\mathsf{r}_N^{5/2}} + \frac{\varepsilon^2\{1+|\mu_{N,\varepsilon}|\}}{\mathsf{r}_N^2} < \frac{1}{64}\{1+|\mu_{N,\varepsilon}|\}^2 \qquad \left(\vartheta_{N,\varepsilon} < \frac{1}{4}\right),$$

$$\frac{\varepsilon^3\{1+|\mu_{N,\varepsilon}|\}^{7/6}}{\mathsf{r}_N^{15/2}} < \frac{\varepsilon\sqrt{1+|\mu_{N,\varepsilon}|}}{\mathsf{r}_N^{5/2}} + \frac{\varepsilon^2}{\mathsf{r}_N^2}\{1+|\mu_{N,\varepsilon}|\} \qquad (\lambda_{N,\varepsilon}<1).$$

These inequalities hold if (but not only if)

$$\max\left\{\frac{\varepsilon\sqrt{1+|\mu_{N,\varepsilon}|}}{\mathsf{r}_N^{5/2}}, \frac{\varepsilon^2\{1+|\mu_{N,\varepsilon}|\}}{\mathsf{r}_N^2}\right\} < \frac{1}{128}\{1+|\mu_{N,\varepsilon}|\}^2 \qquad \left(\vartheta_{N,\varepsilon}<\frac{1}{4}\right),$$

$$\frac{\varepsilon^3}{\mathsf{r}_N^{15/2}}\{1+|\mu_{N,\varepsilon}|\}^{7/6} < \frac{\varepsilon\sqrt{1+|\mu_{N,\varepsilon}|}}{\mathsf{r}_N^{5/2}} \qquad (\lambda_{N,\varepsilon}<1).$$

The first of these inequalities is equivalent to requiring that both $\varepsilon < \mathsf{r}_N^{5/2}\{1+|\mu_{N,\varepsilon}|\}^{3/2}/128$ and $\varepsilon < \mathsf{r}_N\sqrt{1+|\mu_{N,\varepsilon}|}/\sqrt{128}$. Since $1+|\mu_{N,\varepsilon}|>1$, this is true if the first requirement of (81) holds. The second inequality is equivalent to requiring that $1+|\mu_{N,\varepsilon}| < \mathsf{r}_N^{15/2}/\varepsilon^3$.

Finally, we bound $\vartheta_{N,\varepsilon}\mathsf{r}_N$ from below by taking the second term in the numerator of the definition of $\vartheta_{N,\varepsilon}$; we have that

$$\vartheta_{N,\varepsilon}\mathsf{r}_N \geq \frac{\varepsilon^{2/3}\mathsf{r}_N^{1/3}}{\{1+|\mu_{N,\varepsilon}|\}^{1/3}};$$

thus, there is a constant $K>0$ such that

$$\mathfrak{l}(\vartheta_{N,\varepsilon}\mathsf{r}_N) \leq K\left\{\mathfrak{l}(\varepsilon)+\mathfrak{l}(\mathsf{r}_N)+\mathfrak{l}\left(\frac{1}{1+|\mu_{N,\varepsilon}|}\right)\right\}$$

for all ε and r_N in $(0,1)$ and all $N\in\mathbb{N}$. □

Define now the function $\Psi_B^{\varepsilon,N}$ and the constant $K_{(84)}$ by

$$\Psi_B^{\varepsilon,N} \stackrel{\text{def}}{=} \Psi_C^{\varepsilon,N} + \mathsf{U}_P^{N,\varepsilon} \qquad \text{and} \qquad K_{(84)} \stackrel{\text{def}}{=} 2^7 + K_{8.8} + 256(1+|\omega_2|). \tag{84}$$

For each $\gamma\in(0,2/7)$, let $\mathscr{S}_\gamma$ be the collection of $(\varepsilon,N)\in(0,1)\times\mathbb{N}$ such that

$$\varepsilon < \frac{\mathsf{r}_N^{105/(4-14\gamma)}}{K_{(84)}} \qquad |\mu_{N,\varepsilon}| < \varepsilon^\gamma\frac{\mathsf{r}_N^{15/2}}{K_{(84)}\varepsilon^{2/7}}, \qquad \text{and} \qquad \mathsf{r}_N<1. \tag{85}$$

We then have

LEMMA 9.5. *Suppose that* $\{(\varepsilon_k,N_k);\, k\in\mathbb{N}\}$ *is a sequence of elements of* $\mathscr{S}_\gamma$ *such that* $\lim_{k\to\infty}\varepsilon_k=0$, $\lim_{k\to\infty}N_k=\infty$, *and* $\lim_{k\to\infty}\mathsf{r}_k\mathfrak{l}^2(\varepsilon_k)=0$. *Then* $\lim_{k\to\infty}\|\Psi_B^{\varepsilon_k,N_k}\|_{C(\mathbf{S})}=0$ *and*

$$\lim_{k\to\infty}\mathbb{E}^{\varepsilon_k}\left[\left\{\int_{u=s\wedge\mathfrak{e}}^{t\wedge\mathfrak{e}}(\mathscr{L}^{\varepsilon_k}\Psi_B^{\varepsilon_k,N_k})(X_u)du\right\}^-\right]=0$$

for all $t\geq s\geq 0$.

Proof. Collect together Lemmas 7.8, 7.9, 8.8, 9.2, 9.3, and 9.4. Recall also (23). We have that there is a constant $K > 0$ such that

$$\begin{aligned}
(86)\quad &\mathbb{E}^\varepsilon\left[\left\{\int_{s=0}^{t\wedge\mathfrak{e}}(\mathscr{L}^\varepsilon\Psi_B^{\varepsilon,N})(X_s)ds\right\}^-\right] \\
&\geq -K(1+t)\left\{|\boldsymbol{\nu}_N| + \mathsf{r}_N + \{1+|\mu_{N,\varepsilon}|\}^2\frac{\varepsilon^2}{\mathsf{r}_N}\right\} \\
&- K\mathscr{E}^{8.5}(\lambda,N)(1+t)\left\{1+\{1+|\mu_{N,\varepsilon}|\}^2\mathsf{r}_N + \left(\frac{\varepsilon}{\mathsf{r}_N}\right)^2\{1+|\mu_{N,\varepsilon}|\}^5\right\} \\
&- KK_\lambda^{8.5}(1+t)\{1+|\mu_{N,\varepsilon}|\}^{7/3}\left(\frac{\varepsilon}{\mathsf{r}_N}\right)^{2/3} \\
&- K(1+t)\left\{\frac{\varepsilon^{1/3}}{\mathsf{r}_N^{5/6}}\{1+|\mu_{N,\varepsilon}|\}^{1/2} + \frac{\varepsilon^{2/3}}{\mathsf{r}_N^{2/3}}\{1+|\mu_{N,\varepsilon}|\}^{2/3}\right\}\mathfrak{l}(\varepsilon) \\
&- K(1+t)\left\{\varepsilon\mathfrak{l}(\varepsilon) + \mathsf{r}_N\mathfrak{l}^2(\varepsilon) + \mathsf{r}_N\mathfrak{l}^2(\mathsf{r}_N) + \mathsf{r}_N\mathfrak{l}^2\left(\frac{1}{1+|\mu_{N,\varepsilon}|}\right) + \mathscr{E}_N^{8.7}\right\} \\
&- \frac{K(1+t)}{\varepsilon^{17/6}}\exp\left[-\frac{1}{K_{8.8}}\left(\frac{\mathsf{r}_N^{3/2}}{\varepsilon}\right)^{1/2}\right] \\
&- K(1+t)\left\{\frac{\varepsilon^{1/2}\{1+|\mu_{N,\varepsilon}|\}}{\mathsf{r}_N^2} + \frac{\varepsilon^{1/4}\{1+|\mu_{N,\varepsilon}|\}^{3/4}}{\mathsf{r}_N^{11/8}} + \frac{\varepsilon\{1+|\mu_{N,\varepsilon}|\}^2}{\mathsf{r}_N^{11/4}}\right\}\mathfrak{l}(\varepsilon) \\
&- K(1+t)\left\{\sum_{\ell\in\Lambda_P}\left|\hat{\mathsf{u}}_+^{N,\varepsilon}\right| + \sum_{\ell\in\Lambda_W}\left|\hat{\mathsf{u}}_-^{N,\varepsilon}\right|\right\}
\end{aligned}$$

for all $\varepsilon \in (0,1)$ and $N \in \mathbb{N}$ such that

$$\varepsilon < \frac{\mathsf{r}_N^{3/2}}{K_{8.8}}, \quad 1+|\mu_{N,\varepsilon}| \leq \frac{\mathsf{r}_N^2}{\varepsilon^2}, \quad \varepsilon < \frac{\mathsf{r}_N^{5/2}}{128(1+|\omega_2|)}, \quad \text{and} \quad 1+|\mu_{N,\varepsilon}| < \frac{\mathsf{r}_N^{15/2}}{\varepsilon^3}.$$

Fix now $\gamma \in (0, 2/7)$. Let $\mathscr{S}'_\gamma$ be the collection of $(\varepsilon, N) \in (0,1)\times\mathbb{N}$ such that

$$\begin{gathered}
\{1+|\mu_{N,\varepsilon}|\}^2 \leq \varepsilon^{2\gamma}\frac{\mathsf{r}_N}{\varepsilon^2}, \quad 1+|\mu_{N,\varepsilon}| < \varepsilon^\gamma\frac{1}{\mathsf{r}_N}, \quad \{1+|\mu_{N,\varepsilon}|\}^5 \leq \varepsilon^{5\gamma}\frac{\mathsf{r}_N^2}{\varepsilon^2}, \\
\{1+|\mu_{N,\varepsilon}|\}^{7/3} \leq \varepsilon^{7\gamma/3}\frac{\mathsf{r}_N^{2/3}}{\varepsilon^{2/3}}, \quad \{1+|\mu_{N,\varepsilon}|\}^{1/2} \leq \varepsilon^{\gamma/2}\frac{\mathsf{r}_N^{5/6}}{\varepsilon^{1/3}}, \\
\{1+|\mu_{N,\varepsilon}|\}^{2/3} \leq \varepsilon^{2\gamma/3}\frac{\mathsf{r}_N^{2/3}}{\varepsilon^{2/3}}, \quad \{1+|\mu_{N,\varepsilon}|\} \leq \varepsilon^\gamma\frac{\mathsf{r}_N^2}{\varepsilon^{1/2}}, \\
\{1+|\mu_{N,\varepsilon}|\}^{3/4} \leq \varepsilon^{3\gamma/4}\frac{\mathsf{r}_N^{11/8}}{\varepsilon^{1/4}}, \quad \text{and} \quad \{1+|\mu_{N,\varepsilon}|\}^2 \leq \varepsilon^{2\gamma}\frac{\mathsf{r}_N^{11/4}}{\varepsilon}
\end{gathered} \tag{87}$$

and such that

$$\varepsilon < \frac{\mathsf{r}_N^{3/2}}{K_{8.8}}, \quad 1+|\mu_{N,\varepsilon}| \le \frac{\mathsf{r}_N^2}{\varepsilon^2}, \quad \varepsilon < \frac{\mathsf{r}_N^{5/2}}{128(1+|\omega_2|)}, \quad \text{and} \quad 1+|\mu_{N,\varepsilon}| < \frac{\mathsf{r}_N^{15/2}}{\varepsilon^3}.$$

Note that $\lim_{\varepsilon\to 0}\varepsilon^\gamma \mathfrak{l}(\varepsilon) = \lim_{N\to\infty}\mathsf{r}_N\mathfrak{l}^2(\mathsf{r}_N) = 0$. Also note that if $(\varepsilon, N) \in \mathscr{S}'_\gamma$, then

$$\begin{aligned}\frac{1}{\varepsilon^{17/3}}\exp\left[-\frac{1}{K_{8.8}}\left(\frac{\mathsf{r}_N^{3/2}}{\varepsilon}\right)^{1/2}\right] &\le \frac{1}{\varepsilon^{17/3}}\exp\left[-\frac{\{128(1+|\omega_2|\})^{2/5}}{K_{8.8}}\frac{(\varepsilon^{2/5})^{3/4}}{\varepsilon^{1/2}}\right]\\ &\le \frac{1}{\varepsilon^{17/3}}\exp\left[-\frac{\{128(1+|\omega_2|\})^{2/5}}{K_{8.8}}\frac{1}{\varepsilon^{1/5}}\right].\end{aligned}$$

Fix now a sequence (ε_k, N_k) in $\mathscr{S}'_\gamma$ such that $\lim_{k\to\infty}\varepsilon_k = 0$, $\lim_{k\to\infty} N_k = \infty$, and $\lim_{k\to\infty}\mathsf{r}_k\mathfrak{l}^2(\varepsilon_k) = 0$. The definition of $\mathscr{S}'_\gamma$ implies that all of the terms involving $1+|\mu_{N,\varepsilon}|$ tend to zero. The second bound of (87) implies that there is a $K > 0$ such that $\mathfrak{l}(1/\{1+|\mu_{N,\varepsilon}|\}) \le K\mathfrak{l}(\mathsf{r}_N)$ for all $(\varepsilon, N) \in \mathscr{S}'_\gamma$. Thus $\lim_{k\to\infty}\mathsf{r}_N\mathfrak{l}^2(1/\{1+|\mu_{N,\varepsilon}|\}) = 0$. We also clearly have that $\lim_{k\to\infty}|\boldsymbol{\nu}_{N_k}| = \lim_{k\to\infty}\mathsf{r}_{N_k} = 0$. We use (52) on the terms involving $|\hat{\mathsf{u}}_+^{N,\varepsilon}|$ and $|\hat{\mathsf{u}}_-^{N,\varepsilon}|$. We also use Lemmas 8.5 and 8.7 to control the behavior of $\mathscr{E}^{8.5}(\lambda, N)$ and $\mathscr{E}_N^{8.7}$. We get that

$$\varliminf_{k\to\infty}\mathbb{E}^\varepsilon\left[\left\{\int_{u=s\wedge\mathfrak{e}}^{t\wedge\mathfrak{e}}(\mathscr{L}^\varepsilon\Psi_B^{\varepsilon_k,N_k})(X_u)du\right\}^-\right] \ge -K(1+t)\varlimsup_{N\to\infty}|\mathscr{E}^{8.5}(\lambda, N)|.$$

We now take $\lambda \to 0$ to get the claimed result (recalling Lemma 8.5).

We now show that $\mathscr{S}_\gamma \subset \mathscr{S}'_\gamma$. If $\mathsf{r}_N < 1$, then

$$\frac{\mathsf{r}_N^{5/2}}{K_{(84)}} \le \min\left\{\frac{\mathsf{r}_N^{3/2}}{K_{8.8}}, \frac{\mathsf{r}_N^{5/2}}{128(1+|\omega_2|)}\right\} \qquad \text{and} \qquad \frac{\mathsf{r}_N^{15/2}}{\varepsilon^2} \le \min\left\{\frac{\mathsf{r}_N^2}{\varepsilon^2}, \frac{\mathsf{r}_N^{15/2}}{\varepsilon^3}\right\}.$$

Rewriting the requirements on $1+|\mu_{N,\varepsilon}|$, we get that $(\varepsilon, N) \in (0,1)\times\mathbb{N}$ is in $\mathscr{S}'_\gamma$ if

$$1+|\mu_{N,\varepsilon}| < \varepsilon^\gamma \min\left\{\frac{\mathsf{r}_N^{1/2}}{\varepsilon}, \frac{1}{\mathsf{r}_N}, \frac{\mathsf{r}_N^{2/5}}{\varepsilon^{2/5}}, \frac{\mathsf{r}_N^{2/7}}{\varepsilon^{2/7}}, \frac{\mathsf{r}_N^{5/3}}{\varepsilon^{2/3}}, \frac{\mathsf{r}_N}{\varepsilon}, \frac{\mathsf{r}_N}{\varepsilon^{1/2}}, \frac{\mathsf{r}_N^{11/6}}{\varepsilon^{1/3}}, \frac{\mathsf{r}_N^{11/8}}{\varepsilon^{1/2}}\right\} \tag{88}$$

and $\mathsf{r}_N < 1$, $\varepsilon < \mathsf{r}_N^{5/2}/K_{(84)}$ and $1+|\mu_{N,\varepsilon}| < \mathsf{r}_N^{15/2}/\varepsilon^2$.

Picking out the largest exponent of r_N in the denominator and the smallest exponent of ε in the denominator, in (88), we see that $(\varepsilon, N) \in (0,1)\times\mathbb{N}$ is in $\mathscr{S}'_\gamma$ if

$$\varepsilon < \frac{\mathsf{r}_N^{5/2}}{K_{(84)}}, \qquad \mathsf{r}_N < 1, \qquad \text{and} \qquad 1 + |\mu_{N,\varepsilon}| < \min\left\{\frac{\mathsf{r}_N^{15/2}}{\varepsilon^2}, \frac{\mathsf{r}_N^{11/6}}{\varepsilon^{2/7-\gamma}}\right\}.$$

Since γ is by assumption positive, $\varepsilon^{2/7-\gamma} > \varepsilon^2$ for all $\varepsilon \in (0,1)$, so $(\varepsilon, N) \in (0,1) \times \mathbb{N}$ is in $\mathscr{S}'_\gamma$ if

$$\varepsilon < \frac{\mathsf{r}_N^{5/2}}{K_{(84)}}, \qquad \mathsf{r}_N < 1, \qquad \text{and} \qquad 1 + |\mu_{N,\varepsilon}| < \frac{\mathsf{r}_N^{15/2}}{\varepsilon^{2/7-\gamma}}.$$

In order that the upper bound on $1+|\mu_{N,\varepsilon}|$ be non-vacuous, we necessarily must have that $\mathsf{r}_N^{15/2}/\varepsilon^{2/7-\gamma} \geq 1$. Since $K_{(84)} > 2^7$ and $\gamma < 1/7$,

$$\varepsilon < \frac{\mathsf{r}_N^{105/(4-14\gamma)}}{K_{(84)}} \qquad \text{implies that} \qquad \frac{\mathsf{r}_N^{15/2}}{\varepsilon^{2/7-\gamma}} \geq K_{(84)}^{2/7-\gamma} \geq (2^7)^{1/7} \geq 2$$

(note that $\frac{15}{2}/(\frac{2}{7}-\gamma) = 105/(4-14\gamma)$), in which case

$$|\mu_{N,\varepsilon}| < \frac{1}{2}\frac{\mathsf{r}_N^{15/2}}{\varepsilon^{2/7-\gamma}} \qquad \text{implies that} \qquad 1 + |\mu_{N,\varepsilon}| < \frac{\mathsf{r}_N^{15/2}}{\varepsilon^{2/7-\gamma}}.$$

In other words, if

$$\varepsilon < \min\left\{\frac{\mathsf{r}_N^{5/2}}{K_{(84)}}, \frac{\mathsf{r}_N^{105/(4-14\gamma)}}{K_{(84)}}\right\}, \quad \mathsf{r}_N < 1, \quad \text{and} \quad |\mu_{N,\varepsilon}| < \frac{1}{2}\frac{\mathsf{r}_N^{15/2}}{\varepsilon^{2/7-\gamma}},$$

then $(\varepsilon, N) \in \mathscr{S}'_\gamma$. Since $\gamma \in (0, 1/7)$, $105/(4-14\gamma) > 5/2$. Also, $K_{(84)} > 2$. This allows us to see that $\mathscr{S}_\gamma \subset \mathscr{S}'_\gamma$, finishing the proof. $\square$

We finally have

Proof of Proposition 5.2. Fix $\gamma' \in (0, 1/7)$ such that

$$\gamma' < \gamma \qquad \text{and} \qquad \frac{105}{4-14\gamma'} < \frac{105}{4} + \gamma.$$

We want to show that $(\varepsilon_N, N) \in \mathscr{S}_{\gamma'}$ for N large enough. We calculate that

$$\frac{\varepsilon_N}{\mathsf{r}_N^{105/(4-14\gamma')}} = \frac{1}{|\omega_2|^{105/(4-14\gamma')}}\left(\frac{1}{\mathsf{a}_N^{(d)}}\right)^{105/4+\gamma/2-105/(4-14\gamma')},$$

$$\begin{aligned}\frac{|\mu_{N,\varepsilon_N}|}{(\mathsf{r}_N^{15/2}/\varepsilon_N^{2/7-\gamma'})} &\leq |\beth|\frac{|\boldsymbol{\nu}_N|\mathsf{r}_N\varepsilon_N^{2/7-\gamma'}}{\varepsilon_N^2\mathsf{r}_N^{15/2}} \leq |\beth|\frac{|\boldsymbol{\nu}_N|}{\mathsf{r}_N^{13/2}\varepsilon_N^{12/7}} \\ &\leq \frac{|\beth|}{|\omega_2|^{13/2}}\frac{(\mathsf{a}_N^{(d)})^{13/2+12/7(105/4+\gamma/2)}}{\mathsf{a}_{N+1}^{(d)}\mathsf{a}_N^{(d)}} \\ &= \frac{|\beth|}{|\omega_2|^{13/2}}\frac{(\mathsf{a}_N^{(d)})^{721/14+6\gamma/7}}{\mathsf{a}_{N+1}^{(d)}} \leq \frac{|\beth|}{|\omega_2|^{13/2}}\frac{(\mathsf{a}_N^{(d)})^{721/14+\gamma}}{\mathsf{a}_{N+1}^{(d)}}.\end{aligned}$$

Clearly $\mathsf{r}_N < 1$ for N large enough and $\lim_{N\to\infty} \mathsf{r}_N \mathfrak{l}^2(\varepsilon_N) = 0$. □

REMARK 9.6. The origin of the complicated exponents in Theorem 2.10 is the expression on the right of (86), which must tend to zero. This of course can happen for many other sequences $\{\varepsilon_N; N \in \mathbb{N}\}$ (not just (17)). Similarly, we could probably slightly relax the requirements of (85). We have already tested the patience of the reader (and author) enough to make any such refinements ill-considered.

One can also see why we can only look at specific sequences of ε's, rather than the whole continuum. If one starts with Remark 7.3 and requires that $\varepsilon < \mathsf{r}_N^{3/2}$, then one might like to take all ε such that $\mathsf{r}_{N+1}^{3/2} \le \varepsilon < \mathsf{r}_N^{3/2}$; i.e., partition $(0, 1)$ (and hence the choice of ε) into intervals of the form $[\mathsf{r}_{N+1}^{3/2}, \mathsf{r}_N^{3/2})$. One must then consider $\varepsilon \approx \mathsf{r}_{N+1}^{3/2}$. But if $\varepsilon \approx \mathsf{r}_{N+1}^{3/2}$ and $|\boldsymbol{\nu}_N| \approx \mathsf{r}_N \mathsf{r}_{N+1}$, then

$$|\mu_{N,\varepsilon}| \approx \frac{\mathsf{r}_N^2}{\mathsf{r}_{N+1}^2} \gg \frac{\mathsf{r}_N^{15/2}}{\mathsf{r}_{N+1}^{3/7}} \approx \frac{\mathsf{r}_N^{15/2}}{\varepsilon^{2/7}}$$

(since $\mathsf{r}_{N+1}^{11/7} \ll 1 \ll \mathsf{r}_N^{-11/2}$). Thus, if ε is at the left edge of the interval $[\mathsf{r}_{N+1}^{3/2}, \mathsf{r}_N^{3/2})$, the second requirement of (85) is far from being satisfied.

It is important to note that we DO allow $|\mu_{N,\varepsilon}|$ to grow, but not too quickly. Our calculations would definitely be easier (but our results much weaker) if we forced $|\mu_{N,\varepsilon}|$ to stay bounded or to tend to zero; the separation of scales in (49) would become even stronger.

Some of the complexity of the exponents in Theorem 2.10 comes from the comments of Remark 7.3 that the glueing corrector along γ_N not interfere with itself across $\mathcal{N}_N$. To reduce this interference, we would like r_N to be as large as possible. On the other hand, the larger that r_N is, the larger $|\mu_{N,\varepsilon}|$ is. This competition restricts things.

Another perspective on the issue of separation of scales is suggested by Figure 4. Our calculations hinged upon being able to first stochastically average in the small loop on the right-hand side of Figure 4, and then to solve a Poisson equation on the loop and show that it had certain asymptotics. A significantly different approach might be to follow calculations like [FW94] and show that boundary layer calculations around $\partial\mathsf{E}$ reflect the fact that the global invariant measure is Lebesgue measure. Perhaps the mixing calculations like [SK92] might be useful in carrying out such a program. If one would attempt to use [SK92], one would need to bound the effect of noisy errors in the "special flow" (and to show some uniformity in the error bounds of [SK92]).

10. Bounds on the corrector function

Here we prove Proposition 7.2, which is the basic glueing estimate. This will follow from the work of [Sow05], [Sow].

Let's set up our problem in the framework of [Sow05], [Sow]. We start by rewriting things in terms of the geometry defined by $\mathscr{L}$. We start in the usual way. For each $f \in C^\infty(\mathbb{T})$, define $(M_{1,f}g)(x) \overset{\text{def}}{=} \langle df, dg\rangle(x)$ for all $g \in C^\infty(\mathbb{T})$ and $x \in \mathbb{T}$. It is easy to check that $M_{1,f}$ is a derivation of $C^\infty(\mathbb{T})$ (treated as an algebra over $\mathbb{R}$; see [Boo86, p. 39]), so there is a vector field $M_{2,f}$ on $\mathbb{T}$ such that $(M_{1,f}g)(x) = (M_{2,f}(x), \nabla g(x))$ for all $g \in C^\infty(\mathbb{T})$ and $x \in \mathbb{T}$. Furthermore, the map $f \mapsto M_{2,f}$ is linear in f and that $M_{2,f^2}(x) = 2f(x)M_{2,f}(x)$ for all $f \in C^\infty(\mathbb{T})$ and $x \in \mathbb{T}$, so there is in fact a fiber map $\tilde{M} : T\mathbb{T} \to T\mathbb{T}$ such that $M_{2,f}(x) = \tilde{M}\nabla f(x)$ for all $f \in C^\infty(\mathbb{T})$ and $x \in \mathbb{T}$. It is also easy to see that $(\tilde{M}X, Y) = (X, \tilde{M}Y)$ for all X and Y in any common $T_x\mathbb{T}$, and by considering local charts, it is easy to see that $\tilde{M}$ is smooth. The nondegeneracy assumption of (2) also implies that $\tilde{M}|_{T_x\mathbb{T}} > 0$ for all $x \in \mathbb{T}$. Define finally $(X,Y)_G \overset{\text{def}}{=} (X, \tilde{M}^{-1}Y)$ for all X and Y in any common $T_x\mathbb{T}$, where $\tilde{M}^{-1}$ is a smooth inverse of $\tilde{M}$; then $(\cdot,\cdot)_G$ is a Riemannian metric on $\mathbb{T}$. We have that $\nabla_G f = \tilde{M}\bar{\nabla} f$, where ∇_G is the gradient operator with respect to $(\cdot,\cdot)_G$. Thus $(\nabla_G f, \nabla_G g)_G = \langle df, dg\rangle$ for all f and g in $C^\infty(\mathbb{T})$. Next, define the two vector field $\tilde{\mathbf{e}}_1$ and $\tilde{\mathbf{e}}_2$ on $\mathbb{T}$ by requiring that $(\tilde{\mathbf{e}}_i f)(\mathsf{t}(x)) = \frac{\partial(f\circ \mathsf{t})}{\partial x_i}(x)$ for all $x = (x_1, x_2) \in \mathbb{R}^2$, $f \in C^\infty(\mathbb{T})$, and $i \in \{1,2\}$. Define the fiber map $\mathcal{J} : T\mathbb{T} \to T\mathbb{T}$ by $\mathcal{J}X \overset{\text{def}}{=} (X, \tilde{\mathbf{e}}_1)\tilde{\mathbf{e}}_2 - (X, \tilde{\mathbf{e}}_2)\tilde{\mathbf{e}}_1$ for all $X \in T\mathbb{T}$. Define then a second fiber map $\mathcal{J}_G : T\mathbb{T} \to T\mathbb{T}$ as $\mathcal{J}_G X \overset{\text{def}}{=} \tilde{M}^{1/2}\mathcal{J}\tilde{M}^{-1/2}X$ for all $X \in T\mathbb{T}$, where $\tilde{M}^{1/2}$ is a self-adjoint and smooth square root of $\tilde{M}$, and $\tilde{M}^{-1/2}$ is its inverse. Finally, define $\omega_G(X,Y) \overset{\text{def}}{=} (X, \mathcal{J}_G Y)_G$ for all X and Y in any common $T_x\mathbb{T}$. It is easy to check that ω_G is a symplectic form which is related to $(\cdot,\cdot)_G$. Define also $\text{љ} \overset{\text{def}}{=} \frac{d\omega_e}{d\mathsf{t}^*\omega_G}$ and then define $\text{љ}_G \in C^\infty(\mathbb{T})$ by requiring that $\text{љ}_G \circ \mathsf{t} = \text{љ}$. With this setup, for any $x \in \mathsf{t}^{-1}(\mathbf{S} \setminus \mathcal{C}_N)$ and $X \in T_x\mathbb{R}^2$,

$$\begin{aligned}\omega_G(\text{Ч}_N(\mathsf{t}(x)), T\mathsf{t}X) &= \omega_G(T\mathsf{t}\bar{\nabla}_e \mathsf{H}_N(x), T\mathsf{t}X) = \frac{1}{\text{љ}(x)}\omega_e(\bar{\nabla}_e\mathsf{H}_N(x), X)\\ &= \frac{1}{\text{љ}(x)}X\mathsf{H}_N = \frac{1}{\text{љ}(x)}X(\mathsf{H}_N^{\text{loc}}\circ \mathsf{t}) = \frac{1}{\text{љ}_G(\mathsf{t}(x))}\omega_G(\bar{\nabla}_G\mathsf{H}_N^{\text{loc}}(\mathsf{t}(x)), T\mathsf{t}X)\end{aligned}$$

so that $\text{Ч}_N = \frac{1}{\text{љ}_G}\bar{\nabla}_G\mathsf{H}_N^{\text{loc}}$ on $\mathbf{S}\setminus\mathcal{C}_N$. Let Δ_G be the Laplace-Beltrami operator defined by $(\cdot,\cdot)_G$ and define the operators $b_G f \overset{\text{def}}{=} \text{љ}_G\{\mathscr{L}f - \frac{1}{2}\Delta_G f\}$ and $\mathscr{L}_G f \overset{\text{def}}{=} \frac{\text{љ}_G}{2}\Delta_G f + b_G f$ for all $f \in C^\infty(\mathbb{T})$; then b_G is a smooth vector field on $\mathbb{T}$ and $\mathscr{L} = \frac{1}{\text{љ}_G}\mathscr{L}_G$. For all $\varepsilon \in (0,1)$,

$$\mathscr{L}^\varepsilon = \frac{1}{\text{љ}_G}\left\{\frac{1}{\varepsilon^2}\bar{\nabla}_G\mathsf{H}_N^{\text{loc}} + \mathscr{L}_G\right\} - \frac{\boldsymbol{\nu}}{\varepsilon^2}\hat{\text{Ч}}_N$$

(recall (37)).

Let $\vee$ be the standard maximum operator on $\mathbb{R}$. Define

$$\bar{\mathrm{G}}_N \overset{\text{def}}{=} \max\left\{\left(\bigvee_{\ell\in\Lambda} \mathrm{G}_\ell\right), (\tilde{\mathrm{G}}_N + \sum_{\ell\in\Lambda_P} \mathrm{G}_\ell), (\tilde{\mathrm{G}}_N + \sum_{\ell\in\Lambda_W} \mathrm{G}_\ell)\right\}, \qquad \varkappa_N \overset{\text{def}}{=} \sqrt{\frac{\pi}{2\bar{\mathrm{G}}_N}},$$

$$\begin{aligned}\mathcal{M}_N \overset{\text{def}}{=} \bar{\mathrm{G}}_N^{1/4} \Bigg\{ &\left(\frac{\mathsf{r}_N}{\daleth} \sum_{\ell\in\Lambda_P} \mathrm{G}_\ell \dot{f}_\ell(0)\right)^2 \left(\tilde{\mathrm{G}}_N + \sum_{\ell\in\Lambda_P} \mathrm{G}_\ell\right) \\ &+ \left(\frac{\mathsf{r}_N}{\daleth} \sum_{\ell\in\Lambda_W} \mathrm{G}_\ell \dot{f}_\ell(0)\right)^2 \left(\tilde{\mathrm{G}}_N + \sum_{\ell\in\Lambda_W} \mathrm{G}_\ell\right) \\ &+ \sum_{\ell\in\Lambda_P} (\dot{f}_\ell(0) + \mho_N^+)^2 \mathrm{G}_\ell + \sum_{\ell\in\Lambda_W} (\dot{f}_\ell(0) + \mho_N^-)^2 \mathrm{G}_\ell \Bigg\}^{1/2} \\ &+ \max\Bigg\{ \bigvee_{\ell\in\Lambda_P} \left| \dot{f}_\ell(0) + \mho_N^+ - \frac{\mathsf{r}_N}{\daleth} \sum_{\ell\in\Lambda_P} \mathrm{G}_\ell \dot{f}_\ell(0)\right|, \\ &\bigvee_{\ell\in\Lambda_W} \left| \dot{f}_\ell(0) + \mho_N^- - \frac{\mathsf{r}_N}{\daleth} \sum_{\ell\in\Lambda_W} \mathrm{G}_\ell \dot{f}_\ell(0)\right|, \\ &\frac{\mathsf{r}_N}{\daleth} \left| \sum_{\ell\in\Lambda_W} \dot{f}_\ell(0)\mathrm{G}_\ell - \sum_{\ell\in\Lambda_W} \dot{f}_\ell(0)\mathrm{G}_\ell \right| \Bigg\}.\end{aligned}$$

LEMMA 10.1. *There is a* $K_{10.1} > 0$ *such that* $\bar{\mathrm{G}}_N \le \frac{K_{10.1}}{\mathsf{r}_N}$, $\sqrt{\mathsf{r}_N}/K_{10.1} \le \varkappa_N \le K_{10.1}\sqrt{\mathsf{r}_N}$, *and* $\mathcal{M}_N \le K_{10.1}/\mathsf{r}_N^{1/4}$ *for all* N.

Proof. The dominant component of $\bar{\mathrm{G}}_N$ is $\tilde{\mathrm{G}}_N$, which is of order $1/\mathsf{r}_N$. This gives us the first two bounds. The square root and $\vee$ terms in $\mathcal{M}_N$ are both of order 1. □

For each $N \in \mathbb{N}$, define $\mathfrak{E}^*_N(z) \overset{\text{def}}{=} \exp\left[-\frac{1}{2}\sqrt{(\varkappa_N z)^2+1}\right]$ for all $z \in \mathbb{R}$ Also, note that for each $N \in \mathbb{N}$, ξ_N of (45) is zero on $\bigcup_{\ell\in\Lambda} Đ^{\varsigma}_\ell \setminus \mathcal{C}_N$, and that $\sup_{N\in\mathbb{N}} \|\xi_N\|_{C(\mathbf{S}\setminus\mathcal{C}_N)} < \infty$. We can also see that

$$\inf\left\{\boldsymbol{\sigma}_N(x) : x \in \mathbf{S} \setminus \left(\mathcal{C}_N \cup \bigcup_{\ell\in\Lambda} Đ^{\varsigma}_\ell\right), N \in \mathbb{N}\right\}$$

is positive, so there is a constant $K > 0$ such that

$$\|\hat{\mho}_N\| \le K\boldsymbol{\sigma}_N \tag{89}$$

on $\mathbf{S} \setminus \mathcal{C}_N$ for all $N \in \mathbb{N}$.

The heart of the corrector functions lay in the following lemma.

PROPOSITION 10.2. *There are constants $K_{10.2} > 1$ and $\bar{\delta}_{10.2} \in (0,1)$ such that for each $N \in \mathbb{N}$, $\delta \in (0, \bar{\delta}_{10.2})$ and ε in $(0,1)$ such that $\varepsilon < \sqrt{\delta}/K_{10.2}$, there is a function $\Psi_C^{\delta,\varepsilon,N}$ such that $\Psi_C^{\delta,\varepsilon,N} + \hat{F}_N \in C^2(\mathbf{S} \setminus \mathcal{C}_N)$, such that $|\Psi_C^{\delta,\varepsilon,N}(x)| \le K_{10.2}\varepsilon r_N^{-3/4}\mathfrak{E}_N^*\left(\mathsf{H}_N^{\mathrm{loc}}(x)/\varepsilon\right)$ and*

$$\begin{aligned}&\left|(\mathscr{L}^\varepsilon \Psi_C^{\delta,\varepsilon,N})(x)\right| \\ &\quad \le \frac{K_{10.2}}{r_N^{3/4}\sqrt{\delta}}\mathfrak{E}_N^*\left(\frac{\mathsf{H}_N^{\mathrm{loc}}(x)}{\varepsilon}\right) + \frac{K_{10.2}|\boldsymbol{\nu}_N|}{\varepsilon^2 r_N^{1/4}}\mathfrak{E}_N^*\left(\frac{\mathsf{H}_N^{\mathrm{loc}}(x)}{\varepsilon}\right)\boldsymbol{\sigma}_N(x) \\ &\qquad + \frac{K_{10.2}}{\varepsilon r_N^{3/4}}\exp\left[-\left|\frac{\mathsf{H}_N^{\mathrm{loc}}(x)}{\varepsilon\sqrt{\delta}}\right|\right]\boldsymbol{\sigma}_N(x) + \frac{K_{10.2}}{\varepsilon r_N^{3/4}}\exp\left[-\frac{1}{K_{10.2}}\frac{\sqrt{\delta}}{\varepsilon}\right]\end{aligned}$$

for all $x \in \mathcal{N}_N$, and such that

$$\|\nabla_G \Psi_C^{\delta,\varepsilon,N}(x)\|_G \le \frac{K_{10.2}}{r_N^{3/4}}\mathfrak{E}_N^*\left(\frac{\mathsf{H}_N^{\mathrm{loc}}(x)}{\varepsilon}\right) \tag{90}$$

for all $x \in \mathbf{S} \setminus \mathcal{C}_N$ such that $|\mathsf{H}_N^{\mathrm{loc}}(x)| \ge \varepsilon$.

Proof. The proof essentially follows from [Sow]. Namely, there is a constant $K' > 0$ and a $\delta_\circ \in (0,1)$ such that for all $\mathbb{N} \in \mathbb{N}$, $\delta \in (0, \delta_\circ)$, and $\varepsilon \in (0,1)$ such that $\varepsilon \le \sqrt{\delta}/K_1$, there is a function $\Psi_C^{\delta,\varepsilon,N}$ such that $\Psi_C^{\delta,\varepsilon,N} + \hat{F}_N \in C^2(\mathbf{S}\setminus\mathcal{C}_N)$, $|\Psi_C^{\delta,\varepsilon,N}(x)| \le K\varepsilon\mathcal{M}_N \bar{\mathrm{G}}_N^{1/2}\mathfrak{E}_N^*\left(\mathsf{H}_N^{\mathrm{loc}}(x)/\varepsilon\right)$ for all $x \in \mathcal{N}_N$, and such that

$$\begin{aligned}&\left|\frac{1}{\varepsilon^2}(\bar{\nabla}_G \mathsf{H}_N^{\mathrm{loc}}, \nabla_G \Psi_C^{\delta,\varepsilon,N})_G(x) + (\mathscr{L}_G \Psi_C^{\delta,\varepsilon,N})(x)\right| \\ &\qquad \le K_1 \mathcal{M}_N \bar{\mathrm{G}}_N^{1/2}\left\{\frac{1}{\sqrt{\delta}}\mathfrak{E}_N^*\left(\frac{\mathsf{H}_N^{\mathrm{loc}}(x)}{\varepsilon}\right)\right. \\ &\qquad\qquad \left. + \frac{\boldsymbol{\sigma}_N(x)}{\varepsilon}\exp\left[-\left|\frac{\mathsf{H}_N^{\mathrm{loc}}(x)}{\varepsilon\sqrt{\delta}}\right|\right] + \frac{1}{\varepsilon}\exp\left[-\frac{1}{K_1}\frac{\sqrt{\delta}}{\varepsilon}\right]\right\}\end{aligned} \tag{91}$$

for all $x \in \mathcal{N}_N$. Note that in [Sow05], we never explicitly showed that $\Psi_C^{\delta,\varepsilon,N} + \hat{F}_N$ has continuous derivatives of order 2 and less (we did show that $\varepsilon^{-2}(\bar{\nabla}_G \mathsf{H}_N^{\mathrm{loc}}, \nabla_G \Psi_C^{\delta,\varepsilon,N})_G + \mathscr{L}_G \Psi_C^{\delta,\varepsilon,N}$ is continuous). In fact, one can easily see that derivatives of order 2 and less all exist and are smooth except at the critical points (i.e., $\mathfrak{X}$). One can mollify at these points.

We now write that $(\mathscr{L}^\varepsilon \Psi_C^{\delta,\varepsilon,N})(x) = \mathrm{E}_1^{\delta,\varepsilon,N}(x)/\text{љ}_G(x) + \mathrm{E}_2^{\delta,\varepsilon,N}(x)$, where

$$\begin{aligned}\mathrm{E}_1^{\delta,\varepsilon,N}(x) &= \frac{1}{\varepsilon^2}(\bar{\nabla}_G \mathsf{H}_N^{\mathrm{loc}}, \nabla_G \Psi_C^{\delta,\varepsilon,N})_G(x) + (\mathscr{L}_G \Psi_C^{\delta,\varepsilon,N})(x), \\ \mathrm{E}_2^{\delta,\varepsilon,N}(x) &= -\frac{\boldsymbol{\nu}_N}{\varepsilon^2}(\hat{\text{Ч}}_N, \nabla_G \Psi_C^{\delta,\varepsilon,N})_G(x).\end{aligned}$$

We bound $\mathrm{E}_1^{\delta,\varepsilon,N}$ by using (91). From Lemma 6.1, we secondly have that $\hat{\text{Ч}}_N$ is zero in a neighborhood of the $\mathfrak{x}_\ell$'s, so from [Sow] and (89), we have that there is a constant $K_2 > 0$ such that

$$|\mathrm{E}_2^{\delta,\varepsilon,N}(x)| \le \frac{K_2 \mathcal{M}_N |\boldsymbol{\nu}_N|}{\varepsilon^2} \mathfrak{E}_N^* \left(\frac{\mathsf{H}_N^{\mathrm{loc}}(x)}{\varepsilon} \right) \boldsymbol{\sigma}_N(x)$$

on $\mathsf{E} \setminus \mathcal{C}_N$. Combining our estimates and using Lemma 10.1, we get the desired bound on $\mathscr{L}^\varepsilon \Psi_C^{\delta,\varepsilon,N}$.

The bound on $\nabla_G \Psi_C^{\delta,\varepsilon,N}$ follows from [Sow05, Lemma 4.10]; again, we use Lemma 10.1. □

Recall next Lemma 8.3 and (57). Define

$$\Psi_B^{\delta,\varepsilon,N}(x) \stackrel{\mathrm{def}}{=} \mathfrak{c}_\wedge \left(\frac{\mathsf{H}_N^{\mathrm{loc}}(x)}{\varepsilon^{1/2} \mathsf{r}_N^{1/4}} \right) \left\{ \Psi_C^{\delta,\varepsilon,N}(x) + \frac{K_{10.2}}{\upsilon_{(56)} \mathsf{r}_N^{3/4} \sqrt{\delta}} \text{Ъ}^\varepsilon(x) \mathfrak{E}_N^* \left(\frac{\mathsf{H}_N^{\mathrm{loc}}(x)}{\varepsilon} \right) \right\}.$$

Proof of Proposition 7.2. Note that

$$(92) \quad \varepsilon^{1/2} \mathsf{r}_N^{1/4} = \mathsf{r}_N \left(\frac{\varepsilon}{\mathsf{r}_N^{3/2}} \right)^{1/2} \quad \text{and} \quad \frac{\sqrt{\mathsf{r}_N} \left(\varepsilon^{1/2} \mathsf{r}_N^{1/4} \right)}{\varepsilon} = \left(\frac{\mathsf{r}_N^{3/2}}{\varepsilon} \right)^{1/2}.$$

When $\varepsilon / \mathsf{r}^{3/2} < \frac{1}{4}$, $\mathfrak{c}_\wedge(\mathsf{H}_N^{\mathrm{loc}} / (\varepsilon^{1/2} \mathsf{r}_N^{1/4}))$ is zero in a neighborhood of $\mathcal{C}_N$.

We calculate that

$$(\mathscr{L}^\varepsilon \Psi_B^{\delta,\varepsilon,N})(x) = \mathfrak{c}_\wedge \left(\frac{\mathsf{H}_N^{\mathrm{loc}}(x)}{\varepsilon^{1/2} \mathsf{r}_N^{1/4}} \right) \sum_{i=1}^4 \mathrm{E}_i^{\delta,\varepsilon,N}(x) + \sum_{i=5}^7 \mathrm{E}_i^{\delta,\varepsilon,N}(x),$$

where

$$\mathrm{E}_1^{\delta,\varepsilon,N}(x) \stackrel{\mathrm{def}}{=} \left\{ (\mathscr{L}^\varepsilon \Psi_C^{\delta,\varepsilon,N})(x) + \frac{K_{10.2}}{\upsilon_{(56)} \mathsf{r}_N^{3/4} \sqrt{\delta}} (\mathscr{L}^\varepsilon \text{Ъ}^\varepsilon)(x) \mathfrak{E}_N^* \left(\frac{\mathsf{H}_N^{\mathrm{loc}}(x)}{\varepsilon} \right) \right\},$$

$$\mathrm{E}_2^{\delta,\varepsilon,N}(x) \stackrel{\mathrm{def}}{=} \frac{K_{10.2} \text{Ъ}^\varepsilon(x)}{\upsilon_{(56)} \varepsilon \mathsf{r}_N^{3/4} \sqrt{\delta}} \dot{\mathfrak{E}}_N^* \left(\frac{\mathsf{H}_N^{\mathrm{loc}}(x)}{\varepsilon} \right) \left\{ \frac{\boldsymbol{\nu}_N}{\varepsilon^2} \frac{\xi_N(x)}{\beth} + \boldsymbol{\beta}_N(x) \right\},$$

$$\mathrm{E}_3^{\delta,\varepsilon,N}(x) \stackrel{\mathrm{def}}{=} \frac{K_{10.2} \text{Ъ}^\varepsilon(x)}{2 \upsilon_{(56)} \varepsilon^2 \mathsf{r}_N^{3/4} \sqrt{\delta}} \ddot{\mathfrak{E}}_N^* \left(\frac{\mathsf{H}_N^{\mathrm{loc}}(x)}{\varepsilon} \right) \boldsymbol{\sigma}_N(x),$$

$$\mathrm{E}_4^{\delta,\varepsilon,N}(x) \stackrel{\mathrm{def}}{=} \frac{K_{10.2}}{\upsilon_{(56)} \varepsilon \mathsf{r}_N^{3/4} \sqrt{\delta}} (\nabla_G \text{Ъ}^\varepsilon, \nabla_G \mathsf{H}_N^{\mathrm{loc}})_G(x) \dot{\mathfrak{E}}_N^* \left(\frac{\mathsf{H}_N^{\mathrm{loc}}(x)}{\varepsilon} \right),$$

$$\mathrm{E}_5^{\delta,\varepsilon,N}(x) \stackrel{\mathrm{def}}{=} \frac{1}{\varepsilon^{1/2} \mathsf{r}_N^{1/4}} \dot{\mathfrak{c}}_\wedge \left(\frac{\mathsf{H}_N^{\mathrm{loc}}(x)}{\varepsilon^{1/2} \mathsf{r}_N^{1/4}} \right) \left\{ \Psi_C^{\delta,\varepsilon,N}(x) \right.$$
$$\left. + \frac{K_{10.2} \text{Ъ}^\varepsilon(x)}{\upsilon_{(56)} \mathsf{r}_N^{3/4} \sqrt{\delta}} \mathfrak{E}_N^* \left(\frac{\mathsf{H}_N^{\mathrm{loc}}(x)}{\varepsilon} \right) \right\} \left\{ \frac{\boldsymbol{\nu}_N}{\varepsilon^2} \frac{\xi_N(x)}{\beth} + \boldsymbol{\beta}_N(x) \right\},$$

$$\mathrm{E}_6^{\delta,\varepsilon,N}(x) \overset{\text{def}}{=} \frac{1}{2\varepsilon \mathsf{r}_N^{1/2}} \ddot{\mathfrak{c}}_\wedge \left(\frac{\mathsf{H}_N^{\text{loc}}(x)}{\varepsilon^{1/2} \mathsf{r}_N^{1/4}} \right) \left\{ \Psi_C^{\delta,\varepsilon,N}(x) \right.$$
$$\left. + \frac{K_{10.2} \mathfrak{B}^\varepsilon(x)}{\upsilon_{(56)} \mathsf{r}_N^{3/4} \sqrt{\delta}} \mathfrak{E}_N^* \left(\frac{\mathsf{H}_N^{\text{loc}}(x)}{\varepsilon} \right) \right\} \boldsymbol{\sigma}_N(x),$$

$$\mathrm{E}_7^{\delta,\varepsilon,N}(x) \overset{\text{def}}{=} \frac{1}{\varepsilon^{1/2} \mathsf{r}_N^{1/4}} \dot{\mathfrak{c}}_\wedge \left(\frac{\mathsf{H}_N^{\text{loc}}(x)}{\varepsilon^{1/2} \mathsf{r}_N^{1/4}} \right) \left\{ \left(\nabla_G \mathsf{H}_N^{\text{loc}}, \nabla_G \Psi_C^{\delta,\varepsilon,N} \right)_G (x), \right.$$
$$+ \frac{K_{10.2}}{\upsilon_{(56)} \mathsf{r}_N^{3/4} \sqrt{\delta}} \left(\nabla_G \mathsf{H}_N^{\text{loc}}, \nabla_G \mathfrak{B}^\varepsilon \right)_G (x) \mathfrak{E}_N^* \left(\frac{\mathsf{H}_N^{\text{loc}}(x)}{\varepsilon} \right)$$
$$\left. + \frac{K_{10.2} \mathfrak{B}^\varepsilon(x)}{\upsilon_{(56)} \varepsilon \mathsf{r}_N^{3/4} \sqrt{\delta}} \dot{\mathfrak{E}}_N^* \left(\frac{\mathsf{H}_N^{\text{loc}}(x)}{\varepsilon} \right) \boldsymbol{\sigma}_N(x) \right\}.$$

By combining Proposition 10.2 and Lemma 8.3, we have that

$$\mathrm{E}_1^{\delta,\varepsilon,N}(x) \geq \frac{K_{10.2}}{\mathsf{r}_N^{3/4} \sqrt{\delta}} \left\{ \frac{1}{\upsilon_{(56)}} (\mathscr{L}^\varepsilon \mathfrak{B}^\varepsilon)(x) - 1 \right\} \mathfrak{E}_N^* \left(\frac{\mathsf{H}_N^{\text{loc}}(x)}{\varepsilon} \right)$$
$$- \frac{K_{10.2} |\boldsymbol{\nu}_N|}{\varepsilon^2 \mathsf{r}_N^{1/4}} \mathfrak{E}_N^* \left(\frac{\mathsf{H}_N^{\text{loc}}(x)}{\varepsilon} \right) \boldsymbol{\sigma}_N(x)$$
$$- \frac{K_{10.2}}{\varepsilon \mathsf{r}_N^{3/4}} \exp \left[- \left| \frac{\mathsf{H}_N^{\text{loc}}(x)}{\varepsilon \sqrt{\delta}} \right| \right] \boldsymbol{\sigma}_N(x) - \frac{K_{10.2}}{\varepsilon \mathsf{r}_N^{3/4}} \exp \left[- \frac{1}{K_{10.2}} \frac{\sqrt{\delta}}{\varepsilon} \right], \tag{93}$$

and thus by (57), the first term on the right of (93) is bounded from below by

$$- \frac{K_{10.2} (K_{8.3} + 1) \mathfrak{l}(\varepsilon)}{\upsilon_{(56)} \mathsf{r}_N^{3/4} \sqrt{\delta}} \mathfrak{E}_N^* \left(\frac{\mathsf{H}_N^{\text{loc}}(x)}{\varepsilon} \right) \boldsymbol{\sigma}_N(x) - \frac{K_{10.2} K_{8.3} \varepsilon}{\upsilon_{(56)} \mathsf{r}_N^{3/4} \sqrt{\delta}}. \tag{94}$$

We next calculate that

$$\left| \Psi_C^{\delta,\varepsilon,N}(x) + \frac{K_{10.2} \mathfrak{B}^\varepsilon(x)}{\upsilon_{(56)} \mathsf{r}_N^{3/4} \sqrt{\delta}} \mathfrak{E}_N^* \left(\frac{\mathsf{H}_N^{\text{loc}}(x)}{\varepsilon} \right) \right|$$
$$\leq K_{10.2} \left\{ \frac{\varepsilon}{\mathsf{r}_N^{3/4}} + \frac{K_{8.3} \varepsilon^2 \mathfrak{l}(\varepsilon)}{\upsilon_{(56)} \mathsf{r}_N^{3/4} \sqrt{\delta}} \right\} \mathfrak{E}_N^* \left(\frac{\mathsf{H}_N^{\text{loc}}(x)}{\varepsilon} \right)$$
$$\leq \frac{K_{10.2} \varepsilon \mathfrak{l}(\varepsilon)}{\mathsf{r}_N^{3/4}} \left\{ 1 + \frac{K_{8.3} \varepsilon}{\upsilon_{(56)} \sqrt{\delta}} \right\} \mathfrak{E}_N^* \left(\frac{\mathsf{H}_N^{\text{loc}}(x)}{\varepsilon} \right). \tag{95}$$

This immediately implies the first claimed bound on $\Psi_B^{\delta,\varepsilon,N}$.

Simple calculation shows that there is a $K_1 > 0$ such that $\left| \dot{\mathfrak{E}}_N^*(z) \right| \leq K_1 \varkappa_N \mathfrak{E}_N^*(z)$ and $\left| \ddot{\mathfrak{E}}_N^*(z) \right| \leq K_1 \varkappa_N^2 \mathfrak{E}_N^*(z)$ for all $z \in \mathbb{R}$ and $N \in \mathbb{N}$. Keeping

(89) in mind, we thus see that there is thus a constant $K_2 > 0$ such that

$$(96)\quad \begin{aligned} |\mathrm{E}_2^{\delta,\varepsilon,N}(x)| &\le \frac{K_2\varepsilon^2\mathfrak{l}(\varepsilon)\varkappa_N}{\varepsilon\mathsf{r}_N^{3/4}\sqrt{\delta}}\left\{\boldsymbol{\sigma}_N(x)\frac{|\boldsymbol{\nu}_N|}{\varepsilon^2}+1\right\}\mathfrak{E}_N^*\left(\frac{\mathsf{H}_N^{\mathrm{loc}}(x)}{\varepsilon}\right)\\ &\le \frac{K_{10.1}K_2\mathfrak{l}(\varepsilon)|\boldsymbol{\nu}_N|}{\varepsilon\mathsf{r}_N^{1/4}\sqrt{\delta}}\mathfrak{E}_N^*\left(\frac{\mathsf{H}_N^{\mathrm{loc}}(x)}{\varepsilon}\right)\boldsymbol{\sigma}_N(x)+\frac{K_{10.1}K_2\varepsilon\mathfrak{l}(\varepsilon)}{\mathsf{r}_N^{1/4}\sqrt{\delta}},\\ |\mathrm{E}_3^{\delta,\varepsilon,N}(x)| &\le \frac{K_2\varepsilon^2\mathfrak{l}(\varepsilon)\varkappa_N^2}{\varepsilon^2\mathsf{r}_N^{3/4}\sqrt{\delta}}\mathfrak{E}_N^*\left(\frac{\mathsf{H}_N^{\mathrm{loc}}(x)}{\varepsilon}\right)\boldsymbol{\sigma}_N(x)\\ &\le \frac{K_2K_{10.1}^2\mathfrak{l}(\varepsilon)\mathsf{r}_N^{1/4}}{\sqrt{\delta}}\mathfrak{E}_N^*\left(\frac{\mathsf{H}_N^{\mathrm{loc}}(x)}{\varepsilon}\right)\boldsymbol{\sigma}_N(x),\\ |\mathrm{E}_4^{\delta,\varepsilon,N}(x)| &\le \frac{K_2\varepsilon\varkappa_N}{\varepsilon\mathsf{r}_N^{3/4}\sqrt{\delta}}\mathfrak{E}_N^*\left(\frac{\mathsf{H}_N^{\mathrm{loc}}(x)}{\varepsilon}\right)\sqrt{\boldsymbol{\sigma}_N(x)}\\ &\le \frac{K_2K_{10.1}}{\mathsf{r}_N^{1/4}\sqrt{\delta}}\mathfrak{E}_N^*\left(\frac{\mathsf{H}_N^{\mathrm{loc}}(x)}{\varepsilon}\right)\sqrt{\boldsymbol{\sigma}_N(x)}. \end{aligned}$$

Next, note that (recall (92))

$$\begin{aligned} &\left\{\left|\dot{\mathfrak{c}}_\wedge\left(\frac{\mathsf{H}_N^{\mathrm{loc}}(x)}{\varepsilon^{1/2}\mathsf{r}_N^{1/4}}\right)\right|+\left|\ddot{\mathfrak{c}}_\wedge\left(\frac{\mathsf{H}_N^{\mathrm{loc}}(x)}{\varepsilon^{1/2}\mathsf{r}_N^{1/4}}\right)\right|\right\}\mathfrak{E}_N^*\left(\frac{\mathsf{H}_N^{\mathrm{loc}}(x)}{\varepsilon}\right)\\ &\qquad\qquad \le K_{(22)}\mathfrak{E}_N^*\left(\frac{\varepsilon^{1/2}\mathsf{r}_N^{1/4}}{\varepsilon}\right)\\ &\le K_{(22)}\exp\left[-\frac{\varkappa_N}{2}\left(\frac{\varepsilon^{1/2}\mathsf{r}_N^{1/4}}{\varepsilon}\right)\right]\le K_{(22)}\exp\left[-\frac{1}{2K_{10.1}}\left(\frac{\mathsf{r}_N^{3/2}}{\varepsilon}\right)^{1/2}\right] \end{aligned}$$

for all $x \in \mathbf{S}$, $\varepsilon \in (0,1)$, and $N \in \mathbb{N}$. Combining this with (95), we have that there is a $K_3 > 0$ such that

$$\begin{aligned} |\mathrm{E}_5^{\delta,\varepsilon,N}(x)| &\le \frac{K_3\varepsilon^{1/2}\mathfrak{l}(\varepsilon)}{\mathsf{r}_N}\left\{1+\frac{\varepsilon}{\sqrt{\delta}}\right\}\left\{\frac{|\boldsymbol{\nu}_N|}{\varepsilon^2}+1\right\}\exp\left[-\frac{1}{2K_{10.1}}\left(\frac{\mathsf{r}_N^{3/2}}{\varepsilon}\right)^{1/2}\right],\\ |\mathrm{E}_6^{\delta,\varepsilon,N}(x)| &\le \frac{K_3\mathfrak{l}(\varepsilon)}{\mathsf{r}_N^{5/4}}\left\{1+\frac{\varepsilon}{\sqrt{\delta}}\right\}\exp\left[-\frac{1}{2K_{10.1}}\left(\frac{\mathsf{r}_N^{3/2}}{\varepsilon}\right)^{1/2}\right]. \end{aligned}$$

If $\varepsilon < \mathsf{r}_N^{3/2}/4$ (and hence $\varepsilon < \mathsf{r}_N^{3/2}$), then on the support of $\dot{\mathfrak{c}}_\wedge\left(\frac{\mathsf{H}_N^{\mathrm{loc}}}{\varepsilon^{1/2}\mathsf{r}_N^{1/4}}\right)$, we have that

$$|\mathsf{H}_N^{\mathrm{loc}}(x)| \ge \varepsilon^{1/2}\mathsf{r}_N^{1/4} \ge \varepsilon^{1/2}(\varepsilon^{2/3})^{1/4} = \varepsilon^{2/3} \ge \varepsilon,$$

so we can use (90). Thus, when $\varepsilon < \mathsf{r}_N^{3/2}/4$, there is $K_4 > 0$ such that

$$|\mathrm{E}_7^{\delta,\varepsilon,N}(x)| \le \frac{K_4}{\varepsilon^{3/2}\mathsf{r}_N^{1/4}}\left\{\frac{1}{\mathsf{r}_N^{3/4}}+\frac{\varepsilon}{\mathsf{r}_N^{3/4}\sqrt{\delta}} + \frac{\varkappa_N\varepsilon^2\mathfrak{l}(\varepsilon)}{\varepsilon\mathsf{r}_N^{3/4}\sqrt{\delta}}\right\}\exp\left[-\frac{1}{2K_{10.1}}\left(\frac{\mathsf{r}_N^{3/2}}{\varepsilon}\right)^{1/2}\right].$$

Taking the worst of the different combinations (i.e., the smallest exponents of ε and r_N), we get that there is a $K_5 > 0$ such that

$$\text{(97)}\qquad \sum_{i=5}^{7}|\mathrm{E}_i^{\delta,\varepsilon,N}(x)| \le \frac{K_5}{\varepsilon^{3/2}\mathsf{r}_N^{5/4}\sqrt{\delta}}\exp\left[-\frac{1}{2K_{10.1}}\left(\frac{\mathsf{r}_N^{3/2}}{\varepsilon}\right)^{1/2}\right] \le \frac{K_5}{\varepsilon^{7/3}\sqrt{\delta}}\exp\left[-\frac{1}{2K_{10.1}}\left(\frac{\mathsf{r}_N^{3/2}}{\varepsilon}\right)^{1/2}\right]$$

for all $x \in \mathbf{S}\setminus\mathcal{C}_N$ and all ε and δ in $(0,1)$ such that $\varepsilon < \sqrt{\delta}/K_{10.2}$ and such that $\varepsilon < \mathsf{r}_N^{3/2}/4$ (if $\varepsilon < \mathsf{r}_N^{3/2}/4$, then $\varepsilon^{3/2}\mathsf{r}_N^{5/4} \ge \varepsilon^{3/2}(\varepsilon^{2/3})^{5/4} = \varepsilon^{7/3}$).

Let's combine things together. There are exponentially small terms in (93) and (97); these contribute a term of size

$$\frac{1}{\varepsilon\mathsf{r}_N^{3/4}}\exp\left[-\frac{1}{K_{10.2}}\frac{\sqrt{\delta}}{\varepsilon}\right] + \frac{1}{\varepsilon^{7/3}\sqrt{\delta}}\exp\left[-\frac{1}{2K_{10.1}}\left(\frac{\mathsf{r}_N^{3/2}}{\varepsilon}\right)^{1/2}\right].$$

There are also constant terms in (94) and (96); these contribute a term of size

$$\frac{\varepsilon}{\mathsf{r}_N^{3/4}\sqrt{\delta}} + \frac{\varepsilon\mathfrak{l}(\varepsilon)}{\mathsf{r}_N^{1/4}\sqrt{\delta}} \le \frac{\varepsilon\mathfrak{l}(\varepsilon)}{\mathsf{r}_N^{3/4}\sqrt{\delta}}\{1+\sqrt{\mathsf{r}_N}\} \le \{1+\sqrt{|\omega_2|}\}\frac{\varepsilon\mathfrak{l}(\varepsilon)}{\mathsf{r}_N^{3/4}\sqrt{\delta}}.$$

Thirdly, there are terms in (93) and (96) containing both $\boldsymbol{\sigma}_N$ and $\boldsymbol{\nu}_N$; these combine to give a term of size

$$\begin{aligned}\left\{\frac{|\boldsymbol{\nu}_N|}{\varepsilon^2\mathsf{r}_N^{1/4}} + \frac{\mathfrak{l}(\varepsilon)|\boldsymbol{\nu}_N|}{\varepsilon\mathsf{r}_N^{1/4}\sqrt{\delta}}\right\}&\mathfrak{E}_N^*\left(\frac{\mathsf{H}_N^{\mathrm{loc}}(x)}{\varepsilon}\right)\boldsymbol{\sigma}_N(x) \\ &\le \frac{|\boldsymbol{\nu}_N|}{\varepsilon^2\mathsf{r}_N^{1/4}}\left\{1+\frac{\varepsilon}{\sqrt{\delta}}\right\}\mathfrak{l}(\varepsilon)\mathfrak{E}_N^*\left(\frac{\mathsf{H}_N^{\mathrm{loc}}(x)}{\varepsilon}\right)\boldsymbol{\sigma}_N(x) \\ &\le 2\frac{|\boldsymbol{\nu}_N|\mathfrak{l}(\varepsilon)}{\varepsilon^2\mathsf{r}_N^{1/4}}\mathfrak{E}_N^*\left(\frac{\mathsf{H}_N^{\mathrm{loc}}(x)}{\varepsilon}\right)\boldsymbol{\sigma}_N(x) = 2\frac{|\mu_{N,\varepsilon}|\mathfrak{l}(\varepsilon)}{|\gimel|\mathsf{r}_N^{5/4}}\mathfrak{E}_N^*\left(\frac{\mathsf{H}_N^{\mathrm{loc}}(x)}{\varepsilon}\right)\boldsymbol{\sigma}_N(x)\end{aligned}$$

when $\varepsilon < \sqrt{\delta}/K_{10.2}$ (since by assumption $K_{10.2} > 1$, so $\sqrt{\delta}/K_{10.2} < 1$ if $\delta \in (0,1)$). Next, we note that there are terms in (94) and (96) which include $\boldsymbol{\sigma}_N$, but not $\boldsymbol{\nu}_N$; these give a term of size

$$\left\{\frac{\mathfrak{l}(\varepsilon)}{\mathsf{r}_N^{3/4}\sqrt{\delta}}+\frac{\mathfrak{l}(\varepsilon)\mathsf{r}_N^{1/4}}{\sqrt{\delta}}\right\}\mathfrak{E}^*_N\left(\frac{\mathsf{H}_N^{\mathrm{loc}}(x)}{\varepsilon}\right)\boldsymbol{\sigma}_N(x)$$
$$\leq \frac{\mathfrak{l}(\varepsilon)}{\mathsf{r}_N^{3/4}\sqrt{\delta}}\{1+\mathsf{r}_N\}\mathfrak{E}^*_N\left(\frac{\mathsf{H}_N^{\mathrm{loc}}(x)}{\varepsilon}\right)\boldsymbol{\sigma}_N(x)$$
$$\leq \{1+|\omega_2|\}\frac{\mathfrak{l}(\varepsilon)}{\mathsf{r}_N^{3/4}\sqrt{\delta}}\mathfrak{E}^*_N\left(\frac{\mathsf{H}_N^{\mathrm{loc}}(x)}{\varepsilon}\right)\boldsymbol{\sigma}_N(x).$$

Lastly, we have the $\sqrt{\boldsymbol{\sigma}_N}$ term in (96) and the penultimate term on the right of (93). We can now conclude the stated lower bound on $\mathscr{L}^\varepsilon\Psi_B^{\delta,\varepsilon,N}$.

Combine things together. Note that

$$\mathfrak{E}^*_N(z)\leq \exp\left[-2^{-1}|\varkappa_N z|\right]\leq \exp\left[-(2K_{10.1})^{-1}|z|\mathsf{r}_N^{1/2}\right]$$

for all $z\in\mathbb{R}$ and $N\in\mathbb{N}$. □

11. Stochastic averaging; the proof of Lemmas 5.10, 8.7, and 8.5

11.1. Proof of Lemmas 5.10 and 8.7. We appeal to [Sow].

First, for each $N\in\mathbb{N}$, define

$$\mathbf{E}_N(x)\stackrel{\mathrm{def}}{=}\frac{\nabla\mathsf{H}_N^{\mathrm{loc}}}{\|\nabla\mathsf{H}_N^{\mathrm{loc}}\|^2}(x),$$
$$\alpha_N(x)\stackrel{\mathrm{def}}{=}\frac{D^2\mathsf{H}_N^{\mathrm{loc}}(\nabla\mathsf{H}_N^{\mathrm{loc}}(x),\nabla\mathsf{H}_N^{\mathrm{loc}}(x))-D^2\mathsf{H}_N^{\mathrm{loc}}(\bar\nabla\mathsf{H}_N^{\mathrm{loc}}(x),\bar\nabla\mathsf{H}_N^{\mathrm{loc}}(x))}{\|\nabla\mathsf{H}_N^{\mathrm{loc}}(x)\|^4}$$

for all $x\in\mathcal{N}_N$ and all $N\in\mathbb{N}$.

LEMMA 11.1. *There is a $K>1$ such that*

$$\|\text{Ч}_N(x)\|\geq\frac{\sqrt{|\mathsf{H}_N^{\mathrm{loc}}(x)|}}{K},\qquad \|\mathbf{E}_N(x)\|\geq\frac{1}{K},\qquad |\alpha_N(x)|\leq\frac{K}{|\mathsf{H}_N^{\mathrm{loc}}(x)|},$$
$$|\text{Ч}_N\alpha_N(x)|\leq\frac{K}{|\mathsf{H}_N^{\mathrm{loc}}(x)|},\quad |\mathbf{E}_N\alpha_N(x)|\leq\frac{1}{|\mathsf{H}_N^{\mathrm{loc}}(x)|^2},$$
$$\|\nabla\mathsf{H}_N^{\mathrm{loc}}(x)\|\geq\frac{\sqrt{|\mathsf{H}_N^{\mathrm{loc}}(x)|}}{K}$$

for all $x\in\mathcal{N}_N$ and $N\in\mathbb{N}$.

Proof. From [Sow], we have that

$$\|\text{Ч}_N(x)\|=\|\nabla\mathsf{H}_N^{\mathrm{loc}}(x)\|,\quad \|\mathbf{E}_N(x)\|=\frac{1}{\|\nabla\mathsf{H}_N^{\mathrm{loc}}\|},\quad |\alpha_N(x)|\leq\frac{2\|D^2\mathsf{H}_N^{\mathrm{loc}}(x)\|}{\|\nabla\mathsf{H}_N^{\mathrm{loc}}(x)\|^2},$$
$$|\text{Ч}_N\alpha_N(x)|\leq 2\frac{\|D^3\mathsf{H}_N^{\mathrm{loc}}(x)\|}{\|\nabla\mathsf{H}_N^{\mathrm{loc}}(x)\|}+12\frac{\|D^2\mathsf{H}_N^{\mathrm{loc}}(x)\|^2}{\|\nabla\mathsf{H}_N^{\mathrm{loc}}(x)\|^2},$$

$$|\mathbf{E}_N\alpha_N(x)| \leq 2\frac{\|D^3\mathsf{H}_N^{\mathrm{loc}}(x)\|}{\|\nabla\mathsf{H}_N^{\mathrm{loc}}(x)\|^3} + 12\frac{\|D^2\mathsf{H}_N^{\mathrm{loc}}(x)\|^2}{\|\nabla\mathsf{H}_N^{\mathrm{loc}}(x)\|^4}.$$

Since $\|\nabla\mathsf{H}_N^{\mathrm{loc}}\|$, $\|D^2\mathsf{H}_N^{\mathrm{loc}}\|$, and $\|D^3\mathsf{H}_N^{\mathrm{loc}}\|$ are all bounded from above on $\mathcal{N}_N$, uniformly in $\mathbb{N}\in\mathbb{N}$, there is a $K>0$ such that

(98)

$$\|\text{Ч}_N(x)\| = \|\nabla\mathsf{H}_N^{\mathrm{loc}}(x)\|, \qquad \|\mathbf{E}_N(x)\| \geq \frac{1}{K}, \qquad |\alpha_N(x)| \leq \frac{K}{\|\nabla\mathsf{H}_N^{\mathrm{loc}}(x)\|^2}.$$
$$|\text{Ч}_N\alpha_N(x)| \leq \frac{K}{\|\nabla\mathsf{H}_N^{\mathrm{loc}}(x)\|^2}, \qquad |\mathbf{E}_N\alpha_N(x)| \leq \frac{K}{\|\nabla\mathsf{H}_N^{\mathrm{loc}}(x)\|^4}$$

for all $x\in\mathcal{N}_N$ and $N\in\mathbb{N}$.

Let $\tilde{\mathcal{W}}\subset\mathbb{R}^2$ be a neighborhood of $\mathbf{0}_e$ such that $\tilde{\mathcal{W}}\subset\subset\tilde{\mathcal{U}}$. Then

$$\upsilon_1 \stackrel{\text{def}}{=} \inf\left\{\|\text{Ч}_N(x)\| : x\in\mathcal{N}_N\setminus\bigcup_{\ell\in\Lambda}\tilde{\phi}_\ell(\tilde{\mathcal{W}}),\ N\in\mathbb{N}\right\}$$

is positive. Since $\sqrt{|\mathsf{H}_N^{\mathrm{loc}}(x)|} \leq \max\{\sqrt{\mathsf{r}_N},\hbar\} \leq \sqrt{\hbar+|\omega_2|}$ for all $x\in\mathcal{N}_N$ and $N\in\mathbb{N}$, we thus have that

$$\|\text{Ч}_N(x)\| \geq \frac{\upsilon_1}{\sqrt{\hbar+|\omega_2|}}\sqrt{|\mathsf{H}_N^{\mathrm{loc}}(x)|} \tag{99}$$

for all $x\in\mathcal{N}_N\setminus\bigcup_{\ell\in\Lambda}\tilde{\phi}_\ell(\tilde{\mathcal{W}})$. Now recall (21). Since $T\phi_\ell$ is full rank on $\mathcal{U}_\ell$ and $\tilde{\mathrm{B}}_\ell$ is positive on $\tilde{\mathcal{U}}$,

$$\upsilon_2 \stackrel{\text{def}}{=} \inf\left\{\frac{\|\text{Ч}_N(\tilde{\phi}_\ell(x))\|}{\|\bar{\nabla}_e\tilde{\mathsf{H}}(x)\|_e} : x\in\tilde{\mathcal{W}},\ x\neq\mathbf{0}_e, \ell\in\Lambda\right\}$$

is positive. Secondly, note that

$$\inf_{\substack{x\in\mathbb{R}^2\\ \tilde{\mathsf{H}}(x)\neq 0}}\frac{\|\nabla_e\tilde{\mathsf{H}}(x)\|_e}{\sqrt{|\tilde{\mathsf{H}}(x)|}} = \inf_{\substack{(x_1,x_2)\in\mathbb{R}^2\\ x_1x_2\neq 0}}\frac{\sqrt{x_1^2+x_2^2}}{\sqrt{x_1x_2}} = \inf_{\substack{(x_1,x_2)\in\mathbb{R}^2\\ x_1x_2\neq 0}}\sqrt{\frac{x_1}{x_2}+\frac{x_2}{x_1}} = \sqrt{2}.$$

From Lemma 6.6, we can see that

$$\begin{aligned}\|\text{Ч}_N(x)\| &\geq \upsilon_2\|\nabla_e\tilde{\mathsf{H}}(\phi_\ell(x))\|_e \geq \upsilon_2\sqrt{2}\sqrt{|\tilde{\mathsf{H}}(\phi_\ell(x))|}\\ &\geq \upsilon_2\sqrt{2}\sqrt{\left|\tilde{\mathsf{H}}(\phi_\ell(x)) - \left\lfloor\frac{\tilde{\mathsf{H}}(\phi_\ell(x))}{\mathsf{r}_N}+\frac{1}{2}\right\rfloor\mathsf{r}_N\right|} = \upsilon_2\sqrt{2}\sqrt{|\mathsf{H}_N^{\mathrm{loc}}(x)|}\end{aligned}$$

for $x\in\tilde{\phi}_\ell(\tilde{\mathcal{W}})$ and $\ell\in\Lambda$. Combining this and (99), we get that

$$\|\text{Ч}_N(x)\| \geq \min\left\{\frac{\upsilon_1}{\sqrt{\hbar+|\omega_2|}},\upsilon_2\sqrt{2}\right\}\sqrt{|\mathsf{H}_N^{\mathrm{loc}}(x)|}$$

for all $x \in \mathcal{N}_N$ and $N \in \mathbb{N}$. This gives us the last claim. We can then use this in (98) to get the remaining claims. □

Let's first get our averaging estimate in the D_ℓ's.

Proof of Lemma 5.10. We first note that on each $\mathsf{Đ}_\ell$, $\mathfrak{z} = \mathfrak{z}^1$ and $\nabla \mathsf{H}_{T,\ell} = \nabla \mathsf{H}^{\text{loc}}_1$. Thus we can use Lemma 11.1.

Recall that φ is required to have bounded derivatives of all orders on $\mathbf{S} \setminus \mathsf{E}$. Thus there is a constant $K_1 > 0$ such that $|(\text{Ч}\varphi)| \le K_1$, $|(\text{Ч}^2\varphi)| \le K_1$, $|(\mathbf{E}_1\varphi)| \le K_1/\|\nabla \mathsf{H}^{\text{loc}}_1\|$, and $|(\mathbf{E}_1 \text{Ч}\varphi)| \le K_1/\|\nabla \mathsf{H}^{\text{loc}}_1\|$ for all $x \in \mathbf{S} \setminus \mathsf{E}$. We can also compute that

$$\begin{aligned}(\text{Ч}\mathbf{E}_1\varphi) &= \frac{\left(\nabla_{\bar{\nabla}\mathsf{H}^{\text{loc}}_1}\nabla \mathsf{H}^{\text{loc}}_1, \nabla\varphi\right) + \left(\nabla \mathsf{H}^{\text{loc}}_1, \nabla_{\bar{\nabla}\mathsf{H}^{\text{loc}}_1}\nabla\varphi\right)}{\|\nabla \mathsf{H}^{\text{loc}}_1\|^2} \\ &\quad - 2\frac{\left(\nabla \mathsf{H}^{\text{loc}}_1, \nabla\varphi\right)\left(\nabla_{\bar{\nabla}\mathsf{H}^{\text{loc}}_1}\nabla \mathsf{H}^{\text{loc}}_1, \nabla \mathsf{H}^{\text{loc}}_1\right)}{\|\nabla \mathsf{H}^{\text{loc}}_1\|^4}, \\ (\mathbf{E}_1^2\varphi) &= \frac{\left(\nabla_{\nabla\mathsf{H}^{\text{loc}}_1}\nabla \mathsf{H}^{\text{loc}}_1, \nabla\varphi\right) + \left(\nabla \mathsf{H}^{\text{loc}}_1, \nabla_{\nabla\mathsf{H}^{\text{loc}}_1}\nabla\varphi\right)}{\|\nabla \mathsf{H}^{\text{loc}}_1\|^4} \\ &\quad - 2\frac{\left(\nabla \mathsf{H}^{\text{loc}}_1, \nabla\varphi\right)\left(\nabla_{\nabla\mathsf{H}^{\text{loc}}_1}\nabla \mathsf{H}^{\text{loc}}_1, \nabla \mathsf{H}^{\text{loc}}_1\right)}{\|\nabla \mathsf{H}^{\text{loc}}_1\|^6}\end{aligned}$$

for all $x \in \mathbf{S} \setminus \mathsf{E}$. Thus there is a $K_2 > 0$ such that $|\text{Ч}\mathbf{E}_1\varphi| \le K_2/\|\nabla \mathsf{H}^{\text{loc}}_1\|^2$ and $|\mathbf{E}_1^2\varphi| \le K_2/\|\nabla \mathsf{H}^{\text{loc}}_1\|^3$ on $\mathbf{S} \setminus \mathsf{E}$. Using the last claim of Lemma 11.1, there is thus a constant $K_3 > 0$ such that $|\text{Ч}\varphi| \le K_3$, $|\text{Ч}^2\varphi| \le K_3$, $|\mathbf{E}_1\varphi| \le K_3/\sqrt{|\mathsf{H}^{\text{loc}}_1|}$, $|\mathbf{E}_1\text{Ч}\varphi| \le K_3/\sqrt{|\mathsf{H}^{\text{loc}}_1|}$, $|\text{Ч}\mathbf{E}_1\varphi| \le K_3/\sqrt{|\mathsf{H}^{\text{loc}}_1|}$, and $|\mathbf{E}_1^2\varphi| \le K_3/|\mathsf{H}^{\text{loc}}_1|^{3/2}$ on $\mathbf{S} \setminus \mathsf{E}$. Hence from [Sow], we have that there is a $K_4 > 0$ such that

$$\left|(\text{Ч}\Phi^{\mathfrak{z},\lambda}_\varphi) - \{\varphi - (\mathcal{A}\varphi)\}\right| \le K\lambda\mathcal{T}$$

on $\mathbf{S} \setminus \mathsf{E}$, where $\mathcal{T}(x) \stackrel{\text{def}}{=} \inf\{t > 0 : \mathfrak{z}_t(x) = x\}$ for all $x \in \mathbf{S} \setminus \mathsf{E}$, and such that

$$\begin{aligned}|\Phi^{\mathfrak{z},\lambda}_\varphi| &\le \frac{K_4}{\lambda}, \qquad \|D\Phi^{\mathfrak{z},\lambda}_\varphi\| \le \frac{K_4}{\lambda}\left\{\frac{1}{\sqrt{|\mathsf{H}^{\text{loc}}_1|}} + \frac{1}{\sqrt{|\mathsf{H}^{\text{loc}}_1|}}\right\} + \frac{K_4}{\lambda^2}\frac{1}{|\mathsf{H}^{\text{loc}}_1|}, \\ \|D^2\Phi^{\mathfrak{z},\lambda}_\varphi\| &\le \frac{K_4}{\lambda}\left\{\frac{1}{|\mathsf{H}^{\text{loc}}_1|} + \frac{1}{|\mathsf{H}^{\text{loc}}_1|^{1/2+1/2}} + \frac{1}{|\mathsf{H}^{\text{loc}}_1|^{3/2}}\right\} \\ &\quad + \frac{K_4}{\lambda^2}\left\{\frac{1}{|\mathsf{H}^{\text{loc}}_1|^{1+1/2}} + \frac{1}{|\mathsf{H}^{\text{loc}}_1|^2} + \frac{1}{|\mathsf{H}^{\text{loc}}_1|^{1+1/2}}\right\} \\ &\quad + \frac{K_4}{\lambda^3}\left\{\frac{1}{|\mathsf{H}^{\text{loc}}_1|^{1+1}} + \frac{1}{|\mathsf{H}^{\text{loc}}_1|^2}\right\} + K_4\left\{\frac{1}{\sqrt{|\mathsf{H}^{\text{loc}}_1|}} + \frac{1}{|\mathsf{H}^{\text{loc}}_1|}\right\}\|D\Phi^{\mathfrak{z},\lambda}_\varphi\|\end{aligned}$$

on $x \in \mathbf{S} \setminus \mathsf{E}$. Combine things to get the stated result, keeping in mind that λ and $|\mathsf{H}_1^{\mathrm{loc}}|$ are bounded from above but can be arbitrarily close to 0. We also use standard bounds on the orbit time $\mathcal{T}$ (see Lemma 5.1 of [Sow03]). □

We next take up averaging in E. We can average ξ^* along the flow of $\mathfrak{z}^N$; recall that Lemma 6.3 ensures that the orbits of $\mathfrak{z}^N$ are periodic on $\mathbf{S} \setminus \gamma_N$; thus, for every $N \in \mathbb{N}$ and $x \in \mathbf{S} \setminus \gamma_N$, we can define

$$(\mathcal{A}_N \xi^*)(x) \stackrel{\text{def}}{=} \lim_{T \to \infty} \frac{1}{T} \int_{s=0}^{T} \xi^*(\mathfrak{z}_s^N(x)) ds.$$

LEMMA 11.2. *There is a constant $K > 0$ such that for $\lambda \in (0,1)$, $N \in \mathbb{N}$, and*

$$\left| (\text{Ч}_N \Phi_{\xi^*}^{\mathfrak{z}_N,\lambda}) - \{\xi^* - \mathcal{A}_N \xi^*\} \right| \le K\lambda,$$

$$\left| \Phi_{\xi^*}^{\mathfrak{z}_N,\lambda} \right| \le \frac{K}{\lambda \mathsf{r}_N}, \qquad \left\| D\Phi_{\xi^*}^{\mathfrak{z}_N,\lambda} \right\| \le \frac{K}{\lambda^2 \mathsf{r}_N^2 |\mathsf{H}_N^{\mathrm{loc}}|},$$

$$\left\| D^2 \Phi_{\xi^*}^{\mathfrak{z}_N,\lambda} \right\| \le \frac{K}{\lambda^3 \mathsf{r}_N^3 |\mathsf{H}_N^{\mathrm{loc}}|^2}$$

on $\mathcal{N}_N$.

Proof. Since ξ^* is identically zero on the $\mathcal{D}'_\ell$'s, we first note that

$$\sup \Big\{ |\mathbf{E}_N^i \text{Ч}_N^j \xi^*(x)| : \\ x \in \mathcal{N}_N,\ N \in \mathbb{N}, \text{ and } i \text{ and } j \text{ in } \{0,1,2\} \text{ such that } i + j \le 2 \Big\}$$

is finite.

Now set $\check{\lambda}_N \stackrel{\text{def}}{=} \lambda \mathsf{r}_N$. From [Sow], we have that there is a $K > 0$ such that

$$\left| (\text{Ч}_N \Phi_{\xi^*}^{\mathfrak{z}_N,\lambda})(x) - \{\xi^*(x) - (\mathcal{A}_N \xi^*)(x)\} \right| \le K \check{\lambda}_N \int_{z \in \mathfrak{z}_{\mathbb{R}}^N(x)} \frac{|\xi^*(z)|}{\|\text{Ч}_N(x)\|} \mathscr{H}^1(dz)$$

and

$$|\Phi_{\xi^*}^{\mathfrak{z}_N,\lambda}(x)| \le \frac{K}{\check{\lambda}_N},$$

$$\left\| D\Phi_{\xi^*}^{\mathfrak{z}_N,\lambda}(x) \right\| \le \frac{K}{\check{\lambda}_N} \left\{ \frac{1}{\sqrt{|\mathsf{H}_N^{\mathrm{loc}}(x)|}} + 1 \right\} + \frac{K}{\check{\lambda}_N^2} \frac{1}{|\mathsf{H}_N^{\mathrm{loc}}(x)|},$$

$$\left\| D^2\Phi^{\mathfrak{z}_N,\lambda}_{\xi^*}(x)\right\| \le \frac{K}{\check{\lambda}_N}\left\{\frac{1}{|\mathsf{H}^{\text{loc}}_N(x)|}+\frac{1}{\sqrt{|\mathsf{H}^{\text{loc}}_N(x)|}}+1\right\}$$
$$+\frac{K}{\check{\lambda}_N^2}\left\{\frac{1}{|\mathsf{H}^{\text{loc}}_N(x)|^{3/2}}+\frac{1}{|\mathsf{H}^{\text{loc}}_N(x)|^2}+\frac{1}{|\mathsf{H}^{\text{loc}}_N(x)|}\right\}$$
$$+\frac{K}{\check{\lambda}_N^3}\left\{\frac{1}{|\mathsf{H}^{\text{loc}}_N(x)|^2}+\frac{1}{|\mathsf{H}^{\text{loc}}_N(x)|^2}\right\}\left\{\frac{1}{\sqrt{|\mathsf{H}^{\text{loc}}_N(x)|}}+\frac{1}{|\mathsf{H}^{\text{loc}}_N(x)|}\right\}\|D\Phi^{\mathfrak{z}_N,\lambda}_{\xi^*}(x)\|.$$

We note that by the change of variables formula,

$$\lim_{N\to\infty}\sup_{x\in\mathcal{N}_N}\left|\mathsf{r}_N\int_{z\in\mathfrak{z}^N_{\mathbb{R}}(x)}\frac{|\xi^*(z)|}{\|\text{Ч}_N(x)\|}\mathscr{H}^1(dz)-\int_{z\in\mathsf{E}}|\xi^*(z)|\mathscr{H}^2(dz)\right|=0.$$

Picking out the dominant terms under the assumption that λ and $\mathsf{H}^{\text{loc}}_N$ are both bounded from above but may be arbitrarily close to zero, we can fairly easily conclude the claimed results. □

We next need to compare $\mathcal{A}_N\xi^*$ with $(\mathcal{A}\xi^*)([\mathsf{E}])$. We have

LEMMA 11.3. *There is a constant $K>0$ and a sequence $\{\mathscr{E}_n;\, n\in\mathbb{N}\}$ of positive real numbers such that*

$$\left|(\mathcal{A}_N\xi^*)(x)-\frac{\omega_2}{\mathscr{H}^2(\mathsf{E})}\right|\le K\mathsf{r}_N\mathfrak{l}^2(|\mathsf{H}^{\text{loc}}_N(x)|)+\mathscr{E}_n$$

for all $x\in\mathcal{N}_N$ and $N\in\mathbb{N}$ and such that $\lim_{N\to\infty}\mathscr{E}_n=0$.

Proof. For each $\ell\in\Lambda$, let $\zeta_\ell\in C^\infty(\mathbb{T})$ be such that $\operatorname{supp}\zeta_\ell\subset\mathcal{U}_\ell$ and $\zeta_\ell=1$ in a neighborhood of $\mathfrak{x}_\ell$. We first define several quantities. Let $\mathscr{Q}\overset{\text{def}}{=}\{\mathbb{R}_+\times\mathbb{R}_+,\mathbb{R}_+\times\mathbb{R}_-,\mathbb{R}_-\times\mathbb{R}_+,\mathbb{R}_-\times\mathbb{R}_-\}$ be the collection of quadrants in $\mathbb{R}^2$. Define

$$(\mathcal{A}^\circ_N\xi^*)(x)\overset{\text{def}}{=}\mathsf{r}_N\int_{z\in\mathfrak{z}^N_{\mathbb{R}}(x)}\frac{\xi^*(z)}{\|\text{Ч}_N(z)\|}\mathscr{H}^1(dz),\quad x\in\mathcal{N}_N,\ N\in\mathbb{N},$$
$$\mathcal{T}^N(x)\overset{\text{def}}{=}\mathsf{r}_N\int_{z\in\mathfrak{z}^N_{\mathbb{R}}(x)}\frac{1}{\|\text{Ч}_N(z)\|}\mathscr{H}^1(dz),\quad x\in\mathcal{N}_N,\ N\in\mathbb{N},$$
$$\mathcal{T}^N_{\ell,Q}(x)\overset{\text{def}}{=}\mathsf{r}_N\int_{\substack{z\in\mathfrak{z}^N_{\mathbb{R}}(x)\\ z\in\tilde{\phi}_\ell(Q\cap\tilde{\mathcal{U}})}}\frac{\zeta_\ell(z)}{\|\text{Ч}_N(z)\|}\mathscr{H}^1(dz),$$
$$x\in\mathcal{N}_N,\ \ell\in\Lambda,\ N\in\mathbb{N},\ Q\in\mathscr{Q},$$
$$\mathcal{T}^\infty_{\ell,Q}\overset{\text{def}}{=}\int_{\substack{z\in\mathsf{E}\\ z\in\tilde{\phi}_\ell(Q\cap\tilde{\mathcal{U}})}}\zeta_\ell(z)\mathscr{H}^2(dz),\quad \ell\in\Lambda,\ Q\in\mathscr{Q},$$

$$\mathcal{T}_E^N(x) \stackrel{\text{def}}{=} \mathsf{r}_N \int_{z\in\mathfrak{z}_{\mathbb{R}}^N(x)} \left\{1-\sum_{\ell\in\Lambda}\zeta_\ell(z)\right\} \frac{1}{\|\text{Ч}_N(z)\|}\mathscr{H}^1(dz),$$
$$x\in\mathcal{N}_N,\ N\in\mathbb{N},$$
$$\mathcal{T}_E^\infty \stackrel{\text{def}}{=} \int_{z\in\mathsf{E}} \left\{1-\sum_{\ell\in\Lambda}\zeta_\ell(z)\right\}\mathscr{H}^2(dz).$$

Defining $\text{E}_1^N \stackrel{\text{def}}{=} \sup_{x\in\mathcal{N}_N} |(\mathcal{A}_N^\circ\xi^*)(x)-\omega_2|$ and $\text{E}_2^N \stackrel{\text{def}}{=} \sup_{x\in\mathcal{N}_N} |\mathcal{T}_E^N(x)-\mathcal{T}_E^\infty|$ for all $N\in\mathbb{N}$, we have by the change of variables formula that $\lim_{N\to\infty}\text{E}_1^N = \lim_{N\to\infty}\text{E}_2^N = 0$.

We can then write that

$$\begin{aligned}
&(\mathcal{A}_N\xi^*)(x) - \frac{\omega_2}{\mathscr{H}^2(\mathsf{E})} = \frac{(\mathcal{A}_N^\circ\xi^*)(x)}{\mathcal{T}^N(x)} - \frac{\omega_2}{\mathscr{H}^2(\mathsf{E})}\\
&= \frac{\left\{(\mathcal{A}_N^\circ\xi^*)(x)-\omega_2\right\}\mathscr{H}^2(\mathsf{E}) - \omega_2\left\{\mathcal{T}^N(x)-\mathscr{H}^2(\mathsf{E})\right\}}{\mathcal{T}^N(x)\mathscr{H}^2(\mathsf{E})}\\
&= \frac{(\mathcal{A}_N^\circ\xi^*)(x)-\omega_2}{\mathscr{H}^2(\mathsf{E})}\\
&\qquad - \frac{\omega_2}{\mathcal{T}^N(x)\mathscr{H}^2(\mathsf{E})}\left\{\sum_{\ell\in\Lambda,\,Q\in\mathscr{Q}}\left\{\mathcal{T}_{\ell,Q}^N(x)-\mathcal{T}_{\ell,Q}^\infty\right\} + \left\{\mathcal{T}_E^N(x)-\mathcal{T}_E^\infty\right\}\right\}.
\end{aligned}$$

For $N\in\mathbb{N}$ large enough that $\text{E}_2^N \le \mathcal{T}_E^\infty/2$, we thus have that $\mathcal{T}^N(x) \ge \mathcal{T}_E^N(x) \ge \mathcal{T}_E^\infty/2$, so

$$\begin{aligned}
\left|(\mathcal{A}_N\xi^*)(x) - \frac{\omega_2}{\mathscr{H}^2(\mathsf{E})}\right| &\le \frac{2}{\mathcal{T}_E^\infty}\text{E}_1^N + \frac{2|\omega_2|}{\mathcal{T}_E^\infty\mathscr{H}^2(\mathsf{E})}\text{E}_2^N\\
&\qquad + \frac{2|\omega_2|}{\mathcal{T}_E^\infty\mathscr{H}^2(\mathsf{E})}\sum_{\ell\in\Lambda,Q\in\mathscr{Q}}|\mathcal{T}_{\ell,Q}^N(x)-\mathcal{T}_{\ell,Q}^\infty|.
\end{aligned}$$

Let's now set up some machinery for a local comparison of $\mathcal{T}_{\ell,Q}^N$ and $\mathcal{T}_{\ell,Q}^\infty$. Fix $\varphi\in C_c^\infty(\tilde{\mathcal{U}})$. Secondly, let $\mathcal{P}\in C^\infty(\tilde{\mathcal{U}})$ be such that

$$\mathcal{P}(\mathbf{0}_e) = (\partial\mathcal{P}/\partial x_1)(\mathbf{0}_e) = (\partial\mathcal{P}/\partial x_2)(\mathbf{0}_e) = 0,$$
$$\inf\left\{\sum_{i,j\in\{1,2\}}\alpha_i\alpha_j\frac{\partial^2\mathcal{P}}{\partial x_i\partial x_j}(x_1,x_2) : (\alpha_1,\alpha_2)\in\mathbb{R}^2, \alpha_1^2+\alpha_2^2=1,\ (x_1,x_2)\in\tilde{\mathcal{U}}\right\} > 0.$$

Note that thus

$$\inf_{x\in\tilde{\mathcal{U}}\setminus\{\mathbf{0}_e\}}\frac{\mathcal{P}(x)}{\mathbf{n}(x)} > 0. \tag{100}$$

Define then $\amalg(x) \stackrel{\text{def}}{=} \varphi(x)\sqrt{\mathbf{n}(x)/\mathcal{P}(x)}$ for all $x \in \tilde{\mathcal{U}} \setminus \{\mathbf{0}_e\}$, and set

$$\mathcal{T}^*_N(h) \stackrel{\text{def}}{=} \mathsf{r}_N \sum_{k=0}^{\infty} \int_{z=0}^{\infty} \amalg\left(z, \frac{h+k\mathsf{r}_N}{z}\right)\frac{dz}{z}$$

for all $h \in \mathsf{r}_N\mathcal{W} \setminus \{0\}$, and define $\mathcal{T}^*_\infty \stackrel{\text{def}}{=} \int_{x\in\tilde{\mathcal{U}}\cap\mathbb{R}^2_+} \amalg(x)dx$. For each fixed $\ell \in \Lambda$ and $Q \in \mathscr{Q}$, we can find such a φ and $\mathcal{P}$ such that either $\mathcal{T}^N_{\ell,Q}(x) = \mathcal{T}^*_N(|\mathsf{H}^{\text{loc}}_N(x)|)$ for all $x \in \mathcal{N}_N \cap \tilde{\phi}_\ell(Q\cap\tilde{\mathcal{U}})$ or $\mathcal{T}^N_{\ell,Q}(x) = \mathcal{T}^*_N(\mathsf{r}_N - \mathsf{H}^{\text{loc}}_N(x))$ for all $x \in \mathcal{N}_N \cap \tilde{\phi}_\ell(Q\cap\tilde{\mathcal{U}})$; thus either $\mathcal{T}^N_{\ell,Q}(x) - \mathcal{T}^\infty_{\ell,Q} = \mathcal{T}^*_N(|\mathsf{H}^{\text{loc}}_N(x)|) - \mathcal{T}^*_\infty$ for all $x \in \mathcal{N}_N \cap \tilde{\phi}_\ell(Q\cap\tilde{\mathcal{U}})$ or $\mathcal{T}^N_{\ell,Q}(x) - \mathcal{T}^\infty_{\ell,Q} = \mathcal{T}^*_N(\mathsf{r}_N - \mathsf{H}^{\text{loc}}_N(x)) - \mathcal{T}^*_\infty$ for all $x \in \mathcal{N}_N \cap \tilde{\phi}_\ell(Q\cap\tilde{\mathcal{U}})$. Note that both $|\mathsf{H}^{\text{loc}}_N|$ and $\mathsf{r}_N - \mathsf{H}^{\text{loc}}_N$ take values in $(0,\mathsf{r}_N)$ on $\mathcal{N}_N$.

Fix $h \in (0,\mathsf{r}_N)$. Define $\delta_{N,z} \stackrel{\text{def}}{=} \mathsf{r}_N/z$ for all $N \in \mathbb{N}$ and $z > 0$. Then

$$\mathcal{T}^*_N(h) = \int_{z=0}^{\infty} \sum_{k=0}^{\infty} \amalg\left(z, \frac{h}{z} + k\delta_{N,z}\right)\delta_{N,z}dz,$$
$$\mathcal{T}^*_\infty = \int_{(z,u)\in\tilde{\mathcal{U}}\cap\mathbb{R}^2_+} \amalg(z,u)dz\,du.$$

Defining

$$\mathcal{T}^{*,\circ}_N(h) \stackrel{\text{def}}{=} \int_{z=0}^{\infty}\int_{u=0}^{\infty} \amalg\left(z, \frac{h}{z}+u\right)du\,dz = \int_{z=0}^{\infty}\int_{u=h/z}^{\infty} \amalg(z,u)du\,dz,$$

we then have that $\mathcal{T}^*_N(h) - \mathcal{T}^*_\infty = \{\mathcal{T}^*_N(h) - \mathcal{T}^{*,\circ}_N(h)\} + \{\mathcal{T}^{*,\circ}_N(h) - \mathcal{T}^*_\infty\}$. Since $\tilde{\mathcal{U}} \cap \mathbb{R}^2_+ \subset (0,1)^2$,

$$\begin{aligned}|\mathcal{T}^{*,\circ}_N(h) - \mathcal{T}^*_\infty| &\le \|\amalg\|_{C(\tilde{\mathcal{U}})} \int_{z=h}^{1}\int_{u=0}^{h/z} du\,dz \\ &= \|\amalg\|_{C(\tilde{\mathcal{U}})} \int_{z=h}^{1} \frac{h}{z}dz \le \|\amalg\|_{C(\tilde{\mathcal{U}})} h\mathfrak{l}(h).\end{aligned}$$

To study $\mathcal{T}^*_N - \mathcal{T}^{*,\circ}_N$, we write that

$$\begin{aligned}&\mathcal{T}^*_N(h) - \mathcal{T}^{*,\circ}_N(h) \\ &\quad= \int_{z=0}^{1}\sum_{k=0}^{\infty}\int_{u=k\delta_{N,z}}^{(k+1)\delta_{N,z}} \left\{\amalg\left(z, \frac{h}{z}+k\delta_{N,z}\right) - \amalg\left(z, \frac{h}{z}+u\right)\right\}du\,dz.\end{aligned}$$

Proceeding, we study the regularity of $\amalg$. We compute that for all $x = (x_1,x_2) \in \tilde{\mathcal{U}}$,

$$\frac{\partial\amalg}{\partial x_2}(x) = \frac{\partial\varphi}{\partial x_2}(x)\sqrt{\frac{\mathbf{n}(x)}{\mathcal{P}(x)}} + \varphi(x)\frac{x_2}{\sqrt{\mathbf{n}(x)\mathcal{P}(x)}} - \frac{\varphi(x)}{2}\frac{\frac{\partial\mathcal{P}}{\partial x_2}(x)\sqrt{\mathbf{n}(x)}}{\mathcal{P}^{3/2}(x)}.$$

Recall (100) and note that the assumptions on $\mathcal{P}$ imply that there is a $K_1>0$ such that $|\frac{\partial \mathcal{P}}{\partial x_2}(x)| \le K_1\|x\|_e$ for all $x\in\tilde{\mathcal{U}}$. Thus, there is a constant $K_2>0$ such that $|(\partial\amalg/\partial x_2)(x)| \le K_2/\sqrt{\mathbf{n}(x)}$. Thus

$$\begin{aligned}
&|\mathcal{T}^*_N(h)-\mathcal{T}^{*,\circ}_N(h)|\\
&\qquad\le \int_{z=0}^1 \sum_{k=0}^\infty \left\{\int_{u=k\delta_{N,z}}^{(k+1)\delta_{N,z}} \int_{r=k\delta_{N,z}}^{u} \left|\frac{\partial \amalg}{\partial x_2}\left(z,\frac{h}{z}+r\right)\right| dr\,du\right\}dz\\
&\qquad\le \int_{z=0}^1 \sum_{k=0}^\infty \left\{\int_{r=k\delta_{N,z}}^{(k+1)\delta_{N,z}} \left|\frac{\partial \amalg}{\partial x_2}\left(z,\frac{h}{z}+r\right)\right| \{(k+1)\delta_{N,z}-r\}dr\right\}dz\\
&\qquad\le K_2\int_{z=0}^1 \delta_{N,z}\int_{r=0}^\infty \frac{1}{\sqrt{\mathbf{n}\left(z,\frac{h}{z}+r\right)}}\chi_{(0,1)^2}\left(z,\frac{h}{z}+r\right)dr\,dz.
\end{aligned}$$

Consider now the requirement that $\frac{h}{z}+r$ be in $(0,1)$. The assumption that h, r, and z are all positive implies that both h/z and r must be in $(0,1)$. The former in turn implies that $h<z$. Thus

$$\begin{aligned}
|\mathcal{T}^*_N(h)-\mathcal{T}^{*,\circ}_N(h)| &\le K_1\mathsf{r}_N\int_{z=h}^1\int_{r=0}^1 \frac{1}{z\sqrt{\mathbf{n}\left(z,\frac{h}{z}+r\right)}}dr\,dz\\
&= K_1\mathsf{r}_N\int_{z=h}^1 \frac{1}{z}\int_{s=h/z^2}^{h/z^2+1/z}\frac{1}{\sqrt{1+s^2}}ds\,dz,
\end{aligned}$$

where we have used the substitution $s=(h/z+r)/z$ (i.e., $h/z+r=zs$). Note that $(1+s)^2\le 2(1+s^2)$ for all $s>0$, that $s\mapsto 1/(1+s)$ is decreasing on $(0,\infty)$, and that $1/z\le 1/h$ for $z\ge h$. Thus

$$\begin{aligned}
|\mathcal{T}^*_N(h)-\mathcal{T}^{*,\circ}_N(h)| &\le \sqrt{2}K_1\mathsf{r}_N\int_{z=h}^1\frac{1}{z}\int_{s=h/z^2}^{h/z^2+1/z}\frac{1}{1+s}ds\,dz\\
&\le \sqrt{2}K_1\mathsf{r}_N\int_{z=h}^1\frac{1}{z}\int_{s=0}^{1/h}\frac{1}{1+s}ds\,dz\\
&= \sqrt{2}K_1\mathsf{r}_N\left\{\ln h^{-1}\right\}\left\{\ln\left(1+\frac{1}{h}\right)\right\}\le \sqrt{2}K_1\mathsf{r}_N\mathfrak{l}^2(h).
\end{aligned}$$

Thus there is a $K_2>0$ such that $|\mathcal{T}^*_N(h)-\mathcal{T}^{*,\circ}_N(h)|\le K_2\mathfrak{l}^2(h)$ for all $h\in(0,\mathsf{r}_N)$. Combining this and Lemma 11.1, we can find a constant $K_3>0$ such that $|\mathcal{T}^*_N(h)-\mathcal{T}^*_\infty(h)|\le K_2\mathsf{r}_N\left\{h\mathfrak{l}(h)+\mathfrak{l}^2(h)\right\}$ for all $h\in(0,\mathsf{r}_N)$ Note that $\mathsf{r}_N-\mathsf{H}^{\mathrm{loc}}_N\ge|\mathsf{H}^{\mathrm{loc}}_N|$ on $\mathcal{N}_N$. Collecting things together, we have the stated result. $\square$

Proof of Lemma 8.7. Combine Lemmas 11.2 and 11.3. To get the first stated bound, we write that

$$\begin{aligned}\left|(\text{Ч}\Phi^{\mathfrak{z}_N,\lambda}_{\xi^*})(x) - \{\xi^*(x) - (\mathcal{A}\xi^*)([\mathsf{E}])\}\right| \\ \le \left|(\text{Ч}_N\Phi^{\mathfrak{z}_N,\lambda}_{\xi^*})(x) - \{\xi^*(x) - (\mathcal{A}_N\xi^*)(x)\}\right| \\ + |(\mathcal{A}_N\xi^*)(x) - (\mathcal{A}\xi^*)([\mathsf{E}])| + |\boldsymbol{\nu}_N|\left|(\hat{\text{Ч}}_N\Phi^{\mathfrak{z}_N,\lambda}_{\xi^*})(x)\right|.\end{aligned}$$

We use Lemma 11.2 on the first term on the right, Lemma 11.3 on the second, and the regularity of Lemma 11.2 on the third term. □

11.2. Proof of Lemma 8.5. For each $N \in \mathbb{N}$ and $x \in \mathsf{E}$, define

$$\breve{\Phi}^\lambda_N(x) \overset{\text{def}}{=} -\int_{t=0}^\infty e^{-\lambda t}\bar{\xi}^*(\mathbf{p}^N_t(x))dt, \qquad \lambda > 0,$$

$$(\breve{\mathcal{A}}^N_T\bar{\xi}^*)(x) \overset{\text{def}}{=} \frac{1}{T}\int_{t=0}^T \bar{\xi}^*(\mathbf{p}^N_t(x))dt, \qquad T > 0.$$

LEMMA 11.4. *For each $N \in \mathbb{N}$ and $\lambda \in (0,1)$, $\breve{\Phi}^\lambda_N$ is smooth on E, and $\bar{\text{Ч}}^j_N\breve{\Phi}^\lambda_N$ is uniformly continuous on E for each $j \in \{0,1,2,3\}$. Secondly, there is a constant $K > 0$ such that*

$$|\bar{\text{Ч}}^j_N\breve{\Phi}^\lambda_N(x)| \le \frac{K}{\lambda} \tag{101}$$

for all $x \in \mathsf{E}$, $N \in \mathbb{N}$, $\lambda \in (0,1)$ and $j \in \{0,1,2,3\}$. Finally,

$$\left|(\text{Ч}_N\breve{\Phi}^\lambda_N)(x) - \{\xi^*(x) - \beth\boldsymbol{\sigma}(x)\}\right| \le \boldsymbol{\sigma}(x)\int_{t=0}^\infty te^{-t}\left|(\breve{\mathcal{A}}^N_{t/\lambda}\bar{\xi}^*)(x) - \beth\right|dt \tag{102}$$

for all $x \in \mathsf{E}$, $N \in \mathbb{N}$, and $\lambda \in (0,1)$.

Proof. Standard calculations and the observation that the orbits of $\mathbf{p}^N$ are periodic on $\mathcal{N}_N$ imply that $\breve{\Phi}^\lambda_N$ is smooth in E. We can also easily see that

$$(\bar{\text{Ч}}^j_N\breve{\Phi}^\lambda_N)(x) = -\int_{t=0}^\infty e^{-\lambda t}(\bar{\text{Ч}}^j_N\bar{\xi}^*)(\mathbf{p}^N_t(x))dt$$

for all $x \in \mathsf{E}$, $N \in \mathbb{N}$, $\lambda \in (0,1)$, and $j \in \{0,1,2,3\}$. Note that $\bar{\xi}^*$ is smooth on $\overline{\mathsf{E}} \setminus \mathfrak{X}$ and furthermore that it is identically zero in a neighborhood of the points in $\mathfrak{X}$. For each $N \in \mathbb{N}$, the vector field Ч_N is smooth on $\overline{\mathsf{E}} \setminus \mathfrak{X}$. The bounds of (101) and the desired uniform continuity are now fairly easy to see. The bound of (102) follows from the calculations of [Sow]. □

We next claim that the error in (102) is small. Essentially, we construct something like the "special flow" of Arnol'd [Arn91]. The important difference is that this flow has no logarithmic singularities. We also must deal with ambiguities arising at the points of $\mathfrak{X}$.

LEMMA 11.5. *We have* $\lim_{T\to\infty}\overline{\lim}_{N\to\infty}\sup_{x\in\mathsf{E}\setminus\gamma_N}\left|(\breve{\mathcal{A}}^N_T\bar{\xi}^*)(x)-\daleth\right|=0.$

Proof. For each $x\in\bar{\mathsf{E}}\setminus\mathfrak{X}$, let $\{\mathbf{p}_t(x);\, t\in I_x\}$ be the maximal solution of the ODE

$$\dot{\mathbf{p}}_t(x)=\frac{\text{Ч}}{\boldsymbol{\sigma}}(\mathbf{p}_t(x));\, t\in I_x,$$
$$\mathbf{p}_0(x)=x.$$

We now use the transversal of Subsection 6.1. If $\omega_2>0$, set $\mathsf{I}\overset{\text{def}}{=}[0,\omega_2)$, and if $\omega_2<0$, set $\mathsf{I}\overset{\text{def}}{=}(\omega_2,0]$. For each $h\in\mathsf{I}$, define

$$\tau(h)\overset{\text{def}}{=}\inf\left\{t>0: t\in I_x,\ \mathbf{p}_t(\zeta(h))\in\zeta(\mathbb{R})+(\mathbb{Z}\times\{0\})\right\},$$

where we define $\inf\emptyset\overset{\text{def}}{=}\infty$ for consistency. Note that the set $\mathsf{J}\overset{\text{def}}{=}\{h\in\mathsf{I};\,\tau(h)=\infty\}$ of "jumps" of τ (i.e., the bifurcation levels of $\mathbf{p}$) is of cardinality $|\Lambda|$; indeed, each element of J is of the form $\iota\left((\mathsf{H}(\mathfrak{x}^e_\ell)+\langle K,\omega\rangle_{\mathbb{R}^2})/\omega_2\right)\omega_2$ (recall ι of (13)) for some $\ell\in\Lambda$, where K is the unique element of $\mathbb{Z}^2$ such that $\mathfrak{x}^e_\ell+K$ is in the "box" bounded on three sides by $\zeta(\mathbb{R})$ and the unbounded components of $\mathsf{H}^{-1}(0)$ and $\mathsf{H}^{-1}(\omega_2)$, and on the fourth side by $\zeta(\mathbb{R})+(1,0)$ if $\omega_2>0$, and $\zeta(\mathbb{R})-(1,0)$ if $\omega_2<0$ (if $\omega_2>0$, the vector field Ч_N macroscopically points to the right, and if $\omega_2<0$, it points to the left). Note also that

$$\sup_{h\in\mathsf{I}\setminus\mathsf{J}}\tau(h)<\infty. \tag{103}$$

and that τ is uniformly continuous on $\mathsf{I}\setminus\mathsf{J}$. Define $\blacktriangle\overset{\text{def}}{=}\{(t,h): h\in\mathsf{I}\setminus\mathsf{J}, 0\le t<\tau(h)\}$ and define $\eth:\blacktriangle\to\mathsf{E}$ by setting $\eth(t,h)\overset{\text{def}}{=}\mathbf{p}_t(\zeta(h))$ for all $(t,h)\in\blacktriangle$

Let's next extend things by continuity. This cannot be done in a unique way because of the bifurcations at the elements of $\mathfrak{X}$, so we enumerate all ways. Fix $\mathrm{s}\in\{+,-\}^{\mathsf{J}}$. Define $\tau_{\mathrm{s}}:\mathsf{I}\to\mathbb{R}$ by setting $\tau_{\mathrm{s}}(h)\overset{\text{def}}{=}\tau(h)$ if $h\in\mathsf{I}\setminus\mathsf{J}$, and, if $h\in\mathsf{J}$, set $\tau_{\mathrm{s}}(h)\overset{\text{def}}{=}\lim_{h'\searrow h,h'\in\mathsf{I}\setminus\mathsf{J}}\tau(h')$ if $\mathrm{s}_h=+$ and set $\tau_{\mathrm{s}}(h)\overset{\text{def}}{=}\lim_{h'\nearrow h,h'\in\mathsf{I}\setminus\mathsf{J}}\tau(h')$ if $\mathrm{s}_h=-$. Define $\blacktriangle_{\mathrm{s}}\overset{\text{def}}{=}\{(t,h): h\in\mathsf{I}, 0\le t<\tau_{\mathrm{s}}(h)\}$, and define $\eth_{\mathrm{s}}:\blacktriangle_{\mathrm{s}}\to\bar{\mathsf{E}}$ by setting $\eth_{\mathrm{s}}(t,h)\overset{\text{def}}{=}\eth(t,h)$ if $(t,h)\in\blacktriangle$, and for all $t\in[0,\tau_{\mathrm{s}}(h))$ and $h\in\mathsf{J}$, set $\eth_{\mathrm{s}}(t,h)\overset{\text{def}}{=}\lim_{h'\searrow h,h'\in\mathsf{I}\setminus\mathsf{J}}\eth(t,h')$ if $\mathrm{s}_h=+$, and set $\eth_{\mathrm{s}}(t,h)\overset{\text{def}}{=}\lim_{h'\nearrow h,h'\in\mathsf{I}\setminus\mathsf{J}}\eth(t,h')$ if $\mathrm{s}_h=-$. Then $\eth_{\mathrm{s}}(\blacktriangle_{\mathrm{s}})=\bar{\mathsf{E}}$ and $\eth_{\mathrm{s}}$ is a measurable bijection from $\blacktriangle_{\mathrm{s}}$ to $\bar{\mathsf{E}}$. Lastly, define a flow on $\blacktriangle_{\mathrm{s}}$. Let $\{\mathfrak{f}^{\mathrm{s}}_t;\, t\in\mathbb{R}\}$ be the unique flow on $\blacktriangle_{\mathrm{s}}$ such that for all $(t,h)\in\blacktriangle_{\mathrm{s}}$, $\mathfrak{f}^{\mathrm{s}}_s(t,h)=(t+s,h)$ if $0\le t+s<\tau(h)$, and such that $\mathfrak{f}^{\mathrm{s}}_{\tau_{\mathrm{s}}(h)-t}(t,h)=(0,\iota\left((h-\omega_1)/\omega_2\right)\omega_2)$ if $\omega_2>0$, and $\mathfrak{f}^{\mathrm{s}}_{\tau_{\mathrm{s}}(h)-t}(t,h)=(0,\iota\left((h+\omega_1)/\omega_2\right)\omega_2)$ if $\omega_2<0$. Considering all of these flows together, and using (103), we can rather easily see that

$$\lim_{T\to\infty}\sup_{\mathrm{s}\in\{+,-\}^{\mathsf{J}}}\sup_{x\in\blacktriangle_{\mathrm{s}}}\left|\frac{1}{T}\int_{t=0}^{T}\bar{\xi}^*(\eth_{\mathrm{s}}(\mathfrak{f}^{\mathrm{s}}_t(x)))dt - \frac{1}{\mathscr{H}^2(\blacktriangle)}\int_{(t,h)\in\blacktriangle}\bar{\xi}^*(\eth(t,h))dt\,dh\right| = 0, \tag{104}$$

where $\bar{\xi}^*$ is as in (67). We note that

$$\frac{1}{\mathscr{H}^2(\blacktriangle)}\int_{(t,h)\in\blacktriangle}\bar{\xi}^*(\eth(t,h))dt\,dh = \daleth$$

(use the change-of-variables formula; [EG92, Section 3.4.3]).

We now finish the proof and make sure that the limits and suprema are in the right places. Fix now $\delta > 0$ and $T_\delta > 0$ such that

$$\sup_{T\ge T_\delta}\sup_{\mathrm{s}\in\{+,-\}^{\mathsf{J}}}\sup_{x\in\blacktriangle_{\mathrm{s}}}\left|\frac{1}{T}\int_{t=0}^{T}\bar{\xi}^*(\eth_{\mathrm{s}}(\mathfrak{f}^{\mathrm{s}}_t(x)))dt - \daleth\right| < \delta.$$

Fix next $T \ge T_\delta$, an increasing sequence $\{N_k;\, k\in\mathbb{N}\}$ of elements of $\mathbb{N}$, and for each $k\in\mathbb{N}$, a point x_k in $\bar{\mathsf{E}}$. Assume also that $x^* \stackrel{\text{def}}{=} \lim_{k\to\infty} x_k$ exists (it will of course be in $\bar{\mathsf{E}}$). Then it is easy to see that there is an $\mathrm{s}\in\{+,-\}^{\mathsf{J}}$ such that

$$\lim_{k\to\infty}(\breve{\mathcal{A}}^{N_k}_T\bar{\xi}^*)(x_k) = \frac{1}{T}\int_{t=0}^{T}\bar{\xi}^*(\eth_{\mathrm{s}}(\mathfrak{f}^{\mathrm{s}}_t(x)))dt.$$

Combining this with (104), we thus have that

$$\varlimsup_{N\to\infty}\sup_{x\in\mathsf{E}\setminus\gamma_N}\left|(\breve{\mathcal{A}}^{N}_T\bar{\xi}^*)(x) - \daleth\right| \le \delta,$$

which gives the desired result. □

Recall that the point of averaging with respect to $\mathbf{p}^N$ was to average arbitrarily close to γ_N. Let's identify the limits of $\breve{\Phi}^\lambda_N$ at γ_N. Note that these limits depend on whether we are approaching γ_N from "above" (where $\mathsf{H}^{\mathrm{loc}}_N > 0$) or from "below" (where $\mathsf{H}^{\mathrm{loc}}_N < 0$). For each $N\in\mathbb{N}$ and $\lambda > 0$, we define

$$\breve{\Phi}^\lambda_{N,+}(x) \stackrel{\text{def}}{=} \lim_{\substack{x'\to x\\ x'\in\mathsf{E}\\ \mathsf{H}^{\mathrm{loc}}_N(x')>0}} \breve{\Phi}^\lambda_N(x') \qquad \text{for all } x\in\gamma_N\setminus\bigcup_{\ell\in\Lambda_P}\partial\text{Đ}_\ell,$$

$$\breve{\Phi}^\lambda_{N,-}(x) \stackrel{\text{def}}{=} \lim_{\substack{x'\to x\\ x'\in\mathsf{E}\\ \mathsf{H}^{\mathrm{loc}}_N(x')<0}} \breve{\Phi}^\lambda_N(x') \qquad \text{for all } x\in\gamma_N\setminus\bigcup_{\ell\in\Lambda_W}\partial\text{Đ}_\ell.$$

REMARK 11.6. We want to extend $\breve{\Phi}^\lambda_{N,+}$ and $\breve{\Phi}^\lambda_{N,-}$ back into $\mathcal{N}_N$ in a way which allows us to efficiently compute derivatives. This is not too hard away from $\mathfrak{X}$; starting at any point not too near any of the $\mathfrak{x}_\ell$'s, we can simply follow along integral curves of $\nabla\mathsf{H}^{\mathrm{loc}}_N$ until we hit γ_N, and evaluate $\breve{\Phi}^\lambda_{N,+}$ and

$\breve{\Phi}^{\lambda}_{N,-}$ at that point. This is essentially what we shall do in (114). The more complicated part is near the points of $\mathfrak{X}$; if we try to do the same near the $\mathfrak{x}_\ell$'s, we are faced with the fact that $\nabla \mathsf{H}^{\mathrm{loc}}_N$ degenerates at the $\mathfrak{x}_\ell$'s, and the integral curves of $\nabla \mathsf{H}^{\mathrm{loc}}_N$ undergo a bifurcation. Competing against this is the regularity of $\breve{\Phi}^{\lambda}_{N,+}$ and $\breve{\Phi}^{\lambda}_{N,-}$; if these functions are sufficiently flat near the $\mathfrak{x}_\ell$'s, the singularities in the integral curves of $\nabla \mathsf{H}^{\mathrm{loc}}_N$ will have no effect.

To get started, let's locally consider $\breve{\Phi}^{\lambda}_{N,+}$ and $\breve{\Phi}^{\lambda}_{N,-}$ near the $\mathfrak{x}_\ell$'s. For each $\ell \in \Lambda$, define $\breve{\Phi}^{\sim,\lambda}_{N,+,\ell}(x) \stackrel{\text{def}}{=} \breve{\Phi}^{\lambda}_{N,+}(\tilde{\phi}_\ell(x))$ and $\breve{\Phi}^{\sim,\lambda}_{N,-,\ell}(x) \stackrel{\text{def}}{=} \breve{\Phi}^{\lambda}_{N,-}(\tilde{\phi}_\ell(x))$ for all $x \in \phi_\ell(\mathcal{U}_\ell \cap \gamma_N)$, $\lambda \in (0,1)$, and $N \in \mathbb{N}$.

LEMMA 11.7. *There is a constant $K > 0$ such that*

$$\left|\left((\bar{\nabla}_e \tilde{\mathsf{H}})\breve{\Phi}^{\sim,\lambda}_{N,\mathrm{s},\ell}\right)(x)\right| \le \frac{K}{\lambda}\|x\|_e^2 \qquad \textit{and} \qquad \left|\left((\bar{\nabla}_e \tilde{\mathsf{H}})^2\breve{\Phi}^{\sim,\lambda}_{N,\mathrm{s},\ell}\right)(x)\right| \le \frac{K}{\lambda}\|x\|_e^2$$

for all $\mathrm{s} \in \{+,-\}$, $N \in \mathbb{N}$, $x \in \phi_\ell(\tilde{\mathcal{U}}_\ell \cap \gamma_N)$, *and all* $\lambda > 0$.

Proof. Define $\tilde{\boldsymbol{\sigma}}_\ell(x) \stackrel{\text{def}}{=} \boldsymbol{\sigma}(\tilde{\phi}_\ell(x))\tilde{\mathrm{B}}_\ell(x)$ for all $x \in \tilde{\mathcal{U}}$. Then Lemma 11.4 tells us that there is a constant $K > 0$ such that

$$\left|\left(\left(\frac{\bar{\nabla}_e \tilde{\mathsf{H}}}{\tilde{\boldsymbol{\sigma}}_\ell}\right)\breve{\Phi}^{\sim,\lambda}_{N,\mathrm{s},\ell}\right)(x)\right| \le \frac{K}{\lambda} \qquad \text{and} \qquad \left|\left(\left(\frac{\bar{\nabla}_e \tilde{\mathsf{H}}}{\tilde{\boldsymbol{\sigma}}_\ell}\right)^2\breve{\Phi}^{\sim,\lambda}_{N,\mathrm{s},\ell}\right)(x)\right| \le \frac{K}{\lambda}$$

for all $x \in \phi_\ell(\mathcal{U}_\ell \cap \gamma_N)$, $N \in \mathbb{N}$, $\lambda \in (0,1)$, and $\mathrm{s} \in \{+,-\}$. Using some simple calculations, we have that

$$\begin{aligned}
\left(\bar{\nabla}_e \tilde{\mathsf{H}}\right)\breve{\Phi}^{\sim,\lambda}_{N,\mathrm{s},\ell} &= \tilde{\boldsymbol{\sigma}}_\ell\left(\frac{\bar{\nabla}_e \tilde{\mathsf{H}}}{\tilde{\boldsymbol{\sigma}}_\ell}\right)\breve{\Phi}^{\sim,\lambda}_{N,\mathrm{s},\ell}, \\
\left(\bar{\nabla}_e \tilde{\mathsf{H}}\right)^2\breve{\Phi}^{\sim,\lambda}_{N,\mathrm{s},\ell} &= \tilde{\boldsymbol{\sigma}}_\ell^2\left(\frac{\bar{\nabla}_e \tilde{\mathsf{H}}}{\tilde{\boldsymbol{\sigma}}_\ell}\right)^2\breve{\Phi}^{\sim,\lambda}_{N,\mathrm{s},\ell} + (\bar{\nabla}_e \tilde{\mathsf{H}}, \nabla_e \tilde{\boldsymbol{\sigma}}_\ell)_e\left(\frac{\bar{\nabla}_e \tilde{\mathsf{H}}}{\tilde{\boldsymbol{\sigma}}_\ell}\right)\breve{\Phi}^{\sim,\lambda}_{N,\mathrm{s},\ell}.
\end{aligned}$$

We also note that there is a constant $K > 0$ such that $|\tilde{\boldsymbol{\sigma}}_\ell(x)| \le K\|x\|_e^2$ and $\left|(\bar{\nabla}_e \tilde{\mathsf{H}}, \nabla_e \tilde{\boldsymbol{\sigma}}_\ell)_e(x)\right| \le K\|x\|_e^2$ for all $x \in \tilde{\mathcal{U}}$. Combine things to get the stated result. □

Let's next construct the extension from γ_N to $\mathcal{N}_N$. We will do this in two ways, depending on whether or not we are close to the sets $\tilde{\phi}_\ell\{x \in \tilde{\mathcal{U}} : |\tilde{\mathsf{H}}(x)| = 0\}$. Define $\tilde{\mathbf{I}}(x) \stackrel{\text{def}}{=} \frac{1}{2}\{x_2^2 - x_1^2\}$ for all $x = (x_1, x_2) \in \mathbb{R}^2$, and note that

$$4\tilde{\mathbf{I}}^2(x) + 2\tilde{\mathsf{H}}^2(x) = x_1^4 + x_2^4 \tag{105}$$

for all $x = (x_1, x_2) \in \mathbb{R}^2$. Thus there is a $d > 0$ such that the set $\tilde{\Diamond} \stackrel{\text{def}}{=} \left\{x \in \mathbb{R}^2 : |\tilde{\mathbf{I}}(x)| < d \text{ and } |\tilde{\mathsf{H}}(x)| < d\right\}$ is contained in $\tilde{\mathcal{U}}$. Next, define $\maltese_N \stackrel{\text{def}}{=}$

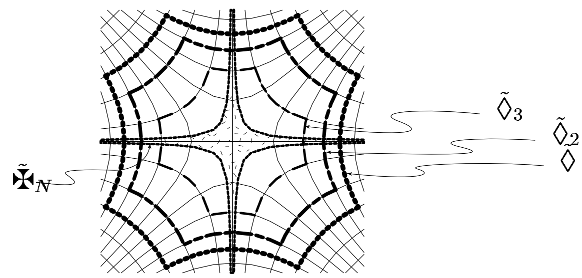


FIGURE 5. Hyperbolic Regions

$\left\{x \in \tilde{\Diamond} : |\tilde{\mathsf{H}}(x)| < \frac{\mathfrak{r}_N}{2}\right\}$. See Figure 5. This starts to formalize some of the thoughts of Remark 11.6. By "near" $\mathfrak{X}$, we mean in one of the $\tilde{\phi}_\ell(\tilde{\Diamond})$'s. The set $\tilde{\maltese}_N$ allows us to further decompose $\tilde{\Diamond}$ as $\tilde{\Diamond} = (\tilde{\Diamond} \setminus \tilde{\maltese}_N) \cup \tilde{\maltese}_N$; in the set $\tilde{\maltese}_N$, we are "very near" a point in $\mathfrak{X}$, while in $\tilde{\Diamond} \setminus \tilde{\maltese}_N$, we are "not very near" a point in $\mathfrak{X}$.

We first consider points which are "very near" $\mathfrak{X}$; i.e., points in $\tilde{\maltese}_N$. Define

$$\breve{\Phi}^{\sim,\lambda,e}_{N,\ell}(x) \overset{\text{def}}{=} \begin{cases} \breve{\Phi}^{\sim,\lambda}_{N,+,\ell}(x_1,0) + \breve{\Phi}^{\sim,\lambda}_{N,+,\ell}(0,x_2) - \breve{\Phi}^{\sim,\lambda}_{N,+,\ell}(\mathbf{0}_e) \\ \qquad \text{if } x \in \phi_\ell(\mathcal{N}_N \cap \mathcal{U}_\ell) \cap \tilde{\maltese}_N \text{ and } \mathsf{H}^{\text{loc}}_N(x) > 0 \\ \breve{\Phi}^{\sim,\lambda}_{N,-,\ell}(x_1,0) + \breve{\Phi}^{\sim,\lambda}_{N,-,\ell}(0,x_2) - \breve{\Phi}^{\sim,\lambda}_{N,-,\ell}(\mathbf{0}_e) \\ \qquad \text{if } x \in \phi_\ell(\mathcal{N}_N \cap \mathcal{U}_\ell) \cap \tilde{\maltese}_N \text{ and } \mathsf{H}^{\text{loc}}_N(x) < 0. \end{cases} \tag{106}$$

for all $\lambda \in (0,1)$, $N \in \mathbb{N}$, and $\ell \in \Lambda$.

LEMMA 11.8. *There is a $K > 0$ such that*

$$\max_{i\in\{1,2\}} \left|\frac{\partial \breve{\Phi}^{\sim,\lambda,e}_{N,\ell}}{\partial x_i}(x)\right| \le \frac{K}{\lambda}\|x\|_e \qquad \textit{and} \qquad \max_{i,j\in\{1,2\}} \left|\frac{\partial^2 \breve{\Phi}^{\sim,\lambda,e}_{N,\ell}}{\partial x_i \partial x_j}(x)\right| \le \frac{K}{\lambda}$$

for all $N \in \mathbb{N}$, $x = (x_1,x_2) \in \phi_\ell(\mathcal{N}_N \cap \mathcal{U}_\ell) \cap \tilde{\maltese}_N$, and $\lambda \in (0,1)$.

Proof. Fix $x = (x_1,x_2) \in \phi_\ell(\mathcal{N}_N \cap \mathcal{U}_\ell) \cap \tilde{\maltese}_N$. Let $\mathrm{s} = +$ if $\mathsf{H}^{\text{loc}}_N(x) > 0$, and let $\mathrm{s} = -$ if $\mathsf{H}^{\text{loc}}_N(x) < 0$. We have that

$$\frac{\partial \breve{\Phi}^{\sim,\lambda,e}_{N,\ell}}{\partial x_1}(x) = \frac{\partial \breve{\Phi}^{\sim,\lambda}_{N,\mathrm{s},\ell}}{\partial x_1}(x_1,0) = \frac{1}{x_1}((\bar{\nabla}_e \tilde{\mathsf{H}})\breve{\Phi}^{\sim,\lambda}_{N,\mathrm{s},\ell})(x_1,0),$$
$$\frac{\partial \breve{\Phi}^{\sim,\lambda,e}_{N,\ell}}{\partial x_2}(x) = \frac{\partial \breve{\Phi}^{\sim,\lambda}_{N,\mathrm{s},\ell}}{\partial x_2}(0,x_2) = -\frac{1}{x_2}((\bar{\nabla}_e \tilde{\mathsf{H}})\breve{\Phi}^{\sim,\lambda}_{N,\mathrm{s},\ell})(0,x_2),$$

$$\begin{aligned}\frac{\partial^2\breve{\Phi}^{\sim,\lambda,e}_{N,\ell}}{\partial x_1^2}(x) &= \frac{\partial^2\breve{\Phi}^{\sim,\lambda}_{N,\mathrm{s},\ell}}{\partial x_1^2}(x_1,0) = -\frac{1}{x_1^2}((\bar{\nabla}_e\tilde{\mathsf{H}})\breve{\Phi}^{\sim,\lambda}_{N,\mathrm{s},\ell})(x_1,0)\\ &\quad+\frac{1}{x_1^2}((\bar{\nabla}_e\tilde{\mathsf{H}})^2\breve{\Phi}^{\sim,\lambda}_{N,\mathrm{s},\ell})(x_1,0),\\ \frac{\partial^2\breve{\Phi}^{\sim,\lambda,e}_{N,\ell}}{\partial x_2^2}(x) &= \frac{\partial^2\breve{\Phi}^{\sim,\lambda}_{N,\mathrm{s},\ell}}{\partial x_2^2}(0,x_2) = \frac{1}{x_2^2}((\bar{\nabla}_e\tilde{\mathsf{H}})\breve{\Phi}^{\sim,\lambda}_{N,\mathrm{s},\ell})(0,x_2)\\ &\quad+\frac{1}{x_2^2}((\bar{\nabla}_e\tilde{\mathsf{H}})^2\breve{\Phi}^{\sim,\lambda}_{N,\mathrm{s},\ell})(0,x_2),\\ \frac{\partial^2\breve{\Phi}^{\sim,\lambda,e}_{N,\ell}}{\partial x_1\partial x_2}(x) &= 0.\end{aligned}$$

The bounds of Lemma 11.7 give us the stated result. □

Let's now consider points which are "not very near" $\mathfrak{X}$; i.e., points in $\tilde{\Diamond}\setminus\maltese_N$. Set $G_h(z)\stackrel{\text{def}}{=}\sqrt{\sqrt{z^2+h^2}-z}$ for all z and h in $\mathbb{R}$, (it is easy to see that this is well-defined for all z and h in $\mathbb{R}$) and set

$$\tilde{\wp}^{\circ}_h(x)\stackrel{\text{def}}{=}(\mathfrak{s}(x_1)G_h(\tilde{\mathbf{I}}(x)),\mathfrak{s}(x_2)G_h(-\tilde{\mathbf{I}}(x)))$$

for all $x=(x_1,x_2)\in\mathbb{R}^2$. Then

$$\tilde{\mathbf{I}}(\tilde{\wp}^{\circ}_h(x))=\tilde{\mathbf{I}}(x)\qquad\text{and}\qquad\tilde{\mathsf{H}}(\tilde{\wp}^{\circ}_h(x))=h \tag{107}$$

for all $x\in\mathbb{R}^2$ and $h\in\mathbb{R}$. For all $x\in\mathbb{R}^2$ and $N\in\mathbb{N}$, now define

$$J_N(x)\stackrel{\text{def}}{=}\left\lfloor\frac{\tilde{\mathsf{H}}(x)}{\mathsf{r}_N}+\frac{1}{2}\right\rfloor\mathsf{r}_N;$$

i.e., $J_N(x)=k\mathsf{r}_N$ if $(k-1/2)\mathsf{r}_N\le\tilde{\mathsf{H}}(x)<(k+1/2)\mathsf{r}_N$. Finally, define $\tilde{\wp}_N(x)\stackrel{\text{def}}{=}\tilde{\wp}^{\circ}_{J_N(x)}(x)$ for all $x\in\mathbb{R}^2$ and set

(108)

$$\breve{\Phi}^{\sim,\lambda,e}_{N,\ell}(x)\stackrel{\text{def}}{=}\begin{cases}\breve{\Phi}^{\sim,\lambda}_{N,+,\ell}(\tilde{\wp}_N(x)) & \text{if } x\in\phi_\ell(\mathcal{U}_\ell\cap\mathcal{N}_N)\cap\tilde{\Diamond}\setminus\maltese_N \text{ and } \mathsf{H}^{\mathrm{loc}}_N(x)>0,\\ \breve{\Phi}^{\sim,\lambda}_{N,-,\ell}(\tilde{\wp}_N(x)) & \text{if } x\in\phi_\ell(\mathcal{U}_\ell\cap\mathcal{N}_N)\cap\tilde{\Diamond}\setminus\maltese_N \text{ and } \mathsf{H}^{\mathrm{loc}}_N(x)<0\end{cases}$$

for all $\lambda\in(0,1)$, $N\in\mathbb{N}$, and $\ell\in\Lambda$. To understand the regularity of $\breve{\Phi}^{\sim,\lambda,e}_{N,\ell}$ on $\phi_\ell(\mathcal{U}_\ell\cap\mathcal{N}_N)\setminus\maltese_N$, let's first prove a general result.

LEMMA 11.9. *Fix $h\in\mathbb{R}\setminus\{0\}$ and $f\in C^2(\mathbb{R}^2)$. Define $F(x)\stackrel{\text{def}}{=}f(\tilde{\wp}^{\circ}_h(x))$ for all $x\in\mathbb{R}^2$. Then for all $x\in\mathbb{R}$,*

$$\max_{i\in\{1,2\}}\left|\frac{\partial F}{\partial x_i}(x)\right|\le\|x\|_e\frac{\left|\left((\bar{\nabla}_e\tilde{\mathsf{H}})f\right)(\tilde{\wp}^{\circ}_h(x))\right|}{\|\tilde{\wp}^{\circ}_h(x)\|_e^2},$$

$$\max_{i,j\in\{1,2\}}\left|\frac{\partial^2 F}{\partial x_i \partial x_j}(x)\right| \le \left\{1+2\frac{\|x\|_e^2}{\|\tilde{\wp}_h^\circ(x)\|_e^2}\right\}\frac{\left|\left((\bar{\nabla}_e\tilde{\mathsf{H}})f\right)(\tilde{\wp}_h^\circ(x))\right|}{\|\tilde{\wp}_h^\circ(x)\|_e^2} + \frac{\|x\|_e^2}{\|\tilde{\wp}_h^\circ(x)\|_e^2}\frac{\left|\left((\bar{\nabla}_e\tilde{\mathsf{H}})^2 f\right)(\tilde{\wp}_h^\circ(x))\right|}{\|\tilde{\wp}_h^\circ(x)\|_e^2}.$$

We also have that

$$((\bar{\nabla}_e\tilde{\mathsf{H}})F)_e(x) = \frac{\|x\|_e^2}{\|\tilde{\wp}_h^\circ(x)\|_e^2}(\bar{\nabla}_e\tilde{\mathsf{H}}, \nabla_e f)_e(\tilde{\wp}_h^\circ(x)) \tag{109}$$

for all $x \in \mathbb{R}^2$.

Proof. First note that $(\nabla_e\tilde{\mathsf{H}}, \nabla_e\tilde{\mathbf{I}})_e \equiv 0$, $\nabla_e\tilde{\mathbf{I}} = -\bar{\nabla}_e\tilde{\mathsf{H}}$, and $\|\nabla_e\tilde{\mathbf{I}}\|_e^2 = \mathbf{n}$. For any $x \in \mathbb{R}^2$ and $X \in T_x\mathbb{R}^2$,

$$\begin{aligned}
T\tilde{\wp}_h^\circ X &= \frac{\left(T\tilde{\wp}_h^\circ X, \nabla_e\tilde{\mathsf{H}}(\tilde{\wp}_h^\circ(x))\right)_e}{\|\nabla_e\tilde{\mathsf{H}}(\tilde{\wp}_h^\circ(x))\|_e^2}\nabla_e\tilde{\mathsf{H}}(\tilde{\wp}_h^\circ(x)) \\
&\quad + \frac{\left(T\tilde{\wp}_h^\circ X, \nabla_e\tilde{\mathbf{I}}(\tilde{\wp}_h^\circ(x))\right)_e}{\|\nabla_e\tilde{\mathbf{I}}(\tilde{\wp}_h^\circ(x))\|_e^2}\nabla_e\tilde{\mathbf{I}}(\tilde{\wp}_h^\circ(x)) \\
&= \frac{X(\tilde{\mathsf{H}}\circ\tilde{\wp}_h^\circ)}{\|\nabla_e\tilde{\mathsf{H}}(\tilde{\wp}_h^\circ(x))\|_e^2}\nabla_e\tilde{\mathsf{H}}(\tilde{\wp}_h^\circ(x)) - \frac{X(\tilde{\mathbf{I}}\circ\tilde{\wp}_h^\circ)}{\|\nabla_e\tilde{\mathbf{I}}(\tilde{\wp}_h^\circ(x))\|_e^2}\bar{\nabla}_e\tilde{\mathsf{H}}(\tilde{\wp}_h^\circ(x)) \\
&= -\frac{X\tilde{\mathbf{I}}}{\mathbf{n}(\tilde{\wp}_h^\circ(x))}\bar{\nabla}_e\tilde{\mathsf{H}}(\tilde{\wp}_h^\circ(x)) = -\frac{(X, \nabla_e\tilde{\mathbf{I}}(x))_e}{\mathbf{n}(\tilde{\wp}_h^\circ(x))}\bar{\nabla}_e\tilde{\mathsf{H}}(\tilde{\wp}_h^\circ(x)).
\end{aligned}$$

Thus for any $x \in \mathbb{R}^2$, $T\tilde{\wp}_h^\circ\bar{\nabla}_e\tilde{\mathbf{I}} \equiv 0$ and

$$\begin{aligned}
T\tilde{\wp}_h^\circ\bar{\nabla}_e\tilde{\mathsf{H}}(x) = -T\tilde{\wp}_h^\circ\nabla_e\tilde{\mathbf{I}}(x) &= \frac{\|\nabla_e\tilde{\mathbf{I}}(x)\|^2}{\mathbf{n}(\tilde{\wp}_h^\circ(x))}\bar{\nabla}_e\tilde{\mathsf{H}}(\tilde{\wp}_h^\circ(x)) \\
&= \frac{\mathbf{n}(x)}{\mathbf{n}(\tilde{\wp}_h^\circ(x))}\bar{\nabla}_e\tilde{\mathsf{H}}(\tilde{\wp}_h^\circ(x)).
\end{aligned}$$

Thus for all $x = (x_1, x_2) \in \mathbb{R}^2$,

$$\begin{aligned}
x_2\frac{\partial F}{\partial x_1}(x) + x_1\frac{\partial F}{\partial x_2}(x) &= (\bar{\nabla}_e\tilde{\mathbf{I}}, \nabla_e F)_e(x) = (T\tilde{\wp}_h^\circ\bar{\nabla}_e\tilde{\mathbf{I}}(x), \nabla_e f(\tilde{\wp}_h^\circ(x)))_e = 0, \\
x_1\frac{\partial F}{\partial x_1}(x) - x_2\frac{\partial F}{\partial x_2}(x) &= (\bar{\nabla}_e\tilde{\mathsf{H}}, \nabla_e F)_e(x) = (T\tilde{\wp}_h^\circ\bar{\nabla}_e\tilde{\mathsf{H}}(x), \nabla_e f(\tilde{\wp}_h^\circ(x)))_e \\
&= \frac{\mathbf{n}(x)}{\mathbf{n}(\tilde{\wp}_h^\circ(x))}(\bar{\nabla}_e\tilde{\mathsf{H}}, \nabla_e f)_e(\tilde{\wp}_h^\circ(x)).
\end{aligned}$$

The second equation is exactly (109). We can simultaneously solve these equations for $\frac{\partial F}{\partial x_1}$ and $\frac{\partial F}{\partial x_2}$. We get that

$$
\begin{aligned}
\frac{\partial F}{\partial x_1}(x) &= \frac{x_1}{x_1^2+x_2^2}\frac{\mathbf{n}(x)}{\mathbf{n}(\tilde{\wp}_h(x))}(\bar{\nabla}_e\tilde{\mathsf{H}},\nabla_e f)_e(\tilde{\wp}_h^\circ(x)) \\
&= x_1\frac{((\bar{\nabla}_e\tilde{\mathsf{H}})f)(\tilde{\wp}_h^\circ(x))}{\mathbf{n}(\tilde{\wp}_h(x))}, \\
\frac{\partial F}{\partial x_2}(x) &= -\frac{x_2}{x_1^2+x_2^2}\frac{\mathbf{n}(x)}{\mathbf{n}(\tilde{\wp}_h(x))}(\bar{\nabla}_e\tilde{\mathsf{H}},\nabla_e f)_e(\tilde{\wp}_h^\circ(x)) \\
&= -x_2\frac{((\bar{\nabla}_e\tilde{\mathsf{H}})f)(\tilde{\wp}_h^\circ(x))}{\mathbf{n}(\tilde{\wp}_h(x))}.
\end{aligned}
\tag{110}
$$

Note that $(\bar{\nabla}_e\tilde{\mathsf{H}},\nabla_e\mathbf{n})_e = -4\tilde{\mathbf{I}}$; hence

$$
\left(\left(\frac{\bar{\nabla}_e\tilde{\mathsf{H}}}{\mathbf{n}}\right)^2 f\right) = \left\{\frac{(\bar{\nabla}_e\tilde{\mathsf{H}})^2 f}{\mathbf{n}^2} - \frac{(\bar{\nabla}_e\tilde{\mathsf{H}},\nabla_e\mathbf{n})_e}{\mathbf{n}^3}(\bar{\nabla}_e\tilde{\mathsf{H}})f\right\}
$$
$$
= \left\{\frac{(\bar{\nabla}_e\tilde{\mathsf{H}})^2 f}{\mathbf{n}^2} + 4\frac{\tilde{\mathbf{I}}}{\mathbf{n}^3}(\bar{\nabla}_e\tilde{\mathsf{H}})f\right\}.
$$

Differentiating (110) again, we get that

$$
\begin{aligned}
\frac{\partial^2 F}{\partial x_1^2}(x) &= \frac{((\bar{\nabla}_e\tilde{\mathsf{H}})f)(\tilde{\wp}_h^\circ(x))}{\mathbf{n}(\tilde{\wp}_h^\circ(x))} + \frac{x_1^2((\bar{\nabla}_e\tilde{\mathsf{H}})^2 f)(\tilde{\wp}_h^\circ(x))}{\mathbf{n}^2(\tilde{\wp}_h^\circ(x))} \\
&\quad + 4\frac{x_1^2\tilde{\mathbf{I}}(\tilde{\wp}_h^\circ(x))((\bar{\nabla}_e\tilde{\mathsf{H}})f)(\tilde{\wp}_h^\circ(x))}{\mathbf{n}^3(\tilde{\wp}_h^\circ(x))}, \\
\frac{\partial^2 F}{\partial x_2^2}(x) &= -\frac{((\bar{\nabla}_e\tilde{\mathsf{H}})f)(\tilde{\wp}_h^\circ(x))}{\mathbf{n}(\tilde{\wp}_h^\circ(x))} + \frac{x_2^2((\bar{\nabla}_e\tilde{\mathsf{H}})^2 f)(\tilde{\wp}_h^\circ(x))}{\mathbf{n}^2(\tilde{\wp}_h^\circ(x))} \\
&\quad + 4\frac{x_2^2\tilde{\mathbf{I}}(\tilde{\wp}_h^\circ(x))((\bar{\nabla}_e\tilde{\mathsf{H}})f)(\tilde{\wp}_h^\circ(x))}{\mathbf{n}^3(\tilde{\wp}_h^\circ(x))}, \\
\frac{\partial^2 F}{\partial x_1\partial x_2}(x) &= -\frac{x_1x_2((\bar{\nabla}_e\tilde{\mathsf{H}})^2 f)(\tilde{\wp}_h^\circ(x))}{\mathbf{n}^2(\tilde{\wp}_h^\circ(x))} - 4\frac{x_1x_2\tilde{\mathbf{I}}(\tilde{\wp}_h^\circ(x))((\bar{\nabla}_e\tilde{\mathsf{H}})f)(\tilde{\wp}_h^\circ(x))}{\mathbf{n}^3(\tilde{\wp}_h^\circ(x))}.
\end{aligned}
$$

Combine things to get the desired result, noting that $4|\tilde{\mathbf{I}}(x)| \le 2\mathbf{n}(x)$ for all $x \in \mathbb{R}^2$. □

We now have

LEMMA 11.10. *There is a* $K > 0$ *such that*

$$
\max_{i\in\{1,2\}}\left|\frac{\partial\breve{\Phi}^{\sim,\lambda,e}_{N,\ell}}{\partial x_i}(x)\right| \le \frac{K}{\lambda}\|x\|_e \qquad \text{and} \qquad \max_{i,j\in\{1,2\}}\left|\frac{\partial^2\breve{\Phi}^{\sim,\lambda,e}_{N,\ell}}{\partial x_i\partial x_j}(x)\right| \le \frac{K}{\lambda}
$$

for all $N \in \mathbb{N}$, $x = (x_1, x_2) \in \phi_\ell(\mathcal{N}_N \cap \mathcal{U}_\ell) \cap \tilde{\Diamond} \setminus \tilde{\maltese}_N$, *and* $\lambda \in (0,1)$, *and such that*

$$\frac{1}{K} \le \frac{\|x\|_e}{\|\tilde{\wp}_N(x)\|_e} \le K \tag{111}$$

for all $x \in \tilde{\Diamond} \setminus \tilde{\maltese}_N$ *and all* $N \in \mathbb{N}$.

Proof. The results stem from combining Lemmas 11.7 and 11.9. We only need to bound $\|x\|/\|\tilde{\wp}_N(x)\|$ for $x \in \tilde{\Diamond} \setminus \tilde{\maltese}_N$. Set $\mathbf{n}_4(x) \overset{\text{def}}{=} x_1^4 + x_2^4$ for all $x = (x_1, x_2) \in \mathbb{R}^2$. Fix $x \in \tilde{\Diamond} \setminus \tilde{\maltese}_N$. From (105) and (107), we have that

$$\frac{\mathbf{n}_4(x)}{\mathbf{n}_4(\tilde{\wp}_N(x))} = \frac{4\tilde{\mathbf{I}}^2(x) + 2\tilde{\mathsf{H}}^2(x)}{4\tilde{\mathbf{I}}^2(\tilde{\wp}_N(x)) + 2\tilde{\mathsf{H}}^2(\tilde{\wp}_N(x))} = \frac{4\tilde{\mathbf{I}}^2(x) + 2\tilde{\mathsf{H}}^2(x)}{4\tilde{\mathbf{I}}^2(x) + 2J_N^2(x)}.$$

We next compute that since $x \notin \tilde{\maltese}_N$, by definition $|\tilde{\mathsf{H}}(x)| > \mathsf{r}_N/2$ and hence $|J_N(x)| \ge \mathsf{r}_N$. We can furthermore compute that

$$|J_N(x)| \ge \left\{\frac{|\tilde{\mathsf{H}}(x)|}{\mathsf{r}_N} - \frac{1}{2}\right\}\mathsf{r}_N = |\tilde{\mathsf{H}}(x)| - \frac{\mathsf{r}_N}{2},$$
$$|J_N(x)| \le \left\{\frac{|\tilde{\mathsf{H}}(x)|}{\mathsf{r}_N} + \frac{1}{2}\right\}\mathsf{r}_N = |\tilde{\mathsf{H}}(x)| + \frac{\mathsf{r}_N}{2},$$

so in fact

$$|\tilde{\mathsf{H}}(x)| \le |J_N(x)| + \frac{\mathsf{r}_N}{2} \le \frac{3}{2}|J_N(x)| \quad \text{and} \quad |\tilde{\mathsf{H}}(x)| \ge |J_N(x)| - \frac{\mathsf{r}_N}{2} \ge \frac{1}{2}|J_N(x)|.$$

Hence $\mathbf{n}_4(x)/\mathbf{n}_4(\tilde{\wp}_N(x)) \in [1/4, 9/4]$, and the claimed result follows by recalling that all norms on $\mathbb{R}^2$ are equivalent. □

The formulae of (106) and (108) extend $\breve{\Phi}^\lambda_{N,\pm}$ from γ_N into $\mathcal{N}_N$ near the $\mathfrak{x}_\ell$'s (more precisely, in the $\tilde{\phi}_\ell(\tilde{\Diamond})$'s). For any point not near one of the $\mathfrak{x}_\ell$'s, we can follow an integral curve of $\nabla \mathsf{H}^{\text{loc}}_N$ until we hit γ_N, and evaluate $\breve{\Phi}^\lambda_{N,+}$ and $\breve{\Phi}^\lambda_{N,-}$ at that point. To combine all of our extensions in a smooth way, we construct a *retract*.

First, we need the following regularity result.

LEMMA 11.11. *There is a constant* $K > 0$ *and a* $\lambda_\circ \in (0,1)$, *and, for each* $\lambda \in (0,1)$, *an* $N_\lambda \in \mathbb{N}$ *such that*

$$\inf_{\substack{\ell \in \Lambda \\ x \in \phi_\ell(\mathcal{N}_N \cap \mathcal{U}_\ell)}} \frac{(\bar{\nabla}_e \tilde{\mathsf{H}}, \nabla_e \breve{\Phi}^{\sim,\lambda,e}_{N,\ell})_e(x)}{\|x\|_e^2} \ge \frac{1}{K}$$
$$\text{or} \quad \inf_{\substack{\ell \in \Lambda \\ x \in \phi_\ell(\mathcal{N}_N \cap \mathcal{U}_\ell)}} \frac{-(\nabla_e \tilde{\mathsf{H}}, \nabla_e \breve{\Phi}^{\sim,\lambda,e}_{N,\ell})_e(x)}{\|x\|_e^2} \ge \frac{1}{K},$$

for all $N \in \mathbb{N}$ *greater than* N_λ.

Proof. Lemma 11.4 and the fact that ξ^* vanishes in the $\mathcal{U}_\ell$'s implies that

$$\left|(\text{Ч}_N\breve{\Phi}^\lambda_N)(x) - \gimel\boldsymbol{\sigma}(x)\right| \le \boldsymbol{\sigma}(x)\int_{t=0}^\infty te^{-t}\left|(\breve{\mathcal{A}}^N_{t/\lambda}\bar{\xi}^*)(x) - \gimel\right| dt$$

for all $\lambda \in (0,1)$, $N \in \mathbb{N}$, and $x \in \bigcup_{\ell\in\Lambda}\phi_\ell(\mathcal{N}_N \cap \mathcal{U}_\ell)$. We now use Lemma 11.5. Thus there is $K > 0$ and a $\lambda_\circ \in (0,1)$, and for each $\lambda \in (0,\lambda_\circ)$, an $N_\lambda \in \mathbb{N}$ such that

$$\inf_{\substack{\ell\in\Lambda \\ x\in\phi_\ell(\mathcal{N}_N\cap\mathcal{U}_\ell)}} \frac{(\text{Ч}_N\breve{\Phi}^\lambda_N)(\tilde{\phi}_\ell(x))}{\|x\|_e^2} \ge \frac{1}{K} \qquad \text{if } \gimel > 0,$$

$$\inf_{\substack{\ell\in\Lambda \\ x\in\phi_\ell(\mathcal{N}_N\cap\mathcal{U}_\ell)}} \frac{-(\text{Ч}_N\breve{\Phi}^\lambda_N)(\tilde{\phi}_\ell(x))}{\|x\|_e^2} \ge \frac{1}{K} \qquad \text{if } \gimel < 0$$

for all $N \in \mathbb{N}$ greater than N_λ. We next use (21) to relate Ч_N to $\bar{\nabla}_e\tilde{\mathsf{H}}$. The final observation is that for all $\lambda \in (0,\lambda_\circ)$ and $N \ge N_0$,

$$((\bar{\nabla}_e\tilde{\mathsf{H}})\breve{\Phi}^{\sim,\lambda,e}_{N,\ell})(x) = ((\bar{\nabla}_e\tilde{\mathsf{H}})\breve{\Phi}^{\sim,\lambda}_{N,\mathrm{s},\ell})(x_1,0) + ((\bar{\nabla}_e\tilde{\mathsf{H}})\breve{\Phi}^{\sim,\lambda}_{N,\mathrm{s},\ell})(0,x_2) \tag{112}$$

for all $x = (x_1,x_2) \in \phi_\ell(\mathcal{N}_N \cap \mathcal{U}_\ell) \cap \maltese_N$, and, by (109),

$$((\bar{\nabla}_e\tilde{\mathsf{H}})\breve{\Phi}^{\sim,\lambda,e}_{N,\ell})(x) = \frac{\|x\|_e^2}{\|\tilde{\wp}_N(x)\|_e^2}((\bar{\nabla}_e\tilde{\mathsf{H}})\breve{\Phi}^{\sim,\lambda}_{N,\mathrm{s},\ell})(\tilde{\wp}_N(x)) \tag{113}$$

if $x \in \phi_\ell(\mathcal{N}_N \cap \mathcal{U}_\ell) \cap \tilde{\Diamond} \setminus \maltese_N$, where in both (112) and (113) we take $\mathrm{s} = +$ if $\mathsf{H}^{\text{loc}}_N(x) > 0$, and $\mathrm{s} = -$ if $\mathsf{H}^{\text{loc}}_N(x) < 0$. To bound this last expression, We use (111). □

Let's next define a vector field. Fix $\lambda \in (0,\lambda_\circ)$ and $N \ge N_\lambda$. For each $\ell \in \Lambda$, define

$$\text{ж}^\lambda_{N,\ell}(x) \overset{\text{def}}{=} -\frac{(T\phi_\ell\nabla\mathsf{H}^{\text{loc}}_N(x), \nabla_e\breve{\Phi}^{\sim,\lambda,e}_{N,\ell}(\phi_\ell(x)))_e\tilde{\mathrm{B}}_\ell(\phi_\ell(x))}{(\bar{\nabla}_e\tilde{\mathsf{H}}, \nabla_e\breve{\Phi}^{\sim,\lambda,e}_{N,\ell})_e(\phi_\ell(x))}$$

for all $x \in \mathcal{N}_N \cap \mathcal{U}_\ell$. For each $\ell \in \Lambda$, Let $\mathfrak{C}_\ell \in C^\infty(\mathbb{T})$ be such that $\mathfrak{C}_\ell(x) = 1$ if $x \in \tilde{\phi}_\ell(\tilde{\Diamond})$ and $\mathfrak{C}_\ell(x) = 0$ if $x \notin \mathcal{U}_\ell$. Define next the vector field

$$\text{Ӝ}^\lambda_N(x) \overset{\text{def}}{=} \frac{\nabla\mathsf{H}^{\text{loc}}_N(x)}{\|\nabla\mathsf{H}^{\text{loc}}_N(x)\|^2} + \sum_{\ell\in\Lambda}\mathfrak{C}_\ell(x)\text{ж}^\lambda_{N,\ell}(x)\frac{\bar{\nabla}\mathsf{H}^{\text{loc}}_N(x)}{\|\nabla\mathsf{H}^{\text{loc}}_N(x)\|^2}$$

for all $x \in \mathcal{N}_N$. Note that $\text{Ӝ}^\lambda_N\mathsf{H}^{\text{loc}}_N \equiv 1$ on $\mathcal{N}_N$ and $\text{Ӝ}^\lambda_N(\breve{\Phi}^{\sim,\lambda,e}_{N,\ell} \circ \phi_\ell) \equiv 0$ on $\mathcal{N}_N \cap \tilde{\phi}_\ell(\tilde{\Diamond})$.

For each $x \in \mathcal{N}_N$, now let $\{\wp^{\lambda,*}_{N,t}(x); t \in I_x\}$ be the maximal solution of the ODE

$$\begin{aligned} \dot{\wp}^{\lambda,*}_{N,t}(x) &= -\mathsf{H}^{\text{loc}}_N(x)\text{Ӝ}^\lambda_N(\wp^{\lambda,*}_{N,t}(x)), \qquad t \in I_x, \\ \wp^{\lambda,*}_{N,0}(x) &= x. \end{aligned}$$

It is easy to see that for all $x \in \mathcal{N}_N$, $[0,1) \subset I_x$, and the limit $\wp^\lambda_N(x) \overset{\text{def}}{=} \lim_{t\nearrow 1} \wp^{\lambda,*}_{N,t}(x)$ exists (see [Sow05, Appendix A]). We finally define

$$\breve{\Phi}^{\mathbf{P};\lambda}_N(x) \overset{\text{def}}{=} \begin{cases} \breve{\Phi}^\lambda_{N,+}(\wp^\lambda_N(x)) & \text{if } \mathsf{H}^{\text{loc}}_N(x) > 0 \\ \breve{\Phi}^\lambda_{N,-}(\wp^\lambda_N(x)) & \text{if } \mathsf{H}^{\text{loc}}_N(x) < 0. \end{cases} \tag{114}$$

We then have

Proof of Proposition 8.5. First, let's bound $\breve{\Phi}^{\mathbf{P};\lambda}_N$ and its derivatives. The bounds on the size of $\breve{\Phi}^{\mathbf{P};\lambda}_N$ come directly from those on $\breve{\Phi}^\lambda_{N,+}$ and $\breve{\Phi}^\lambda_{N,-}$.

To start our analysis of the derivatives of $\breve{\Phi}^{\mathbf{P};\lambda}_N$, define

$$\tilde{\Diamond}_2 \overset{\text{def}}{=} \left\{x \in \mathbb{R}^2 : |\tilde{\mathbf{I}}(x)| < d/2 \text{ and } |\tilde{\mathsf{H}}(x)| < d/2\right\}$$

and $\tilde{\Diamond}_3 \overset{\text{def}}{=} \left\{x \in \mathbb{R}^2 : |\tilde{\mathbf{I}}(x)| < d/4 \text{ and } |\tilde{\mathsf{H}}(x)| < d/4\right\}$ (see Figure 5). We shall show that if $x \in \mathcal{N}_N \cap \tilde{\phi}_\ell(\tilde{\Diamond}_2)$, then $\{\wp^{\lambda,*}_{N,t}(x);\, t \in [0,1)\}$ is contained in $\mathcal{N}_N \cap \tilde{\phi}_\ell(\tilde{\Diamond})$, while if $x \in \mathcal{N}_N \setminus \bigcup_{\ell\in\Lambda} \tilde{\phi}_\ell(\tilde{\Diamond}_2)$, then $\{\wp^{\lambda,*}_{N,t}(x);\, t \in [0,1)\}$ is contained in $\mathcal{N}_N \setminus \bigcup_{\ell\in\Lambda} \tilde{\phi}_\ell(\tilde{\Diamond}_3)$.

Combining Lemmas 11.8, 11.10, and 11.11, we see that there is a constant $K_1 > 0$ such that

$$\begin{aligned} &\sup\left\{|\text{ж}^\lambda_{N,\ell}(x)| : x \in \mathcal{N}_N \cap \mathcal{U}_\ell,\, \ell \in \Lambda\right\} \le \frac{K_1}{\lambda}, \\ &\sup\left\{\|D\text{ж}^\lambda_{N,\ell}(x)\| : x \in (\mathcal{N}_N \cap \mathcal{U}_\ell) \setminus \tilde{\phi}_\ell(\tilde{\Diamond}_3),\, \ell \in \Lambda\right\} \le \frac{K_1}{\lambda^2}, \\ &\sup\left\{\|D^2\text{ж}^\lambda_{N,\ell}(x)\| : x \in (\mathcal{N}_N \cap \mathcal{U}_\ell) \setminus \tilde{\phi}_\ell(\tilde{\Diamond}_3),\, \ell \in \Lambda\right\} \le \frac{K_1}{\lambda^3} \end{aligned} \tag{115}$$

for all $\lambda \in (0,\lambda_\circ)$ and $N \ge N_\lambda$. From the first of these bounds, we know that there is a $K_2 > 0$ such that $|(\text{Ӂ}^\lambda_N(\tilde{\mathbf{I}} \circ \phi_\ell))(x)| \le K_2/\lambda$ for all $x \in (\mathcal{N}_N \cap \mathcal{U}_\ell)$, $\ell \in \Lambda$, $\lambda \in (0,\lambda_\circ)$ and $N \ge N_\lambda$. Fix now $\lambda \in (0,\lambda_\circ)$ and let $N'_\lambda \ge N_\lambda$ be such that $(1 + K_2/\lambda)\mathsf{r}_N < d/4$ for all $N \ge N'_\lambda$.

First consider points in $\mathcal{N}_N \cap \bigcup_{\ell\in\Lambda} \tilde{\phi}_\ell(\tilde{\Diamond}_2)$. Fix $\ell \in \Lambda$ and $x \in \mathcal{N}_N \cap \tilde{\phi}_\ell(\tilde{\Diamond}_2)$ and let $t' \in [0,1]$ be such that $\{\wp^{\lambda,*}_{N,t}(x);\, 0 \le t < t'\} \subset \tilde{\phi}_\ell(\tilde{\Diamond})$. Then

$$\begin{aligned} \left|\tilde{\mathbf{I}}(\phi_\ell(\wp^{\lambda,*}_{N,t}(x)))\right| &\le |\tilde{\mathbf{I}}(\phi_\ell(x))| + (K_2/\lambda)|\mathsf{H}_{\text{loc}}(x)| < \frac{d}{2} + (K_2/\lambda)\mathsf{r}_N < d, \\ \left|\tilde{\mathsf{H}}(\phi_\ell(\wp^{\lambda,*}_{N,t}(x)))\right| &\le |\tilde{\mathsf{H}}(\phi_\ell(x))| + |\mathsf{H}_{\text{loc}}(x)| < \frac{d}{2} + \mathsf{r}_N < d \end{aligned}$$

for all $t \in [0,t')$. Standard arguments thus imply that $\{\wp^{\lambda,*}_{N,t}(x);\, 0 \le t \le 1\}$ is contained in $\overline{\tilde{\phi}_\ell(\tilde{\Diamond})}$, which is in turn contained in $\{x \in \mathbb{T} : \mathfrak{C}_\ell(x) = 1\}$, so $\breve{\Phi}^{\mathbf{P};\lambda}_N(x) = \breve{\Phi}^{\sim;\lambda,e}_{N,\ell}(\phi_\ell(x))$. In light of Lemmas 11.8 and 11.10, this implies the stated regularity of $\breve{\Phi}^{\mathbf{P};\lambda}_N$ on $\mathcal{N}_N \cap \tilde{\phi}_\ell(\tilde{\Diamond}_2)$.

Consider next points in $\mathcal{N}_N \setminus \bigcup_{\ell\in\Lambda} \tilde{\phi}_\ell(\tilde{\Diamond}_2)$. Fix first $\ell \in \Lambda$ and $x \in \mathcal{N}_N \cap \tilde{\phi}_\ell(\tilde{\Diamond} \setminus \tilde{\Diamond}_2)$, and let $t' \in [0,1]$ be such that $\{\wp^{\lambda;*}_{N,t}(x);\, 0 \le t < t'\} \subset \tilde{\phi}_\ell(\tilde{\Diamond})$. Then

$$\begin{aligned}\left|\tilde{\mathbf{I}}(\phi_\ell(\wp^{\lambda;*}_{N,t}(x)))\right| &\ge |\tilde{\mathbf{I}}(\phi_\ell(x))| - (K_2/\lambda)|\mathsf{H}_{\mathrm{loc}}(x)| \ge |\tilde{\mathbf{I}}(\phi_\ell(x))| - (K_2/\lambda)\mathsf{r}_N \\ &> |\tilde{\mathbf{I}}(\phi_\ell(x))| - d/4, \\ \left|\tilde{\mathsf{H}}(\phi_\ell(\wp^{\lambda;*}_{N,t}(x)))\right| &\ge |\tilde{\mathsf{H}}(\phi_\ell(x))| - |\mathsf{H}_{\mathrm{loc}}(x)| > |\tilde{\mathsf{H}}(\phi_\ell(x))| - \mathsf{r}_N \\ &> |\tilde{\mathsf{H}}(\phi_\ell(x))| - d/4\end{aligned}$$

for all $t \in [0, t')$. Thus $\{\wp^{\lambda;*}_{N,t}(x);\, 0 \le t \le t'\} \subset \mathcal{N}_N \setminus \bigcup_{\ell\in\Lambda} \tilde{\phi}_\ell(\tilde{\Diamond}_3)$. Noting that for any $x \in \mathcal{N}_N$ and $t \in [0,1)$, $\wp^{\lambda}_N(x) = \wp^{\lambda}_N(\wp^{\lambda;*}_{N,t}(x))$, we conclude that if $x \in \mathcal{N}_N \setminus \bigcup_{\ell\in\Lambda} \tilde{\phi}_\ell(\tilde{\Diamond}_2)$, then $\{\wp^{\lambda;*}_{N,t}(x);\, 0 \le t \le 1\}$ is contained in $\mathcal{N}_N \setminus \bigcup_{\ell\in\Lambda} \tilde{\phi}_\ell(\tilde{\Diamond}_3)$. Thus, if $x \in \mathcal{N}_N \setminus \bigcup_{\ell\in\Lambda} \tilde{\phi}_\ell(\tilde{\Diamond}_2)$, we can use (115) to bound the behavior of $ж^{\lambda}_{N,\ell}$ and its derivatives along $\{\wp^{\lambda;*}_{N,t}(x);\, 0 \le t \le 1\}$. Standard results on derivative flows of ODE's now allow us to extract bounds on the first and second derivatives of $\breve{\Phi}^{\mathsf{p},\lambda}_N$ on $\mathcal{N}_N \setminus \bigcup_{\ell\in\Lambda} \tilde{\phi}_\ell(\tilde{\Diamond}_3)$ from bounds on the first and second derivatives of $\breve{\Phi}^{\lambda}_{N,+}$ and $\breve{\Phi}^{\lambda}_{N,-}$. Note that derivative flows are linear, so their bounds are exponential in time and the coefficients.

Finally, we turn to (66). We first note that there is a constant $K_3 > 0$ such that for any $f \in C^1(\mathbb{T})$,

$$|f(\wp^{\lambda}_N(x)) - f(x)| \le \frac{K_3 \mathsf{r}_N}{\lambda} \sup_{x\in\mathsf{E}\setminus\gamma_N} \|\nabla f(x)\| \tag{116}$$

for all $x \in \mathcal{N}_N$, $\lambda \in (0, \lambda_\circ)$ and $N \ge N'_\lambda$. The $1/\lambda$ term comes from the first bound of (115). For any $\lambda \in (0,\lambda_\circ)$ and $N \ge N'_\lambda$ and any $x \in \mathcal{N}_N$,

$$\begin{aligned}\left|(\text{Ч}\breve{\Phi}^{\lambda,e}_N)(x) - \{\xi^*(x) - \beth\boldsymbol{\sigma}(x)\}\right| &\le |\boldsymbol{\nu}_N| \left|(\hat{\text{Ч}}_N\breve{\Phi}^{\mathsf{p},\lambda}_N)(x)\right| \\ &+ \left|(\text{Ч}_N\breve{\Phi}^{\mathsf{p},\lambda}_N)(x) - (\text{Ч}_N\breve{\Phi}^{\lambda}_{N,\mathrm{s}})(\wp^{\lambda}_N(x))\right| + \left|\xi^*(x) - \xi^*(\wp^{\lambda}_N(x))\right| \\ &+ |\beth|\left|\boldsymbol{\sigma}(x) - \boldsymbol{\sigma}(\wp^{\lambda}_N(x))\right| \\ &+ \left|(\text{Ч}_N\breve{\Phi}^{\lambda}_{N,\mathrm{s}})(\wp^{\lambda}_N(x)) - \left\{\xi^*(\wp^{\lambda}_N(x)) - \beth\boldsymbol{\sigma}(\wp^{\lambda}_N(x))\right\}\right|.\end{aligned}$$

where we use $\mathrm{s} = +$ if $\mathsf{H}^{\mathrm{loc}}_N(x) > 0$ and $\mathrm{s} = -$ if $\mathsf{H}^{\mathrm{loc}}_N(x) < 0$. The first three terms can be bounded by (116) and the bounds on the regularity of $\breve{\Phi}^{N,\mathrm{s}}$, and the last by combining Lemmas 11.4 and 11.5. □

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Department of Mathematics, University of Illinois at Urbana–Champaign, Urbana, IL 61801, USA

E-mail address: r-sowers@math.uiuc.edu