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ASYMPTOTIC CAPACITIES FOR FINELY DIVIDED BODIES AND STOPPED DIFFUSIONS[1]

BY

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1. Introduction

In R^d, $d \geq 3$, let B_t denote Brownian motion and let D_n be a sequence of closed sets. Let τ_n denote the first hitting time of D_n. We shall give (in Theorems 1.2 and 1.3) conditions under which the stopping times τ_n have, in the sense of [2], a weak limit, denoted by T. More precisely, the sequence τ_n converges *stably* to T, as defined in Section 2 (for a discussion of stable convergence in other contexts see [1]). The limit T is a randomized stopping time, and represents an "exponential killing time", where the rate h is a function of position in R^d. In applications to certain random media problems the sets D_n are to be thought of as consisting of many small bodies, which become ever smaller and more densely distributed as $n \to \infty$. The function h represents a "limiting capacity density" for these bodies (Definition 1.1 below). If f: $R^d \to R$ is bounded Borel and we define

$$u_n(x, t) = E^x\left[f(B_t)\chi_{\{\tau_n > t\}}\right] \tag{1.0}$$

and

$$u(x, t) = E^x\left[f(B_t)\chi_{\{T > t\}}\right], \tag{1.1}$$

then the stable convergence of τ_n to T implies a fairly strong kind of convergence of u_n to u (Theorem 2.2). As is well known, the u_n satisfy the diffusion equations

$$\partial u_n(x, t)/\partial t = \tfrac{1}{2}\Delta u_n(x, t) \quad \text{for } t > 0,\ x \in D_n^c, \tag{1.2}$$

with boundary and initial conditions

$$u_n(x, t) = 0 \quad \text{for } t > 0,\ x \in \partial D_n, \tag{1.3}$$

$$u_n(x,0) = f(x) \quad \text{for } x \in D_n^c. \tag{1.4}$$

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When the D_n have irregular boundaries or f is not continuous conditions (1.3) and (1.4) must of course be stated in a more general form, but the same interpretation is valid.

Since T arises from exponential killing with rate h, it follows that u will also satisfy a diffusion equation

$$(1.5) \quad \partial u(x,t)/\partial t = \tfrac{1}{2}\Delta u(x,t) - h(x)u(x,t) \quad \text{for } t > 0,\ x \in R^d,$$

with initial condition

$$(1.6) \qquad u(x,0) = f(x) \quad \text{for } x \in R^d.$$

Thus the stable convergence of the stopping times τ_n to T can be applied to study the convergence of solutions of such diffusion equations.

This type of problem has been considered for a variety of situations, in [3], [4], [5], [6], [8], [9], [10], [11], [13], and [14]. Additional references are given in [5]. The case we are concerned with here was formulated in [10], and generalized in [3] as follows:

Let $D_i(n)$, $i = 1, \ldots, k(n)$, be compact sets, for each $n = 1, 2 \ldots$. Let there exist a sequence $\rho_n > 0$, $\rho_n \to 0$, such that diameter$(D_i(n)) < \rho_n$ for each i and each n. Let

$$D_n = D_1(n) \cup \cdots \cup D_{k(n)}(n).$$

Let $\lambda_i(n)$ denote the classical equilibrium measure on $D_i(n)$, and let

$$\lambda(n) = \lambda_1(n) + \cdots + \lambda_{k(n)}(n).$$

Suppose the sequence $\lambda(n)$ is bounded in total mass, and that there exists a finite measure λ such that:

$$(1.7) \qquad \lambda(n) \to \lambda \text{ vaguely as } n \to \infty,$$

$$(1.8) \qquad \Sigma^*\langle \lambda_i(n), \lambda_j(n)\rangle \to \langle \lambda, \lambda \rangle \quad \text{as } n \to \infty,$$

where Σ^* denotes the sum over all i and j with $i \neq j$, and for any measures μ and ν,

$$(1.9) \qquad \langle \mu, \nu \rangle = K_d \int\int |x - y|^{-d+2} \mu(dx)\nu(dy),$$

the classical energy inner product. Here K_d is a normalizing factor defined below in (1.12).

Intuitively, we may say that (1.7) specifies a limiting density for the bodies $D_i(n)$, and that (1.8) requires that these bodies be rather evenly spread out (see also lemma 6.1).

Suppose that f is a continuous function with compact support in R^d. Under some extra assumptions in [10], and in general in [3], it was shown:

THEOREM 1.1. *Suppose* (1.7) *and* (1.8) *hold and the limit measure* λ *above has bounded density* h *with respect to Lebesgue measure* m *on* R^d. *Let* u_n *and* u *be defined by* (1.0) *and* (1.1), *respectively, then given any* $\varepsilon > 0$ *and* $t_0 < \infty$ *there exists an integer* n_0 *such that for all* $n \geq n_0$,

$$m\left(\left\{x\colon \sup_{0 \leq t \leq t_0} |u_n(x,t) - u(x,t)| > \varepsilon\right\}\right) < \varepsilon.$$

The function u defined by (1.1) can be expressed in terms of the exponential killing rate h referred to earlier, by

$$u(x,t) = E^x\left[f(B_t)\exp\left(-\int_{[0,t]} h(B_s)\,ds\right)\right]. \tag{1.10}$$

For this reason, Theorem 1.1 is a consequence of the next theorem, Theorem 1.2, which follows from Theorems 2.1, 2.3, and 1.2 of [3]. Theorem 1.2 is a probabilistic statement concerning stable convergence on the sample space of Brownian motion. As explained in [3], Theorem 1.2 implies Theorem 1.1, because stable convergence of stopping times τ_n implies *norm* convergence for the corresponding distributions of B_t on the sets $\{\tau_n > t\}$. A more precise statement of this result is given in Theorem 2.2 of [3] and in Theorem 2.2 in the next section. Of course, Theorem 1.2 also gives information about the convergence of the stopped distributions corresponding to the stopping times τ_n.

It should be noted that, in addition to the setting of Theorem 1.1, Theorem 1.2 can also be applied in situations where no partial differential equation exists, for example to the case of Brownian motion with a general nonanticipating drift.

THEOREM 1.2. *Assume* D_n *satisfies* (1.7) *and* (1.8), *and that the limiting measure* λ *has bounded density* h *with respect to Lebesgue measure* m *on* R^d. *Let* A_t *be the additive functional defined by*

$$A_t = \int_{[0,t]} h(B_s)\,ds. \tag{1.11}$$

Let F_t *be the multiplicative functional* $\exp(-A_t)$, *and let* T *be the randomized stopping time associated with* F_t *as in Section* 2. *Then from any subsequence* $n(k)$, *we can extract another subsequence* $n(k(i))$, *such that, for* m-a.e. x *in* R^d, $\tau_{n(k(i))}$ *converges stably to* T *with respect to* P^x.

Because convergence of stopping times is metrizable (Section 2), Theorem 1.2 immediately shows that if ν is a probability measure on R^d, ν absolutely continuous with respect to m, then the full sequence τ_n converges stably to T, with respect to P^ν. Theorem 1.2 also gives a kind of "in measure" convergence for the full sequence τ_n with respect to P^x for "most" points x (cf. Theorem 1.2 of [3]).

In the present paper we will prove, in Theorem 1.3, weak convergence for a more general setting than that described above in (1.7)–(1.8). Thus Theorem 1.3 implies Theorem 1.2 and hence Theorem 1.1. Our method of proof is new as well as more general and gives a more direct relation (Sections 3 and 4) between the equilibrium measures $\lambda_i(n)$ and the rate h which appears in the limit above.

To state the results of the present paper we need the notion of the classical potential associated with a measure μ, which we write as Pot μ, defined by

(1.12)

$$\text{Pot}\,\mu(x) = k_d\int|x-y|^{-d+2}\mu(dy), \quad \text{where } k_d = \int_0^\infty e^{-1/2t}/[2\pi t]^{d/2}\,dt.$$

For any measure μ, and any function f, we write $f\mu$ to mean the measure γ defined by $\gamma(A) = \int_A f\,d\mu$. Thus Pot $\chi_B\mu$ denotes the potential of the measure $\chi_B\mu$. We denote Lebesgue measure on R^d by m. If μ has a density h with respect to Lebesgue measure we may write Pot h for Pot μ. A property which is true except on a polar set in R^d is said to hold quasi-everywhere (q.e.). Associated with any bounded potential g is an *additive functional* A_t defined by the property that $g(x) = E^x[A_\infty]$ for every x in R^d. We recall that $g \circ B_t$ is a continuous supermartingale (Theorem 2.IX.12 of [7]), and A_t is the increasing process such that $g \circ B_t + A_t$ is a *martingale*. Of course if $g = \text{Pot}\,h$ then

$$A_t = \int_{[0,t]} h \circ B_s\,ds,$$

and a reader who is unfamiliar with the general case will lose little by restricting himself to this situation in what follows. If λ is a measure on R^d such that Pot $\chi_K\lambda$ is bounded on R^d for each bounded ball K, then we can define an additive functional associated with λ, even if Pot λ is ∞ on R^d; we set

$$A_t = \lim_{n\to\infty} A_t(n), \tag{1.13}$$

where $A_t(n)$ is the additive functional associated with Pot $\chi_{K(n)}\lambda$, for any sequence of balls $K(n)$ such that $K(n)\uparrow R^d$. To see that the limit in (1.13) exists finite, we note that for any fixed bounded open set U, for all sufficiently large n and m the functions Pot $\chi_{K(n)}\lambda$ − Pot $\chi_{K(m)}\lambda$ are harmonic on U. Thus

$$\text{Pot}\,\chi_{K(n)}\lambda(B_t) - \text{Pot}\,\chi_{K(m)}\lambda(B_t)$$

is a martingale with respect to any initial point x in U, for times up until the first exit time of U. Since

$$\text{Pot}\,\chi_{K(n)}\lambda(B_t) - \text{Pot}\,\chi_{K(m)}\lambda(B_t) + A_t(n) - A_t(m)$$

is a martingale for all time, it follows that $A_t(n) - A_t(m)$ is constant for time intervals such that B_t is in U. In particular, from an initial point x in U, $A_t(n) = A_t(m)$ for all times t up until the first exit time of U. Thus from *any* initial point, and for any fixed t, the value of $A_t(n)$ is eventually independent of n. This shows the existence of a finite limit in (1.13).

For W compact, we write $c(W)$ for the classical capacity of W.

DEFINITION 1.1. Let D_n be any sequence of closed sets in R^d. The *total capacity measure* for the sequence D_n is defined to be the minimal measure λ such that

$$\limsup_{n\to\infty} c(D_n \cap W) \le \lambda(W)$$

for every compact set W in R^d. It is shown in Section 3 that λ exists. If every subsequence of D_n has the *same total capacity measure*, we will say that the sequence D_n has a *limiting capacity measure* λ.

In Section 4 we give a construction of the limiting capacity measure λ for a sequence D_n. Using this construction, in Section 5 we prove:

THEOREM 1.3. *Let D_n be a sequence of closed sets. Suppose D_n has limiting capacity measure λ, such that* Pot $\chi_K\lambda$ *is bounded on R^d for each bounded ball K. Let A_t be the additive functional associated with λ, let F_t be the multiplicative functional* $\exp(-A_t)$, *and let T be the randomized stopping time constructed from F as in Section 2. Let ν be any finite measure on R^d that does not charge polar sets. Then from any subsequence $n(k)$, we can extract another subsequence $n(k(i))$, such that, for ν-*a.e. *x in R^d, $\tau_{n(k(i))}$ converges stably to T with respect to P^x.*

Just as in the observation after Theorem 1.2, Theorem 1.3 implies weak convergence for the full sequence τ_n with respect to P^ν, for any probability ν on R^d which does not charge polar sets. Again, Theorem 1.3 implies a more general version of Theorem 1.1, in which the limit $u(x,t)$ is defined by

$$u(x,t) = E^x[f(B_t)\exp(-A_t)], \tag{1.14}$$

instead of (1.10).

The proof that Theorem 1.3 implies Theorem 1.2 is completed in Section 6, where it is shown that if (1.7) and (1.8) hold then the sequence D_n has a limiting capacity measure λ.

In connection with Theorem 1.3, it should be noted that the variational methods of [5] give a general compactness principle applicable to parabolic equations (U. Mosco, private communication). If these variational methods are combined with the stopping time compactness proved in [2], a compactness principle for multiplicative functionals can be obtained, for a much wider class of multiplicative functionals than those studied in Theorem 1.3. It seems unclear, however, whether the convergence result given in Theorem 1.3 can be extended to a wider class. The existence of a limiting capacity measure λ appears to depend on geometrical properties of the sets D_n, in particular on the fact that the sets D_n become rather sparse as $n \to \infty$.

Finally, we note that the method of proof used here gives an explicit criterion for convergence of $\tau(n)$ with respect to P^x or a particular x, as is seen in Lemma 5.1.

2. Notation

We shall follow the definitions and notation of [3], Section 2. We will usually use as our basic sample space for Brownian motion the space C of continuous functions ω from $[0, \infty)$ into R^d, with the usual topology on C of uniform convergence on compacts. We define B_t: $C \to R^d$ by $B_t(\omega) = \omega(t)$ for all ω in C. We define $\mathscr{F}_t = \sigma(B_s: s \leq t)$, $\mathscr{G}_t = \mathscr{F}_{t+}$, $\mathscr{F}_\infty = \mathscr{G}_\infty = \mathscr{G}$. Let $\mathscr{B}_1$ denote the Borel sets of $[0,1]$ and let $\mathscr{B}$ denote the Borel sets of $[0, \infty]$.

A *randomized stopping time* T is a map T: $C \times [0,1] \to [0, \infty]$ such $T(\omega, \cdot)$ is left continuous and increasing, $T(\omega, 0) = 0$, and T is a $\mathscr{G}_t \times \mathscr{B}_1$-stopping time. An ordinary stopping time τ can be regarded as a randomized stopping time T defined by $T(\omega, a) = \tau(\omega)$ for $a > 0$. Associated with T is the *stopping time probability measure* F: $C \times \mathscr{B} \to [0,1]$, defined by

$$F(\omega, [0, t]) = \sup\{a: T(\omega, a) \leq t\}. \tag{2.1}$$

Then

$$F(\cdot, [0, t]) \text{ is } \mathscr{G}_t\text{-measurable for each } t. \tag{2.2}$$

We will sometimes write $F(\cdot, (t, \infty])$ as $F((t, \infty])$ or F_t. In our application F_t will turn out to be a multiplicative functional, as intuitively it must, but this property will not be explicitly used.

We can recover T from F by

$$T(\omega, a) = \inf\{t: F(\omega, [0, t]) \geq a\}. \tag{2.3}$$

Given any F for which (2.2) holds, T defined by (2.3) is a randomized stopping time, and (2.1) holds. Thus the notions of randomized stopping time and stopping time measure are equivalent.

Probabilities and expectations involving a randomized stopping time should clearly use the probability $P \times m_1$, where m_1 is Lebesgue measure on $[0,1]$. However, we will usually only write P explicitly, since the meaning is clear from the context.

For any probability P on $(C, \mathscr{G})$, and any randomized stopping time T, we write Q for the distribution on $C \times [0,\infty]$ of the map (identity, T) from $C \times [0,1]$ to $C \times [0,\infty]$. Let F be the stopping time measure associated with T. Then clearly

$$\int Y(\omega) f(t) Q(d\omega, dt) = \int Y \int f(t) F(dt)\, dP.$$

We often write $Q = P \otimes F$. The measure Q obtained in this way is called a *stopped process*. It is proved in [1] and [2] that:

THEOREM 2.1. *A weak limit of stopped processes with a common marginal P on C is again a stopped process.*

Since a sequence $Q_n = P \otimes F_n$ is always tight, this gives a useful compactness principle.

DEFINITION 2.1. If T_n, T are randomized stopping times with stopping time measures F_n, F, respectively, and P is a probability on $(C, \mathscr{B})$, then we say that *T_n converges stably to T, with respect to P*, if $P \otimes F_n$ converges weakly to $P \otimes F$.

It is easy to show that $T_n \to T$ stably if and only if $T_n|_{A\times[0,1]}$ converges to $T|_{A\times[0,1]}$ in distribution for every A in $\mathscr{G}$.

The relevance of this sample space weak convergence for state space questions is shown by the following result:

THEOREM 2.2. *Let T_n, T be randomized stopping times, and let F_n, F be the associated stopping time measures. Let μ be a probability measure on R^d, and let P^μ denote the Wiener measure with initial distribution μ. Suppose T_n converges stably to T with respect to P^μ. Let Y be bounded and $\mathscr{G}$-measurable on C. Let t be such that $P^\mu(T = t) = 0$. Define ν_n, ν, signed measures, by*

$$\int g\, d\nu_n = E^\mu[Yg(B_t)F_n((t,\infty])], \quad \int g\, d\nu = E^\mu[Yg(B_t)F((t,\infty])],$$

for all g bounded Borel on R^d. Then ν_n converges to ν in total variation norm as $n \to \infty$.

Theorem 2.2 is proved in [3] with $Y = 1$. The present version has the same proof.

In Sections 4 and 5 we will need the resolvent potentials Pot^α, defined for $\alpha \geq 0$, μ a measure on R^d, by

$$\mathrm{Pot}^\alpha \mu(x) = \iint_{[0,\infty)} e^{-\alpha t} \varphi_t(x, y) \mu(dy)\, dt, \tag{2.4}$$

where $\varphi_t(x, y)$ is the usual Brownian transition density. We note that $\mathrm{Pot}^0 = \mathrm{Pot}$, where Pot is defined in (1.12).

Let P_t be the Markov semigroup for Brownian motion. Let $\mu_t = \mu P_t$. Clearly

$$\mathrm{Pot}\, \mu_t(x) + \iint_{[0,t]} \varphi_s(x, y)\, ds\, \mu(dy) = \mathrm{Pot}\, \mu(x).$$

Let V be an "exponential α-killing time" so that $P^\mu(V > t) = e^{-\alpha t}$ for all $t \geq 0$ and every μ, and V is independent of the process $(B_t\colon t \geq 0)$. Given μ, let ν_μ denote the distribution of B_V wth respect to P^μ. Then

$$\nu_\mu = \int_{[0,\infty)} \alpha e^{-\alpha t} \mu_t\, dt,$$

and hence

$$\mathrm{Pot}^\alpha \mu = \mathrm{Pot}\, \mu - \mathrm{Pot}\, \nu_\mu \tag{2.5}$$

Let $\mu = \delta_x$, the Dirac measure concentrated at x. Let η be a third measure. Integrating (2.5) with respect to η and using symmetry,

$$\mathrm{Pot}^\alpha \eta(x) = \mathrm{Pot}\, \eta(x) - \int \mathrm{Pot}\, \eta\, d\nu_\mu.$$

That is,

$$\mathrm{Pot}^\alpha \eta(x) = E^x[\mathrm{Pot}\, \eta(B_0) - \mathrm{Pot}\, \eta(B_V)]. \tag{2.6}$$

One can also deduce (2.6) from (2.4) by computing that

$$E^x[\mathrm{Pot}\, \eta(B_V)] = \iint_{[0,\infty)} (1 - e^{\alpha t}) \varphi_t(x, y) \eta(dy)\, dt.$$

3. Limiting capacities

We study a sequence D_n of closed subsets of R^d. We will study the limiting capacity measure for D_n (Definition 1.1).

Notation. For any compact set K in R^d, ψ_K will denote the *equilibrium measure* on K. As noted earlier, $c(K) = \psi_K(K)$ will denote the *capacity* of K.

DEFINITION 3.1. For every $W \subset R^d$, W compact, let

$$\gamma(W) = \limsup_{n\to\infty} c(D_n \cap W). \tag{3.1}$$

LEMMA 3.1. *For every Z, W compact, $\gamma(W \cup Z) \le \gamma(W) + \gamma(Z)$.*

Proof.

$$\begin{aligned}\gamma(W \cup Z) &= \limsup_{n\to\infty} c(D_n \cap [W \cup Z]) \\ &= \limsup_{n\to\infty} c([D_n \cap W] \cup [D_n \cap Z]) \\ &\le \limsup_{n\to\infty} \{c(D_n \cap W) + c(D_n \cap Z)\} \\ &\le \limsup_{n\to\infty} c(D_n \cap W) + \limsup_{n\to\infty} c(D_n \cap Z) \\ &= \gamma(W) + \gamma(Z).\end{aligned}$$

This proves Lemma 3.1.

DEFINITION 3.2. For any subset A of R^d, let

$$\beta(A) = \sup\left\{\sum_{i=1}^{n} \gamma(K_i)\colon K_1, \dots, K_n \text{ compact disjoint subsets of } A\right\}.$$

Obviously $\beta(\varnothing) = 0$ and β is montone, i.e.,

$$A \subset B \Rightarrow \beta(A) \le \beta(B), \tag{3.2}$$

and $\beta \ge \gamma$ on compacts.

LEMMA 3.2. *For any A, B open sets in R^d,*

$$\beta(A \cup B) \le \beta(A) + \beta(B). \tag{3.3}$$

Proof. Let K be compact, $K \subset A \cup B$. Then $K - A$ and $K - B$ are disjoint compact sets, with $K - A \subset B$ and $K - B \subset A$. Hence there exist sets U, V open, disjoint, such that $K - A \subset V$ and $K - B \subset U$. Then $K - U, K - V$ are compact, $K - V \subset A$, and $K - U \subset B$, and $K = (K - V) \cup (K - U)$.

We have shown that any compact subset K of $A \cup B$ can be written as the union of a compact subset W of A and a compact subset Z of B.

Now let $K_1, \ldots, K_n$ be compact disjoint subsets of $A \cup B$. Write $K_i = W_i \cup Z_i$, where W_i is a compact subset of A and Z_i is a compact subset of B. Then

$$\sum_{i=1}^{n} \gamma(K_i) \leq \sum_{i=1}^{n} \{\gamma(W_i) + \gamma(Z_i)\} \leq \beta(A) + \beta(B).$$

This proves the lemma.

Remark 3.1. β is actually strongly subadditive on open sets, i.e., if A and B are open, then $\beta(A \cup B) + \beta(A \cap B) \leq \beta(A) + \beta(B)$. The proof is nearly the same as that of Lemma 3.2, and does not use the strong subadditivity of capacities.

LEMMA 3.3. *Let A_n be open, A_n increasing, $A = \bigcup_{n=1}^{\infty} A_n$. Then*

$$\lim_{n \to \infty} \beta(A_n) = \beta(A). \tag{3.4}$$

Proof. Follows at once from the definitions.

DEFINITION 3.3. For any subset A of R^d, let

$$\beta^*(A) = \inf\{\beta(G) : A \subset G, G \text{ open}\}. \tag{3.5}$$

β^* will be called the *total capacity set function* for the sequence D_n.

Remark 3.2. Let $\mathcal{M}$ be a collection of bounded open sets forming a base for the usual topology on R^d and closed under finite unions. Then for B open, $\beta(B)$ is easily seen to be determined by the values of γ on $\mathcal{M}'$, the collection of closures of sets in $\mathcal{M}$. Hence β^* is completely determined by the values of γ on $\mathcal{M}'$.

LEMMA 3.4. (i) $\beta^*(G) = \beta(G)$, *for G open*;
(ii) β^* *is an outer measure*;
(iii) $\gamma \leq \beta^*$ *on compact sets.*

Proof. Follows immediately from Lemmas (3.2) and (3.3).

LEMMA 3.5. *Let A and B be disjoint subsets of R^d. Then*

$$\beta(A \cup B) \geq \beta(A) + \beta(B). \tag{3.6}$$

Proof. Again, this lemma follows at once from the definition.

COROLLARY. *β^* is countably additive on disjoint open sets.*

DEFINITION 3.4. Let

$$(3.7)\quad \mathscr{H} = \{A: \beta^*(B) = \beta^*(A \cap B) + \beta^*(A^c \cap B) \text{ for all } B \subset R^d\},$$

$$(3.8)\quad \mathscr{L} = \{A: \beta^*(\partial A) = 0\}.$$

Here ∂A denotes the boundary of A.

By the extension theorem of Caratheodory, $\mathscr{H}$ is a σ-algebra and β^* is countably additive on $\mathscr{H}$.

LEMMA 3.6. (i) *$\mathscr{H} = \{A: \beta^*(G) = \beta^*(A \cap G) + \beta^*(A^c \cap G)$ for all open $G \subset R^d\}$;*

(ii) *$\mathscr{L}$ is an algebra;*

(iii) *$\mathscr{L} \subset \mathscr{H}$.*

Proof. (i) Let $\mathscr{G} = \{A: \beta^*(G) = \beta^*(A \cap G) + \beta^*(A^c \cap G)$ for all open $G \subset R^d\}$. Clearly $\mathscr{H} \subset \mathscr{G}$. Let A be in $\mathscr{G}$. Then for any subset B of R^d, consider G open, $B \subset G$. Then

$$\beta(G) = \beta^*(G) = \beta^*(A \cap G) + \beta^*(A^c \cap G) \geq \beta^*(A \cap B) + \beta^*(A^c \cap B).$$

Taking the infimum over G, and noting that β^* is subadditive proves (i).

(ii) This follows at once from the subadditive property of β^*.

(iii) Let A be in $\mathscr{L}$. Let G be open. Let U denote the interior of A, V the interior of A^c, and let B denote the boundary of A. Then

$$\beta^*(U \cap G) \leq \beta^*(A \cap G) \leq \beta^*(U \cap G) + \beta^*(B \cap G) = \beta^*(U \cap G)$$

by the corollary to Lemma 3.5. Thus $\beta^*(A \cap G) = \beta^*(U \cap G)$, and in the same way $\beta^*(A^c \cap G) = \beta^*(V \cap G)$. Hence

$$\begin{aligned}\beta^*(G) &\geq \beta^*([U \cap G] \cup [V \cap G]) = \beta^*(U \cap G) + \beta^*(V \cap G)\\ &= \beta^*(A \cap G) + \beta^*(A^c \cap G),\end{aligned}$$

and hence A is in $\mathscr{H}$ by (i). This proves Lemma 3.6.

LEMMA 3.7. *$\mathscr{H}$ contains the Borel sets.*

Proof. This follows immediately from the corollary to Lemma 3.5 via Proposition 32, p. 285 of [12].

Since the total capacity set function for a sequence is induced by a measure λ (by Lemma 3.7), we have shown the existence of the *total capacity measure* for the sequence. We are particularly interested in the case in which all subsequences of D_n induce the *same* total capacity measure λ, and, following Definition 1.1, we shall refer to λ in this case as the *limiting capacity measure* of the sequence D_n.

We have:

LEMMA 3.8. *Let $\mathscr{M}$ be a collection of bounded open sets forming a base for the usual topology on R^d and closed under finite unions. Suppose D_n is a sequence of closed sets such that $\lim_{n\to\infty} c(D_n \cap \overline{U}) = \gamma(\overline{U})$ for all U in $\mathscr{M}$. Then D_n has a limiting capacity measure λ. Also, if, in addition to the condition on $\{D_n\}$, λ is Radon measure, then*

$$\lim_{n\to\infty} c(D_n \cap W) = \gamma(W) \quad \textit{for all compact } W \textit{ in } \mathscr{L}. \tag{3.9}$$

Proof. The first statement follows at once from Remark 3.2. To prove (3.9), given compact W in $\mathscr{L}$, let U be its interior. Choose U_k in $\mathscr{M}$, $\overline{U}_k \subset U$, U_k increasing to U. Then by Lemma 3.1 we have $\gamma(W) \le \gamma(\overline{U}_k) + \gamma(W - U_k)$, so

$$\begin{aligned}\gamma(\overline{U}_k) &\ge \gamma(W) - \gamma(W - U_k) \ge \gamma(W) - \lambda(W - U_k)\\ &\ge \liminf c\left(D_n \cap \overline{U}_k\right) = \gamma\left(\overline{U}_k\right) \quad \text{by hypothesis.}\end{aligned}$$

Thus $\liminf_{n\to\infty} c(D_n \cap W) \ge \gamma(\overline{U}_k) \ge \gamma(W) - \lambda(W - U_k)$. Since $\lambda(W - U_k) \to 0$ as $k \to \infty$ the lemma follows.

Remark 3.3 If D_n is a sequence of closed sets with a Radon total capacity measure then we can choose a countable base $\mathscr{M} \subset \mathscr{L}$ closed under finite unions. By choosing a subsequence of D_n and re-labelling we can make Lemma 3.8 applicable, and thus obtain a subsequence which has a limiting capacity measure. Of course, if the original sequence D_n already has a limiting capacity measure, the subsequence will have the same limiting capacity measure as the original sequence, and after re-labelling will satisfy (3.9).

4. Resolvent capacities

In this section we will prove a new formula for the limiting capacity measure defined in Section 1. In what follows we use the notation of Section 3. Let D_n be a sequence of closed sets in R^d, $d \ge 3$. We assume that all the sets D_n are *contained in a single compact set* D. We also assume that the sequence D_n has a finite limiting capacity measure λ, and equation (3.9) holds.

Associated with the resolvent potential Pot^{α} there is a corresponding notion of α-equilibrium measure and α-capacity. The α-equilibrium measure $\psi_{\alpha,K}$ for a compact set K is the unique measure on K such that $\mathrm{Pot}^{\alpha}\psi_{\alpha,K} = 1$ q.e. on K. The α-capacity $c^{\alpha}(K)$ of K is just $\psi_{\alpha,K}(K)$. We note that $c^{\alpha}(K)$ increases with α.

DEFINITION 4.1. For any compact K in R^d, any $\alpha \geq 0$, define $\gamma^{\alpha}(K), \zeta^{\alpha}(K)$ by

$$(4.1)\quad \gamma^{\alpha}(K) = \limsup_{n\to\infty} c^{\alpha}(D_n \cap K), \quad \zeta^{\alpha}(K) = \liminf_{n\to\infty} c^{\alpha}(D_n \cap K).$$

We note that γ^0 is the γ of (3.1), and that $\gamma^{\alpha}, \zeta^{\alpha}$ are nondecreasing in α.

DEFINITION 4.2. Let $\mathscr{L}$ be the collection of sets defined in Definition 3.4. A *locally finite $\mathscr{L}$-partition* $\mathscr{P}$ of R^d is a collection of disjoint sets in $\mathscr{L}$ such that each bounded subset of R^d is contained in a finite union of members of $\mathscr{L}$.

LEMMA 4.1. *Let $\alpha \geq 0$, K compact. Then*

$$(4.2)\qquad \gamma^{\alpha}(K) \leq \lambda(K).$$

Proof. Given $\varepsilon > 0$, we can find $r > 0$ such that

$$(1+\varepsilon)\mathrm{Pot}^{\alpha}\delta_y(x) \geq \mathrm{Pot}\,\delta_y(x) \text{ for } |x-y| < r,$$

where δ_y denotes the Dirac measure concentrated at y. It follows easily that if W is compact with diameter $< r$, then

$$(1+\varepsilon)\mathrm{Pot}^{\alpha}\psi_W \geq \mathrm{Pot}\,\psi_W = 1 \quad \text{q.e. on } W,$$

so $(1+\varepsilon)c(W) \geq c^{\alpha}(W)$. Let $\mathscr{P}$ be a locally finite $\mathscr{L}$-partition such that every set A in $\mathscr{P}$ has diameter less than r. Then

$$c^{\alpha}(D_n \cap K) \leq \sum_{\mathscr{P}} c^{\alpha}(D_n \cap K \cap \bar{A}) \leq \sum_{\mathscr{P}} (1+\varepsilon)c(D_n \cap K \cap \bar{A}).$$

Hence

$$\gamma^{\alpha}(K) \leq \sum_{\mathscr{P}}(1+\varepsilon)\gamma(K\cap\bar{A}) \leq \sum_{\mathscr{P}}(1+\varepsilon)\lambda(K\cap\bar{A})$$
$$= \sum_{\mathscr{P}}(1+\varepsilon)\lambda(K\cap A),$$

so $\gamma^{\alpha}(K) \leq (1+\varepsilon)\lambda(K)$. This proves Lemma 4.1.

LEMMA 4.2. *Let W_i, $i = 1, \dots, k$, be compact disjoint sets in $\mathscr{L}$. Let $W = W_1 \cup \cdots \cup W_k$. Then*

$$\lim_{\alpha \to \infty} \zeta^\alpha(W) \geq \gamma(W_1) + \cdots + \gamma(W_k). \tag{4.3}$$

Proof. Let $K_1, \dots, K_k$ be any compact disjoint sets, $K = K_1, \dots, K_k$. Let

$$r = \min_{i \neq j} d(K_i, K_j).$$

For any $\varepsilon > 0$, there exists $a > 0$ such that

$$\operatorname{Pot}^\alpha \delta_y(x) < \varepsilon \operatorname{Pot} \delta_y(x)$$

for all $\alpha \geq a$ whenever $|x - y| \geq r$. Then $\sum_{i=1}^k \operatorname{Pot}^\alpha \psi_{K_i} \leq 1 + k\varepsilon$ everywhere on K, for $\alpha \geq a$, so $\sum_{i=1}^k c(K_i) \leq (1 + k\varepsilon) c^\alpha(K)$. Letting $K_i = D_n \cap W_i$, we have

$$\sum_{i=1}^k c(D_n \cap W_i) \leq (1 + k\varepsilon) c^\alpha(D_n \cap W).$$

Letting $n \to \infty$ and using (3.9) gives

$$\sum_{i=1}^k \gamma(W_i) \leq (1 + k\varepsilon) \zeta^\alpha(W) \leq (1 + k\varepsilon) \lim_{\alpha \to \infty} \zeta^\alpha(W).$$

This proves Lemma 4.2.

LEMMA 4.3. *For any open set U,*

$$\lambda(U) = \sup\{\zeta^\alpha(W) \colon \alpha \geq 0, W \text{ in } \mathscr{L}, W \text{ compact}, W \subset U\} \tag{4.4}$$

Proof. Since $\mathscr{L}$ contains a base for the topology of R^d, it is easy to see that for U open, the function β of Definition 3.2 satisfies

$$\beta(U) = \sup\left\{\sum_{i=1}^k \gamma(W_i) \colon W_1, \dots, W_k \text{ in } \mathscr{L}, \text{compact, disjoint subsets of } U\right\}.$$

Lemma 4.3 then follows from Lemmas 4.1 and 4.2.

LEMMA 4.4. *Let K, W be compact, $W \subset K$. Then $\psi_{\alpha, K}(W) \leq c^\alpha(W)$.*

Proof. We have $\operatorname{Pot}^\alpha \psi_{\alpha, W} = 1$ q.e. on W. Also $\operatorname{Pot}^\alpha \nu \leq \operatorname{Pot}^\alpha \psi_{\alpha, K} \leq 1$ on K, where ν denotes the restriction of $\psi_{\alpha, K}$ to W. Thus $\operatorname{Pot}^\alpha \nu \leq \operatorname{Pot}^\alpha \psi_{\alpha, W}$ q.e.

on W. Hence

$$\mathrm{Pot}^{\alpha}\nu \leq \mathrm{Pot}^{\alpha}\psi_{\alpha, W} \quad \text{on } R^d,$$

by the domination principle, and so $\nu(R^d) \leq \psi_{\alpha, W}(R^d)$, proving the lemma.

DEFINITION 4.3. Let $\psi_{\alpha, n}$ denote the α-equilibrium measure on D_n.

LEMMA 4.5. *Fix $\alpha \geq 0$ and let λ^{α} be any weak limit point of $\psi_{\alpha, n}$, as $n \to \infty$. Then for K compact, U open, $K \supset U$, we have*

$$(4.5) \qquad \gamma^{\alpha}(K) \geq \lambda^{\alpha}(U).$$

Proof. $c^{\alpha}(D_n \cap K) \geq \psi_{\alpha, n}(K) \geq \psi_{\alpha, n}(U)$, by Lemma 4.4. Lemma 4.5 follows at once.

As a consequence of Lemma 4.5 we see that if V is open, U bounded open, and $\overline{U} \subset V$, then $\lambda(V) \geq \gamma^{\alpha}(\overline{U}) \geq \lambda^{\alpha}(U)$. Hence we have the corollary

$$(4.6) \qquad \lambda \geq \lambda^{\alpha}.$$

On the other hand, clearly

$$(4.7) \qquad \lambda^{\alpha}(D) \geq \zeta^{\alpha}(D),$$

where D is the compact set containing all the D_n. By (4.4), $\zeta^{\alpha}(D) \uparrow \lambda(R^d)$ as $\alpha \to \infty$. Thus we have proved:

THEOREM 4.1. *Let D_n be a sequence of closed sets in R^d, contained in a single compact set D, having a finite limiting capacity measure λ, and such that equation* (3.9) *holds. Let λ^{α} be a weak limit point of $\psi_{\alpha, n}$ for each α. Then λ^{α} converges to λ in total variation norm as $\alpha \to \infty$.*

5. Convergence

LEMMA 5.1. *Let D_n be a sequence of closed sets contained in a compact set D, having a finite limiting capacity measure λ, and such that* (3.9) *holds. Suppose* Pot λ *is bounded on R^d, with associated additive functional A_t, let $F_t = \exp(-A_t)$, and let T be the randomized stopping time constructed from F as in Section* 2. *Suppose λ^i are measures such that*

$$(5.1) \qquad \mathrm{Pot}\,\psi_{i,n} \to \mathrm{Pot}\,\lambda^i \ m\text{-a.e. as } n \to \infty, \text{ for } i = 1, 2, \ldots,$$

where $\psi_{i,n}$ is given in Definition 4.3. *Let x be a point in R^d such that*

$$(5.2) \qquad \mathrm{Pot}\,\psi_{i,n}(x) \to \mathrm{Pot}\,\lambda^i(x) \text{ as } n \to \infty, \text{ for } i = 1, 2, \ldots.$$

Then τ_n converges stably to T with respect to P^x.

Proof. To prove convergence, it is enough to show that any stable limit point S of the sequence τ_n must equal T, P^x-a.e.. Thus let S be such a limit point, and let G be its stopping time measure. Choosing a subsequence again and relabelling, we may assume that τ_n converges stably to S with respect to P^x.

Let N denote the set where (5.1) fails to hold. We have x in N^c and $m(N) = 0$. Since (5.1) holds on N^c for each i, a simple argument shows that on N^c, for each i and for each $\alpha \geq 0$

$$\text{Pot}^\alpha \psi_{i,n} \text{ converges to } \text{Pot}^\alpha \lambda^i \text{ pointwise as } n \to \infty. \tag{5.3}$$

Since $\text{Pot}^\alpha \psi_{\alpha,n}$ is the α-equilibrium potential of D_n, we see that $\text{Pot}^\alpha \psi_{\alpha,n}(y)$ is simply the probability starting from y of hitting D_n before being killed by the "α-killing". Thus

$$\text{Pot}^\alpha \psi_{\alpha,n}(y) = \int_0^\infty \alpha e^{-\alpha t} P^y(\tau_n < t)\, dt. \tag{5.4}$$

For y in N^c, for each $\alpha \geq 0$,

$$\lim_{n \to \infty} \int_0^\infty \alpha e^{-\alpha t} P^y(\tau_n \leq t)\, dt = \text{Pot}^\alpha \lambda^\alpha(y) \leq \text{Pot}^\alpha \lambda(y). \tag{5.5}$$

As $\alpha \to \infty$, $\text{Pot}^\alpha \lambda(y)$ converges to 0. Thus if y is in N^c

$$\lim_{t \to 0} \limsup_{n \to \infty} P^y(\tau_n < t) = 0. \tag{5.6}$$

Equation (5.6) and the fact that $t + \tau_n \circ \theta_t = \tau_n$ on $\{\tau_n > t\}$ show easily that for every $t \geq 0$, if y is in N^c then

$$P^y(S = t) = 0. \tag{5.7}$$

Let j be fixed. Let us redefine our Brownian motion temporarily on a new sample space Ω, rich enough that we may define a sequence $V(1), V(2), \ldots$ of times such that for any y in R^d, with respect to P^y the $V(k)$ are independent, and are together independent of $\sigma(B_t\colon t \geq 0)$ and of any randomized stopping time such as T and S previously considered, and for each k, $P^y(V(k) > t) = e^{-jt}$ for every $t \geq 0$.

Let $r \geq 0$ be fixed. Let Y be bounded and $\mathscr{G}_r$-measurable, $Y \geq 0$. Define $\gamma_{n,j,k}$ for $n > 0$, $j, k \geq 0$, by

$$\int h\, d\gamma_{n,j,k} = E^x\left[Y h\left(B_{R(k)}\right) \chi_{\{\tau_n > R(k)\}}\right], \tag{5.8}$$

for all h bounded Borel on R^d, where $R(k) = r + V(1) + \cdots + V(k)$, $R(0) = r$.

Let $\gamma_{j,k}$ be defined by

$$(5.9) \qquad \int h\,d\gamma_{j,k} = E^x\big[Yh(B_{R(k)})\chi_{\{S>R(k)\}}\big],$$

for all h bounded Borel on R^d.

Using Theorem 2.2 we find easily that for every j, k,

(5.10) $\gamma_{n,j,k}$ converges in total variation norm to $\gamma_{j,k}$ as $n \to \infty$.

Since $\gamma_{j,k}(N) = 0$ for all $k \geq 0$, and $\mathrm{Pot}^j\psi_{j,n} \leq 1$ on R^d, we see by (5.3) that

$$(5.11) \qquad \int \mathrm{Pot}^j\psi_{j,n}\,d\gamma_{n,j,k} \to \int \mathrm{Pot}^j\lambda^j\,d\gamma_{j,k} \quad \text{as } n \to \infty.$$

On $\{\tau_n > R(k)\}$, $\tau_n = R(k) + \tau_n \circ \theta_{R(k)}$, so if $A(k,n)$ denotes $\{R(k) < \tau_n \leq R(k+1)\}$, B denotes $\{\tau_n > R(k)\}$, and C denotes $\{0 < \tau_n \leq V(k+1)\}$, an easy application of the strong Markov property gives

$$\begin{aligned} E^x\big[Y\chi_{A(k,n)}\big] &= E^x\big[Y\chi_B\chi_C \circ \theta_{R(k)}\big] \\ &= \int P^y(0 < \tau_n \leq V(1))\gamma_{n,j,k}(dy). \end{aligned}$$

Also, $P^y(0 \leq \tau_n \leq V(1)) = \mathrm{Pot}^j\psi_{j,n}(y)$.

Thus, using (5.11) and (5.7), $E^x[Y\chi_{A(k,n)}]$ converges to $\int \mathrm{Pot}^j\lambda^j\,d\gamma_{j,k}$ as $n \to \infty$. It follows easily from (5.7) that

$$E^x\big[Y\chi_{A(k,n)}\big] \text{ converges to } E^x\big[Y\chi_{\{R(k)<S\leq R(k+1)\}}\big].$$

Thus

$$(5.12) \qquad E^x\big[Y\chi_{\{R(k)<S\leq R(k+1)\}}\big] = \int \mathrm{Pot}^j\lambda^j\,d\gamma_{j,k}.$$

By definition, $\int \mathrm{Pot}^j\lambda^j\,d\gamma_{j,k} = E^x[Y\chi_{\{S>R(k)\}}\mathrm{Pot}^j\lambda^j(B_{R(k)})]$. Thus, by (2.6) and the strong Markov property (noting that Y is $\mathscr{G}_{R(k)}$-measurable), we have

$$\begin{aligned} (5.13) \quad E^x\big[Y\chi_{\{R(k)<S\leq R(k+1)\}}\big] = E^x\big[Y\chi_{\{S>R(k)\}}\big(&\mathrm{Pot}\,\lambda^j(B_{R(k)}) \\ &- \mathrm{Pot}\,\lambda^j(B_{R(k+1)})\big)\big] \end{aligned}$$

Hence, summing on k,

$$(5.14) \quad E^x\big[Y\chi_{\{r<S<\infty\}}\big] = E^x\big[Y\chi_{\{S>r\}}\big(\mathrm{Pot}\,\lambda^j(B_r) - \mathrm{Pot}\,\lambda^j(B_{L(j)})\big)\big],$$

where $L(j)$ is the first $R(k)$ such that $R(k) \geq S$, $L(j) = \infty$ if $S = \infty$, $\text{Pot}\, \lambda^j(B_\infty) = 0$. (Note $R(k)$ depends on j.)

As $j \to \infty$, $\lambda^j \to \lambda$ in norm. As functions of t, $\text{Pot}\, \lambda^j(B_t)$ and $\text{Pot}\, \lambda(B_t)$ are almost surely continuous on $[0, \infty]$, where we set

$$\text{Pot}\, \lambda^j(B_\infty) = \text{Pot}\, \lambda(B_\infty) = 0.$$

A straightforward argument shows that almost surely,

$$\text{Pot}\, \lambda^j(B_t) \to \text{Pot}\, \lambda(B_t)$$

uniformly over t in $[0, \infty]$. Also, for any $\delta > 0$, as $j \to \infty$,

$$P^x(L(j) > S + \delta) \to 0.$$

Hence, letting $j \to \infty$ in (5.14),

$$(5.15) \quad E^x\left[Y\chi_{\{r<S<\infty\}}\right] = E^x\left[Y\chi_{\{S>r\}}(\text{Pot}\, \lambda(B_r) - \text{Pot}\, \lambda(B_S))\right].$$

The right side of (5.15) is

$$\begin{aligned} E^x\left[Y\chi_{\{S>r\}}(A_S - A_r)\right] &= E^x\left[Y\int_{(r,\infty]}(A_t - A_r)G(dt)\right] \\ &= E^x\left[Y\int_{(r,\infty)} G_t\, dA_t\right] \end{aligned}$$

by Fubini. Thus

$$(5.16) \qquad E^x[Y(G_r - G_\infty)] = E^x\left[Y\int_{(r,\infty)} G_t\, dA_t\right].$$

The remainder of the proof is the same as that of Theorem 2.1 of [3]. Let Z_r denote $G_r + \int_{[0,r]} G_t\, dA_t$. By (5.16), Z_r is a right continuous martingale with respect to the fields of Brownian motion, hence is continuous almost surely, and, since Z_r is increasing, Z_r is accordingly constant for all r, P^x-a.e.. Thus P^x-a.e.,

$$(5.17) \qquad G_r - G_\infty = \int_{(r,\infty)} G_t\, dA_t \text{ for all } r.$$

For any fixed path ω such that (5.17) holds, we see easily that $G_r = \exp(-A_r)$ for all r. This proves Lemma 5.1.

LEMMA 5.2. *Let all the hypotheses of Theorem* 1.3 *hold, and in addition let all the sets D_n be contained in a single compact set D. Then the conclusion of Theorem* 1.3 *holds.*

Proof. We use the notation of Section 4. By choosing a subsequence and relabelling, we may assume that (3.9) holds, and that

(5.18) $\psi_{i,n}$ converges weakly to λ^i as $n \to \infty$, for $i = 1, 2, \dots$.

Then $\lambda^i \to \lambda$ in total variation norm, by Theorem 4.1.

Clearly we may assume that Lebesgue measure m is absolutely continuous with respect to the given measure ν. Replacing ν if necessary by another measure which is mutually absolutely continuous with respect to ν, we may assume that Pot ν is bounded and continuous on R^d. It is easy to see then that, for each i, Pot $\psi_{i,n}$ converges to Pot λ^i in $L^1(\nu)$-norm, as $n \to \infty$. Thus, passing to yet another subsequence and relabelling, we may assume that there is a set N in R^d with $\nu(N) = 0$, such that on N^c, for each i,

(5.19) Pot $\psi_{i,n}$ converges to Pot λ^i pointwise as $n \to \infty$.

The proof of Lemma 5.2 is then completed by Lemma 5.1, applied to x in N^c.

Proof of Theorem 1.3. By relabelling we may assume that the given subsequence is the whole subsequence. Choose $U(j)$ bounded open, $U(j) \uparrow R^d$, $\lambda(\partial U(j)) = 0$. Let $\tau(j, n)$ be the first hitting time of $\overline{U}(j) \cap D_n \equiv D(n, j)$. Let $\lambda(j) = \lambda \chi_{U(j)}$. Let $A_t(j)$ be the additive functional defined by $\lambda(j)$. Let $F(j)$ be the stopping measure defined by $F_t(j) = \exp(-A_t(j))$, $T(j)$ the randomized stopping time associated with $F(j)$.

It is easy to show that the sequence $(D(n, j))_{n=1,2,\dots}$ has limiting capacity measure $\lambda(j)$. By Lemma 5.2, and the Cantor diagonal process, we can choose a subsequence n_i such that for ν-a.e. x, for every j, $\tau(n_i, j) \to T(j)$ stably with respect to P^x as $i \to \infty$, for each j. We will say that x is *good* if these relations hold for all j. Clearly ν-a.e. x is good. We will complete the proof of the theorem by showing that if x is good then $\tau(n_i) \to T$ stably with respect to P^x as $i \to \infty$. Relabelling once more, we may assume that the subsequence n_i is the whole sequence, so that for x good we have $\tau(n, j) \to T(j)$ stably with respect to P^x as $n \to \infty$ for each j. Fix x good, and fix Y bounded, $\mathscr{G}$-measurable on C, f continuous on $[0, \infty]$. We must show that

(5.20) $$E^x[Yf \circ \tau_n] \to E^x[Yf \circ T] \quad \text{as } n \to \infty,$$

where as usual we denote expectation on the product space $C \times [0, 1]$ by the same symbol as ordinary expectation on C.

Clearly to prove (5.20) we may assume that $f(\infty) = 0$. Let σ_j denote the first hitting time of $U(j)^c$. For any t, j, n,

$$\begin{aligned}
&|E^x[Yf \circ \tau_n] - E^x[Yf \circ \tau(n, j)]| \\
&= \left|E^x\left[Y(f \circ \tau_n - f \circ \tau(n, j))\chi_{\{\tau_n > \sigma_j\}}\right]\right| \\
&\le \left|E^x\left[Y(f \circ \tau_n - f \circ \tau(n, j))\chi_{\{\tau_n \ge t\}}\right]\right| \\
&\quad + \left|E^x\left[Y(f \circ \tau_n - f \circ \tau(n, j))\chi_{\{\sigma_j < t\}}\right]\right| \\
&\le 2\|Y\|\ \|f\|_{[t,\infty]} + \|Y\|\ \|f\| P^x(\sigma_j < t),
\end{aligned}$$

where both norms are sup norms and $\|f\|_{[t,\infty]}$ means the sup norm of f as a function on the set $[t, \infty]$. Choosing t large, and then choosing j_0, we see that for any $\delta > 0$ there exists j_0 such that for every $j \ge j_0$,

$$(5.21) \qquad |E^x[Yf \circ \tau_n] - E^x[Yf \circ \tau(n, j)]| \le \delta \quad \text{for every } n.$$

Clearly for every $t \in [0, \infty)$ and every $\omega \in C$, $A_t(j)(\omega) = A_t(\omega)$ for all sufficiently large j. Thus for every ω, the measure $F(j)(\omega)$ on $[0, \infty]$ converges weakly to $F(\omega)$ as $j \to \infty$. Thus $E^x[Yf \circ T(j)] \to E^x[Yf \circ T]$ as $j \to \infty$. Hence for any $\delta > 0$, there exists j_1 such that for all $j \ge j_1$,

$$(5.22) \qquad |E^x[Yf \circ T(j)] - E^x[Yf \circ T]| \le \delta.$$

Applying (5.21) and (5.22) for a $j \ge \max(j_0, j_1)$ easily gives (5.20) and completes the proof of Theorem 1.3.

6. Existence of limiting capacities

LEMMA 6.1. *Let $D_i(n)$ be a closed set in R^d, for $i = 1, \ldots, k(n)$, $n = 1, 2, 3, \ldots$, and let $\nu(n)$ be a finite measure on $D_i(n)$ for each i and each n. Let*

$$D_n = D_1(n) \cup \cdots \cup D_{k(n)}(n)$$

for each i and each n. Suppose that there exists $\rho_n > 0$, such that $\rho_n \to 0$ and diameter$(D_i(n)) < \rho_n$ *for each i and each n. Let ν be a finite measure such that $\nu(n) \to \nu$ vaguely. Define U, L and E by*

$$(6.1) \quad U = \limsup_{n \to \infty} \Sigma^* \left\langle \nu_i(n), \nu_j(n) \right\rangle, \quad L = \liminf_{n \to \infty} \Sigma^* \left\langle \nu_i(n), \nu_j(n) \right\rangle,$$

where Σ^* *denotes the sum over* $i \neq j$, $i, j = 1, 2, \ldots, k(n)$;

$$E = \lim_{r \downarrow 0} \limsup_{n \to \infty} \Sigma^r \big\langle \nu_i(n), \nu_j(n) \big\rangle, \tag{6.2}$$

where Σ^r *denotes the sum over* $i \neq j$, $i, j = 1, 2, \ldots, k(n)$, *i and j such that* $|x - y| \leq r$ *for every* $x \in D_i(n)$, $y \in D_j(n)$.

Then

$$L \geq \langle \nu, \nu \rangle, \tag{6.3}$$

and

$$U = \langle \nu, \nu \rangle + E. \tag{6.4}$$

Proof. Let $A(r) = \{(x, y) \colon |x - y| \leq r\}$, $B(r) = A(r)^c$. If $\rho_n < r$ then clearly

$$\Sigma^r \big\langle \nu_i(n), \nu_j(n) \big\rangle \leq K_d \Sigma^* \int_{A(r)} |x - y|^{-d+2} \nu_i(n)(dx) \nu_j(n)(dy), \tag{6.5}$$

$$\begin{aligned} &\int_{B(r)} |x - y|^{-d+2} \nu(n)(dx) \nu(n)(dy) \\ &\qquad = \Sigma^* \int_{B(r)} |x - y|^{-d+2} \nu_i(n)(dx) \nu_j(n)(dy), \end{aligned} \tag{6.6}$$

and

$$k_d \Sigma^* \int_{A(r)} |x - y|^{-d+2} \nu_i(n)(dx) \nu_j(n)(dy) \leq \Sigma^{3r} \big\langle \nu_i(n), \nu_j(n) \big\rangle. \tag{6.7}$$

For every $c < \langle \nu, \nu \rangle$, there exists $r > 0$ such that

$$k_d \int_{B(r)} |x - y|^{-d+2} \nu(dx) \nu(dy) > c. \tag{6.8}$$

Hence there exists n_0 such that for every $n \geq n_0$,

$$k_d \int_{B(r)} |x - y|^{-d+2} \nu(n)(dx) \nu(n)(dy) > c. \tag{6.9}$$

Thus for every $n \geq n_0$, if $\rho_n < r$ then by (6.5) and (6.6),

$$\Sigma^* \big\langle \nu_i(n), \nu_j(n) \big\rangle \geq c + \Sigma^r \big\langle \nu_i(n), \nu_j(n) \big\rangle. \tag{6.10}$$

Hence

$$L \geq \langle \nu, \nu \rangle, U \geq \langle \nu, \nu \rangle + E. \tag{6.11}$$

For every $\varepsilon > 0$, for every $r > 0$, there exists n_0 such that for every $n \geq n_0$,

$$k_d \int_{B(r)} |x - y|^{-d+2} \nu(n)(dx) \nu(n)(dy) < \langle \nu, \nu \rangle + \varepsilon. \tag{6.12}$$

For every $n \geq n_0$, if $\rho_n < r$ then by (6.6),

$$k_d \Sigma^* \int_{B(r)} |x - y|^{-d+2} \nu_i(n)(dx) \nu_j(n)(dy) < \langle \nu, \nu \rangle + \varepsilon. \tag{6.13}$$

Thus for every $n \geq n_0$, if $\rho_n < r$ then by (6.7),

$$\Sigma^* \big\langle \nu_i(n), \nu_j(n) \big\rangle < \langle \nu, \nu \rangle + \varepsilon + \Sigma^{3r} \big\langle \nu_i(n), \nu_j(n) \big\rangle. \tag{6.14}$$

Hence for every $\varepsilon > 0$, for every $r > 0$,

$$U \leq \langle \nu, \nu \rangle + \varepsilon + \limsup_{n \to \infty} \Sigma^{3r} \big\langle \nu_i(n), \nu_j(n) \big\rangle, \tag{6.15}$$

and so

$$U \leq \langle \nu, \nu \rangle + E. \tag{6.16}$$

This proves Lemma 6.1.

We note that Lemma 6.1 implies that $\lim_{n \to \infty} \Sigma^* \langle \nu_i(n), \nu_j(n) \rangle = \langle \nu, \nu \rangle$ if and only if $E = 0$.

LEMMA 6.2. *Let $D_i(n), \nu_i(n), D_n, \nu(n), \nu$ satisfy all the hypotheses of Lemma* 1.

(i) *Suppose that*

$$\text{Pot}\, \nu_i(n) \geq 1 \text{ q.e. } on\ D_i(n)\ for\ all\ i\ and\ all\ n. \tag{6.17}$$

Then (D_n) has a finite total capacity measure λ, and $\lambda \leq \nu$.

(ii) *Suppose that* (6.17) *holds, and that also*

$$\text{Pot}\, \nu_i(n) = 1\ \nu_i(n)\text{-a.e., } for\ all\ i\ and\ all\ n, \tag{6.18}$$

that ν has finite energy, and that

$$\lim_{n \to \infty} \Sigma^* \big\langle \nu_i(n), \nu_j(n) \big\rangle = \langle \nu, \nu \rangle, \tag{6.19}$$

where Σ^* is defined as in Lemma 6.1. *Then $\lambda = \nu$.*

Proof. (i) Let W be compact. Let U be bounded open, such that $W \subset U$. Let $\delta > 0$ be such that if $d(x, W) \le \delta$ then $x \in U$. Let n_0 be such that $\rho_n < \delta$ for every $n \ge n_0$.

Let $I(n) = \{i\colon D_i(n) \cap W \ne \varnothing\}$. Let $\psi(n)$ denote the equilibrium measure for $W \cap D(n)$. We have $\operatorname{Pot} \psi(n) \le 1 \le \operatorname{Pot} \Sigma_{I(n)} \nu_i(n)$ q.e. on $W \cap D_n$. Thus

$$c(W \cap D_n) \le \sum_{I(n)} \|\nu_i(n)\|.$$

If $n \ge n_0$ then $\nu_i(n)(U^c) = 0$ for every $i \in I(n)$. Hence $c(W \cap D_n) \le \nu(n)(U)$ for all $n \ge n_0$. Thus $\limsup_{n \to \infty} c(W \cap D_n) \le \nu(\overline{U})$. Since this is true for all U, $\limsup_{n \to \infty} c(W \cap D_n) \le \nu(W)$. Hence the total capacity measure λ is finite and $\lambda \le \nu$, so (i) is proved.

(ii) $\lambda \le \nu$ by (i). Let ε be given. By Lemma 6.1, there exists $r > 0$ and n_0 such that for every $n \ge n_0$,

$$\Sigma^r \langle \nu_i(n), \quad \nu_j(n) \rangle < \varepsilon, \tag{6.20}$$

where Σ^r is defined as in Lemma 6.1.

Consider any bounded open sets $U_1, \dots, U_p$, with $\overline{U}_1, \dots, \overline{U}_p$ disjoint and

$$\operatorname{diameter}(\overline{U}_k) < r, \quad k = 1, \dots, p.$$

Let $V = U_1 \cup \cdots \cup U_p$. Let $W_1, \dots, W_p$ be compact, disjoint, such that

$$\overline{U}_k \subset \operatorname{interior}(W_k), \quad \operatorname{diameter}(W_k) < r, \quad k = 1, \dots, p.$$

Since $\rho_n \to 0$, there exists $n_1 \ge n_0$ such that for every $n \ge n_1$, if $D_i(n) \cap \overline{U}_k = \varnothing$ then $D_i(n) \subset W_k$. Let $I(k, n) = \{i\colon D_i(n) \cap \overline{U}_k \ne \varnothing\}$. Let

$$\varphi(k, n) = \sum_{I(k, n)} \nu_i(n).$$

Clearly, support $\varphi(k, n) \subset W_k$ for $n \ge n_1$. Let $\varepsilon(k, n) = \Sigma \langle \nu_i(n), \nu_j(n) \rangle$, where the sum is over $i, j \in I(k, n)$, $i \ne j$. Let $\varepsilon(n) = \varepsilon(1, n) + \cdots + \varepsilon(p, n)$. By (6.20), $\varepsilon(n) < \varepsilon$ for $n \ge n_1$. Clearly

$$\Sigma \langle \varphi(k, n), \varphi(k, n) \rangle \le \varepsilon(k, n) + \|\varphi(k, n)\|. \tag{6.21}$$

Let $\psi(k, n)$ be the equilibrium measure on $W_k \cap D_n$. We have

$$(\langle \psi(k, n), \varphi(k, n) \rangle)^2 \le \langle \psi(k, n), \psi(k, n) \rangle \langle \varphi(k, n), \varphi(k, n) \rangle,$$

so for $n \ge n_1$, $\|\varphi(k, n)\|^2 \le \|\psi(k, n)\| (\|\varphi(k, n)\| + \varepsilon(k, n))$. Hence for $n \ge n_1$,

$$\|\varphi(k, n)\| \le \|\psi(k, n)\| + \varepsilon(k, n). \tag{6.22}$$

Summing on k, for $n \geq n_1$, $\nu(n)(V) \leq \|\psi(1, n)\| + \cdots + \|\psi(p, n)\| + \varepsilon$, so

$$\nu(V) \leq \gamma(W_1) + \cdots + \gamma(W_p) + \varepsilon.$$

Thus $\nu(V) \leq \lambda(W_1) + \cdots + \lambda(W_p) + \varepsilon \leq \|\lambda\| + \varepsilon$. Hence $\|\nu\| \leq \|\lambda\|$, and (ii) follows. This proves Lemma 6.2.

7. Almost uniform convergence

In this section we shall note an extra fact about the convergence proved in Theorem 1.3, for the situation discussed in Section 6. We begin with a simple result concerning real analysis.

LEMMA 7.1. *Let $f \in L^p(m)$, for some $p > 1$, $f \geq 0$, $\int f^p\, dm = c$ Let μ be a finite measure on R^d. Suppose support $f \subset B(0, R)$, and, for every $x \in R^d$, $\mu(B(x, 2R)) < \varepsilon$. Then*

$$\int (\mu * f)^p\, dm \leq \|\mu\| \int f^p\, dm\, \varepsilon^{p/q}, \tag{7.1}$$

where $1/p + 1/q = 1$.

Proof. Let $g = \mu * f$. We have

$$\begin{aligned}\int g^p\, dm &= \int g g^{p-1}\, dm \\ &= \int \mu(dx)\left[\int f_x g^{p-1}\, dm\right] \\ &= \int \mu(dx)\left[\int f_x (\mu_x * f)^{p-1}\, dm\right],\end{aligned}$$

where $f_x(\cdot) = f(\cdot - x)$ and $\mu_x = \mu \chi_{B(x,2R)}$. Also

$$\begin{aligned}\int \left((\mu_x * f)^{p-1}\right)^q dm &= \int \left((\mu_x/\|\mu_x\|) * f\right)^{(p-1)q} dm\, \|\mu_x\|^{(p-1)q} \\ &\leq \int (\mu_x/\|\mu_x\|) * f^{(p-1)q}\, dm\, \|\mu_x\|^{(p-1)q} \\ &= \int f^{(p-1)q}\, dm\, \|\mu_x\|^{(p-1)q} \\ &= \int f^p\, dm\, \|\mu_x\|^p \\ &\leq \int f^p\, dm\, (\varepsilon^p).\end{aligned}$$

Hence

$$\int g^p\, dm \le \int \mu(dx) \|f_x\|_p \|(\mu_x * f)^{p-1}\|_q \le \int \mu(dx) \|f\|_p (\|f\|_p)^{p/q} (\varepsilon^{p/q}),$$

so $\int g^p\, dm \le \|\mu\| \int f^p\, dm(\varepsilon^{p/q})$. This proves Lemma 7.1.

LEMMA 7.2. *Let $D_i(n)$, $\nu_i(n)$, D_n, $\nu(n)$, ν satisfy all the hypotheses of Lemma 6.2(ii). Also, assume $\|\nu(n)\| \le K < \infty$ for all n. Then for every $\delta > 0$ there exists a and n_0 such that for every $n \ge n_0$,*

$$m(\{x: |x| > a, \operatorname{Pot} \nu(n)(x) > \delta\}) < \delta. \tag{7.2}$$

Proof. Let $f_R(x) = k|x|^{-d+2}$ for $|x| < R$, $f_R(x) = 0$ for $|x| \ge R$, where k is defined in (1.1). Let $\Phi_R\mu = \operatorname{Pot} \mu - f_R * \mu$, for any measure μ. Choose R so that

$$|\Phi_R\mu(x)| \le (\delta/2K)\|\mu\|, \tag{7.3}$$

for any measure μ, where we use the total variation norm.

Fix $p > 1$ such that $\int (f_R)^p\, dm < \infty$. Let ε be such that

$$2\varepsilon(2R + \rho_n)^{-d+2} + K\rho_n^{-d+2} < \eta, \tag{7.4}$$

where

$$(\delta/2)^p K \int (f_R)^p\, dm\, (\eta)^{p/q} < \delta. \tag{7.5}$$

Since $\langle \nu, \nu \rangle < \infty$, it is easy to see that there exists b, n_0 such that for every $n \ge n_0$,

$$\Sigma_1 < \varepsilon, \tag{7.6}$$

where Σ_1 denotes the sum of $\langle \nu_i(n), \nu_j(n) \rangle$ over all i and j such that $i \ne j$ and $D_i(n) - B(0, b)$ is nonempty. Clearly

$$\|\nu_i(n)\| \le \rho_n^{-d+2} \quad \text{for all } i, n. \tag{7.7}$$

By (7.6), if $n \ge n_0$ and $|x| > b + 2R$ then

$$\Sigma_2 < 2\varepsilon(2R + \rho_n)^{-d+2}, \tag{7.8}$$

where Σ_2 denotes the sum of $\|\nu_i(n)\|\ \|\nu_j(n)\|$ over all i and j such that $i \ne j$

and

$$D_i(n) \cap B(x,2R) \quad \text{and} \quad D_j(n) \cap B(x,2R)$$

are both nonempty. Clearly

$$[\nu(n)(B(x,2R))]^2 \le \Sigma_2 + \Sigma_3, \tag{7.9}$$

where Σ_3 denotes the sum of $\|\nu_i(n)\|^2$ over all i such that $D_i(n) \cap B(x,2R)$ is nonempty. By (7.8) and (7.9), for $n \ge n_0$ and $|x| > b + 2R$,

$$[\nu(n)(B(x,2R))]^2 \le 2\varepsilon(2R + \rho_n)^{-d+2} + K\rho_n^{-d+2}. \tag{7.10}$$

Thus, by (7.4), for $n \ge n_0$ and $|x| > b + 2R$,

$$\nu(n)(B(x,2R)) < \eta. \tag{7.11}$$

Let $\tilde{\nu}(n) = \nu(n)\chi_{B(0,\,b+4R)}$. By (7.11), for every x in R^d, for $n \ge n_0$,

$$\tilde{\nu}(n)(B(x,2R)) < \eta. \tag{7.12}$$

Let $a = b + 5R$. For $n \ge n_0$,

$$\begin{aligned}&\{x\colon |x| > a, \operatorname{Pot}\nu(n) > \delta\}\\ &\quad = \{x\colon |x| > a, f_R * \nu(n)(x) > \delta/2\} = \{x\colon |x| > a, f_R * \tilde{\nu}(n) > \delta/2\}.\end{aligned}$$

Hence

$$\begin{aligned}m(\{x\colon |x| > a, \operatorname{Pot}\nu(n) > \delta\} &\le (2/\delta)^p \int (f_R * \tilde{\nu}(n))^p\, dm\\ &\le (2/\delta)^p K \int (f_R)^p\, dm (\eta)^{p/q} < \delta,\end{aligned}$$

by Lemma 7.1 and (7.5). This proves Lemma 7.2.

As a consequence of Lemma 7.2, we see that for a sufficiently large, for all x except those points in a set of small Lebesgue measure, $P^x(\tau(n) < \infty)$ is small. This shows that we can translate the almost everywhere convergence for a subsequence given by Theorem 1.2 into almost uniform convergence on all of R^d, not just on bounded subsets of R^d.

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