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A NOTE ON LOWER BOUNDS OF MARTINGALE MEASURE DENSITIES

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Dedicated to the memory of Joe Doob

ABSTRACT. For a given element $f \in L^1$ and a convex cone $C \subset L^\infty$, $C \cap L^\infty_+ = \{0\}$, we give necessary and sufficient conditions for the existence of an element $g \geq f$ lying in the polar of C. This polar is taken in $(L^\infty)^*$ and in L^1. In the context of mathematical finance the main result concerns the existence of martingale measures whose densities are bounded from below by a prescribed random variable.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. Consider a convex cone $C \subset L^\infty = L^\infty(\Omega, \mathcal{F}, \mathbf{P})$, satisfying the condition

$$C \cap L^\infty_+ = \{0\}, \tag{1.1}$$

where L^∞_+ is the non-negative orthant of L^∞. Typically, C consists of random variables, dominated by stochastic integrals $\int_0^T H_t\, dS_t$ (compare [4]). Here $S = (S_t)_{0 \leq t \leq T}$ is a semimartingale, describing the stock-price process and $H = (H_t)_{0 \leq t \leq T}$ is a predictable S-integrable process, belonging to some class of admissible trading strategies. Assumption (1.1) is usually referred to as the no-arbitrage condition. Note that the cases of transaction costs, portfolio constraints and infinitely many assets can also be incorporated in this framework.

Furthemore, consider the polar of C, taken in $L^1 = L^1(\Omega, \mathcal{F}, \mathbf{P})$:

$$\{y \in L^1 : \int_\Omega xy\, d\mathbf{P} \leq 0,\ x \in C\}. \tag{1.2}$$

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For the case of a bounded process S, the set (1.2) is generated by densities of absolutely continuous martingale measures. In this note we discuss the following question:

(Q) Let $f \in L^1$. Under what conditions does there exist an element $g \in L^1$ in the polar of C such that $g \geq f$?

In fact, this question concerns the existence of a martingale measure $\mathbf{Q}$ whose density is bounded from below by the prescribed random variable f up to a multiplicative constant $\alpha > 0$: $d\mathbf{Q}/d\mathbf{P} \geq \alpha f$.

Sometimes it is useful to take the polar of C in $(L^\infty)^*$, the dual space of L^∞; see, e.g., [3]. In our case it also appears that an easier answer to the question (Q) can be given if g is allowed to lie in $(L^\infty)^*$; see Corollary 1 below and [8]. The answer to this question in precise terms is given in Corollary 2.

Our results are essentially the following. Regard $f \in L^1$ as a functional on L^∞, defined by the formula

$$\langle x, f\rangle = \int_\Omega xf \, d\mathbf{P}.$$

Then the existence of the desired element g is equivalent to the boundedness of f from above on a certain subset of the cone C. If g is allowed to be an element of $(L^\infty)^*$, this subset may be chosen as

$$C_1 = \{x \in C : x^- \leq 1 \text{ a.s.}\},$$

where $x^- = \max\{-x, 0\}$. If we seek $g \in L^1$, such a subset should be somewhat bigger:

$$C_V = \{x \in C : x^- \in V\},$$

where V is a neighbourhood of zero in the Mackey topology $\tau(L^\infty, L^1)$.

2. Answer to the question (Q)

We find it natural to examine the problem in a somewhat more general context. Let (X, τ) be a locally convex-solid Riesz space. This means that X is a vector lattice, endowed with a topology τ whose local base consists of convex solid sets; see [1] for details. For an element $x \in X$, its positive part, negative part and absolute value are denoted by x^+, x^- and $|x|$. The set $V \subset X$ is called solid if the conditions $x \in V$, $|y| \leq |x|$ imply that $y \in V$.

Consider a convex cone $C \subset X$, such that

$$C \cap X_+ = \{0\}, \tag{2.1}$$

where $X_+ = \{x \in X : x \geq 0\}$. Let V be a solid subset of X. Put

$$C_V = \{x \in C : x^- \in V\}.$$

Using the implication

$$x \leq y \Longrightarrow x^- \geq y^-, \tag{2.2}$$

it is elementary to check that

$$C_V = C \cap (V + X_+). \tag{2.3}$$

Denote by X^* the topological dual of X with the order induced by the dual cone $X_+^* = \{\xi \in X^* : \langle x, \xi\rangle \geq 0,\ x \in X_+\}$. The polar of C is taken in X^*:

$$C^\circ = \{\xi \in X^* : \langle x, \xi\rangle \leq 0, \quad x \in C\}.$$

We use the customary notation $\sigma(X^*, X)$ for the weak-star topology and $|\sigma|(X, X^*)$ for the coarsest locally convex-solid topology on X, compatible with the duality $\langle X, X^*\rangle$ [1]. The polar of an arbitrary set $A \subset X$ is defined as follows:

$$A^\circ = \{\xi \in X^* : \langle x, \xi\rangle \leq 1, \quad x \in A\}.$$

THEOREM 1. *Let (X, τ) be a locally convex-solid Riesz space. Assume that there exists a $\sigma(X^*, X)$-compact set $\Gamma \subset X_+^*$ such that the convex cone generated by Γ is $\sigma(X^*, X)$-dense in X_+^*. Let $C \subset X$ be a convex cone satisfying (2.1). Then for any $f \in X^*$ the following conditions are equivalent:*

(i) *There exists a convex solid τ-neighbourhood of zero V such that*

$$\sup_{x\in C_V} \langle x, f\rangle < +\infty, \quad C_V = \{x \in C : x^- \in V\}.$$

(ii) *There exists $g \in C^\circ$ such that $g \geq f$.*

Proof. (ii) $\Longrightarrow$ (i). Consider the convex solid $|\sigma|(X, X^*)$-neighbourhood of zero

$$V = \{x \in X : \langle |x|, g - f\rangle \leq 1\}.$$

Let $x \in C_V$. Then

$$\langle x, f\rangle = \langle x, g\rangle + \langle x, f - g\rangle \leq \langle -x, g - f\rangle \leq \langle x^-, g - f\rangle \leq 1.$$

(i) $\Longrightarrow$ (ii). Let Γ' be the $\sigma(X^*, X)$-closed convex hull of the set $\Gamma \cup \{0\}$. Consider the $\sigma(X^*, X)$-compact convex set

$$\Pi = (V - X_+)^\circ + \Gamma' = (V^\circ \cap X_+^*) + \Gamma'$$

and put

$$\lambda = \sup_{x\in C_V} \langle x, f\rangle. \tag{2.4}$$

If the condition (ii) is false, we may apply the Hahn-Banach theorem [9, Chap. II, Th. 9.2] to separate the sets $f + \lambda\Pi$ and C° by an element $x \in X$:

$$\sup_{\eta\in C^\circ} \langle x, \eta\rangle < \inf_{\zeta\in f+\lambda\Pi} \langle x, \zeta\rangle.$$

Since C° is a cone, we get $\langle x, \eta\rangle \leq 0$, $\eta \in C^\circ$. Thus, $x \in C^{\circ\circ} = \operatorname{cl} C$ by the bipolar theorem [9, Chap. IV, Th. 1.5], where $\operatorname{cl} C$ is the closure of C in any topology, compatible with the duality $\langle X, X^*\rangle$, and

$$\langle x, f\rangle + \lambda \inf_{\zeta\in\Pi} \langle x, \zeta\rangle > 0. \tag{2.5}$$

Furthemore, since $\inf_{\zeta\in\Pi}\langle x,\zeta\rangle \le 0$, we conclude that $\langle x, f\rangle > 0$ and $x \notin X_+$. Indeed, for any τ-neighbourhood of zero W take an element $y_W \in (\mu x + W \cap V) \cap C$, $\mu > 0$. If $x^- = 0$, then $y_W \ge z_W$ for some $z_W \in V$. By (2.2) and the solidness of V we have $y_W^- \in V$. Thus, $\mu x \in \operatorname{cl} C_V$ for any $\mu > 0$ and we obtain a contradiction, since $\langle x, f\rangle > 0$ and f must be bounded (from above) on $\operatorname{cl} C_V$.

Moreover, $\inf_{\zeta\in\Pi}\langle x,\zeta\rangle < 0$, because otherwise x is non-negative on Γ and consequently on X_+^*. In other words, $x \in X_+$, which we just have seen to be wrong. So, we may normalize x such that $\inf_{\zeta\in\Pi}\langle x,\zeta\rangle = -1$ and

$$\langle x, f\rangle > \lambda \tag{2.6}$$

by (2.5). Noting that $-\Pi^\circ \subset -(V - X_+)^{\circ\circ} = \operatorname{cl}(V + X_+)$, we get

$$x \in -\Pi^\circ \cap \operatorname{cl} C \subset \operatorname{cl}(V + X_+) \cap \operatorname{cl} C \subset \operatorname{cl} C_V. \tag{2.7}$$

To prove the last inclusion in (2.7) note that αx is an interior point of $V + X_+$ for all $\alpha \in [0, 1)$; see, e.g., [9, Chap. II]. For fixed $0 \le \alpha < 1$ let W be a τ-neighbourhood of zero such that $\alpha x + W \subset V + X_+$. Since $\alpha x \in \operatorname{cl} C$, the set $(\alpha x + W) \cap C$ is non-empty. By (2.3) this means that $\alpha x \in \operatorname{cl} C_V$ for each $0 \le \alpha < 1$ and therefore also for $\alpha = 1$.

Clearly, relations (2.6), (2.7) yield the desired contradiction to (2.4), which completes the proof. □

The conditions of Theorem 1 are satisfied for any Banach lattice X (with the norm topology τ) since we can take $\Gamma = B_{X^*} \cap X_+^*$, where B_{X^*} is the unit ball of X^*. Moreover, in this case, we can consider only one neighbourhood of zero $V = B_X$ in condition (i). The corresponding result for the space L^∞ with the norm topology is formulated below.

Corollary 1. *For any element $f \in (L^\infty)^*$ the following conditions are equivalent:*

(i) $\sup_{x\in C_1}\langle x, f\rangle < +\infty, \quad C_1 = \{x \in C : x^- \le 1 \text{ a.s.}\}$.
(ii) *There exists $g \in (L^\infty)^*$ such that $g \ge f$ and $g \in C^\circ$.*

As a second example, the Mackey topology $\tau(L^\infty, L^1)$ is locally convex-solid (see [2, Section 11]) and the set

$$\Gamma = \{x \in L_+^\infty : \|x\|_{L^\infty} \le 1\} \subset L_+^1$$

is $\sigma(L^1, L^\infty)$-compact (weakly compact in L^1). Thus, Theorem 1 is valid for the space $(L^\infty, \tau(L^\infty, L^1))$. To make this result more concrete, we recall another description of the topology $\tau(L^\infty, L^1)$.

A function $\varphi : [0,\infty) \mapsto [0,\infty)$ is called an N-function if it is convex and

$$\lim_{t\to+0}\frac{\varphi(t)}{t} = 0, \quad \lim_{t\to+\infty}\frac{\varphi(t)}{t} = \infty.$$

It follows that φ is non-decreasing and continuous. Let $\|x\|_\varphi$ denote the Luxemburg norm (see, e.g., [6]):

$$\|x\|_\varphi = \inf\{\lambda > 0 : \int_\Omega \varphi(|x|/\lambda)\, d\mathbf{P} \le 1\}.$$

It is known that the Mackey topology $\tau(L^\infty, L^1)$ is generated by the family of Luxemburg norms $\{\|\cdot\|_\varphi : \varphi \in \Phi_N\}$, where Φ_N is the collection of all N-functions (see [7]).

In addition, this topology is generated by sets

$$\mu \bigcap_{k=1}^{\infty} U_{\varepsilon_k}, \quad U_{\varepsilon_k} = \{x : \mathbf{P}(|x| \ge k) \le \varepsilon_k\}, \quad k = 1, \dots, \infty, \quad \mu > 0,$$

where $(\varepsilon_k)_{k=1}^\infty$ is any positive sequence. Indeed, for any sequence $\varepsilon_k > 0$ there exists an N-function φ satisfying the conditions

$$\varphi(t) \ge \max_{1 \le i \le k} \{1/\varepsilon_i\}, \quad t \ge k.$$

If $\|x\|_\varphi \le 1$, then

$$\mathbf{P}(|x| \ge k) = \int_{\{|x| \ge k\}} d\mathbf{P} \le \varepsilon_k \int_{\{|x| \ge k\}} \varphi(|x|)\, d\mathbf{P} \le \varepsilon_k.$$

Conversily, for any N-function φ put $\varepsilon_k = k^{-2}/\varphi(k+1)$. If $x \in \bigcap_{k=1}^\infty U_{\varepsilon_k}$, then

$$\begin{aligned}\|x\|_\varphi &\le \int_{|x|<1} \varphi(|x|)\, d\mathbf{P} + \sum_{k=1}^{\infty} \int_{k \le |x| < k+1} \varphi(|x|)\, d\mathbf{P} \\ &\le \varphi(1) + \sum_{k=1}^{\infty} \varphi(k+1) \mathbf{P}\{|x| \ge k\} \le \varphi(1) + \sum_{k=1}^{\infty} k^{-2}.\end{aligned}$$

We collect these results in the following corollary, which gives the answer to the question (Q).

COROLLARY 2. *For any element $f \in L^1$ the following conditions are equivalent:*

(i) *There exists a sequence $\varepsilon_k > 0$ such that*

$$\sup\{\langle x, f\rangle : x \in \bigcap_{k=1}^{\infty} C^{\varepsilon_k}\} < \infty, \quad C^{\varepsilon_k} = \{x \in C : \mathbf{P}(x^- \ge k) \le \varepsilon_k\}.$$

(ii) *There exists an N-function φ such that*

$$\sup_{x \in C_\varphi} \langle x, f\rangle < \infty, \quad C_\varphi = \{x \in C : \|x^-\|_\varphi \le 1\}.$$

(iii) *There exists a convex solid $\tau(L^\infty, L^1)$-neighbourhood of zero V such that*

$$\sup_{x\in C_V} \langle x, f\rangle < +\infty, \quad C_V = \{x \in C : x^- \in V\}.$$

(iv) *There exists $g \in L^1$ such that $g \geq f$ and $g \in C^\circ$.*

The equivalence between (iii) and (iv) follows from Theorem 1. The two other equivalencies are implied by the properties of the Mackey topology $\tau(L^\infty, L^1)$, presented above.

3. Examples

Recall that $(L^\infty)^*$ may be identified with the space of all bounded finitely additive measures μ on $\mathcal{F}$ with the property that $\mathbf{P}(A) = 0$ implies that $\mu(A) = 0$ [5]. Our first example shows that in the context of Corollary 1, in general, it is not possible to find the element $g \in (L^\infty)^*$ already in L^1 even if $f \in L^\infty$.

Example 1. Let $\Omega = [0, 1]$, suppose $\mathcal{F}$ consists of all Lebesgue measurable sets and let $\mathbf{P}$ be the Lebesgue measure. Consider a purely finitely additive measure $\mu : \mathcal{F} \mapsto \{0, 1\}$ such $\mu(I) = 1$ for any open interval $I \subset (0, 1)$ containing $1/2$ (see [10]). It follows that $\mu\{|t - 1/2| \geq \delta\} = 0$ for all $\delta > 0$. Put

$$C = \{x \in L^\infty : \int_\Omega x\, d(\mathbf{P} + \mu) \leq 0\}.$$

The element $f = 1 \in (L^\infty)^* \cap L^\infty$ is bounded on the set C_1, defined in Corollary 1:

$$\langle x, 1\rangle = \int_\Omega x\, d\mathbf{P} \leq -\int_\Omega x\, d\mu \leq 1, \quad x \in C_1,$$

and it is dominated by the element of $C^\circ \subset (L^\infty)^*$ corresponding to the measure $\mathbf{P} + \mu$. However, f is unbounded on any set $\bigcap_{k=1}^\infty C^{\varepsilon_k}$, defined in Corollary 2(i).

To show this, consider a sequence $x_n \in L^\infty$, defined by the formulas

$$x_n(t) = n, \quad |t - 1/2| \geq \varepsilon_n/2, \quad x_n(t) = -n, \quad |t - 1/2| < \varepsilon_n/2,$$

$n \geq 1$, $t \in [0, 1]$. Without loss of generality, we may assume that $\varepsilon_k > 0$ monotonically tends to 0. Evidently, $x_n \in \bigcap_{k=1}^\infty C^{\varepsilon_k}$:

$$\int_\Omega x_n\, d(\mathbf{P} + \mu) = \int_0^1 x_n(t)\, dt - n = -2n\varepsilon_n \leq 0,$$

$$\mathbf{P}(x_n^- \geq k) = 0, \quad n < k; \quad \mathbf{P}(x_n^- \geq k) = \varepsilon_n \leq \varepsilon_k, \quad n \geq k.$$

But

$$\langle x_n, 1\rangle = \int_0^1 x_n(t)\, dt = n(1 - 2\varepsilon_n) \to +\infty, \quad n \to \infty.$$

Hence, by Corollary 2, $f = 1$ cannot be dominated by any element of $C^\circ \cap L^1$.

The next examples are in a more financial spirit. Note that in both of them the cone C is a subspace. This is not essential: passing to $C - L^\infty_+$, the results still hold true.

EXAMPLE 2. We consider a slight modification of an example given in [4, Remark 6.5.2]. Let $\Omega = \mathbb{N}$, the sigma-algebra $\mathcal{F}_0$ is generated by the sets $(\{2n-1, 2n\})_{n=1}^\infty$, and let $\mathcal{F} = \mathcal{F}_1$ be the power set of Ω. Define the probability measure $\mathbf{P}$ on $\mathcal{F}$ by $\mathbf{P}\{2n-1\} = \mathbf{P}\{2n\} = 2^{-n-1}$. Let the asset prices $(S_t)_{t=0}^1$ at times 0 and 1 be $S_0 \equiv 0$, and

$$S_1(2n-1) = 1, \quad S_1(2n) = -2^{-n}, \quad n \in \mathbb{N}.$$

Let the cone C be generated by the elements $\gamma(S_1 - S_0)$ in L^∞, where γ is an $\mathcal{F}_0$-measurable random variable. As usual, γ may be interpreted as an investor's portfolio at time $t = 0$. Then the set C consists of the possible investor's gains at time $t = 1$. Evidently, the no-arbitrage condition (1.1) is satisfied.

We claim that for any $f \in L^1_+$ the conditions of Corollaries 1 and 2 are equivalent and that there exists an element $g \geq f$, $g \in C^\circ \cap L^1$, if and only if

$$\sum_{n=0}^{\infty} f(2n-1) < \infty. \tag{3.1}$$

It suffices to show that condition (3.1) implies condition (iv) of Corollary 2 and that condition (i) of Corollary 1 implies (3.1). Assume that (3.1) is satisfied and put

$$g(2n-1) = \max\{f(2n-1), 2^{-n} f(2n)\}, \quad g(2n) = 2^n g(2n-1), \quad n \in \mathbb{N}.$$

Then $g \in L^1(\mathbf{P})$ and $g \geq f$. Computing the conditional expectation,

$$\mathbf{E}_{\mathbf{P}}(gS_1|\mathcal{F}_0)(2n-1) = (g(2n)S_1(2n) + g(2n-1)S_1(2n-1))/2^{n+1} = 0,$$

we see that $g \in C^\circ$.

Now assume that condition (i) of Corollary 1 is satisfied. Put $\gamma(2n-1) = \gamma(2n) = 2^n$. Then $\gamma S_1 \in C_1$ and

$$\langle \gamma S_1, f \rangle = \sum_{n=1}^{\infty} (f(2n-1)/2 - 2^{-n-1} f(2n)) < +\infty.$$

Since $f \in L^1(\mathbf{P})$, we have $\sum_{n=1}^\infty 2^{-n-1} f(2n) < +\infty$ and condition (3.1) holds true.

For the cone considered in Example 2, there is no difference between the conditions of Corollaries 1 and 2 (in contrast to Example 1, which did not allow for a financial interpretation). Below we consider a market with infinitely many assets, where these conditions are different and the following is true:

$$(f + L^1_+) \cap C^\circ = \emptyset, \quad (f + (L^\infty)^*_+) \cap C^\circ \neq \emptyset \tag{3.2}$$

for some $f \in L^1_+$.

Example 3. Consider the same probability space $(\Omega, \mathcal{F}, \mathbf{P})$ as in Example 1. Let $(A_n)_{n=1}^\infty$, $A_n \subset [0, 1/2]$, be a sequence of independent events with probabilities $\mathbf{P}(A_n) = 1/2^n$. To construct such a sequence take independent random variables $\xi_n : \Omega \mapsto \{0, 1\}$ such that $\mathbf{P}(\xi_n = 1) = 1/2^{n-1}$ and put

$$A_n = \{\xi_n^{-1}(1)\}/2 = \{t \in [0, 1/2] : \xi_n(2t) = 1\}.$$

Furthemore, put $b_0 = 1/2$, $b_n = b_{n-1} + 4^{-n}$, $n \geq 1$, and consider the sequence of intervals $B_n = (b_{n-1}, b_n] \subset (1/2, 5/6]$. The sets B_n are mutually disjoint and disjoint from $\bigcup_{n=1}^\infty A_n$. Let

$$f = \sum_{n=1}^\infty 2^n I_{B_n} + I_{[0,1/2]} + I_{[5/6,1]}.$$

Clearly, $f \in L^1_+(\mathbf{P})$.

Now we introduce a countable sequence of asset price increments,

$$x_n = S_1^n - S_0^n = 2^n I_{B_n} - I_{A_n}, \quad n \in \mathbb{N}$$

at times 0 and 1. We assume that the processes $(S_t^n)_{t=0}^1$ are adapted to the filtration $(\mathcal{F}_0, \mathcal{F}_1)$, where $\mathcal{F}_1 = \mathcal{F}$ and $\mathcal{F}_0$ is trivial. The portfolios γ^n are non-random, since they are assumed to be $\mathcal{F}_0$-measurable.

Let C be the linear subspace of L^∞ spanned (algebraically) by x_n. The elements of C describe the investor's gains, obtained by trading in a finite collection of assets. The condition $\mathbf{E}_\mathbf{P}(x_n) = 0$ implies that C is disjoint from $L^\infty_+ \setminus \{0\}$.

Let $z = \sum_{n \in J} \gamma^n x_n$ be any element of C_1. Here J is a finite subset of $\mathbb{N}$ and γ^n are some constants. By the definition of C_1 we have

$$z = \sum_{n \in J} \gamma^n (2^n I_{B_n} - I_{A_n}) \geq -1, \text{ a.s.}$$

Considering this inequality on the sets B_n and $\bigcap_{n \in J} A_n$, we get

$$-\gamma^n 2^n \leq 1, \quad \sum_{n \in J} \gamma^n \leq 1.$$

It follows that condition (i) of Corollary 1 is satisfied:

$$\langle z, f \rangle = \sum_{n \in J} \gamma^n \Big(2^n \int_{B_n} f \, d\mathbf{P} - \int_{A_n} f \, d\mathbf{P}\Big)$$
$$= \sum_{n \in J} \gamma^n (1 - 2^{-n}) \leq 1 + \sum_{n \in J} 4^{-n} \leq 4/3.$$

To show that condition (i) of Corollary 2 fails, consider any sequence $\varepsilon_k > 0$, $k \geq 1$, and assume that f is bounded from above by a constant β on the set $\bigcap_{k=1}^{\infty} C^{\varepsilon_k}$. Define natural numbers m, $n_1, \dots, n_m$ as follows:

$$m > \beta + 1, \quad \sum_{i=1}^{m} \frac{1}{2^{n_i}} \leq \min\{\varepsilon_1, \dots, \varepsilon_m\}.$$

We have

$$\mathbf{P}(x_{n_1} + \cdots + x_{n_m} \leq -k) = 0, \quad k > m,$$

and

$$\mathbf{P}(x_{n_1} + \cdots + x_{n_m} \leq -k) \leq \mathbf{P}\left(\bigcup_{i=1}^{m} \{x_{n_i} \leq -1\}\right) \leq \sum_{i=1}^{m} \frac{1}{2^{n_i}} \leq \varepsilon_k, \quad k \leq m.$$

Thus $x_{n_1} + \cdots + x_{n_m} \in \bigcap_{k=1}^{\infty} C^{\varepsilon_k}$ and we obtain a contradiction:

$$\begin{aligned}\langle x_{n_1} + \cdots + x_{n_m}, f \rangle &= \sum_{i=1}^{m} \left(2^{n_i} \int_{B_{n_i}} f \, d\mathbf{P} - \int_{A_{n_i}} f \, d\mathbf{P} \right) \\ &= m - \sum_{i=1}^{m} 2^{-n_i} \geq m - 1 > \beta.\end{aligned}$$

Note also that if ν is the non-negative finitely additive measure corresponding to an element $g \in C^{\circ}$, $g \geq f$, then

$$\nu(A_n) = \langle I_{A_n}, g \rangle = 2^n \langle I_{B_n}, g \rangle \geq 2^n \langle I_{B_n}, f \rangle = 1.$$

Hence, ν is not countably additive.

Finally, we mention that it would be interesting to determine if the relations (3.2) can hold true for the case of finitely many assets.

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