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ON BURKHOLDER'S SUPERMARTINGALES

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In memory of J.L. Doob

ABSTRACT. For $0 < p < \infty$, put

$$Y_t(c,p) = Y = B_t^{*(p-2)}[B_t^2 - t] + cB_t^{*p}, \quad t > 0,$$

where B_t is a Brownian Motion and $B_t^* = \max_{0 \le s \le t} |B_s|$. Then for $0 < p \le 2$, Y is a submartingale if and only if $c \ge \frac{2-p}{p}$, while for $2 \le p < \infty$, Y is a supermartingale if and only if $c \le \frac{2-p}{p}$. This extends results of Burkholder. The first of these assertions implies a strong version of some of the Burkholder-Gundy inequalities.

1. Introduction

Let B_t, $t \ge 0$, be the standard Brownian motion started at 0. Let $\mathcal{F}_t = \sigma(B_s, s \le t)$, $t \ge 0$. For a function f on $[0,\infty)$ define $f^*(t) = \sup_{0 \le s \le t} |f(s)|$, $0 \le t < \infty$. For $p > 0$ and $c \in (-\infty, \infty)$ define the process $Y_t = Y_t(c,p)$, $t \ge 0$, by $Y_0 = 0$ and

$$Y_t = B_t^{*(p-2)}[B_t^2 - t] + cB_t^{*p}, \quad t > 0.$$

We will prove:

THEOREM 1.1.

(i) *If $0 < p \le 2$, then Y is a submartingale if and only if $c \ge \frac{2-p}{p}$.*

(ii) *If $p \ge 2$, then Y is a supermartingale if and only if $c \le \frac{2-p}{p}$.*

Throughout this paper, stopping time, martingale, submartingale, and supermartingale will always mean with respect to $\mathcal{F}_t$, $t \ge 0$. For the values not covered by Theorem 1.1, Y is neither a submartingale or a supermartingale. Burkholder proved, in [3], that Y is a submartingale if $1 < p \le 2$ and if $c \ge \gamma_p$, where the explicitly given constants γ_p exceed $\frac{2-p}{p}$ except for $p = 2$. He also proved a version of this result for the class of all martingales, the focus of [3].

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A famous theorem of Burkholder and Gundy (see [2]) states that, for $p > 0$, there are positive constants a_p and A_p such that for all stopping times τ with respect to the filtration of B_t, $t > 0$, we have both

$$\mathrm{E}\,\tau^{p/2} \le A_p\,\mathrm{E}\,B_\tau^{*p}, \text{ and} \tag{1}$$

$$\mathrm{E}\,B_\tau^{*p} \le a_p\,\mathrm{E}\,\tau^{p/2}. \tag{2}$$

Theorem 1.1 and the fact that $|B_t| \le B_t^*$ immediately give strong versions of (1) for $0 < p < 2$. It suffices to consider bounded and strictly positive stopping times τ. Then $\mathrm{E}\,Y_\tau \ge 0$, since Y is a submartingale, and this yields

$$\mathrm{E}\,\frac{\tau}{B_\tau^{*(2-p)}} \le \frac{2}{p}\,\mathrm{E}\,B_\tau^{*p}, \quad 0 < p < 2. \tag{3}$$

Such ratio inequalities, including some for general discrete time martingales, go back to Garsia [5]. See [1]. To see that (3) implies the $0 < p < 2$ range of (1), with $A_p = (\frac{2}{p})^{p/2}$, use Holder's inequality:

$$\begin{aligned}
\mathrm{E}\,\tau^{p/2} &= \mathrm{E}\left(\frac{\tau^{p/2}}{B_\tau^{*(2/p)(2-p)}} \cdot B_\tau^{*(2/p)(2-p)}\right) \\
&\le \left[\mathrm{E}\left(\frac{\tau^{p/2}}{B_\tau^{*(p/2)(2-p)}}\right)^{2/p}\right]^{p/2} \left[\mathrm{E}\left(B_\tau^{*(p/2)(2-p)}\right)^{2/(2-p)}\right]^{(2-p)/2} \\
&\le \left(\frac{2}{p}\,\mathrm{E}\,B_\tau^{*p}\right)^{p/2} (\mathrm{E}\,B_\tau^{*p})^{(2-p)/2} \\
&= \left(\frac{2}{p}\right)^{p/2} \mathrm{E}\,B_\tau^{*p}.
\end{aligned}$$

The inequality (3) implies not only the $p < 2$ cases of (1), but also, roughly, that τ cannot be too large where B_τ is small. The inequalities (1)–(3) generalize to inequalities for continuous martingales. See [2], [4], and [6]. As has been noted, in [3] Burkholder is mainly concerned with the analogs of (1) for the exponents $1 \le p \le 2$ for the class of all martingales. (These analogs are not true for $p < 1$.) Burkholder's method for extracting (1)-like inequalities from his submartingales, which is very different from that just given, would yield the $0 < p \le 2$ cases of (1) from our Theorem 1.1 with the same constants $A_p = (\frac{2}{p})^{p/2}$ which we obtained. Conversely, our method together with Burkholder's analogs of the submartingales Y will give analogs of (3) for the class of all martingales, for $1 \le p < 2$.

One key to our proof of Theorem 1.1 is the following. Suppose $\tau \le \eta$ are two stopping times with respect to the Brownian filtration $\mathcal{F}_t$, $t \ge 0$, where $\mathcal{F}_t$ is the completion of $\sigma(B_s, s \le t)$, such that $B_\tau^* = B_\eta^*$. Let Y be as in Theorem 1.1, and define Z by $Z_t = 0$ if $t \le \tau$; $Z_t = Y_t - Y_\tau$ if $t \in [\tau, \eta]$; and

$Z_t = Z_\eta$ if $t > \eta$. Then Z is a martingale, since

$$Z_t = \int_0^t f(s) dM_s,$$

where $f(s)$ is the adapted integrand $B_\tau^{*(p-2)} I(\tau \le s \le \eta)$ and M_s is the Lévy martingale $B_s^2 - s$. Thus whether Y is a submartingale or supermartingale depends only on what happens at those times t such that $B_t = B_t^*$. We will give the proof of Theorem 1.1(i) in detail. The proof of Theorem 1.1(ii) is a little easier and very similar, using Lemmas 2.1 and 2.2 which hold for all positive p, and is not given.

2. Proof of Theorem 1.1

Let P_a and E_a denote probability and expectation associated with a process distributed as $B_t + a$, $t \ge 0$.

Lemma 2.1. *Let* $0 < \varepsilon < 1$, $a > 0$, *and* $q > 0$. *Let* $\tau_a = \tau_{a,\varepsilon} = \inf\{t \ge 0 : |X_t - a| = \varepsilon a\}$. *Then*

(i) $\mathrm{E}_a X_{\tau_a}^* - a = a\varepsilon \log 2$,

(ii) $\mathrm{E}_a X_{\tau_a}^{*q} - a^q = \varepsilon q a^q \log 2 + a^q f_q(\varepsilon)$, *where* $f_q(\varepsilon) = o(\varepsilon)$ *as* $\varepsilon \downarrow 0$, *and*

(iii) $\mathrm{E}_a X_{\tau_a}^{*(q-2)} X_{\tau_a}^2 - a^q = \varepsilon a^q (q-2) \log 2 + a^q g_q(\varepsilon)$, *where* $g_q(\varepsilon) = o(\varepsilon)$ *as* $\varepsilon \downarrow 0$.

Proof. (i) First we prove (i) for $a = 1$, temporarily dropping both subscripts on $\tau_{1,\varepsilon}$ and the subscripts on P_1, E_1. Now

$$\begin{aligned} \mathrm{E}\, X_\tau^* - 1 &= \mathrm{E}(X_\tau^* - 1) I(X_\tau = 1+\varepsilon) + \mathrm{E}(X_\tau^* - 1) I(X_\tau = 1-\varepsilon) \\ &= \frac{1}{2}\varepsilon + \mathrm{E}(X_\tau^* - 1) I(X_\tau = 1-\varepsilon). \end{aligned}$$

If $0 < x < \varepsilon$,

$$\begin{aligned} P(X_\tau^* \ge 1+x, X_\tau = 1-\varepsilon) &= P(X \text{ hits } 1+x \text{ before it hits } 1-\varepsilon \text{ and} \\ &\qquad \text{then hits } 1-\varepsilon \text{ before it hits } 1+\varepsilon) \\ &= \frac{\varepsilon}{\varepsilon + x} \cdot \frac{\varepsilon - x}{2\varepsilon}. \end{aligned}$$

Thus

$$\begin{aligned} \mathrm{E}(X_\tau^* - 1) I(X_\tau = 1-\varepsilon) &= \int_0^\varepsilon P(X_\tau^* \ge 1+x, X_\tau = 1-\varepsilon) dx \\ &= \frac{1}{2} \int_0^\varepsilon \frac{\varepsilon - x}{\varepsilon + x} dx = \varepsilon \log 2 - \varepsilon/2, \end{aligned} \tag{4}$$

providing the $a = 1$ case. To prove (i) for other a an almost identical computation suffices. However it is even easier to note that under P_1 the process

aX_{t/a^2}, $0 \le t \le \tau_1$, has the same distribution as X_t, $0 \le t \le \tau_a$, under P_a, so that, under P_a, $X^*_{\tau_a} - a$ has the same distribution as $a(X^*_{\tau_1} - 1)$ under P_1.

(ii) We have $(y+1)^q - 1 = qy + r_q(y)$, where $r_q(y) = o(y)$ as $y \to 0$. So, using the last sentence of the proof of (i) just above, we obtain

$$\begin{aligned} \mathrm{E}_a(X^{*q}_{\tau_a} - a^q) &= \mathrm{E}_a[(X^*_{\tau_a} - a) + a]^q - a^q \\ &= \mathrm{E}_1\{a[(X^*_{\tau_1} - 1) + 1]\}^q - a^q \\ &= a^q\{q\,\mathrm{E}_1(X^*_{\tau_1} - 1) + \mathrm{E}_1\, r_q(X^*_{\tau_1} - 1)\}. \end{aligned}$$

Now $|X^*_{\tau_1} - 1| \le \varepsilon$, and this together with part (i) of this lemma proves (ii), with $f_q(\varepsilon) = \mathrm{E}_1\, r_q(X^*_{\tau_1} - 1)$.

(iii) Using again the last line of the proof of (i), we note the joint distribution of $(X_{\tau_a}, X^*_{\tau_a})$ under P_a is the same as the joint distribution of $(aX_{\tau_1}, aX^*_{\tau_1})$ under P_1, so that

$$\mathrm{E}_a\, X^{*(q-2)}_{\tau_a} X^2_{\tau_a} - a^q = a^q(\mathrm{E}_1\, X^{*(q-2)}_{\tau_1} X^2_{\tau_1} - 1).$$

Now

$$\begin{aligned} &\mathrm{E}_1\, X_{\tau_1}{}^{*(q-2)} X^2_{\tau_1} \\ &\quad = \mathrm{E}_1\, X^{*(q-2)}_{\tau_1} X^2_{\tau_1} I(X_{\tau_1} = 1+\varepsilon) + \mathrm{E}_1\, X^{*(q-2)}_{\tau_1} X^2_{\tau_1} I(X_{\tau_1} = 1-\varepsilon) \\ &\quad = \frac{1}{2}(1+\varepsilon)^q + (1-\varepsilon)^2\, \mathrm{E}_1\, X^{*(q-2)}_{\tau_1} I(X_{\tau_1} = 1-\varepsilon) \\ &\quad = \frac{1}{2}(1+\varepsilon)^q + (1-\varepsilon)^2 [\mathrm{E}_1\, X^{*(q-2)}_{\tau_1} - \mathrm{E}_1\, X^{*(q-2)}_{\tau_1} I(X_{\tau_1} = 1+\varepsilon)] \\ &\quad = \frac{1}{2}(1+\varepsilon)^q + (1-\varepsilon)^2 \{[\varepsilon q \log 2 + f_q(\varepsilon) + 1] - \frac{1}{2}(1+\varepsilon)^{q-2}\}, \end{aligned}$$

where $f_q(\varepsilon) = o(\varepsilon)$, using part (ii) of this lemma. Now, again using

$$\lim_{y \to 0}((1+y)^\alpha - 1)/y = \alpha,$$

for any fixed α, it is easy to finish the proof of (iii). □

We put, for $a > 0$,

$$S_a = \inf\{t > 0 : |B_t| = a\}$$

and

$$\gamma_{a,\varepsilon} = \gamma_a = \inf\{t > S_a : |B_t - a| = \varepsilon\}.$$

It follows from the fact that $B_t^2 - t$ is a martingale that

$$\mathrm{E}(\gamma_a - S_a | \mathcal{F}_{S_a}) = \mathrm{E}(B^2_{\gamma_a} - B^2_{S_a} | \mathcal{F}_{S_a}) = \varepsilon^2. \tag{5}$$

LEMMA 2.2.

(i) $\mathrm{E}\sup_{0\le t\le a}|Y_t(c,p)|<\infty$, $a>0$,

(ii) $\mathrm{E}[Y_{\gamma_a}(c,p)-Y_{S_a}(c,p)|\mathcal{F}_{S_a}]=\varepsilon(cp+p-2)a^p\log 2+a^p(g_p(\varepsilon)+cf_p(\varepsilon))+a^{p-2}S_a(\varepsilon(2-p)\log 2-f_{p-2}(\varepsilon))-\Theta_p(\varepsilon)a^p$, *where* f *and* g *are as in Lemma 2.1, and* $\Theta_p(\varepsilon)=o(\varepsilon)$ *as* $\varepsilon\downarrow 0$.

Proof. We prove (i) for $a=1$ and $0<p<2$ only. The proof of (i) is immediate for $p\ge 2$. For $0<p<2$, (i) follows from

$$\mathrm{E}\sup_{0\le t\le 1}\frac{t}{B_t^{*(2-p)}}<\infty.$$

Now

$$\begin{aligned}\mathrm{E}\sup_{0\le t\le 1}\frac{t}{B_t^{*(2-p)}}&\le\sum_{k=0}^{\infty}\mathrm{E}\sup_{2^{-(k+1)}\le t\le 2^{-k}}\frac{t}{B_t^{*(2-p)}}\\&\le\sum_{k=0}^{\infty}\mathrm{E}\frac{2^{-k}}{B_{2^{-(k+1)}}^{*(2-p)}}\\&=\sum_{k=0}^{\infty}\mathrm{E}\frac{2^{-k}2^{(k+1)(2-p)/2}}{B_1^{*(2-p)}}<\infty.\end{aligned}$$

To prove (ii), recall that $|B_{S_a}|=B^*_{S_a}=a$, and write, using the notation of Lemma 2.1,

$$\begin{aligned}\mathrm{E}[Y_{\gamma_a}(c,p)-Y_{S_a}(c,p)|\mathcal{F}_{S_a}]&=[\mathrm{E}_a X_{\tau_a}^2X_{\tau_a}^{*(p-2)}-a^p]\\&\quad+[c(\mathrm{E}_a X_{\tau_a}^{*p}-a^p)]\\&\quad+[S_a a^{p-2}-S_a\,\mathrm{E}_a X_{\tau_a}^{*(p-2)}-\mathrm{E}_a\,\tau_a X_{\tau_a}^{*(p-2)}]\\&=:[I]+[II]+[III].\end{aligned}$$

Now

$$[I]+[II]=\varepsilon(cp+(p-2))a^p\log 2+a^p(g_p(\varepsilon)+cf_p(\varepsilon)),$$

where g_p and f_p are as in Lemma 2.1. To evaluate III, note that, by Lemma 2.1(ii),

$$-S_a a^{p-2}+S_a\,\mathrm{E}_a X_{\tau_a}^{*(p-2)}=S_a a^{p-2}((2-p)\varepsilon-f_{2-p}(\varepsilon))-\mathrm{E}_a\,\tau_a X_{\tau_a}^{*(p-2)}.$$

The $\Theta_p(\varepsilon)$ in the statement of this lemma is $-\mathrm{E}_a\,\tau_a X_{\tau_a}^{*(p-2)}$. Note that for $0<p\le 2$,

$$\mathrm{E}_a\,\tau_a X_{\tau_a}^{*(p-2)}\le(a-a\varepsilon)^{p-2}\,\mathrm{E}_a\,\tau_a=(a-a\varepsilon)^{p-2}a^2\varepsilon^2,$$

while for $p\ge 2$,

$$0\le\mathrm{E}_a\,\tau_a X^{*(p-2)}\le(a+a\varepsilon)^{p-2}a^2\varepsilon^2.$$ □

We use the following well known characterizations of sub- and supermartingales to prove Theorem 1.1.

LEMMA 2.3. *Suppose $Z = Z_t$, $t \geq 0$, has continuous paths, is adapted to $\mathcal{F}_t$, $t \geq 0$, and that $\mathrm{E}\, Z_t^* < \infty$, $t \geq 0$.*

(i) *Then Z is a submartingale if and only if for every pair of stopping times $\eta_1 \leq \eta_2$ such that $\mathrm{E}\, Z_{\eta_2}^* < \infty$, $\mathrm{E}\, Z_{\eta_1} \leq \mathrm{E}\, Z_{\eta_2}$.*

(ii) *Also, Z is a submartingale if and only if for stopping times as in (i) $\mathrm{E}(Z_{\eta_2} | \mathcal{F}_{\eta_1}) \geq Z_{\eta_1}$.*

Proof of Theorem 1.1. We know that $Y(c,p)$ is continuous (from Lemma 2.2(i)). Furthermore

$$\mathrm{E} \sup_{0 \leq t \leq S_a} |Y_t(c,p)| < \infty, \tag{6}$$

for all c, p, a. For $p \geq 2$ this is immediate. For $0 < p < 2$ it follows from

$$\mathrm{E} \sup_{0 \leq t \leq S_a} \frac{t}{B_t^{*(2-p)}} < \infty.$$

Now

$$\mathrm{E} \sup_{0 \leq t \leq S_a} \frac{t}{B_t^{*(2-p)}} \leq \mathrm{E} \sup_{0 \leq t \leq 1} \frac{t}{B_t^{*(2-p)}} + \mathrm{E} \sup_{1 \leq t \leq \max(S_a,1)} \frac{t}{B_t^{*(2-p)}}.$$

The first summand is finite, by Lemma 2.2(i). And

$$\mathrm{E} \sup_{1 \leq t \leq \max(S_a,1)} \frac{t}{B_t^{*(2-p)}} \leq \mathrm{E} \frac{1}{B_1^*} \mathrm{E}(\max(S_a,1) | \mathcal{F}_1) \leq \mathrm{E} \frac{1}{B_1^*} \mathrm{E}\, S_{2a} < \infty.$$

Recall that we prove Theorem 1.1(i) only. The proof of the "only if" part of Theorem 1.1(i) follows from Lemma 2.3(ii), with $\eta_1 = S_1$ and $\eta_2 = \gamma_1 (= \gamma_{1,\varepsilon})$ for small enough ε. For we have, using Lemma 2.2(ii),

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{-1} [\mathrm{E}(Y_{\gamma_1}(c,p) | \mathcal{F}_{S_1}) - Y_{S_1}(c,p)] = [cp + (p-2) + S_1(2-p)] \log 2,$$

and this limit is uniform on $\{S_1 < y\}$ for every $y > 0$. Now $c < \frac{2-p}{p}$ implies $cp + (p-2) < 0$, and so on $\{0 < S_1 < [(p-2) + cp]/(2-p)\}$, this limit is negative. Thus there is a fixed ε where

$$P[\mathrm{E}(Y_{\gamma_1}(c,p) | \mathcal{F}_{S_1}) < Y_{S_1}(c,p)] > 0.$$

To prove the "if" part of Theorem 1.1(i), first note that it suffices to prove that $Y_t(\frac{2-p}{p}, p)$, $t \geq 0$, is a submartingale, since if $c < \frac{2-p}{p}$,

$$Y_t(c,p) = Y_t\left(\frac{2-p}{p}, p\right) + \left[\frac{2-p}{p} - c\right] B_t^*,$$

and B_t^* is nondecreasing and thus a submartingale. For the remainder of this paper we put $W_t = Y_t(\frac{2-p}{p}, p)$, where $0 < p \leq 2$.

To show that W is a submartingale, it suffices to show that if $\theta_1 \le \theta_2$ are stopping times satisfying $\theta_2 \le S_a$ for some a, then $\mathrm{E}\, W_{\theta_1} \le \mathrm{E}\, W_{\theta_2}$. This follows from Lemma 2.3(i), and the fact that $\lim_{a\to\infty} S_a = \infty$, so that if θ_i satisfies $\mathrm{E}\, W^*_{\theta_i} < \infty$, $i = 1, 2$, then

$$\lim_{a\to\infty} \mathrm{E}\, W_{\min(\theta_i, S_a)} = \mathrm{E}\, W_{\theta_i}, \quad i = 1, 2.$$

Without loss of generality, we may take $a = 1$, and will prove Theorem 1.1(i) by showing that, for any stopping times $\alpha \le \beta \le S_1$,

$$\mathrm{E}\, W_\alpha \le \mathrm{E}\, W_\beta. \tag{7}$$

To prove (7), let $0 < \delta < 1$ and let $\varepsilon \in (0, 1)$ be so small that

$$\varepsilon(2 - p) \log 2 + f_{p-2}(\varepsilon) \ge 0, \tag{8}$$

where f_{p-2} is the function in the statement of Lemma 2.1. Put

$$\begin{aligned}
\gamma_0(\delta) = \gamma_0 &= \inf\{t : B^*_t = \delta\},\\
\eta_0(\delta, \varepsilon) = \eta_0 &= \inf\{t \ge \gamma_0 : |B_t - \delta| = \varepsilon\delta\},\\
\gamma_1 &= \inf\{t \ge \eta_0 : B_t = B^*_t\} \quad (\text{note } \gamma_1 = \eta_0 \text{ if } B_{\eta_0} > B_{\gamma_0}),\\
\eta_1 &= \inf\{t \ge \gamma_1 : |B_t - B^*_{\gamma_1}| = \varepsilon B^*_{\gamma_1}\},\\
&\;\;\vdots\\
\gamma_k &= \inf\{t \ge \eta_{k-1} : B_t = B^*_t\},\\
\eta_k &= \inf\{t \ge \gamma_k : |B_t - B^*_{\gamma_k}| = \varepsilon B^*_{\gamma_k}\}.
\end{aligned}$$

Let $M = \min\{k : B^*_{\eta_k} \ge 1\}$. Note $B^*_{\eta_M} \le 1 + \varepsilon < 2$. We have

$$\frac{1}{2}\varepsilon B^*_{\gamma_k} \le \mathrm{E}(B^*_{\eta_k} - B^*_{\gamma_k} | \mathcal{F}_{\gamma_k}) \le \varepsilon B^*_{\gamma_k}, \tag{9}$$

and

$$\mathrm{E}(\eta_k - \gamma_k | \mathcal{F}_{\gamma_k}) = \varepsilon^2 (B^*_{\gamma_k})^2, \tag{10}$$

since the conditional distribution of $\eta_k - \gamma_k$ given $\mathcal{F}_{\gamma_k}$ is the distribution of $\tau_{B^*_{\gamma_k}, \varepsilon B^*_{\gamma_k}}$, noting that $B^*_{\eta_k} - B^*_{\gamma_k} = \varepsilon B^*_{\gamma_k}$ on $\{|B_{\eta_k}| > |B_{\gamma_k}|\}$. Thus

$$\mathrm{E} \sum_{k=0}^{M} \mathrm{E}(\eta_k - \gamma_k) \le 2\varepsilon\, \mathrm{E} \sum_{k=0}^{M} (B^*_{\eta_k} - B^*_{\gamma_k}) = 2\varepsilon\, \mathrm{E}(B^*_M - B^*_{\gamma_0}) \le 4\varepsilon.$$

Now by Lemma 2.2(ii) and (8), if $a^- = \min(a,0)$,

$$
\begin{aligned}
(11)\qquad & \mathrm{E}\sum_{k=0}^{M} \mathrm{E}(W_{\eta_k} - W_{\gamma_k}|\mathcal{F}_{\gamma_k})^- \\
&= \mathrm{E}\sum_{k=0}^{\infty} \mathrm{E}(W_{\eta_k} - W_{\gamma_k}|\mathcal{F}_{\gamma_k})^- I(k \le M) \\
&\ge \mathrm{E}\sum_{k=0}^{\infty} \left[{B^*_{\gamma_k}}^{p} \left(g_p(\varepsilon) + \left(\frac{2-p}{p}\right) f_p(\varepsilon) - \Theta_p(\varepsilon) \right)^- \right] I(k \le M) \\
&\ge \sum_{k=0}^{\infty} \Gamma_p(\varepsilon) P(k \le M),
\end{aligned}
$$

where $\Gamma_p(\varepsilon) \to 0$ as $\varepsilon \downarrow 0$, since $B^*_{\gamma_k} \le 1$ if $k \le M$. Using (9) and the facts that

$$\sum_{k=0}^{\infty} (B^*_{\eta_k} - B^*_{\gamma_k}) I(k \le M) \le 2$$

and $B^*_{\gamma_k} \ge \delta$ for $k \ge 0$, we have

$$
\begin{aligned}
2 &\ge \mathrm{E}\sum_{k=0}^{\infty} \mathrm{E}(B^*_{\eta_k} - B^*_{\gamma_k}|\mathcal{F}_{\gamma_k}) I(k \le M) \\
&\ge \mathrm{E}\sum_{k=0}^{\infty} \frac{1}{2}\varepsilon B^*_{\gamma_k} I(k \le M) \\
&\ge \frac{1}{2}\varepsilon\delta \sum_{k=0}^{\infty} P(k \le M).
\end{aligned}
$$

Together with (11) this implies

$$(12)\qquad \liminf_{\varepsilon \downarrow 0} \mathrm{E}\sum_{k=0}^{M} E(W_{\eta_k} - W_{\gamma_k}|\mathcal{F}_{\gamma_k})^- \ge 0.$$

Now define $\alpha(\delta,\varepsilon)$ by

$$\alpha(\delta,\varepsilon) = \begin{cases} S_\delta (= \gamma_0) & \text{on } \{\alpha \le S_\delta\}, \\ \eta_k & \text{on } \{\gamma_k < \alpha \le \eta_k\},\ k \ge 0, \\ \alpha & \text{on } \{\eta_k < \alpha \le \gamma_k\},\ k \ge 0. \end{cases}$$

Similarly define $\beta(\delta,\varepsilon)$. Then

$$\alpha(\delta,\varepsilon) \le \beta(\delta,\varepsilon) \le S_2.$$

We have $\alpha(\delta,\varepsilon) \geq \max(\alpha,\delta)$, and

$$\mathrm{E}(\alpha(\delta,\varepsilon) - \max(\alpha,\delta)) \leq \sum_{k=0}^{M} \mathrm{E}(\gamma_k - \eta_k) I(k \leq M) \to 0 \text{ as } \varepsilon \to 0. \tag{13}$$

Also $\mathrm{E}(\beta(\delta,\varepsilon) - \max(\beta,\delta)) \to 0$ as $\varepsilon \to 0$. Since $B_t^2 - t$ is a martingale, and since B_t^* does not change on $[\eta_k, \gamma_k]$, we have that if $T_1 \leq T_2$ are stopping times,

$$\mathrm{E}(W_{\min(T_2,\eta_k)} | \mathcal{F}_{T_1}) = W_{T_1} \text{ on } \{\gamma_k \leq T_1 \leq \eta_k\}, \tag{14}$$

using the remarks at the end of Section 1.

On $\{\alpha(\delta,\varepsilon) = \eta_k\}$, recalling that $\beta(\delta,\varepsilon)$ cannot be in (γ_j, η_j) for any j,

$$\begin{aligned} W_{\beta(\delta,\varepsilon)} - W_{\alpha(\delta,\varepsilon)} = &\sum_{j=k+1}^{M} (W_{\eta_j} - W_{\gamma_j}) I(\beta(\delta,\varepsilon) > \gamma_j) \\ &+ \sum_{j=k}^{M} (W_{\min(\gamma_{j+1},\beta(\delta,\varepsilon))} - W_{\eta_j}) I(\beta(\delta,\varepsilon) > \eta_j). \end{aligned}$$

Thus, if $\{\alpha(\delta,\varepsilon) = \eta_k\} = F_k$,

$$\begin{aligned} &\mathrm{E}(W_{\beta(\delta,\varepsilon)} - W_{\alpha(\delta,\varepsilon)}) I(F_k) \\ &\quad = \sum_{j=k+1}^{M} \mathrm{E}\,\mathrm{E}(W_{\eta_j} - W_{\gamma_j} | \mathcal{F}_{\gamma_j}) I(\beta(\delta,\varepsilon) > \gamma_j) I(F_k) \\ &\qquad + \sum_{j=k}^{M} \mathrm{E}\,\mathrm{E}(W_{\min(\gamma_{j+1},\beta(\delta,\varepsilon))} - W_{\eta_j} | \mathcal{F}_{\eta_j}) I(\beta(\delta,\varepsilon) > \eta_j) I(F_k). \end{aligned}$$

All the conditional expectations in the second sum equal zero, while the first sum exceeds $\sum_{k=0}^{M} \mathrm{E}(W_{\eta_k} - W_{\gamma_k} | \mathcal{F}_{\gamma_k})^-$. Thus using (12) we have

$$\liminf_{\varepsilon \downarrow 0} \mathrm{E}(W_{\beta(\delta,\varepsilon)} - W_{\alpha(\delta,\varepsilon)}) I(F_k) \geq 0. \tag{15}$$

Similarly if

$$\begin{aligned} G_k &= \{\eta_k < \alpha(\delta,\varepsilon) \leq \gamma_k\}, \quad 1 \leq k \leq M-1, \\ G_0 &= \{\alpha(\delta,\varepsilon) = \gamma_0\}, \end{aligned}$$

then

$$\liminf_{\varepsilon \downarrow 0} \mathrm{E}(W_{\beta(\delta,\varepsilon)} - W_{\alpha(\delta,\varepsilon)}) I(G_k) \geq 0. \tag{16}$$

The path continuity of W and (6) and (11) give

$$\mathrm{E}\, W_{\beta(\delta,\varepsilon)} \to \mathrm{E}\, W_{\min(\beta,\delta)}$$

and

$$\mathrm{E}\, W_{\alpha(\delta,\varepsilon)} \to \mathrm{E}\, W_{\min(\alpha,\delta)}$$

as $\varepsilon \to 0$, so adding all the terms of (15) and (16) gives

$$\mathrm{E}\, W_{\min(\beta,\gamma_0(\delta))} \geq \mathrm{E}\, W_{\min(\alpha,\gamma_0(\delta))}, \quad 0 < \delta < 1.$$

Now we let $\delta \to 0$ and again (6), (11), and path continuity give (7).

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