ILLINOIS JOURNAL OF MATHEMATICS
Volume 31, Number 3, Fall 1987

VERTICAL ORDER IN THE HILBERT CUBE

BY

DENNIS J. GARITY AND DAVID G. WRIGHT

1. Introduction

The relation between the vertical order of a compact set X in E^n and the tameness of the set X has been investigated by F. Tinsley, J. Walsh, and D. Wright ([9], [10] and [8]). The results obtained when $n = 3$ differ from the results obtained for $n > 3$. In particular, a wild one dimensional subset of E^3 must have uncountable vertical order whereas there are wild compacta of any dimension less than $n - 1$ in E^n, $n > 3$, of finite vertical order.

In the Hilbert Cube, the concept of a Z set takes the place of the concept of tameness, at least for sets with infinite codimension. For details, see [1] and [2]. We investigate the role that vertical order plays in the Hilbert Cube. For a large class of subsets of the Hilbert Cube with infinite codimension, namely the weakly infinite dimensional subsets, we show that such subsets are Z sets if they have countable vertical order. Thus, vertical order in the Hilbert Cube seems to be similar to vertical order in E^3.

In Section 3 we set forth the known results on Z sets, weakly infinite dimensional sets, and codimension that we will need. Section 4 is devoted to the proof that subsets of the Hilbert Cube with finite codimension are strongly infinite dimensional. Section 5 contains the main results on vertical order.

2. Definitions and notation

The k-cell I^k will be represented as the product $I_1 \times \cdots \times I_k$ where each I_i is the closed interval $[-1,1]$. We let $\operatorname{Int}(I^k)$ denote $\{x \in I^k |$ for each $i, |x_i| < |\}$ and we let ∂I^k denote the boundary of $I^k, I^k \setminus \operatorname{Int}(I^k)$. We identify $I^m \times I^n$ with I^{m+n} by identifying

$$((x_1, \ldots, x_m),(y_1, \ldots, y_n))$$

with

$$(x_1, \ldots, x_m, y_1, \ldots, y_n).$$

For a fixed k, A_i represents the face of I^k determined by $x_i = -1$ and B_i represents the face determined by $x_i = 1$. We let Σ^{m-1}, $1 \le m < k$, denote the

Received October 18, 1985

$(m-1)$ sphere in $I^k = I^m \times I^{k-m}$ given by $\partial I^m \times (0,\ldots,0)$. We also let $\Sigma^{k-1} = \partial I^k$. Let B_+^n, B_-^n be the n-cells defined by taking all points of Σ^n for which the $(n+1)$-*st* coordinate $x_{n+1} \geq 0$, $x_{n+1} \leq 0$, respectively.

We represent the Hilbert Cube, Q, as $\prod_{i=1}^{\infty} I_i$ and we let $Q_k = \prod_{i+k}^{\infty} I_i$. The opposite faces A_i and B_i of Q are defined in the same way that they are defined in I^k. Let Σ^{k-1} denote the subset of $Q = I^k \times Q_{k+1}$ given by $\partial I^k \times (0,0,\ldots)$.

All spaces will be subsets of I^k or of Q. A collection $\{(C_i, D_i) | i \in J\}$ of pairs of disjoint closed subsets of a space is an *essential family* in X if whenever $\{S_i | i \in J\}$ is a collection of closed subsets of X such that S_i separates C_i from D_i in X, then $\bigcap_{i\in J} S_i \neq \varnothing$. A space is *strongly infinite dimensional* if it has a countably infinite essential family. A space is *weakly infinite dimensional* if it is not strongly infinite dimensional. (Notice that by our definition, finite dimensional sets are weakly infinite dimensional. This is not standard but helps us to concisely state our results.) A space is *countable dimensional* if it is a countable union of finite dimensional subsets. For more information, see [5] and [4].

A closed subset of A of Q is said to have *codimension* $\geq k$ if $H_q(U, U \setminus A) = 0$ for $0 \leq q < k$ and for all open subsets U of Q. The homology is taken with integer coefficients. The subset A has *codimension* k if it has codimension $\geq k$, but does not have codimension $\geq k+1$. The subset A has *infinite codimension* if it has codimension $\geq k$ for all k. For a discussion of codimension, see [2].

A closed subset A of Q is a Z set if there exist maps from Q to $Q \setminus A$ that are arbitrarily close to the identity [1]. Let p be the projection from Q onto Q_2. A subset X of Q has *vertical order* $\leq k$ if the cardinality of $p^{-1}(x) \cap X$ is $\leq k$ for each $x \in Q_2$. The subset X of Q has *countable vertical order over a subset* A of Q_2 if $p^{-1}(a) \cap X$ is countable for each $a \in A$. X has *countable vertical order* if it has countable vertical order over all of Q_2.

3. *Z* Sets infinite codimension and countable to one closed maps

Results in [1] show that Z sets are standardly embedded in the Hilbert Cube. That is, if X_1 and X_2 are subsets of Q that are homeomorphic and are Z sets, then there is a homeomorphism $h\colon Q \to Q$ so that $h(X_1) = X_2$. The following result of Daverman and Walsh gives one method of detecting Z sets.

THEOREM 3.1 [2, p. 419]. *A closed subset A of an ANR X is a Z set if and only if A has infinite codimension and is a* 1-*LCC subset of X.*

The subset A of X is 1-LCC if for each $a \in A$ and for each neighborhood U of a in X there exists a neighborhood V of a such that every map f: $S^1 \to V \setminus A$ extends to a map $\tilde{f}$: $B^2 \to U \setminus A$.

In investigating sets X with countable vertical order in Q, we will need to know what the projection $p\colon X \to Q_2$ does to such sets. The following theorem gives the results we will need.

THEOREM 3.2. *If X is a closed subset of Q with countable vertical order and if X is countable dimensional (weakly infinite dimensional), then $p(X)$ is countable dimensional (weakly infinite dimensional).*

This theorem follows directly from results in [6] and [7] which show that countable dimensionality and weak infinite dimensionality are preserved by countable to one maps on compacta.

4. Finite codimension subsets in Q

The main result of this section is that closed subsets of Q that have finite codimension are strongly infinite dimensional. This, combined with results from Section 3, shows that the infinite codimension of weakly infinite dimensional closed subsets of Q is preserved under projection onto Q_2 if the original subset has countable vertical order.

LEMMA 4.1. *Let X be a closed subset of I^k so that for some positive integer $m < k$,*

$$X \subset (\operatorname{Int} I^m) \times I^{k-m}.$$

Furthermore, suppose that inclusion induced homomorphism

$$H_{m-1}(\Sigma^{m-1}) \to H_{m-1}(I^k \setminus X)$$

is non-trivial. If S is a closed subset of X so that $X \cap A_{m+1}$ and $X \cap B_{m+1}$ are separated in X by S, then the inclusion induced homomorphism $H_m(\Sigma^m) \to H_m(I^k \setminus S)$ is non-trivial.

Proof. Since $X \cap A_{m+1}$ and $X \cap B_{m+1}$ are separated in X by S, $X \setminus S$ can be written as the union of two disjoint sets U_1, U_2 that are open in X and such that

$$X \cap A_{m+1} \subset U_1 \quad \text{and} \quad X \cap B_{m+1} \subset U_2.$$

Set $F_1 = U_1 \cup S$ and $F_2 = U_2 \cup S$. Then F_1 and F_2 are closed subsets of I^k so that $X = F_1 \cup F_2$, $F_1 \cap F_2 = S$, $F_1 \cap B_+^m = \varnothing$, and $F_2 \cap B_-^m = \varnothing$. The proof now follows easily from the following commutative diagram using the

Mayer-Vietoris Theorem and reduced homology.

$$\begin{array}{ccccccc}
0 = H_m(B^m_+) \oplus H_m(B^m_-) & \longrightarrow & H_m(\Sigma^m) & \xrightarrow{\cong} & H_{m-1}(\Sigma^{m-1}) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \text{non-trivial}} & & \\
H_m(I^k \setminus F_1) \oplus H_m(I^k \setminus F_2) & \to & H_m(I^k \setminus S) & \to & H_{m-1}(I^k \setminus X) & \to &
\end{array}$$

THEOREM 4.2. *Let X be a compactum contained in* $\operatorname{Int} I^m \times I^{k-m}$ *such that the inclusion induced homomorphism $H_{m-1}(\Sigma^{m-1}) \to H_{m-1}(I^k \setminus X)$ is non-trivial. Then setting $A_i' = A_i \cap X$ and $B_i' = B_i \cap X$, the collection $\{(A_i', B_i') | m < i \leq k\}$ is an essential $(k - m)$ family for X.*

Proof. Let S_i be closed sets in X that separate A_i' and B_i' in X, $m < i \leq k$. The lemma implies that the inclusion induced homomorphism

$$H_m(\Sigma^m) \to H_m\big(I^k \setminus S_{m+1}\big)$$

is non-trivial. Since S_{m+2} separates A'_{m+2} from B'_{m+2}, $S_{m+1} \cap A'_{m+2}$ and $S_{m+1} \cap B'_{m+2}$ are separated in S_{m+1} by $S_{m+1} \cap S_{m+2}$. The lemma then shows that the inclusion induced homomorphism

$$H_{m+1}(\Sigma^{m+1}) \to H_{m+1}\big(I^k \setminus (S_{m+1} \cap S_{m+2})\big)$$

is non-trivial. Continuing in this manner and using induction we find that the inclusion induced homomorphism

$$H_{k-1}(\Sigma^{k-1}) \to H_{k-1}\left(I^k \setminus \bigcap_{i=m+1}^{k} S_i\right)$$

is non-trivial. But $H_{k-1}(I^k)$ is trivial. So $\bigcap_{i=m+1}^{k} S_i \neq \varnothing$.

THEOREM 4.3. *Let X be a compactum in* $\operatorname{Int} I^m \times Q_{m+1}$ *such that the inclusion induced homomorphism $H_{m-1}(\Sigma^{m-1}) \to H_{m-1}(Q \setminus X)$ is non-trivial. Then X is strongly infinite dimensional.*

Proof. Let $A_i' = A_i \cap X$ and $B_i' = B_i \cap X$. We will show that $\{(A_i', B_i') | i > m\}$ is an essential family for X. Let $\{S_i | i > m\}$ be a collection of closed subsets of X such that S_i separates A_i' from B_i' in X. Let n be an integer greater than m. Set I_0^n to be the subset of $Q = I^n \times Q_{n+1}$ given by $I^n \times (0, 0, 0, \ldots)$. Let $X' = X \cap I_0^n$ and $S_i' = S_i \cap I_0^n$. Applying the previous theorem to X' in I_0^n we have

$$\bigcap\{S_i' | m < i \leq n\} \neq \varnothing.$$

Therefore, $\bigcap\{S_i | m < i \leq n\} \neq \varnothing$. The compactness of Q then implies that

$$\bigcap\{S_i | i > m\} \neq \varnothing,$$

and the theorem is proved.

LEMMA 4.4. *Let X be a nowhere dense compactum in Q. If $\Pi_n(U \setminus X)$ is trivial for all contractible open sets U in Q and all nonnegative integers n, then X has infinite codimension.*

Proof. It will suffice to show that a map from a finite polyhedron into Q can be approximated by a map whose image misses X. This fact follows easily by induction on the skeleta of the polyhedron.

THEOREM 4.5. *If X is a compactum in the Hilbert Cube that has finite codimension, then X is strongly infinite dimensional.*

Proof. We assume that X has finite codimension in the Hilbert Cube Q_4. By identifying Q_4 with $\{(0,0,0)\} \times Q_4 \subset I^3 \times Q_4 = Q$, we obtain an embedding of X in Q. The codimension of X in Q is still finite (but larger by three) and has the properties that X is nowhere dense and $\Pi_n(U \setminus X)$ is trivial for every contractible open set U and $n = 0, 1$. Since X has finite codimension, we know by Lemma 3.1 that there is a contractible open set U and an integer $n > 1$ so that $\Pi_n(U \setminus X)$ is not trivial. We assume that n is minimal so that $\Pi_k(U \setminus X)$ is trivial for $k < n$. By the Hurewicz Isomorphism Theorem, the contractibility of U, and the Z-set approximation theorem [1], there is a map $f\colon I^{n+1} \to U$ so that f is a Z-embedding $f(\partial I^{n+1})$ misses X and is nontrivial homologically in $U \setminus X$.

Identifying I^{n+1} with $I^{n+1} \times (0,0,0,\dots)$ in $I^{n+1} \times Q_{n+2} = Q$, it is easy to extend f to an embedding $h\colon Q \to U$ so that $h(\partial I^{n+1} \times Q_{n+2}) \cap X = \varnothing$. By Theorem 4.3, $h(Q) \cap X$ is strongly infinite dimensional, and we see that X itself is strongly infinite dimensional.

Note. If we are content to work with compact countable dimensional subsets of Q instead of weakly infinite dimensional subsets, there is an inductive argument that shows directly that such subsets have infinite codimension. As in [2], finite dimensional subsets of Q have infinite codimension. Any compact countable dimensional space X has large transfinite inductive dimension *trInd* [3]. An inductive argument shows that X has a countable basis $\{U_i\}$ so that the boundary of each U_i, $\mathrm{Bd}(U_i)$ has infinite codimension. This together with Corollary 2.4 from [2] shows that X has infinite codimension.

5. Vertical order

LEMMA 5.1. *If a compact subset X of Q_2 has codimension $\geq k$ in Q_2, then $X \times I$ has codimension $\geq k$ in Q.*

Proof. This follows in exactly the same way as the proof of Lemma 2.2 in [2] where the result is proved when X has infinite codimension. The proof uses a Mayer-Vietoris argument.

THEOREM 5.2. *Let X be a compact subset of Q and p: $Q \to Q_2$ be projection. If $p(X)$ has codimension ≥ 2 in Q_2 and if $\dim(X \cap p^{-1}(q)) \leq 0$ for each $q \in Q_2$, then X is 1-LCC in Q.*

Proof. Let f be a map of B^2 into Q. It suffices to show that f can be approximated arbitrarily closely by a map $\tilde{f}$ so that $\tilde{f}(B^2) \cap X = \varnothing$. Since $p(x)$ has codimension ≥ 2 in Q_2, $p(X) \times I$ has codimension ≥ 2 in Q by the previous lemma. We may thus assume without loss of generality that $f^{-1}(p(X) \times I)$ is a Cantor set C in the interior of B^2.

For each point $p \in C$ we find a small contractible open set U_p in Q_2 so that $I \times U_p$ contains $f(p)$ and $(\{w\} \times U_p) \cap X = \varnothing$ for some $w \in I$ with w_1 differing from $(f(p))$ by a small preassigned number. Using compactness, we find a finite number of pairwise disjoint disks $D_1, D_2, \ldots, D_k$ in B^2 whose interiors cover C so that the diameters of $f(D_i)$ are small, $f(D_i) \subset I \times U_p$ for some $p \in C$.

Let $\tilde{f}$ equal f on the complement of the D_i. Extend $\tilde{F}$ to each D_i by using a vertical homotopy to the level $\{w\} \times Q_2$, where w is as above, and then sending the rest of D_i into $U_p \times \{w\}$.

THEOREM 5.3. *Let X be a weakly infinite dimensional compact subset of Q that has countable vertical order. Then X is a Z set.*

Proof. By Theorem 3.1, it suffices to show that X has infinite codimension and is 1-LCC. By theorem 3.2, $p(X)$ is weakly infinite dimensional and thus Theorem 4.5 implies both $p(X)$ has infinite codimension in Q_2 and X has infinite codimension in Q. Theorem 5.2 now implies that X is 1-LCC in Q.

COROLLARY 5.4. *Suppose X is a weakly infinite dimensional compact subset of Q. Let $F = \{x \in Q_2 | X$ has uncountable vertical order over $x\}$. If F is a countable union of Z sets, then X is a Z set.*

Proof. Let p: $Q \to Q_2$ be projection. Let f: $B^2 \to Q$ be a map. By a slight adjustment, we may assume $p \circ f(B^2)$ lies in a Hilbert Cube Q_2' in $Q_2 \setminus F$. Let $Q' = I_1 \times Q_2'$, and let $X' = X \cap Q'$. By the previous theorem, X' is a Z set in Q'. Since $f(B^2) \subset Q'$, f can be approximated arbitrarily closely by a map $\tilde{f}$ so that

$$\tilde{f}(B^2) \subset Q' \setminus X' \subset Q \setminus X.$$

Thus X is 1-LCC in Q and is a Z set.

COROLLARY 5.5. *If X is a wild finite dimensional subset of Q, then X has uncountable order over an uncountable subset of Q_2.*

If X is a compact subset of Q that has infinite codimension and countable vertical order, it is not necessarily true that $p(X)$ must have codimension ≥ 2 in Q_2. For example, choose $X = A_1$ or B_1. Hence, the technique of Theorem 5.3 will not apply to any such subset of Q. However, the following conjecture still seems reasonable.

Conjecture. Let X be a compact subset of Q that has infinite codimension and countable vertical order (or vertical order two). Then X is a Z set.

REFERENCES

1. T.A. CHAPMAN, *Lectures on Hilbert cube manifolds*, Amer. Math. Soc. C.B.M.S. regional conference series, No. 28, Providence, RI, 1976.
2. R.J. DAVERMAN and J.J. WALSH, *Cech homology characterizations of infinite dimensional manifolds*, Amer. J. Math., vol. 103 (1981), pp. 411–435.
3. R. ENGELKING, "Transfinite Dimension: a survey and some problems" in *Surveys in general topology*, Academic Press, New York, 1980, pp. 131–161.
4. D.J. GARITY and R.M. SCHORI, *Infinite dimensional dimension theory*, Topology Proc., vol. 10 (1985), pp. 59–74.
5. L.R. RUBIN, R.M. SCHORI and J.J. WALSH, *New dimension theory techniques for constructing infinite dimensional examples*, Topology and Appl., vol. 10 (1979), pp. 93–102.
6. E.G. SKLYARENKO, *Some Remarks on spaces having an infinite number of dimensions*, Dokl. Akad. Nauk. SSSR, vol. 126 (1959), pp. 1203–1206 (Russian).
7. ______, *Two theorems on infinite dimensional spaces*, Dokl. Akad. Nauk. SSSR, vol. 143 (1962), pp. 1053–1056 = Soviet Math. Dokl., 1962, pp. 547–550.
8. F. TINSLEY and D.G. WRIGHT, *On vertical order of one-dimensional compacta in E^3*, Canad. J. Math., vol. 36 (1984), pp. 520–528.
9. J.J. WALSH and D.G. WRIGHT, *Taming compacta in E^4*, Proc. Amer. Math. Soc., vol. 86 (1982), pp. 641–645.
10. D.G. WRIGHT, *Geometric taming of compacta in E^n*, Proc. Amer. Math. Soc., vol. 86 (1982), pp. 646–648.

OREGON STATE UNIVERSITY
CORVALLIS, OREGON
BRIGHAM YOUNG UNIVERSITY
PROVO, UTAH