ILLINOIS JOURNAL OF MATHEMATICS
Volume 31, Number 3, Fall 1987

ON INNER FUNCTIONS WITH DERIVATIVE IN BERGMAN SPACES

BY

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Introduction

Let U be the unit disk in the complex plane $\mathbf{C}$ and f a function holomorphic in U (abbreviated $f \in H(U)$). For any α, $-1 < \alpha < \infty$, and $p, 0 < p < \infty$, we define

$$M_p(r, f)^p = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta,$$

where $0 \leq r < 1$,

$$\|f\|_{H^p} = \sup_{0<r<1} M_p(r, f), \quad \text{and} \quad \|f\|_{p,\alpha}^p = \int_0^1 (1-r)^\alpha M_p(r, f)^p \, dr.$$

If $\|f\|_{H^p} < \infty$ then f is said to belong to the Hardy Space H^p, and if $\|f\|_{p,\alpha} < \infty$, then f is said to belong to the weighted Bergman Space $A^{p,\alpha}$. H^p can be viewed as $A^{p,-1}$. H^∞ is the collection of bounded analytic functions, and $A^{1,\alpha}$ for $\alpha = (1/p) - 2$ is often referred to as B^p, $0 < p < 1$. An inner function is an element ϕ of H^∞ such that $|\phi(e^{i\theta})| = 1$ almost everywhere on ∂U with respect to one-dimensional Lebesgue measure; [7] has the basic facts about inner functions.

Several authors have considered the problem of necessary and sufficient conditions that $\phi' \in A^{p,\alpha}$ for various p, α; in particular, this problem was treated extensively in [1] and [3]. In [1], the following theorem was proved:

THEOREM. *Let ϕ be an inner function*, $1 \leq p \leq 2, \alpha > -1$.
(a) *If $\alpha > p - 1$, then $\phi' \in A^{p,\alpha}$.*
(b) *If $p - 2 < \alpha < p - 1$, then $\phi' \in A^{p,\alpha}$ iff $\phi' \in A^{1,\alpha-p+1}$.*
(c) *If $\alpha \leq p - 2$, $p > 1$, then $\phi' \in A^{p,\alpha}$ iff ϕ is a finite Blaschke product.*

In [13] Verbitsky announced a significant generalization of this result, including, in particular, an extension to $1 \leq p < \infty$, without, however, supplying proofs. In this paper we extend the above result to $1 \leq p < \infty$, by a general method which is also used to provide alternative proofs to some other results in [1] and [10].

Received January 27, 1986.

The basic approach is to make use of the notion of the "approximating Blaschke product" B_ϕ for an inner function ϕ developed by Cohn in [5] and [6]; we get an equivalent condition for $\phi' \in A^{p,\alpha}$ in terms of the zeroes $\{z_k\}$ of B_ϕ, from which the results follow in a straightforward way. Along the way we make use of a recent result of Luecking [11] which characterizes the positive measures μ on U for which there is a $C > 0$ such that $(\int_U |f|^q \, d\mu)^{1/q} \leq C\|f\|_{p,\alpha}$ for all $f \in A^{p,\alpha}, 0 < q < p$; his result is that this occurs iff

$$k(z) = \mu(D_\varepsilon(z))/m_\alpha(D_\varepsilon(z)) \in L^a \quad \text{for } 1/s + q/p = 1,$$

where $D_\varepsilon(z)$ is the pseudo-hyperbolic disk around $z \in U$ having radius ε, and

$$m_\alpha = (1 - |z|)^\alpha \, dm(z),$$

for dm two dimensional Lebesgue measure on U. (Actually, we use another equivalent condition stated later in his paper.) We also use a result which provides an equivalent condition for

$$\sum |f(z_k)|(1 - |z_k|) \leq C\|f\|_{p,\alpha}$$

to hold for all $f \in A^{p,\alpha}$, where $\{z_k\}$ is a Blaschke sequence; this condition is different from Luecking's and although apparently "well known", may not previously have appeared with proof. This result together with some duality notions used in its statement are presented first in the following.

In what follows we will refer to weighted Bergman Spaces as "Bergman Spaces," and variously write A for $A^{p,\alpha}$ and $\| \;\; \|_A$ for $\| \;\; \|_{p,\alpha}$. We also write $f \doteq g$ as meaning the existence of constants $A, B > 0$ such that $Ag(x) \leq f(x) \leq Bg(x)$ for all x in an appropriate domain.

I would like to express may thanks to Pat Ahern and Bill Cohn for many useful and encouraging conversations on this subject.

Let A be a Bergman space with norm $\| \;\; \|_A$ and X another space of functions analytic in U. We define $X = A^*$ as follows: for every continuous linear functional Λ on A there is a unique

$$g(z) = \sum_{k=0}^{\infty} b_k z^k \in X$$

such that if $f(z) = \sum_{k=0}^{\infty} a_k z^k \in A$, then

$$\Lambda(f) = \lim_{\rho \to 1^-} \sum_{k=0}^{\infty} a_k \bar{b}_k \rho^k,$$

and conversely, if $g \in X$ is fixed, then

$$\Lambda_g(f) = \lim_{\rho \to 1^-} \sum_{k=0}^{\infty} a_k \bar{b}_k \rho^k$$

exists for all $f \in A$ and Λ_g defines a continuous linear functional on A.

Let $\{z_k\}_{k=1}^{\infty} \subset U$ be a sequence satisfying $\sum_{k=1}^{\infty}(1 - |z_k|) < \infty$. Then $\{z_k\}_{k=1}^{\infty}$ is called a Blaschke sequence, and the function

$$B(z) = \prod_{k=1}^{\infty} \frac{|z_k|}{z_k} \frac{z_k - z}{1 - \bar{z}_k z}$$

converges uniformly on compact subsets of U; B is called the Blaschke product corresponding to $\{z_k\}$. The Blaschke product satisfies $|B(z)| < 1$ for $z \in U$ with $|B(e^{i\theta})| = 1$ almost everywhere with respect to Lebesgue measure on ∂U, and it has zero set $\{z_k\}$. If in addition there is a $\delta > 0$ such that for all k,

$$\sum_{\substack{j \neq k \\ j=1}}^{\infty} \left| \frac{z_j - z_k}{1 - \bar{z}_j z_k} \right| \geq \delta,$$

then the sequence $\{z_k\}$ is said to be uniformly separated. In all of this we can assume $|z_1| \leq |z_2| \leq |z_3| \leq \cdots$.

Given an arbitrary sequence $\{z_k\} \subset U$, $0 < p < \infty$, and a space A, one can define the linear operator T_p on A by $T_p(f) = (1 - |z_k|^2)^{1/p} f(z_k)$. Since Carleson's interpolation theorem [7, p. 149] states that, for $0 < p < \infty$, $T_p(H^p) = l^p$ if and only if $\{z_k\}$ is uniformly separated, the Blaschke product B formed with a uniformly separated sequence is called an interpolating Blaschke product, abbreviated i.b.p.

With these introductory notions defined, we begin the following lemma:

1. LEMMA. *If $B(z) = \sum_{k=0}^{\infty} b_k z^k$ is an interpolating Blaschke product with zero set $\{z_k\}_{k=1}^{\infty}$, where $z_k \neq 0$ for all k, and if $f(z) = \sum_{k=0}^{\infty} a_k z^k \in H(U)$, then for $0 < \rho < 1$,*

$$\sum_{k=1}^{\infty} a_k \bar{b}_k \rho^k = \frac{f(0)}{B(0)} - \sum_{k=1}^{\infty} \frac{f(\rho z_k)(1 - |z_k|^2)}{|z_k| b_k(z_k)},$$

where

$$b_k(z) = \prod_{\substack{j=1 \\ j \neq k}}^{\infty} \frac{|z_j|}{z_j} \frac{z_j - z}{1 - \bar{z}_j z}.$$

Proof. This is the standard application of the residue theorem to the integral representation

$$\sum_{k=0}^{\infty} a_k \bar{b}_k \rho^k = \frac{1}{2\pi} \int_0^{2\pi} f(\rho e^{i\theta}) \overline{B(e^{i\theta})}\, d\theta.$$

We now list some elementary properties of Bergman spaces which we will need.

2. LEMMA. *For $A = A^{p,\alpha}$ a Bergman space with $\alpha > -1$, $0 < p < \infty$, and for $A = H^p$, $0 < p < \infty$, we have*:

(a) *Polynomials $\sum_{k=0}^{n} a_k z^k$ are dense in A.*
(b) *If $f_\rho(z) = f(\rho z)$ for $0 < \rho < 1$, then $\|f_\rho\|_A \to \|f\|_A$ as $\rho \to 1^-$.*
(c) *$H^\infty \subset A$ and if $g \in H^\infty$, $f \in A$, then $\|gf\|_A \le \|g\|_\infty \|f\|_A$.*

Proof. These are standard facts.

3. LEMMA. *Let A be a Bergman space, $(\alpha > -1, 0 < p < \infty)$, possibly H^p, and let $\{z_k\}_{k=1}^{\infty}$ be any sequence in U, and let $f \in A$. Then there is a constant $C > 0$ such that*

$$\sum_{k=1}^{\infty} |f(z_k)|(1 - |z_k|^2) \le C\|f\|_A$$

iff there is a constant C' such that

$$|f(0)| + \sum_{k=1}^{\infty} |f(z_k)|(1 - |z_k|^2) \le C'\|f\|_A.$$

Proof. The second statement trivially implies the first with $C = C'$, so assume the first statement. Since $M_q(r, f)$ is increasing in r for any q, $|f(0)| < M_q(r, f)$ for all $r, 0 \le r \le 1$. From this follows the inequality $|f(0)| \le (1 + \alpha)^{1/p}\|f\|_{p,\alpha}$, and adding this inequality to the first statement gives the result for $A^{p,\alpha}$ with $C' = C + (1 + \alpha)^{1/p}$; for H^p a similar argument gives the result with $C' = C + 1$.

4. LEMMA. *Let $\{z_k\}$ be a uniformly separated sequence, and B be the Blaschke product formed with $\{z_k\}$, excluding 0 if $0 \in \{z_k\}$. Let A be a Bergman space. Then there is a constant $C > 0$ such that $\sum_{k=1}^{\infty}|f(z_k)|(1 - |z_k|^2) \le C\|f\|_A$ for all $f \in A$ iff $B \in A^*$; in other words, T_1: $A \to l^1$ is bounded iff $B \in A^*$.*

Proof. First assume T_1: $A \to l^1$ is bounded, and suppose initially that $z_k \ne 0$ for all k. Then by Lemma 2 there is a constant C' such that

$$|f(0)| + \sum_{k=1}^{\infty} |f(z_k)|(1 - |z_k|^2) \le C'\|f\|_A \quad \text{for all } f \in A.$$

Then if B is as above and f is a polynomial, we have by Lemma 1, and uniform separability of $\{z_k\}$ that

$$\begin{aligned}\left|\lim_{\rho\to1^-}\int_0^{2\pi} f(\rho e^{i\theta})\overline{B(e^{i\theta})}\frac{d\theta}{2\pi}\right| &= \left|\frac{f(0)}{B(0)} - \lim_{\rho\to1^-}\sum_{k=1}^{\infty}\frac{f(\rho z_k)(1-|z_k|^2)}{|z_k|b_k(z_k)}\right| \\ &= \left|\frac{f(0)}{B(0)} - \sum_{k=1}^{\infty}\frac{f(z_k)(1-|z_k|^2)}{|z_k|b_k(z_k)}\right| \\ &\le C_1\left[|f(0)| + \sum_{k=1}^{\infty}|f(z_k)|(1-|z_k|^2)\right] \\ &\le C_2\|f\|_A.\end{aligned}$$

Thus

$$\Lambda_B(f) = \lim_{\rho\to1^-}\int_0^{2\pi} f(\rho e^{i\theta})\overline{B(e^{i\theta})}\frac{d\theta}{2\pi}$$

defines a continuous linear functional on the polynomials, which then extends to a continuous linear functional on A, since the polynomials are dense in A. Thus by the uniqueness of representation of continuous linear functionals, $B \in A^*$. If one of z_k is 0, then the proof is the same, with $C' = C$.

For the other direction, assume $\{z_k\}$ uniformly separated, $z_k \neq 0$ initially, and A is a mixed norm space with $B \in A^*$. Then since $H^\infty \subset A$, we have that there is a $C > 0$ such that for all $f \in H^\infty$,

$$\left|\frac{f(0)}{B(0)} - \sum_{k=1}^{\infty}\frac{f(z_k)(1-|z_k|^2)}{|z_k|b_k(z_k)}\right| \le C\|f\|_A.$$

Now let $f \in H^\infty$ be fixed; then $\{0\} \cup \{z_k\}$ is a uniformly separated sequence. Thus by Theorem A in the introduction there is a $g_f \in H^\infty$ with the property that

$$g_f(z_k)\frac{f(z_k)}{|z_k|b_k(z_k)} = -\frac{|f(z_k)|}{|z_k||b_k(z_k)|} \quad\text{and}\quad g_f(0)\frac{f(0)}{B(0)} = \frac{|f(0)|}{|B(0)|},$$

i.e., g_f has unit modulus at each z_k and has the effect of rotating each term of the sum into its negative modulus. Furthermore, by a remark in [12, p. 18], there is a constant $\nu(\{z_k\})$ such that $\|g_f\|_\infty \le \nu(\{z_k\})$ for all $f \in A$. Now substituting $g_f f$ in for f in the previous inequality and using 2(c) we get

$$\begin{aligned}\frac{|f(0)|}{|B(0)|} + \sum_{k=1}^{\infty}\frac{|f(z_k)|(1-|z_k|^2)}{|z_k||b_k(z_k)|} &\le C\|g_f f\|_A \\ &\le C\|g_f\|_\infty\|f\|_A \\ &\le C_\nu(\{z_k\})\|f\|_A.\end{aligned}$$

Thus for some C we have

$$|f(0)| + \sum_{k=1}^{\infty} |f(z_k)|(1 - |z_k|^2) \leq C\|f\|_A \quad \text{for all } f \in H^\infty;$$

hence T_1 is bounded on H^∞. But then for any $f \in A$, if $f_\rho(z) = f(\rho z)$, $0 < \rho < 1$, then

$$\sum_{k=1}^{\infty} |f(\rho z_k)|(1 - |z_k|^2) \leq C\|f_\rho\|_A.$$

Hence by Fatou's Lemma along with 2(b) we get

$$\begin{aligned}\sum_{k=1}^{\infty} |f(z_k)|(1 - |z_k|^2) &\leq \varliminf_{\rho \to 1} \left[\sum_{k=1}^{\infty} |f(\rho z_k)|(1 - |z_k|^2) \right] \\ &\leq C \varliminf_{\rho \to 1} \|f_\rho\|_A = C\|f\|_A.\end{aligned}$$

Finally if $z_k = 0$ for some k, the above argument gives the boundedness of T_1 corresponding to $\{z_k\}$-$\{0\}$; the boundedness of T_1 corresponding to $\{z_k\}$ follows from Lemma 3. This completes the proof.

We point out that it is clear that the proof works in a more general context than that of Bergman spaces, since only certain properties of these spaces were used. Specifically, the proof holds for any Banach space of analytic functions satisfying the conditions of Lemma 2 (including a weakening of 2(c)), with some alteration in the statement allowing for the requirement in lemma 1 that $z_k \neq 0$ for all k.

Before stating the main theorem we state some facts about the "approximating Blaschke product" for an arbitrary inner function; this notion together with proofs for the facts listed below are found in [5], [6]. Let ϕ be an arbitrary inner function, $0 < \delta < 1$, and $R(\delta)$ be the "Carleson region" constructed in [9, p. 342, Theorem 5.1]. Then $\Gamma = U \cap \partial R$ is a countable union of arcs or radial segments:

$$\Gamma = \bigcup_n \gamma_n, \quad \gamma_n = [a_n, b_n]$$

with

$$\sigma_1 \leq \left| \frac{a_n - b_n}{1 - \bar{a}_n b_n} \right| \leq \sigma_2$$

for some constants $0 < \sigma_1, \sigma_2 < 1$. Let w_n be the midpoint of γ_n. Then $\{w_n\}$

is a uniformly separated sequence; call B_ϕ the i.b.p. formed from $\{w_n\}$. We then have:

5. LEMMA. *Let ϕ and B_ϕ be as above. Then* $1 - |\phi(z)| \doteq 1 - |B_\phi(z)|$.

Proof. See [6, p. 12–13].

We now come to the main theorem.

6. THEOREM. *Let ϕ be an inner function, with B_ϕ an approximating Blaschke product with zero set $\{z_k\}$. If $-1 < \alpha < p - 1$ where $p \geq 1$, then*

$$\phi' \in A^{p,\alpha} \text{ if and only if } \sum_k (1 - |z_k|)^{\alpha - p + 2} < \infty.$$

Proof. First assume $p = 1$. We note that by [2, Theorem 6] and our Lemma 5, $\phi' \in A^{1,\alpha}$ iff

$$\int_0^1 \int_0^{2\pi} \left(\frac{1 - |B_\phi(re^{i\theta})|}{1 - r} \right) d\theta \, (1 - r)^\alpha \, dr < \infty.$$

Now since B_ϕ is an i.b.p. we have

$$1 - |B_\phi(re^{i\theta})| \doteq (1 - r^2) \sum_k \frac{(1 - |z_k|^2)}{|1 - \bar{z}_k re^{i\theta}|^2}$$

(see [14, pp. 30–31]). Hence a calculation gives

$$\begin{aligned}
\int_0^1 (1 - r)^\alpha \int_0^{2\pi} \sum_k \frac{(1 - |z_k|^2)}{|1 - \bar{z}_k re^{i\theta}|^2} \, d\theta \, dr
&= \sum_k (1 - |z_k|^2) \int_0^1 \int_0^{2\pi} \frac{d\theta}{|1 - \bar{z}_k re^{i\theta}|^2} (1 - r)^\alpha \, dr \\
&= \sum_k (1 - |z_k|^2) \int_0^1 \frac{(1 - r)^\alpha}{1 - |z_k|^2 r^2} \, dr \\
&\doteq \sum_k (1 - |z_k|)(1 - |z_k|)^\alpha = \sum_k (1 - |z_k|)^{1+\alpha}.
\end{aligned}$$

For $p > 1$ we use a different approach. By applications of [2, Theorem 6] and our Lemma 5, $\phi' \in A^{p,\alpha}$ iff $B'_\phi \in A^{p,\alpha}$. Assume $B_\phi(0) \neq 0$. Now it is clear that $B'_\phi \in A^{p,\alpha}$ iff $D^1 B_\phi \in A^{p,\alpha}$, where

$$D^\beta f(z) = \sum_{k=0}^{\infty} (k + 1)^\beta a_k z^k \quad \text{for } f(z) = \sum_{k=0}^{\infty} a_k z^k \in H(U), \quad -\infty < \beta < \infty.$$

Now by [8, Theorem 6], $D^1 B_\phi \in A^{p,\alpha}$ iff $D^{\gamma+1} B_\phi \in A^{p,\gamma}$, where $\gamma = \alpha/(1-p) > -1$. However, the identification $(A^{p',\gamma})^* = \{f \in H(U)\colon D^{\gamma+1} f \in A^{p,\gamma}\}$ (see [4, p. 54]) allows us to summarize the above by saying

$$\phi' \in A^{p,\alpha} \quad \text{iff} \quad B_\phi \in (A^{p',\gamma})^* \quad \text{where } \frac{1}{p} + \frac{1}{p'} = 1.$$

Now by our Lemma 4,

$$B_\phi \in (A^{p',\gamma})^* \quad \text{iff} \quad \sum |f(z_k)|\left(1 - |z_k|^2\right) \le C\|f\|_{A^{p',\gamma}} \quad \text{for all } f,$$

where $C > 0$ is a fixed constant. If $B_\phi(0) = 0$, then this equivalence follows by noting that

$$B'_\phi \in A^{p,\alpha} \quad \text{iff} \quad \left(B_\phi/z\right)' \in A^{p,\alpha},$$

and repeating the above argument with B_ϕ replaced by B_ϕ/z.

Our final step is to translate the boundedness of $T_1\colon A^{p',\gamma} \to l^1$ into an equivalent summability condition on $\{z_k\}$; this can be done by an appeal to the main theorem [11], mentioned in the introduction. The above equivalence can be written as

$$B_\phi \in (A^{p',\gamma})^* \quad \text{iff} \quad \int_U |f|\, d\mu \le C\|f\|_{A^{p',\gamma}}$$

where $\mu = \sum_k (1 - |z_k|)\delta_{z_k}$, a positive measure on U. Thus we can proceed as follows: select an $\varepsilon, 0 < \varepsilon < 1$, and let $D_\varepsilon(w)$ denote the pseudo-hyperbolic disk of radius ε around w. Select a sequence $\{w_n\}$ such that $D_{\varepsilon/2}(w_n)$ are disjoint but $D_\varepsilon(w_n)$ covers U. Then, following the remarks at the beginning of Section 3 and also in Section 4 of [11] we get the right hand side of the above equivalence occurs iff

$$\sum_n \left[\sum_{z_k \in D_\varepsilon(w_n)} (1 - |z_k|)^{(p'-2-\gamma)/p'} \right]^{p'/(p'-1)} < \infty.$$

But since $\{z_k\}$ are uniformly separated, hence separated, if ε is sufficiently small this is equivalent to

$$\sum_k (1 - |z_k|)^{(p'-2-\gamma)/(p'-1)} = \sum_k (1 - |z_k|)^{\alpha+2-p} < \infty,$$

and we are done.

Remark. In [5], Cohn proves that, for $\frac{1}{2} < p < 1$, $\phi' \in H^p$ iff $\sum (1 - |z_k|)^{1-p} < \infty$, where again $\{z_k\}$ are the zeroes of B_ϕ. This can be considered as a companion to the above result by taking $\alpha = -1$, $\frac{1}{2} < p < 1$.

As a consequence of Theorem 6, we have the following collection of results on the derivative of an inner function.

7. THEOREM. *Let* ϕ *be an inner function*, $1 \le p < \infty$, $\alpha > -1$.
(a) *If* $\alpha > p - 1$, *then* $\phi' \in A^{p,\alpha}$.
(b) *If* $p - 2 < \alpha < p - 1$, *then* $\phi' \in A^{p,\alpha}$ *iff* $\phi' \in A^{1,\alpha-p+1}$.
(c) *If* $\alpha \le p - 2$, $p > 1$, *then* $\phi' \in A^{p,\alpha}$ *iff* ϕ *is a finite Blaschke product.*

Proof. (a) For this we repeat the proof in [1], simply noting that in fact only $p > 0$ is required:

$$\int_0^{2\pi}\int_0^1 |\phi'(re^{i\theta})|^p (1-r)^\alpha \, dr\, d\theta \le \int_0^{2\pi}\int_0^1 (1-r)^{\alpha-p}\, dr\, d\theta < \infty$$

if $\alpha - p > -1$.

(b) By Theorem 6, $\phi' \in A^{p,\alpha}$ iff the approximating Blaschke product B_ϕ has zeroes $\{z_k\}$ satisfying

$$\sum_k (1 - |z_k|)^{\alpha-p+2} < \infty.$$

But this can be written as $\Sigma_k(1-|z_k|)^{(\alpha-p+1)-1+2}$, thus $\phi' \in A^{p,\alpha}$ iff $\phi' \in A^{1,\alpha-p+1}$, here $\alpha > p-2$ assures $\alpha - p + 1 > -1$.

(c) By Theorem 6, if $\phi' \in A^{p,\alpha}$ for $\alpha \le p-2$ then B_ϕ must be a finite Blaschke product. But then

$$|\phi'(re^{i\theta})| \le \frac{1 - |\phi(re^{i\theta})|^2}{1-r^2} \doteq \frac{1 - |B_\phi(re^{i\theta})|^2}{1-r^2} = 0(1).$$

Thus $\phi' \in H^1$ so ϕ is continuous up to ∂U [7, Theorem 3.11]; this implies ϕ is a finite Blaschke product.

Remark. If $p = 1$, and $\alpha = p - 2 = -1$ in (c) above, we still may have $\phi' \in A^{1,-1}$ iff ϕ is a finite Blaschke product if we interpret $A^{1,-1} = H^1$. Part (c) appeared with different proof in [10], as Theorem 1.1.

Next we observe that Theorem 6 also provides an alternative proof for part of [1, Theorem 6.2].

8. THEOREM. *If* $\phi(z) = \Sigma_{n=0}^\infty a_n z^n$, $\frac{1}{2} < s < 1$, *then the following are equivalent*:
(a) $\phi' \in H^s$;
(b) $\phi' \in B^{1/(2-s)}$;
(c) $\Sigma_n |a_n|^2 n^s < \infty$.

Proof. The equivalence of (a) and (b) is as follows: by the remark after Theorem 6, $\phi' \in H^s$ iff B_ϕ has zeroes $\{z_k\}$ satisfying $\Sigma(1 - |z_k|)^{1-s} < \infty$. But by 6(a) this occurs iff $\phi' \in A^{1,-s} = B^{1/(2-s)}$. For the equivalence of (b) and (c), $\phi' \in B^{1/(2-s)}$ iff $\phi' \in A^{2,1-s}$ by 7(b). But a straight forward calculation shows that the right hand side occurs iff $\Sigma_{n=0}^{\infty}|a_n|^2 n^s < \infty$.

We conclude by providing a proof of a theorem of Ahern stated in [10, p. 7].

9. THEOREM. *If ϕ is an inner function with $\phi' \in A^{p,p-3/2}$ for some $p > \frac{1}{2}$ then ϕ is a Blaschke product.*

Proof. As before, let B_ϕ be the approximating Blaschke product for ϕ; then $B_\phi' \in A^{p,p-3/2}$. Now if $\{z_k\}$ is the zero set for B_ϕ, since $\{z_k\}$ are uniformly separated, there is $\mu, 0 < \mu < 1$, such that

$$D_k = \left\{z \in U: |z - z_k| < \mu\left(1 - |z_k|^2\right)\right\}$$

are disjoint [15, p. 6]. Then since $|B_\phi'(z_k)| \doteq (1 - |z_k|^2)^{-1}$, we have

$$\begin{aligned}
\infty &> \int_0^{2\pi}\int_0^1 |B_\phi'(re^{i\theta})|^p(1-r)^{p-3/2}\,dr\,d\theta \\
&\geq \sum_k \int_{D_k} |B_\phi'(z)|^p(1-|z|)^{p-3/2}\,dm(z) \\
&\doteq \sum_k (1-|z_k|)^{p-3/2}\int_{D_k}|B_\phi'(z)|^p\,dm(z) \\
&\geq \sum_k (1-|z_k|)^{p-3/2}|B_\phi'(z_k)|^p m(D_k) \\
&\doteq \sum_k (1-|z_k|)^{p-3/2}(1-|z_k|)^{-p}(1-|z_k|)^2 \\
&= \sum_k (1-|z_k|)^{1/2}.
\end{aligned}$$

But then $\Sigma_k(1 - |z_k|)^{1/2} < \infty$ implies $B_\phi' \in A^{1,-1/2} = B^{2/3}$. Thus $\phi' \in B^{2/3}$ as before, but by [3, Theorem 3] this implies ϕ is a Blaschke product.

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