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THE PROBABILITY OF ESCAPING INTERFERENCE

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The tradition of the University of Illinois Weekly Probability Luncheon began in the late 1950s under the auspices of Joe Doob. This meeting has continued to bring together probabilists and others for informal conversation. We (the authors) started working together at one of these probability luncheons. We think that Joe Doob, as a mainstay of the Urbana-Champaign Saturday hike, might have been interested in an application of this paper—finding the probability of hiking across a one-lane bridge without meeting anyone trying to cross in the opposite direction.

Abstract. Consider two independent sequences of travelers arriving at opposite ends of a one-lane shared pathway. Each traveler attempts to traverse the entire pathway to the opposite end. An attempt fails if the traveler collides with an opposing traveler. In a collision, both opposing travelers are annihilated. We study the probability that a traveler manages to traverse the entire length of the one-lane shared pathway unobstructed. The dynamics of the travelers include the possibility of acting as bodyguards and "running interference" for a more recent arrival traveling in the same direction.

This model was developed to address some questions in the theory of crystal growth. It may have possible applications in Particle Physics as well as to traffic at a one-lane bridge. This paper develops some properties of the model while focusing on the probability that a traveler crosses the entire pathway without interference.

1. Introduction

The model that we developed for crystal growth [7] can be described intuitively in terms of a one-lane bridge. Vehicles arrive at the ends of the bridge according to two independent Poisson processes, then travel across the bridge with constant speed. If a vehicle collides with one traveling in the opposite direction, then both vehicles annihilate each other and vanish from the bridge. We consider the probability that a vehicle makes it across the entire length

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of the one-lane bridge without colliding with a vehicle traveling the opposite direction.

This model was developed [7] to describe crystal growth dynamics observed at the Center for Atmospheric Research [6]. Crystal layers form on an existing crystal seed and have a greater tendency to initiate at the borders of this seed. We simplified our model to the case of a one-dimensional crystal seed with crystal layers initiating only at the ends. The travelers in the shared pathway model represent the "wave fronts" of forming crystal layers. A collision of two travelers represents the completion of a crystal layer by the joining of two opposing crystal wave fronts. Crystal layers can also complete by a crystal wave front crossing the length of the crystal seed without meeting an opposing crystal wave front.

Unlike the classical Johnson-Mehl-Kolmogorov model [8], [5], our crystal growth model considers initiation of new crystal layers only on boundaries of a seed crystal as opposed to initiation from random points in a given region. Our results differ from the limiting results for this classical model [9], [11], [1], [2], [4] in that we do not study behavior as the region in which the crystal forms becomes sufficiently large. The main result of our previous paper is the derivation of a closed form expression for equilibrium growth rate of the crystal in terms of the initiation rates. This derivation involves an ergodic theorem for an embedded Markov process.

In Section 2, we recast the crystal growth model of our previous paper in a more general model. Properties of this model are developed in Section 3. These properties include the positive-recurrence property of the model, a bound on the time until the process is arbitrarily close to equilibrium, and a discrete-time version of the process. We use the discrete-time version of the shared-pathway process to define and derive some of the probabilities of escaping interference.

The probabilities of escaping interference are considered according to the initial distribution of the model. The fixed-initial-point probabilities are explored in Section 4 and the equilibrium ones are in Section 5. In both sections, we consider both unilateral and bilateral probabilities. A unilateral event depends only on arrivals at a given endpoint of the interval, irrespective of the arrivals at the other end of the interval. A bilateral event ignores right-left distinctions and only depends on the first arrival regardless of the side of the arrival. A probability of a unilateral (respectively bilateral) event will be referred to as a unilateral (respectively bilateral) probability.

The potential of new applications in particle physics colliders and in traffic at a one-lane bridge inspired us to generalize our crystal growth model. The generalization is rather intuitively easy to understand, yet the indirect effects of travelers moving in the same directions make this model non-trivial. Discussion on our results and areas for future research are in Section 6.

2. The shared-pathway model

In this section, we describe the one-lane shared-pathway model. We first give a general description and later become more specific. We define a shared-pathway process precisely from an initial distribution and two Poisson processes. These two processes represent arrivals to the ends of the shared pathway. The state space of the shared-pathway process is derived from the Poisson processes. This shared pathway process is finally defined, taking values in the derived state space.

We represent the shared pathway by the unit interval $[0, 1]$. Arrivals at the endpoints (at 0 and 1) of the shared pathway occur according to two independent Poisson processes with rate λ on the left and rate μ on the right. A new arrival attempts to traverse the shared pathway by traveling the pathway at unit speed. This "traveler" eventually either completely traverses the shared pathway or collides with an opposing "traveler". In the case of a collision, the two opposing travelers annihilate each other and vanish from the state of the shared-pathway process. Thus, the shared-pathway process keeps track of travelers that are still actively trying to traverse the shared pathway but ignores those that have already either completely traversed the shared pathway or that have collided with an opposing traveler.

Let A_λ and A_μ be two independent Poisson processes with rates λ and μ, respectively. These two processes represent arrivals to either side of the shared pathway: A_λ represents the arrivals to the left-side of the pathway, and A_μ represents the travelers arriving to the right-side of the shared pathway. We construct the shared-pathway process S from A_λ and A_μ as follows.

Let $(L_k)_{k\geq 1}$ and $(R_k)_{k\geq 1}$ be the sequence of jump times of A_λ and A_μ, respectively. The four sets $\{\sum_{k=1}^{\infty} L_k < \infty\}$, $\{\sum_{k=1}^{\infty} R_k < \infty\}$, $\{\sum_{k=1}^{i} L_k = 1 + \sum_{k=1}^{j} R_k$ for some $i, j\}$, and $\{1 + \sum_{k=1}^{i} L_k = \sum_{k=1}^{j} R_k$ for some $i, j\}$ have probability zero. Without loss of generality, we discard these sets from the product probability space of (A_λ, A_μ). The consequence of this discard is that for all $t \geq 0$, both $A_\lambda(t)$ and $A_\mu(t)$ are finite and that no arrival occurs at exactly the same time as an opposite-side traveler crosses the shared pathway. These consequences are convenient for the construction of S.

The state of the shared-pathway process S at any particular time is represented by a pair of finite sequences. These pairs will be elements of some set $E_{m,n} = \{((x_1, \ldots, x_m), (y_n, \ldots, y_1))$ with $m, n \geq 0$, $0 \leq x_1 < \cdots < x_m < y_n < \cdots < y_1 \leq 1\}$. If either $m = 0$ or $n = 0$ or both, then the corresponding finite sequence is written as the empty set. More precisely, these special cases of $E_{m,n}$ can be written as

$$
\begin{aligned}
E_{0,0} &= \{(\emptyset, \emptyset)\} \\
E_{0,n\neq 0} &= \{(\emptyset, (y_n, \ldots, y_1)) \text{ with } 0 \leq y_n < \cdots < y_1 \leq 1\} \\
E_{m\neq 0,0} &= \{((x_1, \ldots, x_m), \emptyset) \text{ with } 0 \leq x_1 < \cdots < x_m \leq 1\}
\end{aligned}
$$

Any element of the set $E_{m,n}$ is a shared-pathway state with m travelers at positions $x_1, \ldots, x_m$ moving towards the right and n travelers at positions $y_n, \ldots y_1$ moving towards the left. A new arrival on the left is indicated by $x_1 = 0$; a new arrival on the right by $y_1 = 1$. Define E as the union $\bigcup_{m,n\geq 0} E_{m,n}$. Let $\mathcal{H}$ be the σ-field generated by the Borel subsets of each $E_{m,n}$ as a subset of $[0,1]^{m+n}$. The space $(E, \mathcal{H})$ will be the state space of the shared-pathway process.

We define the shared-pathway process S from the processes A_λ and A_μ on the time interval $[0,\infty)$ with state space $(E, \mathcal{H})$ in the following manner. Starting at any initial state $S(0) \in E$, the path of S is completely deterministic until the first arrival time: the x_i increase at unit speed and the y_j decrease at unit speed until an arrival, a collision, or a traversing occurs. Each collision ($x_m = y_n$) or traversing ($x_m = 1$ or $y_n = 0$) causes a jump in the state of S. If there is a collision, then the values x_m and y_n vanish from the state of $S(t)$ and m and n both decrease by 1. If there is a traversing to the right, then x_m vanishes from $S(t)$, m decreases by 1 and n remains at 0. If there is a traversing to the left, then $y_n = 0$ vanishes from $S(t)$, n decreases by 1 and m remains at 0. The remaining x_i continue to increase at unit speed and the remaining y_j continue to decrease at unit speed until an arrival, a traversing, or a collision occurs.

We sometimes refer to an arrival to the left side (respectively, right side) of the shared pathway as a left-arrival (respectively, right-arrival). At the time of a left-arrival the state changes from $(x_1, \ldots, x_m)$ to $(0, x_1, \ldots, x_m)$ and m increases to $m+1$. At the time of a right-arrival, the state changes from $(y_n, \ldots, y_1)$ to $(y_n, \ldots, y_1, 1)$ and n increases to $n+1$. The paths of S are completely deterministic between arrivals: the x_i increase at unit speed and the y_j decrease at unit speed until an arrival, a traversing, or a collision occurs.

This defines the shared-pathway process S for the time interval $[0,\infty)$ with state space $(E, \mathcal{H})$. This process is completely determined by its initial value $S(0)$ and the two Poisson processes A_λ and A_μ.

3. Properties of the shared-pathway process

In this section, we develop properties of the shared-pathway process. We establish the positive-recurrence property of the model, prove the existence of a unique equilibrium distribution, and bound the difference between the distribution of S and its equilibrium. We wish that we knew more about the equilibrium probability distribution of the shared-pathway process.

Two discrete-time embedded processes give insight into the continuous-time shared-pathway process. The first embedded process S^* is the sequence of states of S observed at arrival times. The discrete-time S^* has a unique equilibrium measure and converges to this equilibrium with an exponential bound.

The second embedded process X is the exit-position process: it records the position in the interval $[0, 1]$ at which the exits (collisions or crossings) take place. This discrete-time process is developed in detail in [7]. Some of the results for the exit-position process are stated here. Of particular interest is the explicit equilibrium measure for X. These two embedded processes will be used to state and derive the probabilities in Section 5.

LEMMA 1. *The process S is time-homogeneous, strong Markov, and uniformly positive recurrent.*

Proof. See [7, Theorem 2.1]. □

Thus, S must have a unique equilibrium distribution which we call η. With any initial distribution ζ, the distribution of the shared-pathway process approaches η as $t \to \infty$. In fact, there is a bound on the speed on this convergence.

THEOREM 2 (The Speed of Reaching Equilibrium). *Fix an initial distribution ζ on E. Let S^ζ be a copy of S starting with initial distribution ζ. There exists a copy S^η of S that starts with initial distribution η such that for any $t \geq 0$,*

$$\|S^\zeta(t) - S^\eta(t)\|_{\text{tot}} \leq 2(1 - e^{-(\lambda+\mu)})^{t-1}$$

where $\|\cdot\|_{\text{tot}}$ is the total variation norm.

Proof. Define S^η as a copy of S that starts with initial distribution η and has the same arrivals as S^ζ. Define T^ζ as the first time that the two processes S^ζ and S^η meet: $T^\zeta = \inf_{t>0}\{S^\eta(t) = S^\zeta(t)\}$. If T^ζ is infinite, then S^η and S^ζ are never equal. If T^ζ is finite, then S^η and S^ζ are the same after time T^ζ.

If f is any bounded measurable function on E, then for all $t \geq 0$,

$$\begin{aligned} E[f(S^\eta(t)) - f(S^\zeta(t))] &= E[1_{\{t \leq T^\zeta\}}(f(S^\eta(t)) - f(S^\zeta(t)))] \\ &\qquad + E[1_{\{t > T^\zeta\}}(f(S^\eta(t)) - f(S^\zeta(t)))] \\ &\leq 2\|f\|_{\sup} P(t \leq T^\zeta). \end{aligned}$$

Thus, $\|S^\zeta(t) - S^\eta(t)\|_{\text{tot}} \leq 2P(t \leq T^\zeta)$.

Let A be the process of arrival times of S^ζ and S^η. Whenever A has its first inter-arrival time greater than one, the processes S^ζ and S^η meet in the set $E_{0,0}$. Let W be the number of inter-arrival intervals of A of duration less than one before the first such interval of duration greater than or equal to one. Since these durations are independent and exponentially distributed with parameter $\lambda + \mu$, it follows that for $k \geq 0$, $P(W \geq k) = (1 - e^{-(\lambda+\mu)})^k$. By the definition of W, $T^\zeta < 1 + W$. Thus, $P(t < T^\zeta) \leq P(t < 1 + W)$. Therefore,

$$\|S^\zeta(t) - S^\eta(t)\|_{\text{tot}} \leq 2P(t < 1 + W) \leq (1 - e^{-(\lambda+\mu)})^{t-1}.$$ □

Notice that the rate of convergence is independent of ζ and depends only on the sum $\lambda+\mu$. In spite of its elementary character, and the fact that we largely ignored the possibility of S^ζ meeting S^η at any point other than the element of $E_{0,0}$, Theorem 2 gives some access to S^η. If $\lambda+\mu=1$ and $t=60$, then $2(1-e^{-(\lambda+\mu)})^{t-1}\approx 3.25\cdot 10^{-4}$. Thus, if time is measured in minutes, it should not be necessary to wait more than an hour to develop practical equilibrium, no matter the initial distribution (possibly unknown).

It seems difficult to derive analytical properties of the equilibrium distribution η of the shared-pathway process S. The only explicit result on the value of η that we have obtained is the value (see [7])

$$\eta(E_{0,0})=\begin{cases} e^{-\lambda}(1+\lambda)(1+2\lambda)^{-1} & \text{if } \lambda=\mu,\\ e^{-\mu}(\lambda e^{\lambda-\mu}-\mu)(\lambda e^{2(\lambda-\mu)}-\mu)^{-1} & \text{if } \lambda\neq\mu.\end{cases}$$

We next turn our attention to an embedded process S^* of S. The S^* is the sequence of states of S observed at arrival times. Let $(t_k)_{k\geq 1}$ be the ordered sequence of event times of $A_\lambda+A_\mu$. We define S^* as $S^*=(S^*_k)_{k\geq 1}$ where $S^*_k=S(t_k)$ for $k\geq 1$. The state space of S^* is the set E^* which is the subset of E including only the states with $x_1=0$ or $y_1=1$.

Lemma 3. *The discrete-time process S^* is uniformly positive recurrent.*

Proof. Since S is strong Markov and the increasing sequence $(t_k)_{k\geq 1}$ is a sequence of stopping times of S, the embedded discrete-time S^* is a time-homogeneous Markov process. For any sample path of S, the set $\{t\geq 0 : S(t)\in E_{0,0}\}$ is a disjoint union of intervals. Call these 0-intervals. The 0-intervals begin with an exit due to a traversal or a collision and end with an arrival time. If $t_1>1$, then t_1 ends such an interval of S and $S^*_1\in\{((0),\emptyset),(\emptyset,(1))\}$. For every $k>1$, if $t_k-t_{k-1}>1$, then t_k ends a 0-interval of S and $S^*_k\in\{((0),\emptyset),(\emptyset,(1))\}$. Since $(t_k-t_{k-1})_{k\geq 1}$ is a set of independent, identically distributed random variables with $P(t_k-t_{k-1}>1)=e^{-(\lambda+\mu)}$, the expected passage times of S^* to the set $\{((0),\emptyset),(\emptyset,(1))\}$ have uniformly bounded expectation. Thus, S^* is uniformly positive recurrent. □

Since S^* is uniformly positive recurrent, it has a unique stationary probability measure ν [10, Theorem 7.1]. This measure is technically defined only on E^*, but it can be extended to all of E by defining $\nu(E\setminus E^*)=0$. Like the continuous-time S process, the discrete-time process S^* approaches equilibrium in a bounded way.

Corollary 4 (The Speed of Reaching Equilibrium of the Embedded Process). *Fix an initial distribution ζ on E^*. Let S^*_ζ be a copy of S^* starting with initial distribution ζ. There exists a copy S^*_ν of S^* such that for any $k\geq 1$,*

$$\|S^*_\zeta(k)-S^*_\nu(k)\|_{\text{tot}}\leq 2(1-e^{-(\lambda+\mu)})^k$$

where $\|\cdot\|_{\mathrm{tot}}$ *is the total variation norm.*

At this point, we know that an equilibrium measure η for S exists and an equilibrium measure ν for S^* exists. These two measures are related via the PASTA property [12]. One advantage of ν over η is the ability to express equilibrium probabilities of escaping interference in terms of arrivals. Once expressed in terms of arrivals, some explicit derivations can be made using another discrete-time process X embedded in S.

Consider the ordered set of "exit points" of S, that is the set of the points in the interval $[0, 1]$ at which a collision or a traversal takes place, ordered by the time of the event. We derive a discrete-time exit process X from S in such a way that X_n is the n^{th} element of the set of exit points. In our previous paper, we developed some properties of X including an explicit form for its equilibrium distribution π and its transition function. In this paper, the explicit form of π and the equilibrium distribution of X is used to derive a closed form expression for the equilibrium probabilities of escaping interference.

4. The clean-slate escaping probabilities

This first section on the probability of escaping interference considers the simplest possible initial value for S. The "clean-slate" probabilities assume that the shared-pathway process starts with no initial travelers: $S(0) \in E_{0,0}$. Let P^0 denote the probability inherited from S with $S(0) \in E_{0,0}$. The simplicity of the P^0-probabilities of escaping interference indicates that these derivations be considered without the extra complications inherent in considering other initial distributions for S.

Theorem 5 (The Clean-Slate Unilateral Probability). *The P^0-probability that the first arrival on the left escapes interference is*

$$\frac{\lambda}{\lambda+\mu}e^{-\mu} + \frac{\mu}{\lambda+\mu}e^{-(\lambda+2\mu)}.$$

Proof. The probability that the first arrival to the shared pathway is a left-arrival is $\lambda/(\lambda+\mu)$. This initial left-arrival escapes interference if and only if the first right-arrival arrives after this left-arrival traverses ($R_1 > L_1 + 1$ given $R_1 > L_1$). The difference $R_1 - L_1$ is exponentially distributed with rate μ, so the probability that $R_1 - L_1$ is greater than one is $e^{-\mu}$.

The probability that there are exactly $k \geq 1$ right-arrivals before the first left-arrival is $(\mu/(\lambda+\mu))^k(\lambda/(\lambda+\mu))$. In this case, the first left-arrival escapes interference if and only if it arrives after the previous right-arrival has completely traversed the shared pathway ($L_1 > R_k + 1$ given $L_1 > R_k$) and the next right-arrival arrives after this left-arrival traverses ($R_{k+1} >$

$L_1 + 1$ given $R_{k+1} > L_1$). Since there are no arrivals in the interval (L_1, R_k), $L_1 - R_k$ is exponential with rate $\lambda + \mu$. Thus, the probability that $L_1 - R_k$ is greater than one is $e^{-(\lambda+\mu)}$. There may be left-arrivals, though, in the interval (R_{k+1}, L_1). By the memoryless property of exponential random variables, $R_{k+1} - L_1$ is exponentially distributed with rate μ, and the probability that $R_{k+1} - L_1$ is greater than one is $e^{-\mu}$. Thus, the P^0-probability that the next left-arrival escapes interference is

$$\frac{\lambda}{\lambda+\mu}e^{-\mu} + \sum_{k=1}^{\infty}\left(\frac{\mu}{\lambda+\mu}\right)^k \frac{\lambda}{\lambda+\mu}e^{-(\lambda+\mu)}e^{-\mu}.$$

(This is also proved in [7, Theorem 4.2].) □

THEOREM 6 (The Clean-Slate Bilateral Probability). *The P^0-probability that the first arrival escapes interference is $\frac{\lambda}{\lambda+\mu}e^{-\mu} + \frac{\mu}{\lambda+\mu}e^{-\lambda}$.*

Proof. The probability that the first arrival is on the left is $\lambda/(\lambda+\mu)$. In this case, this first arrival escapes interference if and only if the first right-arrival arrives at least one time unit after the first left-arrival ($R_1 > L_1+1$ given $R_1 > L_1$). By the memoryless property of exponential random variables, $R_1 - L_1$ is exponentially distributed with rate μ. Thus, the probability that the first arrival escapes interference given that it is on the left is $e^{-\mu}$. By similar reasoning, the probability that the first arrival is on the right is $\mu/(\lambda+\mu)$ and the probability that the first arrival escapes interference given that it is on the right is $e^{-\lambda}$. The theorem follows. □

5. The equilibrium probabilities of escaping interference

Recall that the discrete-version S^* of the shared-pathway process has equilibrium distribution ν. In this section, we consider the P^ν-probabilities of escaping interference. We derive the bilateral probability and approximate the unilateral probability. The proof of the bilateral probability relies on a result derived in [7]. The main idea of the proof is threefold: 1) express the desired probability as a limit of two sums, 2) express these sums in terms of the exit process X, and 3) use the explicit equilibrium distribution of X for calculation. This same method of proof does not work in the unilateral case. Besides the approximation and bounds for the unilateral probability in this paper, there is also a simulation of the unilateral probability for different values of λ and μ [7, Table 1]. The following ratio

$$\Pi(\lambda,\mu) = \begin{cases} 1/(1+2\lambda) & \text{if } \lambda = \mu, \\ (\mu-\lambda)/[\mu e^{2(\mu-\lambda)} - \lambda] & \text{if } \lambda \neq \mu, \end{cases}$$

was developed in [7] from the position process X and will be used to describe the probabilities in this section.

LEMMA 7. *The P^ν-probability that the first arrival is on the left and escapes interference is $\frac{\lambda}{\lambda+\mu}\Pi(\lambda,\mu)$.*

Proof. Let $K(n)$ be the cardinality of the first n arrivals which arrive on the left and escape interference. The limit as $n \to \infty$ of the ratio of $K(n)$ to the number of the first n arrivals that are on the left is P^η-almost surely equal to $\Pi(\lambda,\mu)$ [7, Lemma 4.1]. Since the limit as $n \to \infty$ of the ratio of the number of left-arrivals to total arrivals is $\lambda/(\lambda+\mu)$ P^η-almost surely,

$$\lim_{n\to\infty} \frac{K(n)}{n} = \frac{\lambda}{\lambda+\mu}\Pi(\lambda,\mu)$$

P^η-almost surely. We can apply the expectation under the assumption that S^* is in equilibrium to the above equation. By Lebesgue's Dominated Convergence Theorem,

$$\lim_{n\to\infty} E^\eta\left[\frac{K(n)}{n}\right] = \frac{\lambda}{\lambda+\mu}\Pi(\lambda,\mu)$$

Notice that $E^\eta[K(n)]$ is the sum from $k=1$ to n of the P^η-probability that the k^{th} arrival is on the left and escapes interference. As k gets large, this probability approaches the P^ν probability by Corollary 4. Thus, the P^ν-probability that the first arrival is on the left and escapes interference is $\lambda\Pi(\lambda,\mu)/(\lambda+\mu)$. $\square$

COROLLARY 8. *The P^ν-probability that the first arrival is on the right and escapes interference is $\frac{\mu}{\lambda+\mu}\Pi(\mu,\lambda)$.*

Proof. A direct consequence of swapping λ and μ in Lemma 7. $\square$

THEOREM 9 (The Equilibrium Unilateral Probability). *The value $\frac{\lambda}{\lambda+\mu}\Pi(\lambda,\mu) + \frac{\mu}{\lambda+\mu}\Pi(\mu,\lambda)e^{-2\mu}$ is less than the P^ν-probability that the first arrival on the left escapes interference, and their difference is bounded above by*

$$\frac{\mu}{\lambda+\mu}\left(1-e^{-(\lambda+\mu)}\right)\left(\frac{\lambda}{\lambda+\mu}+e^{-\lambda-2\mu}-\frac{\lambda}{\lambda+\mu}e^{-(\lambda+\mu)}\right)$$

Proof. Let A be the event that the first arrival is on the left and escapes interference. Let B be the event that the first arrival is on the right, it escapes interference, and the first arrival on the left escapes interference. Let C be the event that the first arrival is on the right, it does not escape interference, and the first arrival on the left escapes interference. Then, the probability that we seek is $P^\nu(A\cup B\cup C)$. Since these three events are disjoint, we seek the sum $P^\nu(A)+P^\nu(B)+P^\nu(C)$. The probability of event A is derived in Lemma 7. Thus, $P^\nu(A) = \lambda\Pi(\lambda,\mu)/(\lambda+\mu)$.

The equilibrium probability that the first arrival is on the right and eventually escapes interference is $\mu\Pi(\mu,\lambda)/(\lambda+\mu)$ by Corollary 8. In event B, the first arrival on the right escapes interference; thus, the first left-arrival must happen after time t_1+1. In order for this left-arrival to escape interference, there must be no right-arrivals within one time unit of the left-arrival time. The conditional probability that the first left-arrival escapes interference given that the first arrival is on the right and this right-arrival escapes interference is $e^{-2\mu}$. Therefore, $P^\nu(B)=\mu\Pi(\mu,\lambda)e^{-2\mu}/(\lambda+\mu)$.

It is with the probability of C, that we do not have an exact answer. If the first arrival is on the right side of the shared pathway and this arrival has a collision (but this collision is not with the first left-arrival), then it collides with an initial right-traveler. In this case, the effect of the initial distribution of S at time 0 is to produce a right-traveler to collide with the first right-arrival and to possibly produce other initial right-travelers that might have the effect of acting as "bodyguards", clearing the path for the first left-arrival to cross the shared pathway. Thus, the probability of C is bounded above by the probability that the first arrival is on the right at some time $R_1<1$, the first left-arrival is at some time $L_1>R_1$ and there are no right-arrivals in the time interval $[1\vee(L_1-1),L_1+1)$. Thus

$$\begin{aligned}P^\nu(C)&\le\frac{\mu}{\lambda+\mu}\int_0^1(\lambda+\mu)e^{-(\lambda+\mu)R}\int_0^\infty\lambda e^{-\lambda L}e^{-\mu(L\vee 2)}\,dL\,dR\\&=\frac{\mu}{\lambda+\mu}\left[\frac{\lambda}{\lambda+\mu}\left(1-e^{-(\lambda+\mu)}\right)^2+e^{-\lambda-2\mu}(1-e^{-(\lambda+\mu)})\right]\\&=\frac{\mu}{\lambda+\mu}\left(1-e^{-(\lambda+\mu)}\right)\left(\frac{\lambda}{\lambda+\mu}+e^{-\lambda-2\mu}-\frac{\lambda}{\lambda+\mu}e^{-(\lambda+\mu)}\right).\end{aligned}$$

Thus, the equilibrium probability that the first left arrival eventually escapes interference is bounded below by $P^\nu(A)+P^\nu(B)$ and above by the sum of $P^\nu(A)+P^\nu(B)$ and the upper bound on $P^\nu(C)$. □

The bound on the unilateral probability is not as satisfying as an exact result would be. Simulation results are available [7, Table 1]. An exact result might be possible if we had an explicit form for η.

THEOREM 10 (The Equilibrium Bilateral Probability). *The P^ν-probability that the first arrival escapes interference is* $\frac{\lambda}{\lambda+\mu}\Pi(\lambda,\mu)+\frac{\mu}{\lambda+\mu}\Pi(\mu,\lambda)$.

Proof. From Lemma 7, the P^ν-probability that the first arrival is on the left and escapes interference is $\lambda\Pi(\lambda,\mu)/(\lambda+\mu)$. By similar reasoning, the P^ν-probability that the first arrival is on the right and escapes interference is $\mu\Pi(\mu,\lambda)/(\lambda+\mu)$. Thus, by conditioning on the side of arrival, we find that the P^ν-probability that the first arrival traverses the shared pathway without interference is $(\lambda\Pi(\lambda,\mu)+\mu\Pi(\mu,\lambda))(\lambda+\mu)$. □

6. Discussion and conclusions

We derived three probabilities of escaping interference and bounds on a fourth probability. Both the shared pathway process and its discrete-time version converge in a bounded way to their equilibrium distributions. We are left, though, with many directions for future research. The exact structure of the equilibrium distributions seems elusive. Perhaps simulations could be used to get more insight into these equilibrium distributions.

Generalizations of the shared pathway process would be interesting to study. We have studied a slight generalization of when the speeds of the travelers are not the unit speed and may be depend on the side of arrival [7, Section 5]. The generalization that would be most useful for crystal growth theory would be one in which the one-dimensional shared pathway line segment is replaced by a two-dimensional triangle of shared pathways or even to a three-dimensional configuration of shared pathways.

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