Illinois Journal of Mathematics
Volume 50, Number 2, Summer 2006, Pages 395–411
S 0019-2082

CHAOTIC REPRESENTATION PROPERTY OF CERTAIN AZÉMA MARTINGALES

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Dedicated to the memory of J.L. Doob

ABSTRACT. An open problem in martingale theory is to characterize the normal martingales having the chaotic representation property. This property is shown to hold for certain families of multidimensional Azéma martingales.

Il a sans doute fallu autant de génie aux créateurs du calcul infinitésimal pour expliciter la notion si simple de dérivée, qu'à leurs successeurs pour faire tout le reste. L'invention des temps d'arrêt par Doob est tout à fait comparable. These lines are excerpted from the book [4] dedicated to J.L. Doob by C. Dellacherie and P.A. Meyer. No one will deny that inventing stopping times was a stroke of genius; all the more so because they are useless when not considered in their proper context: what Doob had to invent was in fact the triangle *filtrations–martingales–stopping times;* all three of them are simultaneously needed, any two are not sufficient to build upon.

Martingale theory is still far from having revealed all its secrets; among other open questions stands the problem of characterizing the martingales which have the chaotic representation property. No progress has been made on this question in recent years; the scope of this note will be limited to recalling the definitions and giving some new examples. These examples are borrowed from an interesting set of processes, the *multidimensional Azéma martingales*, which has been little explored so far (in particular, only the two-dimensional ones have been classified).

1. The problem of chaotic representation

We shall work in continuous time; all filtrations are implicitly assumed to be right-continuous and completed; all martingales and semimartingales are right-continuous and left-limited. If $X = (X_t)_{t\geqslant 0}$ and $Y = (Y_t)_{t\geqslant 0}$ are two (semi)martingales, recall that their covariation (colloquially called their

Received June 6, 2005; received in final form February 27, 2006.
2000 *Mathematics Subject Classification.* 60G44.

bracket) is the process $[X,Y]$ such that $[X,Y]_t$ is the limit, in a suitable sense, of the sums $\sum_i (X_{t_{i+1}} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i})$ along dyadic subdivisions of $[0,t]$. By convention, all brackets and stochastic integrals will be taken null at time 0 (the above sum does not include the term $X_0 Y_0$).

We shall be interested in martingales taking their values in $\mathbb{R}^n$, where $n \geqslant 1$; their "martingaleness" only refers to the affine structure of $\mathbb{R}^n$, but the next definition involves its Euclidean structure. δ^{ij} is the usual Kronecker symbol.

Definition. An $\mathbb{R}^n$-valued martingale $X = (X^1, \ldots, X^n)$ is called *normal* if the processes $X^i_t X^j_t - \delta^{ij} t$ (or, equivalently, the processes $[X^i, X^j]_t - \delta^{ij} t$) are martingales.

More generally, one could define normal martingales taking values in a Euclidean affine space.

An n-dimensional Brownian motion is an example of a normal martingale; conversely, by a well known theorem due to Lévy, if a normal martingale is continuous, it must be a Brownian motion. In dimension 1, a compensated, standard Poisson process $N_t - t$ is a (discontinuous) normal martingale.

If X is a normal martingale in $\mathbb{R}^n$ and f a square integrable function from the "simplex" $C_k = \{(t_1, \ldots, t_k) \,:\, 0 < t_1 < \ldots < t_k\}$ to the space of k-linear forms on $\mathbb{R}^n$, the real-valued multiple stochastic integral

$$I = \int_{0<t_1<\ldots<t_k} f(t_1, \ldots, t_k)\, \mathrm{d}X_{t_1} \ldots \mathrm{d}X_{t_k}$$

can be defined and verifies the isometry property $\|I\|_{\mathrm{L}^2(\Omega)} = \|f\|_{\mathrm{L}^2(C_k, \mathrm{Lebesgue})}$. The closed linear subset of $\mathrm{L}^2(\Omega)$ consisting of such k-tuple integrals is called the *k-th chaos* of the martingale X; moreover, for $k \neq \ell$, the k-th chaos and the ℓ-th one are orthogonal subspaces in the Hilbert space $\mathrm{L}^2(\Omega)$. The closure of the sum of all these subspaces is called the *chaotic space* associated to X; its elements are the random variables $U \in \mathrm{L}^2(\Omega)$ admitting a (unique) orthogonal expansion of the form

$$U = \sum_{k \geqslant 0} \int_{0<t_1<\ldots<t_k} f_k(t_1, \ldots, t_k)\, \mathrm{d}X_{t_1} \ldots \mathrm{d}X_{t_k}\ , \tag{1}$$

with $\|U\|^2_{\mathrm{L}^2(\Omega)} = \sum_k \|f_k\|^2_{\mathrm{L}^2(C_k, \mathrm{Lebesgue})}$. (By convention, the zeroth chaos consists of all constant random variables.)

Definition. A normal martingale X is said to have the *chaotic representation property* if its chaotic space is the Hilbert space $\mathrm{L}^2\big(\Omega, \sigma(X), \mathbb{P}\big)$, consisting of all square-integrable functionals of X.

All this, and much more, can be found in Chapter XXI of Dellacherie-Maisonneuve-Meyer [5]; their presentation is restricted to the one-dimensional case, but the extension to $n > 1$ is straightforward.

A well-known theorem of Wiener [17] (who borrowed the word 'chaos' from the polynomial chaos) asserts that the chaotic representation property holds when X is a Brownian motion. (Actually, his chaoses were slightly different from the modern ones, which were only later introduced by Itô [10].) A similar theorem, also due to Wiener [17], holds for compensated Poisson processes; he called this the 'discrete chaos'.

Whether a given normal martingale X has the chaotic representation property or not depends only on the law of X: if X has the chaotic representation property, so does also any other martingale with the same law. The problem of chaotic representation consists in characterizing (the laws of) the normal martingales having the chaotic representation property. At first sight, it might seem that filtrations are very far in the background, the filtration generated by X being the only one to enter the picture. Contrary to this impression, my feeling is that the solution to this problem should involve some kind of "purity", that is, the fact that the filtration generated by X is equal to another, a priori smaller, filtration. Some hints in this direction can be found in Théorème 5 of [7] and Proposition 6 of [8], and in their proofs. Lemma 2 and Corollary 2 below also describe a situation where the chaotic representation property stems from X being adapted to one of its sub-filtrations.

2. Azéma martingales

As explained by Dellacherie, Maisonneuve and Meyer in [5], a necessary condition for X to have the chaotic representation property is the (weaker) *previsible representation property:* every r.v. $U \in \mathrm{L}^2\big(\sigma(X)\big)$ has the form

$$U = \mathbb{E}[U] + \int_0^\infty H_s \,\mathrm{d}X_s \ ,$$

where H is previsible (for the filtration of X) and satisfies

$$\mathbb{E}\Big[\int_0^\infty H_s^2 \,\mathrm{d}s\Big] = \mathrm{Var}[U] < \infty$$

(in the multidimensional case, H is valued in the dual of $\mathbb{R}^n$).

In turn, for a normal martingale X, the previsible representation property entails the existence of previsible processes Φ_k^{ij} such that the martingales $[X^i, X^j]_t - \delta^{ij} t$ have the form $\sum_k \int_0^t \Phi_k^{ij}(s) \,\mathrm{d}X_s^k$. This system of relations, often written in differential form

$$\mathrm{d}[X^i, X^j]_t = \delta^{ij} \,\mathrm{d}t + \sum_k \Phi_k^{ij}(t) \,\mathrm{d}X_t^k \ ,$$

is called a *structure equation.* One usually works in the filtration generated by X, with the previsible processes $\Phi_k^{ij}(t)$ explicitly given as functionals of the path of X on the interval $[0,t)$. The unknown in such an equation is not the process X itself, but its law; it is implicitly assumed that X must be a martingale.

An interesting case of structure equations is when the previsible processes $\Phi_k^{ij}(t)$ are of the form $\phi_k^{ij}(X_{t-})$, for some Borel functions $\phi_k^{ij}:\mathbb{R}^n\to\mathbb{R}$. Then, under some uniqueness assumptions, the martingale X is a Markov process (see Phan [13] and [14]), with an infinitesimal generator that can be written explicitly (see Proposition 3.3.1 in Taviot [16]). But we shall need the explicit form of the generator only in a particular instance which is much simpler than the general case (Lemma 3).

The most elementary example of a structure equation is

$$\mathrm{d}[X^i,X^j]_t=\delta^{ij}\,\mathrm{d}t\ ,$$

whose solution is n-dimensional Brownian motion; it is unique (in law) by Lévy's theorem. More generally, for $c_i\in\mathbb{R}$, the structure equation

$$\mathrm{d}[X^i,X^j]_t=\begin{cases}0 & \text{if } i\neq j,\\ \mathrm{d}t+c_i\,\mathrm{d}X_t^i & \text{if } i=j\end{cases}$$

has a unique solution, whose components X^i are independent; X^i is a Brownian motion if $c_i=0$, and, if $c_i\neq 0$, X^i is a sum of jumps with amplitude c_i, occurring at Poisson times with intensity $\mathrm{d}t/c_i^2$, compensated by the constant drift $-\,\mathrm{d}t/c_i$.

Definition. A normal martingale X is called an *Azéma martingale* if its initial value X_0 is deterministic and if X satisfies a structure equation of the form

$$\mathrm{d}[X^i,X^j]_t=\delta^{ij}\,\mathrm{d}t+\sum_k\phi_k^{ij}(X_{t-})\,\mathrm{d}X_t^k\ ,$$

where $\phi_k^{ij}:\mathbb{R}^n\to\mathbb{R}$ are *affine* functions.

One of the reasons why such processes are worth being studied is that they occur naturally in quantum stochastic calculus, as solutions to linear (this means affine, of course!) quantum SDEs on Bosonic Fock space; see Meyer [12]. Another reason is that the chaotic representation property has been established for some Azéma martingales, but not all; the remaining ones provide a good testing-bench to study this property.

Besides Brownian motions and compensated Poisson processes, the oldest known example of an Azéma martingale is the one-dimensional process

$$X_t=\operatorname{sgn}B_t\,\sqrt{2(t-g_t)}\,, \tag{2}$$

where B is a Brownian motion started at 0 and g_t the last zero of B on $[0,t]$; Azéma has shown in Proposition 118 of [3] that, up to a constant factor, X is the optional projection of B on the filtration of $\operatorname{sgn} B$ (in particular, X is a martingale); X is also the solution (unique in law) to the structure equation $\mathrm{d}[X,X]_t = \mathrm{d}t - X_{t-}\,\mathrm{d}X_t$.

LEMMA 1. *Let X be an n-dimensional Azéma martingale; for $k \geqslant 0$, denote by χ_k the k-th chaos associated to X.*

For all $0 < t_1 < \ldots < t_q$ and all polynomials $P : (\mathbb{R}^n)^q \to \mathbb{R}$, the random variable $U = P(X_{t_1}, \ldots, X_{t_q})$ belongs to the space $\bigoplus_{k=0}^{d} \chi_k$, where d denotes the total degree of P. (A fortiori, U is in $\mathrm{L}^2(\mathbb{P})$.)

More precisely, there exists an algorithm, with inputs the initial value X_0, the times $t_1, \ldots, t_q$, the polynomial P and the coefficients of the structure equation of X (that is, the affine functions ϕ_k^{ij}), and with output the f_k in the chaotic expansion (1) *of U.*

Proof. When $n = 1$, this lemma is established in Lemme 7 of [6], for structure equations a little more general than Azéma martingales. (The existence of the algorithm is not mentioned in [6], but the proof of this Lemme 7 does construct such an algorithm.) In higher dimensions the proof is exactly the same, with more cumbersome notation. □

LEMMA 2. *Let Y be a martingale of the form $\ell \circ X$, where X is an Azéma martingale in $\mathbb{R}^n$ and ℓ a linear mapping from $\mathbb{R}^n$ to some vector space E. Suppose that, for each $t > 0$ and each μ in the dual of E, the r.v. $\mathrm{e}^{\mu Y_t}$ is integrable. Then:*

(1) *Y is unique in law: if X' is any other Azéma martingale with the same structure equation and the same initial condition as X, $\ell \circ X'$ has the same law as Y.*
(2) *The space $\mathrm{L}^2\big(\sigma(Y)\big)$ is included in the chaotic space of X.*
(3) *If furthermore X is measurable with respect to $\sigma(Y)$, it is unique in law and it has the chaotic representation property.*

Proof. For fixed $0 < t_1 < \ldots < t_q$, Lemma 1 says that any r.v. of the form $P(Y_{t_1}, \ldots, Y_{t_q})$, with P a polynomial, belongs the chaotic space of X; moreover its expectation (equal to its component f_0 in the zeroth chaos), can be computed in terms of the map ℓ, the structure equation and the initial condition Y_0.

As the law of $(Y_{t_1}, \ldots, Y_{t_q})$ is a probability on E^q having all exponential moments, the r.v. of the above form are dense in $\mathrm{L}^2\big(\sigma(Y_{t_1}, \ldots, Y_{t_q})\big)$; therefore that space must be included in the chaotic space of X. Since q, $t_1, \ldots, t_q$ are arbitrary, this establishes the second part of the statement.

Similarly, by exponential integrability, the r.v. $\exp\big(\mathrm{i}\,(\mu_1 Y_{t_1} + \ldots + \mu_q Y_{t_q})\big)$ can be approximated in L^1 by the polynomials $\sum_{p=0}^{r} \mathrm{i}^p (\ldots)^p/p!$; so, in the limit, its expectation is expressed in terms of ℓ, the structure equation and the initial condition; this shows uniqueness of Y.

The third conclusion is a trivial consequence of the first two ones. □

A very important case is when Y is X itself:

Corollary 1. *Let X be an Azéma martingale; suppose that, for each $t>0$ and each μ in the dual of $\mathbb{R}^n$, the r.v. $\mathrm{e}^{\mu X_t}$ is integrable. Then:*

(1) *Uniqueness holds: any other Azéma martingale with the same structure equation and the same initial condition has the same law as X.*
(2) *X possesses the chaotic representation property.*

Corollary 1 applies in particular when each X_t is bounded; as we shall see, this case already yields many instances of chaotic representation. As for an example, the process given by (2) is obviously bounded by $\sqrt{2t}$; so it has the chaotic representation property and is the unique solution to its structure equation. More generally, the most general one-dimensional Azéma martingale obeys the structure equation

$$\mathrm{d}[X,X]_t = \mathrm{d}t + (\alpha + \beta X_{t-})\,\mathrm{d}X_t \;; \tag{3}$$

it can be shown that if $-2 \leqslant \beta < 0$, X_t is bounded, by

$$\Big| X_t + \frac{\alpha}{\beta} \Big| \leqslant \sqrt{\Big(X_0 + \frac{\alpha}{\beta}\Big)^2 + \frac{2}{-\beta}\, t}$$

(this will be generalized to multidimensional martingales by Lemma 4), and if $\beta = 0$, X has all exponential moments because it is a Brownian motion or a compensated Poisson process. All in all, for $\beta \in [-2,0]$, uniqueness and the chaotic representation property hold. (In fact, existence and uniqueness hold for all real α and β; and the chaotic representation property also holds when $\beta < -2$ and $\alpha + \beta X_0 \neq 0$. When $\beta > 0$, and when $\beta < -2$ and $\alpha + \beta X_0 = 0$, the chaotic representation is an open problem.)

3. Multidimensional examples

Another case when Corollary 1 has been put to work involves some bidimensional Azéma martingales $X = (Y,Z)$ driven by a structure equation of the form

$$\begin{cases} \mathrm{d}[Y,Y]_t = \mathrm{d}t + (aY_{t-} + bZ_{t-} + c)\,\mathrm{d}Y_t, \\ \mathrm{d}[Y,Z]_t = 0, \\ \mathrm{d}[Z,Z]_t = \mathrm{d}t + (\alpha Y_{t-} + \beta Z_{t-} + \gamma)\,\mathrm{d}Z_t \,. \end{cases}$$

It is shown in [2] that X is bounded if the coefficients a, b, α and β verify certain inequalities; uniqueness and chaotic representation property then follow by Corollary 1.

This bidimensional result will now be generalized, by using exponential integrability instead of boundedness in Corollary 1, and by going to higher dimensions. For $n > 2$, we shall work with a structure equation whose coefficients are linked by some algebraic condition, which cannot be seen when $n \leqslant 2$; this condition is technically sufficient to make the proofs work, I do not know if it could be dispensed of. It says that the $n \times n$ matrix of coefficients which corresponds to $\left(\begin{smallmatrix} a & b \\ \alpha & \beta \end{smallmatrix}\right)$ in the above 2-dimensional case is the product of a diagonal and a symmetric matrix. It will be convenient to have a name for this family of martingales:

Definition. An Azéma martingale will be called *good* if it is governed by a structure equation of the form

$$\mathrm{d}[X^i, X^j]_t = \begin{cases} 0 & \text{if } i \neq j, \\ \mathrm{d}t + \psi_i(X_{t-})\,\mathrm{d}X^i_t & \text{if } i = j, \end{cases} \tag{4}$$

where $\psi_i : \mathbb{R}^n \to \mathbb{R}$ are n affine functions, given by

$$\psi_i(x) = c_i - a_i \sum_{j=1}^{n} b_{ij} x^j \ , \tag{5}$$

with real coefficients $a_i,\ b_{ij},\ c_i$ satisfying the following relations: $a_i \geqslant 0$, the matrix (b_{ij}) is symmetric and positive (but not necessarily definite positive), and $a_i b_{ii} \leqslant 2$.

The latter inequality says that the coefficient of x^i in ψ_i belongs to the interval $[-2, 0]$. In the one-dimensional case, this means that the coefficient β in (3) is in this interval (known in the trade as the "good interval", whence the name 'good Azéma martingale').

Notice that (4) implies that the jumps ΔX^i verify $\left(\Delta X^i_t\right)^2 = \psi_i(X_{t-})\,\Delta X^i_t$; hence, whenever the process X^i jumps, its jump must be equal to $\psi_i(X_{t-})$.

A structure equation such as (4) always has solutions. (This still holds when the ψ_i in (4) are merely required to be continuous, instead of affine; it is a particular case of Théorème 4.0.2 of Taviot [16].)

> Readers familiar with the "double symmetry" properties of the coefficients of structure equations (see [1]) can remark that these properties are automatically satisfied in (4), because (4) is so to speak doubly diagonal: first, one has $[X^i, X^j] = 0$ for $i \neq j$, and second, the previsible representation of $[X^i, X^i]_t - t$ involves $\mathrm{d}X^i_t$ only and not $\mathrm{d}X^j_t$ for $j \neq i$.
>
> It can also be observed that the symmetry $b_{ij} = b_{ji}$ has nothing to do with the double-symmetry properties; it is an altogether additional hypothesis.

The next lemma gives the infinitesimal generator of a good Azéma martingale. The canonical basis of $\mathbb{R}^n$ is called $(\varepsilon_1, \ldots, \varepsilon_n)$; D_i stands for $\partial/\partial x^i$, and D_{ii} for $\partial^2/(\partial x^i)^2$.

LEMMA 3. *Let X be a martingale governed by a structure equation of the form*

$$\mathrm{d}[X^i, X^j]_t = \begin{cases} 0 & \text{if } i \neq j, \\ \mathrm{d}t + \Phi^i_t \, \mathrm{d}X^i_t & \text{if } i = j, \end{cases}$$

where Φ^i are previsible processes. If $f : \mathbb{R}^n \to \mathbb{R}$ is a C^2 function, one has

$$f(X_t) = f(X_0) + \sum_{i=1}^n \int_0^t (\Lambda^i_s f)(X_{s-}) \, \mathrm{d}X^i_s + \int_0^t (L_s f)(X_s) \, \mathrm{d}s \, ,$$

with

$$(\Lambda^i_s f)(x) = \int_0^1 D_i f(x + \theta \Phi^i_s \varepsilon_i) \, \mathrm{d}\theta = \begin{cases} \dfrac{f(x + \Phi^i_s \varepsilon_i) - f(x)}{\Phi^i_s} & \text{if } \Phi^i_s \neq 0, \\ D_i f(x) & \text{if } \Phi^i_s = 0, \end{cases}$$

and

$$\begin{aligned} (L_s f)(x) &= \sum_{i=1}^n \int_0^1 D_{ii} f(x + \theta \Phi^i_s \varepsilon_i)(1-\theta) \, \mathrm{d}\theta \\ &= \sum_{i=1}^n \begin{cases} \dfrac{f(x + \Phi^i_s \varepsilon_i) - f(x) - \Phi^i_s D_i f(x)}{(\Phi^i_s)^2} & \text{if } \Phi^i_s \neq 0, \\ \frac{1}{2} D_{ii} f(x) & \text{if } \Phi^i_s = 0. \end{cases} \end{aligned}$$

In particular, if X is driven by (4)*, Λ^i_s and L^i_s do not depend on s and ω, and they are given by*

$$(\Lambda^i f)(x) = \int_0^1 D_i f\big(x + \theta \psi_i(x) \varepsilon_i\big) \, \mathrm{d}\theta = \begin{cases} \dfrac{f(x + \psi_i(x) \varepsilon_i) - f(x)}{\psi_i(x)} & \text{if } \psi_i(x) \neq 0, \\ D_i f(x) & \text{if } \psi_i(x) = 0, \end{cases}$$

and

$$\begin{aligned} (Lf)(x) &= \sum_{i=1}^n \int_0^1 D_{ii} f(x + \theta \psi_i(x) \varepsilon_i)(1-\theta) \, \mathrm{d}\theta \\ &= \sum_{i=1}^n \begin{cases} \dfrac{f(x + \psi_i(x) \varepsilon_i) - f(x) - \psi_i(x) D_i f(x)}{\psi_i(x)^2} & \text{if } \psi_i(x) \neq 0, \\ \frac{1}{2} D_{ii} f(x) & \text{if } \psi_i(x) = 0. \end{cases} \end{aligned}$$

So this L is the infinitesimal generator of the process X.

Proof. The first part of the lemma just copies Proposition 3.3.1 of Taviot [16] in the diagonal case we are interested in; the second part follows by replacing Φ^i_s with $\psi_i(X_{s-})$. □

We now start focusing on good Azéma martingales. If X is such a process, the symbols a_i, b_{ij} and c_i will always denote the coefficients of its structure equation featuring in (5). The same letter B will be used to denote the symmetric, positive $n \times n$ matrix (b_{ij}), and the linear map from $\mathbb{R}^n$ to itself associated to that matrix.

Proposition 1. *Let X be a good Azéma martingale. The martingale BX with values in* $\operatorname{Range} B$ *has all exponential moments.*

Proof. The elements of $\mathbb{R}^n$ will be considered as column vectors, and ${}^t x$ will stand for the transpose of x; so ${}^t x B x = \sum_{ij} b_{ij} x^i x^j$. (This contradicts the notation used earlier, for instance in the definition of normal martingales, where row vectors were used for typographical simplicity.)

For $x \in \mathbb{R}^n$, set $f(x) = {}^t x B x$, and observe that

$$f(x+\rho\varepsilon_i) = f(x) + 2\rho \sum_j b_{ij} x^j + \rho^2 b_{ii} \ .$$

If $b_{ii} > 0$, this quadratic function of ρ has the property that $f(x+\rho\varepsilon_i) \leqslant f(x)$ for all ρ between 0 and $-\frac{2}{b_{ii}} \sum_j b_{ij} x^j$. Since $0 \leqslant a_i \leqslant \frac{2}{b_{ii}}$, one has, for all $\theta \in [0,1]$,

$$f(x+\rho\varepsilon_i) \leqslant f(x) \qquad \text{for } \rho = -\theta a_i \sum_j b_{ij} x^j \ . \tag{6}$$

This remains true when $b_{ii} = 0$, because in that case $b_{ij} = 0$ for all j by positivity of B.

Consider now the function $g(x) = \sqrt{1+f(x)}$. One has

$$D_i g = \frac{\sum_j b_{ij} x^j}{g} \qquad \text{and} \qquad D_{ii} g = \frac{b_{ii}}{g} - \frac{\left(\sum_j b_{ij} x^j\right)^2}{g^3} \ ;$$

hence

$$\|\nabla g\|^2 = \sum_i (D_i g)^2 = \frac{{}^t x B^2 x}{1 + {}^t x B x} \leqslant \frac{{}^t x B^2 x}{{}^t x B x} \leqslant C_B$$

where C_B is the largest eigenvalue of B; in the sequel, C_B is a generic notation for constants depending only upon B. So g is Lipschitz, and

$$g(y+z) \leqslant g(y) + C_B \|z\| \ .$$

Fix an arbitrary $\mu > 0$, and set $e(x) = \mathrm{e}^{\mu g(x)}$. The latter inequality becomes

$$e(y+z) \leqslant e(y) \exp(C_{B,\mu} \|z\|) \ ; \tag{7}$$

also, (6) propagates from f to g to e, yielding

$$\forall \theta \in [0,1] \qquad e(x - \theta a_i \textstyle\sum_j b_{ij} x^j \varepsilon_i) \leqslant e(x) \ .$$

Applying now (7) to $y = x - \theta a_i \sum_j b_{ij} x^j \varepsilon_i$ and to $z = \theta c_i \varepsilon_i$, one finds

$$e\big(x + \theta \psi_i(x) \varepsilon_i\big) \leqslant e(x - \theta a_i \textstyle\sum_j b_{ij} x^j \varepsilon_i) \exp(C_{B,\mu} |c_i|) \leqslant C_{B,c,\mu}\, e(x) \ , \tag{8}$$

uniformly in $\theta \in [0,1]$.

As $\|Bx\| \leqslant \sqrt{\lambda f(x)} \leqslant \sqrt{\lambda}\, g(x)$, where λ is the largest eigenvalue of B, the Proposition will be proved if we establish that $e(X_t)$ is in L^1. This will be done with the help of Lemma 3, for which we need the derivatives of e:

$$\begin{aligned} D_i e &= \mu\, e\, D_i g \;; \\ D_{ii} e &= \mu\, e\, D_{ii} g + \mu^2\, e\, \big(D_i g\big)^2 \leqslant \mu\, e\, \frac{b_{ii}}{g} + \mu^2 e\, \|\nabla g\|^2 \\ &\leqslant (\mu\, b_{ii} + \mu^2 C_B)\, e \leqslant C_{B,\mu}\, e \;. \end{aligned} \tag{9}$$

The next step is to import from Lemma 3 the change of variable formula

$$\begin{aligned} e(X_t) = e(X_0) &+ \sum_i \int_0^t \Big[\int_0^1 D_i e\big(X_{s-} + \theta\psi_i(X_{s-})\varepsilon_i\big)\,\mathrm{d}\theta\Big]\,\mathrm{d}X^i_s \\ &+ \sum_i \int_0^t \Big[\int_0^1 D_{ii} e\big(X_s + \theta\psi_i(X_s)\varepsilon_i\big)(1-\theta)\,\mathrm{d}\theta\Big]\,\mathrm{d}s \;. \end{aligned}$$

Let K be a compact containing X_0; call T the first time when $X \notin K$, and X^T the martingale X stopped at T. Notice that $X_s \in K$ for $s < T$, and recall that whenever X^i jumps, its jump is $\psi_i(X_-)\varepsilon_i$; so, by continuity of ψ_i, the process X^T is bounded. Write

$$\begin{aligned} e(X^T_t) = e(X_0) &+ \sum_i \int_0^{t\wedge T} \Big[\int_0^1 D_i e\big(X^T_{s-} + \theta\psi_i(X^T_{s-})\varepsilon_i\big)\,\mathrm{d}\theta\Big]\,\mathrm{d}X^i_s \\ &+ \sum_i \int_0^{t\wedge T} \Big[\int_0^1 D_{ii} e\big(X^T_s + \theta\psi_i(X^T_s)\varepsilon_i\big)(1-\theta)\,\mathrm{d}\theta\Big]\,\mathrm{d}s \;. \end{aligned}$$

As X is normal, the stochastic integrals in the above formula are square-integrable martingales on $[0,t]$, and they are killed by taking expectations:

$$\mathbb{E}[e(X^T_t)] = e(X_0) + \mathbb{E}\Big[\int_0^{t\wedge T} \sum_i \Big(\int_0^1 D_{ii} e\big(X^T_s + \theta\psi_i(X^T_s)\varepsilon_i\big)(1-\theta)\,\mathrm{d}\theta\Big)\,\mathrm{d}s\Big] \;.$$

Now, by (9), $D_{ii} e\big(X^T_s + \theta\psi_i(X^T_s)\varepsilon_i\big)$ in this formula can be majorized by $C_{B,\mu} e\big(X^T_s + \theta\psi_i(X^T_s)\varepsilon_i\big)$; in turn, by (8), this is dominated by $C_{B,c,\mu} e(X^T_s)$, and we have $\mathbb{E}[e(X^T_t)] = e(X_0) + \mathbb{E}\big[\int_0^{t\wedge T} U_s\,\mathrm{d}s\big]$ with $U_s \leqslant C_{B,c,\mu} e(X^T_s)$. Putting $V_s = U_s\, \mathbf{1}_{\{s\leqslant T\}} \leqslant C_{B,c,\mu} e(X^T_s)$, we end up with

$$\mathbb{E}[e(X^T_t)] = e(X_0) + \int_0^t \mathbb{E}[V_s]\,\mathrm{d}s \qquad \text{where} \quad V_s \leqslant C_{B,c,\mu} e(X^T_s) \,.$$

Since X^T is bounded, so is also $\mathbb{E}[e(X^T_t)]$, and we have a differential inequality for $\mathbb{E}[e(X^T_t)]$ whose solution is $\mathbb{E}[e(X^T_t)] \leqslant e(X_0)\,\mathrm{e}^{C_{B,c,\mu} t}$. The constant does not depend on K; so, taking $K = \mathrm{Ball}(0,r)$ and letting $r \to \infty$, Fatou's lemma gives in the limit $\mathbb{E}[e(X_t)] \leqslant e(X_0)\,\mathrm{e}^{C_{B,c,\mu} t}$, and $e(X_t)$ belongs to L^1. □

COROLLARY 2. *If X is a good Azéma martingale, $Y = BX$ satisfies the conclusions of Lemma 2. In particular, if Y generates the same σ-field as X, uniqueness and the chaotic representation property hold for X too.*

Proof. By Proposition 1, Y satisfies the hypothesis of Lemma 2. $\square$

An example of the situation described in Corollary 2 is the case of the Dunkl martingale, which was introduced by Rösler and Voit in [15], and whose chaotic representation property is established by Gallardo and Yor in [9]. They are interested in a certain finite set $R_+ \subset \mathbb{R}^m$, whose elements (called the positive roots) have Euclidean norm $\sqrt{2}$; to each $\alpha \in R_+$ is associated a coefficient $k(\alpha) \geqslant 0$. In the present setting, it is more convenient to deal with the set

$$T \;=\; \left\{\sqrt{k(\alpha)}\,\alpha \;:\; \alpha \in R_+\,,\; k(\alpha) > 0\right\} \;\subset\; \mathbb{R}^m \setminus \{0\}\;.$$

A fundamental property is that any two vectors in T are not colinear; T also has some further properties, inherited from the rich algebraic-geometric structure of R_+, but these further properties are irrelevant for our purpose.

We shall be interested in a good Azéma martingale X in the larger space $\mathbb{R}^m \oplus \mathbb{R}^T$, which has dimension $n = m + |T|$. Each vector $v \in \mathbb{R}^m \oplus \mathbb{R}^T$ is made of a vector $v' = (v^1, \ldots, v^m) \in \mathbb{R}^m$ and a family $(v^\tau,\ \tau \in T)$ of real numbers; so we have two sets of indices, i which ranges from 1 to m, and τ, which ranges over T. Define a linear map Q from $\mathbb{R}^m \oplus \mathbb{R}^T$ to $\mathbb{R}^m$ by $Qv = v' + \sum_\tau v^\tau \tau$. The structure equation of X is defined by (4), with

$$\begin{cases} \psi_i(v) = 0 & \text{for } i \in \{1, \ldots, m\}, \\ \psi_\tau(v) = -\dfrac{2}{<\tau,\tau>} <\tau, Qv> & \text{for } \tau \in T. \end{cases}$$

It is easy to see that X is good, that is, ψ_i and ψ_τ have the form (5): it suffices to take the coefficients c null, to let a be given by $a_i = 0$ and $a_\tau = 2/<\tau,\tau>$, and to take the $n \times n$ matrix B equal to the Gram matrix of the system of n vectors of $\mathbb{R}^m$ consisting of the canonical basis of $\mathbb{R}^m$ and the vectors $\tau \in T$. B is positive because it is a Gram matrix (but its rank is m, so it is not definite positive); and the diagonal coefficients are $a_i b_{ii} = 0$ and $a_\tau b_{\tau\tau} = 2$.

One easily sees that the range of B consists of all vectors $v \in \mathbb{R}^m \oplus \mathbb{R}^T$ such that $v^\tau = <\tau, v'>$ for each τ; moreover, $B = J \circ Q$, where $J \,:\, \mathbb{R}^m \to \operatorname{Range} B$ is the isomorphism such that, for $w \in \mathbb{R}^m$, $v = Jw$ is given by $v' = w$ and $v^\tau = <w, \tau>$.

Proposition 1 and Corollary 2 say that BX, or, what amounts to the same, QX, has all exponential moments, is unique in law, and all functionals of QX are in the chaotic space of X. This process $D = QX$ is called the *Dunkl martingale* by Gallardo-Yor [9]. It is the sum, in $\mathbb{R}^m$, of X' and $\sum_\tau X^\tau \tau$; the part X' turns out to be a Brownian motion (because $\psi_i = 0$ for $i \in \{1, \ldots, m\}$), and the rest is a compensated sum of jumps. So X' is the continuous part of the martingale D, and $X^\tau \tau$ is the compensated sum of all jumps of D that are parallel to τ (this is where non-colinearity is used). Consequently, X' and X^τ are adapted to the filtration generated by D, and uniqueness and the chaotic representation property hold for X by Corollary 2.

The proof by Gallardo and Yor that D has all exponential moments is much shorter and easier than in the general case of Proposition 1, because the Euclidean norm of D is continuous; more precisely, $\|D\|$ is a Bessel process. Continuity is due to the fact that the jumps of D corresponding to a given $\tau \in T$ are symmetries with respect to the hyperplane $\tau^\perp$. This symmetry property is easily read on the structure equation of X, because, when a jump occurs, one component X^τ of X jumps with amplitude $\psi_\tau(X_-)$; so D jumps with amplitude $\psi_\tau(X_-)\,\tau = -(2/{<}\tau,\tau{>})\,{<}\tau, D_-{>}\,\tau$, which means that D is symmetric to D_- with respect to the hyperplane $\tau^\perp$.

By Corollary 2, uniqueness and the chaotic representation property still hold if the coefficient 2 in the structure equation is replaced by any value in the interval $[0,2]$; in that case, QX jumps to the corresponding intermediate point on the segment linking QX_- to its mirror symmetric point. An interesting particular case is the mid-value 1; then $U = QX$ jumps to its orthogonal projection on the hyperplane, in the same way (and for the same reason) as X in (2) always jumps back to the origin. The simplest non-trivial example of this situation is the 2-dimensional process $X = (Y, Z)$ with structure equation

$$\begin{cases} \mathrm{d}[Y,Y]_t = \mathrm{d}t, \\ \mathrm{d}[Y,Z]_t = 0, \\ \mathrm{d}[Z,Z]_t = \mathrm{d}t - \dfrac{1}{\sqrt{2k}}(Y_{t-} + \sqrt{2k}\,Z_{t-})\,\mathrm{d}Z_t\ ; \end{cases}$$

$U = QX = Y + \sqrt{2k}\,Z$ is the one-dimensional Markov process with generator

$$Lf(u) = \tfrac{1}{2}\,f''(u) + 2k \begin{cases} \dfrac{f(0) - f(u) + u\,f'(u)}{u^2} & \text{if } u \neq 0, \\ \tfrac{1}{2}\,f''(u) & \text{if } u = 0. \end{cases}$$

Whenever U jumps (this occurs with intensity $2k\,\mathrm{d}t/U_{t-}^2$), it jumps from U_{t-} to $U_t = 0$; between jumps, $|U|$ behaves as a Bessel process with dimension $4k+1$.

The most straightforward (albeit quite useful) application of Proposition 1 is when B is non-degenerate:

Corollary 3. *Let X be a good Azéma martingale. If the matrix B is definite positive, X has all exponential moments, it has the chaotic representation property, and uniqueness holds for its structure equation.*

Proof. The linear map $B : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism; so by Proposition 1, X satisfies the hypothesis of Corollary 1. □

Unexpectedly, the coefficients a_i in the structure equation (5) of a good Azéma martingale play no rôle in Proposition 1 and its corollaries. They are just required to satisfy the constraint $0 \leqslant a_i \leqslant 2/b_{ii}$, but do not even appear in the constants featuring in the exponential integrability of BX. This may indicate that the definition of good Azéma martingales, where the coefficient of x^j in ψ_i has been factorized as $-a_i b_{ij}$ with B symmetric, could reflect an underlying structure of these processes. But we shall now see that the a_i do come into play when one deals with boundedness of X, instead of exponential integrability.

LEMMA 4. *Let X be a good Azéma martingale; suppose that all its coefficients a_i in* (5) *are strictly positive, and put*

$$h(x) = \tfrac{1}{2} \sum_{ij} b_{ij} x^i x^j - \sum_i \frac{c_i}{a_i} x^i .$$

The process $h \circ X$ is bounded above on all finite time-intervals; moreover, if $a_i b_{ii} = 2$ for all i, this process is deterministic, and equal to $h(X_0) + \frac{1}{2}(\operatorname{Tr} B)\, t$.

Proof. Consider the semimartingale

$$Y_t = h(X_t) + \sum_i \Big(\frac{1}{a_i} - \frac{b_{ii}}{2}\Big) [X^i, X^i]_t .$$

Using $[X^i, X^j] = 0$ for $i \neq j$, one has

$$\begin{aligned} \mathrm{d}Y_t = \tfrac{1}{2}\Big(\sum_{ij} b_{ij} X^i_{t-}\, \mathrm{d}X^j_t + \sum_{ij} b_{ij} X^j_{t-}\, \mathrm{d}X^i_t + \sum_i b_{ii}\, \mathrm{d}[X^i, X^i]_t\Big) \\ - \sum_i \frac{c_i}{a_i}\, \mathrm{d}X^i_t + \sum_i \Big(\frac{1}{a_i} - \frac{b_{ii}}{2}\Big)\, \mathrm{d}[X^i, X^i]_t \,; \end{aligned}$$

the first two sums are equal by symmetry of B, and the coefficients of $\mathrm{d}[X^i, X^i]$ add up to $1/a_i$, so

$$\mathrm{d}Y_t = \sum_{ij} b_{ij} X^j_{t-}\, \mathrm{d}X^i_t - \sum_i \frac{c_i}{a_i}\, \mathrm{d}X^i_t + \sum_i \frac{1}{a_i}\, \mathrm{d}[X^i, X^i]_t .$$

Replacing $\mathrm{d}[X^i, X^i]_t$ by its value $\mathrm{d}t + (c_i - a_i \sum_j b_{ij} X^j_{t-})\, \mathrm{d}X^i_t$ taken from the structure equation, one obtains $\mathrm{d}Y_t = \big(\sum_i 1/a_i\big)\, \mathrm{d}t$; this gives the identity

$$h(X_t) = h(X_0) + \Big(\sum_i \frac{1}{a_i}\Big)\, t - \sum_i \Big(\frac{1}{a_i} - \frac{b_{ii}}{2}\Big) [X^i, X^i]_t .$$

The lemma is proved by observing that the bracket term $(1/a_i - b_{ii}/2)[X^i, X^i]_t$ in the right-hand side is always $\geqslant 0$, because $a_i > 0$ and $a_i b_{ii} \leqslant 2$; so one has

$$h(X_t) \leqslant h(X_0) + \Big(\sum_i \frac{1}{a_i}\Big)\, t \,;$$

moreover, the bracket term vanishes if $a_i b_{ii} = 2$; consequently, in that case

$$h(X_t) = h(X_0) + \Big(\sum_i \frac{1}{a_i}\Big)\, t = h(X_0) + \Big(\sum_i \frac{b_{ii}}{2}\Big)\, t\ . \qquad \square$$

Corollary 4. *Let X be a good Azéma martingale. If $a_i > 0$ for all i and if B is definite positive, X is bounded on each interval $[0,t]$.*

Proof. As B is definite positive, the set $\{x \in \mathbb{R}^n : h(x) \leqslant c\}$ is bounded (it is a solid ellipsoid). By Lemma 4, on each finite interval $[0,t]$, X takes its values in such a set. $\square$

Proposition 2. *Let X be a good Azéma martingale with $a_i > 0$ for all i, and such that the vector $(c_1/a_1, \ldots, c_n/a_n)$ belongs to the subspace* Range B*. All exponential moments of X_t are finite (and, by Corollary 1, uniqueness and the chaotic representation property hold).*

Proof. As the vector with components c_i/a_i belongs to the range of B, there exists a vector $w = (w^1, \ldots, w^n)$ such that $c_i/a_i = \sum_j b_{ij} w^j$; and the function h of Lemma 4 verifies $2h(x) = \sum_{ij} b_{ij}(x^i - w^i)(x^j - w^j) - \sum_{ij} b_{ij} w^i w^j$. Calling λ the largest eigenvalue of B, one has for all x

$$\|B(x-w)\|^2 \leqslant \lambda \sum_{ij} b_{ij}(x^i - w^i)(x^j - w^j) = \lambda\big(2h(x) + \sum_{ij} b_{ij} w^i w^j\big)\ ;$$

hence, as a consequence of Lemma 4, the process $B(X - w)$ is bounded on finite intervals. In turn, so is also BX, and so are also the processes $\psi_i(X)$, which are affine functions of BX by (5).

The rest of the proof is similar to (and technically simpler than) the proof of Proposition 1. Fixing $\mu > 0$ and an index i, and putting $e(x) = \exp(\mu x^i)$, we shall show that the r.v. $e(X_t)$ is integrable. All first and second partial derivatives of e vanish, except $D_i e = \mu e$ and $D_{ii} e = \mu^2 e$; so the change of variable formula from Lemma 3 gives

$$\begin{aligned} e(X_t) = e(X_0) &+ \mu \int_0^t \Big[\int_0^1 e\big(X_{s-} + \theta\psi_i(X_{s-})\varepsilon_i\big)\,\mathrm{d}\theta\Big]\,\mathrm{d}X_s^i \\ &+ \mu^2 \int_0^t \Big[\int_0^1 e\big(X_s + \theta\psi_i(X_s)\varepsilon_i\big)(1-\theta)\,\mathrm{d}\theta\Big]\,\mathrm{d}s\ . \end{aligned}$$

Now, on a fixed time interval $[0,\tau]$, $\psi_i \circ X$ is bounded, so

$$e\big(X_s + \theta\psi_i(X_s)\varepsilon_i\big) \leqslant C(\tau)\, e(X_s)\ .$$

With the same localization arguments as in the proof of Proposition 1, this implies $\mathbb{E}[e(X_t)] \leqslant e(X_0)\exp\big[\frac{1}{2}\mu^2 C(\tau) t\big]$ for $t \in [0,\tau]$; hence $e(X_t)$ is in L^1. (But, because $C(\tau)$ is of the order of $\sqrt{\tau}$, the estimate so obtained for $\mathbb{E}[e(X_t)]$ grows as $\mathrm{e}^{Ct^{3/2}}$ for large t.) $\square$

Proposition 3. *Let X be a good Azéma martingale such that $a_i b_{ii} = 2$ for all i, and $\operatorname{rank} B = n-1$. Uniqueness and the chaotic representation property hold for X.*

Proof. Consider the vector with components c_i/a_i (the denominators cannot vanish because $a_i b_{ii} = 2$). It decomposes as a sum $v+Bw$, with $v \in \operatorname{Ker} B$ and $Bw \in \operatorname{Range} B$. The case when $v = 0$ is taken care of by Proposition 2, so we assume $v \neq 0$; since $\operatorname{rank} B = n-1$, the subspace $\operatorname{Ker} B$ is one-dimensional and consists of all multiples of v. Call $\pi : \mathbb{R}^n \to \operatorname{Range} B$ the orthogonal projection on $\operatorname{Range} B$. The function h introduced in Lemma 4 can be written

$$h(x) = \tfrac{1}{2}\sum_{ij} b_{ij}(x^i - w^i)(x^j - w^j) - \tfrac{1}{2}\sum_{ij} b_{ij} w^i w^j - \sum_i v^i x^i \ .$$

In this formula, the first sum depends on x via πx only. Hence the identity $h(X_t) = C + \frac{1}{2}(\operatorname{Tr} B)\, t$ from Lemma 4 has the form $<v, X_t> = f(t, \pi X_t)$, which says that the projection $\pi^\perp X_t$ of X_t on $\operatorname{Ker} B$ is a deterministic function of t and πX_t. Finally, $X = \pi X + \pi^\perp X$ is adapted to the filtration generated by πX, and the Proposition is a consequence of Corollary 2. □

We have seen several good Azéma martingales having the chaotic representation property, for various reasons: boundedness, exponential integrability, or some kind of purity. A natural question is: Do all good Azéma martingales have the chaotic representation property? And if the answer is positive, is there a common underlying scheme that includes all cases?

Another possible direction for further studies would be to investigate what happens for "almost good" Azéma martingales. What we mean by this is an Azéma martingale whose structure equation is a limit of structure equations of good Azéma martingales, in the following sense. Say that a matrix (m_{ij}) is good if it has the form $m_{ij} = a_i b_{ij}$ with $a_i \geqslant 0$, (b_{ij}) symmetric and $\geqslant 0$, and $m_{ii} \leqslant 2$. An almost good matrix is defined as a limit of good matrices, and an almost good Azéma martingale as an Azéma martingale with a structure equation of type (4), where

$$\psi_i(x) = c_i - \sum_{j=1}^{n} m_{ij} x^j \ , \tag{10}$$

with (m_{ij}) an almost good matrix (instead of a good one for good Azéma martingales). It turns out that the set of good $n \times n$ matrices is not closed, so this extension is strict. For instance, as observed by G. Letac [11], any matrix which is block-triangular, with diagonal blocks square and good (or, for that matter, almost good), is almost good; and so is also any matrix obtained from such matrices by changing the order of the coordinates in $\mathbb{R}^n$. In particular, any triangular matrix with diagonal entries in the interval $[0, 2]$ is almost good.

4. Another example

Good Azéma martingales are reasonable candidates for the chaotic representation property. What about non good ones? (By political correctness, we just call them 'non good'.) If one sticks to Azéma martingales having the diagonal form (4) and (10), but with a matrix (m_{ij}) which is not almost good, there seems to be little hope of establishing the chaotic representation by methods similar to the ones used so far. (The chaotic representation property might hold for completely different reasons, though; this would be most interesting!)

In the world of non diagonal Azéma martingales, there probably exist large families of processes for which boundedness, or the existence of all exponential moments, gives the chaotic representation property via Corollary 1. Here is just one example.

Fix a parameter α such that $0 < \alpha \leqslant 1$, and consider the 2-dimensional Azéma martingale $X = (Y, Z)$ driven by the structure equation

$$\begin{cases} \mathrm{d}[Y,Y]_t = \mathrm{d}t - 2\alpha Y_{t-}\,\mathrm{d}Y_t - \alpha Z_{t-}\,\mathrm{d}Z_t, \\ \mathrm{d}[Y,Z]_t = \qquad - \alpha Z_{t-}\,\mathrm{d}Y_t, \\ \mathrm{d}[Z,Z]_t = \mathrm{d}t \qquad\qquad\qquad - \alpha Z_{t-}\,\mathrm{d}Z_t\ . \end{cases}$$

This structure equation admits solutions starting from any initial condition, because it belongs to type (II) in the classification of 2-dimensional Azéma martingales given in [2]. As the equation is not of type (I), no rotation of the axes can make it diagonal, and a fortiori, no rotation of the axes can make it good nor almost good. But uniqueness in law and the chaotic representation property hold, by Corollary 1, because X is bounded on compacts. This stems from the following simple computation: the structure equation yields

$$\mathrm{d}[Y,Y]_t + \mathrm{d}[Z,Z]_t = 2\,\mathrm{d}t - \alpha(2Y_{t-}\,\mathrm{d}Y_t + 2Z_{t-}\,\mathrm{d}Z_t)\ .$$

Replacing $2Y_{t-}\,\mathrm{d}Y_t$ by $\mathrm{d}Y_t^2 - \mathrm{d}[Y,Y]_t$ and $2Z_{t-}\,\mathrm{d}Z_t$ by $\mathrm{d}Z_t^2 - \mathrm{d}[Z,Z]_t$ gives

$$\alpha(\mathrm{d}Y_t^2 + \mathrm{d}Z_t^2) = 2\,\mathrm{d}t - (1-\alpha)\big(\mathrm{d}[Y,Y]_t + \mathrm{d}[Z,Z]_t\big)\ ,$$

which implies

$$\alpha(Y_t^2 + Z_t^2) = \alpha(Y_0^2 + Z_0^2) + 2t - (1-\alpha)\big([Y,Y]_t + [Z,Z]_t\big)\ ;$$

and boundedness is obtained by observing that $(1-\alpha)\big([Y,Y]_t + [Z,Z]_t\big) \geqslant 0$.

The case when $\alpha = 1$ and $Y_0 = Z_0 = 0$ corresponds to an interesting process: for fixed t, X_t is uniformly distributed on the circle $y^2 + z^2 = 2t$ (this illustrates the arcsine law); the process $(Z_t)_{t\geqslant 0}$ is (in law) the one-dimensional Azéma martingale given by (2); between jumps of X the component Y remains constant and the motion of X is parallel to the z-axis; each jump brings X back to the y-axis, more precisely to one of the two points with abscissae $y = \pm\sqrt{2t}$ on this axis.

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