ILLINOIS JOURNAL OF MATHEMATICS
Volume 31, Number 3, Fall 1987

NEW CRITERIA FOR A DECOMPOSABLE OPERATOR

BY

RIDGLEY LANGE AND SHENGWANG WANG

1. Introduction

Since the original definition of decomposable operator on Banach space was introduced by C. Foias in 1963 [4], many characterizations of such operators have been given ([3], [8], [11]). In this paper we give four new characterizations of decomposable operators. The first of these (Theorem 2.3 (ii)) is a much simpler version of a previously known criterion [11]. Our second equivalent (Theorem 2.3 (iii)) seems to be new.

A third criterion (Theorem 2.3 (iv)) for a decomposable operator generalizes a theorem of the first author [7] and can be expressed as follows: Let T be a bounded linear operator on the complex Banach space X. In 1959, E. Bishop showed [2] that if X is reflexive and T and its adjoint T^* both have property (β) (see below), then T has an "asymptotic spectral decomposition" [9]. R. Lange [7] eventually proved the stronger result that T is decomposable. The converse is also true: if T is decomposable, then T and T^* both satisfy property (β) ([3], [5]). Thus Theorem 2.3(iv) generalizes the result of [7] to arbitrary Banach spaces.

Our fourth equivalent condition (Theorem 2.3(v)) is formally weaker than (iv) in that (iv) implies (v) in an obvious way (see Theorem 2.3), but our proof of this theorem is constructed so as to infer decomposability from this "weaker" property.

To accomplish this requires generalizing some results of E. Bishop [2] from reflexive to arbitrary Banach spaces. Most of these conclusions follow in a routine way in §3. But the crucial Lemma 3.2 requires more care in its proof, while our main result requires the use of a theorem in [3].

We give the statement of our theorem in the next section and its proof in §4.

2. Main result

In order to state our principal theorem, we recall some definitions and notations.

2.1 DEFINITION. Let T be a bounded linear operator on a complex Banach space X. We say that T has the spectral decomposition property (abbreviated

Received October 17, 1985.

SDP) if for every open cover $\{G_i\}_{i=1}^n$ of $\sigma(T)$, there corresponds a system of T-invariant subspaces $\{X_i\}_{i=1}^n$ such that:

(1) $X = X_1 + X_2 + \cdots + X_n$;

(2) $\sigma(T|X_i) \subset G_i$ $(1 \leq i \leq n)$.

T is said to be decomposable if each X_i $(1 \leq i \leq n)$ in Definition 2.1 is spectral maximal [4]. It has been proved by the first author [6] and independently by E. Albrecht [1] and B. Nagy [10] that T has the SDP if and only if T is decomposable.

2.2 Definition. We say that T has property (β) if for any sequence $\{f_n\colon G \to X\}$ of analytic functions, $(\lambda - T)f_n(\lambda) \to 0$ (as $n \to \infty$) in the strong topology of X and uniformly on every compact subset of G, it follows that $f_n(\lambda) \to 0$ in the strong topology of X and uniformly on every compact subset of G.

It is easily seen that every operator T with property (β) has the single valued extension property (abbreviated SVEP); i.e., for every analytic function $f\colon \omega_f \to X$ defined on an open $\omega_f \subset C$, the condition $(\lambda - T)f(\lambda) \equiv 0$ implies $f \equiv 0$.

Furthermore, every decomposable operator or equivalently, every operator with the SDP has property (β) [5] and hence has the SVEP.

For a T-invariant subspace Y, T/Y will stand for the coinduced operator of T on the quotient space X/Y.

2.3 Theorem. *For an operator T, the following assertions are equivalent*:

(i) *T is decomposable.*

(ii) *For every pair of open discs G and H with $\overline{G} \subset H$, there exist T-invariant subspaces X_G, X_H such that*

(a) $X = X_G + X_H$,

(b) $\sigma(T|X_H) \subset H$, $\sigma(T|X_G) \subset C - \overline{G}$.

(iii) *For every pair of open discs G and H with $\overline{G} \subset H$, there exist T-invariant subspaces Y and Z such that*

(a) $\sigma(T|Y) \subset C - G$, $\sigma(T/Y) \subset H$,

(b) $\sigma(T|Z) \subset H$, $\sigma(T/Z) \subset C - G$.

(iv) *T and its adjoint T^* have property (β).*

(v) *T has property (β), and T^* has the SVEP such that for every closed F, $X_T(F)$ is closed.*

3. Preliminaries

In this section, we shall adopt some facts and notations from E. Bishop's seminal paper [2]. A couple U_1 and U_2 of an unbounded and a bounded Cauchy domain, respectively, related by $U_2 = C - \overline{U}_1$ are referred to as complementary simple sets. Let W_1 be the set of analytic functions from U_1 to X which vanish at ∞, and W_2 be the set of analytic functions from U_2 to X^*. Using the seminorms

$$\|f\|_{K_1} = \max\{\|f(\lambda)\|: f \in W_1, \lambda \in K_1, K_1\ (\subset U_1) \text{ is compact}\},$$
$$\|g\|_{K_2} = \max\{\|g(\lambda)\|: g \in W_2, \lambda \in K_2, K_2\ (\subset U_2) \text{ is compact}\},$$

one can define a locally convex topology on W_1 and W_2, respectively. For $i = 1, 2$, let V_i be the subset of W_i on which every function can be extended continuously to $\overline{U}_i$. For $f \in V_1$, $g \in V_2$, define

$$\|f\|_{V_1} = \sup\{\|f(\lambda)\|: \lambda \in U_1\},$$
$$\|g\|_{V_2} = \sup\{\|g(\lambda)\|: \lambda \in U_2\},$$

and note that $(V_1, \|\cdot\|_{V_1})$ and $(V_2, \|\cdot\|_{V_2})$ are Banach spaces. For $x \in X$, $\lambda \in U_2$ and $\mu \in U_1$, define

$$\alpha(x\,\lambda, \mu) = (\mu - \lambda)^{-1}x.$$

For fixed $x \in X$ and $\lambda \in U_2$, $\alpha(x, \lambda, \cdot)$ is called an elementary element of V_1. Let V be the subspace of V_1 spanned by the elementary elements of V_1. For $f \in V_1$ and $g \in V_2$, with continuous extensions to the boundary $\Gamma = \partial U_1 = \partial U_2$, the bilinear form

$$\Phi(f) = \langle f, g\rangle = \frac{1}{2\pi i}\int_\Gamma < f(\lambda), g(\lambda) > d\lambda \tag{3.1}$$

is jointly continuous.

Now, let U_1, U_2 be complementary simple sets. With V, V_i and W_i $(i = 1,2)$ as defined above, then there exists a linear manifold Y in W_2 and norm on Y such that [2]:

(i) Y is a Banach space isometrically isomorphoric to V^*.

(ii) $V_2 \subset Y$.

(iii) The imbeddings $V_2 \to Y$ and $Y \to W_2$ are continuous.

(iv) The bilinear form between V and V_2 defined by (3.1) can be extended to a bilinear form between V and Y in conjunction with the isometric isomorphism between Y and V^* asserted by (i).

Let τ be the operator from V to X defined by $\tau f = \lim_{\lambda\to\infty}\lambda f(\lambda)$; then its adjoint, as an operator from X^* to Y $(= V^*)$, satisfies $\tau^* x^* = x^*$ (see [2]).

In [2], E. Bishop also defined the operator H on V by

$$(Hf)(\lambda) = \tau f - (\lambda - T)f(\lambda),$$

and proved that its adjoint H^* satisfies

$$(H^*g)(\mu) = -(\mu - T^*)g(\mu)$$

for every $g \in V^*$.

For fixed $\eta > 0$, define the following norms on X, X^* respectively [2]:

$$\{x\} = \{x\}_\eta = \inf\left\{\left(\eta\|f\|^2 + \|Hf\|^2\right)^{1/2},\quad f \in V,\quad \tau f = X\right\}, \tag{3.2}$$

$$\{x^*\} = \{x^*\}_\eta = \inf\left\{\left(\|g\|^2 + \eta^{-1}\|H^*g - \tau^*x^*\|^2\right)^{1/2},\quad g \in V^*\right\}. \tag{3.3}$$

Then the norms (3.2), (3.3) are equivalent to the original ones on X, X^*, respectively, and the dull space of X with the norm (3.2) is exactly X^* with the norm (3.3).

3.1 LEMMA [2]. *Let*

$$N = \{x \in X\colon \textit{for every } \epsilon > 0, \textit{ there exists } f \in V \textit{ such that } \|Hf\| < \epsilon \textit{ and } \tau f = x\},$$

and

$$M = \{x^* \in X^*\colon \textit{there exists } g \in V^* \textit{ such that } H^*g = \tau^*x^*\}.$$

Then

$$N = \{x \in X\colon \{x\}_\eta \to 0 \textit{ as } \eta \to 0\},$$

and

$$M = \{x^* \in X^*\colon \textit{there exists fixed } R^* > 0 \textit{ such that for all } \eta > 0,\ \{x^*\}_\eta \le R^*\}.$$

Although in his paper [2], Bishop assumed that the underlying space X was reflexive, by a careful reading of his proofs, one can see that all the facts mentioned above actually remain valid for X non-reflexive. We shall use them without any further reference. (In [2], M above was denoted by M_0.)

In addition, we still need an extension of [2] which plays an essential role in the proof of our main theorem.

3.2 LEMMA. *In the notation of Lemma* 3.1, *we have* $N^\perp = \overline{M}^w$, *the weak* closure of* M.

Proof. For every fixed $x \in N$ and fixed $x^* \in M$, it is evident by Lemma 3.1 that

$$|\langle x, x^*\rangle| \le \{x\}_\eta\{x^*\}_\eta \to 0 \quad (\text{as } \eta \to 0)$$

and hence $N^{\perp} \supset M$. Since $N^{\perp}$ is closed in the weak* topology, it follows that $N^{\perp} \supset \overline{M}^{w}$.

To prove the opposite inclusion, let $x \notin N$. Then there exists $\eta_n \to 0$ and $a > 0$ such that $\{x\}_{\eta_n} > a$. By the Hahn-Banach theorem, for every n, there exists $x_n^* \in X^*$ such that $x_n^* \perp N$, $\{x_n^*\}_{\eta_n} < 1$ and

$$|\langle x, x_n^* \rangle| > a. \tag{3.4}$$

From the definition of $\{x^*{}_n\}$, for every η, there exists g_n such that

$$\|g_n\|^2 + \frac{1}{\eta_n}\|H^*g_n - \tau^*x_n^*\|^2 < 1.$$

Evidently, $\{g_n\}$ is bounded in the norm topology and so is $\{x_\eta^*\}$ by the following inequalities:

$$\begin{aligned}\|x_n^*\| &= \|\tau^*x_n^*\| \\ &< \|\tau^*x_n^* - H^*g_n\| + \|H^*g_n\| \\ &< \eta_n^{1/2}\left(1 - \|g_n\|^2\right)^{1/2} + \|H^*g_n\|.\end{aligned}$$

Thus $\{(x_n^*, g_n)\}$ is a bounded sequence in the product space $X^* \times V^*$ and hence it has at least one cluster point (x^*, g) in the Weak* topology. Since

$$\|H^*g_n - \tau^*x_n^*\| < \eta_n^{1/2},$$

we have $H^*g = \tau^*x^*$ and hence $x^* \in M$. On the other hand, (3.4) implies

$$|\langle x, x^* \rangle| \geq a > 0. \tag{3.5}$$

As x ($\notin N$) is arbitrary, it follows from (3.5) that the preannihilator ${}^{\perp}M \subset N$ or equivalently, $N^{\perp} \subset \overline{M}^{w}$. Lemma 3.2 is thus proved.

4. Proof of the main theorem

The conclusion will be reached through the sequence of implications: (i) $\to$ {(ii) or (iii)} $\to$ (iv) $\to$ (v) $\to$ (i).

(i) $\to$ (ii). Evident by Definition 2.1.

(i) $\to$ (iii). Since T is decomposable, for every pair of open discs G and H with $\overline{G} \subset H$, put $Y = X_T(C - G)$, $Z = X_T(\overline{H}_1)$ where H_1 is an open disc with the property $\overline{G} \subset H_1 \subset \overline{H}_1 \subset H$, one can easily see that Y, Z satisfy (iii, a), (iii, b) respectively. Actually, since

$$X = X_T(C - G) + X_T(\overline{H}_1) = Y + Z,$$

T/Y ($= T/X_T(C - G)$) is similar to the operator

$$[T|X_T(\overline{H}_1)]/[X_T(C-G) \cap X_T(\overline{H}_1)] = [T|X_T(\overline{H}_1)]/X_T[(C-G) \cap \overline{H}_1]$$

and hence

$$\sigma(T/Y) = \sigma\{[T|X_T(\overline{H}_1)]/X_T[(C-G) \cap \overline{H}_1]\} \subset \overline{H}_1 \subset H.$$

With the evident inclusion $\sigma(T|Y) \subset C - G$, (iii, a) thus follows; (iii, b) follows in the routine way.

(iii) → (iv). It follows from [3, Theorem 5.8] and (iii, a) that T has property (β).

To prove that T^* has property (β), let Z be the subspace satisfying (iii, b), then

$$\sigma(T^*|Z^\perp) = \sigma(T/Z) \subset C - G,$$
$$\sigma(T^*/Z^\perp) = \sigma(T|Z) \subset H.$$

Again by [3, Theorem 5.8], T^* has property (β).

(ii) → (iv). Put $Y = X_G$, where X_G is the subspace in (ii), then

$$\sigma(T|Y) = \sigma(T|X_G) \subset C - G.$$

Next, we prove that $\sigma\ (T/Y) \subset H$. Since $X = X_G + X_H$, T/Y ($= T/X_G$) is similar to $[T|X_H]/[X_H \cap Y]$ and hence

$$\sigma(T/Y) = \sigma\{[T|X_H]/[X_H \cap Y]\} \subset H.$$

So Y satisfies condition (iii, a). By the previous proof, T has property (β).

To prove that T^* has property (β), let H' be an open disc satisfying $\overline{G} \subset H' \subset \overline{H}' \subset H$, then $X_{H'}$ exists such that

(4.0) (a) $X = X_G + X_{H'}$ (b) $\sigma(T|X_{H'}) \subset H'$, $\sigma(T|X_G) \subset C - G$.

Since T has property (β), the spectral manifolds $X_T(\overline{H}')$ and $X_T(C - G)$ are closed, hence it follows from (4.0, b) that

$$X_G \subset X_T(C-G), \qquad X_{H'} \subset X_T(\overline{H}')$$

and

$$X = X_T(C-G) + X_T(\overline{H}')$$

by (4.0, a). Let $Z = X_T(\overline{H}')$; then $\sigma\ (T|Z) \subset \overline{H} \subset H$. Since T/Z is similar to

$[T|X_T(C-G)]/[X_T(C-G)\cap X_T(\overline{H}')]$, we have

$$\sigma(T/Z)=\sigma\{[T|X_T(C-G)]/[X_T(C-G)\cap X_T(\overline{H}')]\}\subset C-G.$$

By the proof that (iii) → (iv), T^* has property (β).

(iv) → (v). Evident.

(v) → (i). Let G be open and k be a closed neighborhood of ∞ such that $G\cup K^\circ=C$, where K° is the interior of K. Choose complementary simple sets U_1,U_2 with U_1 unbounded and U_2 bounded such that $U_1\supset C-G$, $\overline{U}_1\supset K^\circ$. Evidently, $G\supset\overline{U}_2\supset U_2\supset C-K$. Let N and M be as defined in Lemma 3.1, then we claim that

$$\text{(4.1)}\qquad \text{(a) } N\subset\overline{X_T(G)},\quad \text{(b) } \overline{M}^w\subset X^*_{T^*}(K).$$

To prove (4.1, a), let $x\in N$. For every n, there exists $f_n\in V$ such that

$$\|Hf_n\|<\frac{1}{n},\quad \tau f_n=x$$

or equivalently,

$$\text{(4.2)}\qquad \|(\lambda-T)f_n(\lambda)-x\|<\frac{1}{n}\quad\text{for }\lambda\in U_1.$$

Since T has property (β), it follows from the proof of [3, Proposition 5.6] that $\{f_n(\lambda)\}$ converges uniformly on every compact set contained in U_1. Let $f(\lambda)=\lim_{n\to\infty}f_n(\lambda)$. Then (4.2) implies that

$$(\lambda-T)f(\lambda)=x\quad\text{for }\lambda\in U_1$$

and hence $\sigma_T(x)\subset C-U_1=\overline{U}_2\subset G$. Clearly, (4.1, a) follows from this last inclusion.

To prove (4.1, b), let $x^*\in M$, so that there exists $g\in V^*$ such that

$$\text{(4.3)}\qquad (\mu-T^*)g(\mu)=\tau^*x^*=x^*\quad\text{for }\mu\in U_2.$$

In particular, (4.3) holds for $\mu\in C-K$ and hence $x^*\in X^*{}_{T^*}(K)$ or equivalently, $M\subset X^*_{T^*}(K)$. Since $X^*_{T^*}(K)$ is closed in the weak* topology by [3, Theorem 9.3] and hence $\overline{M}^w\subset X^*_{T^*}(K)$, (4.1, b) follows.

From (4.1) and Lemma 3.2, we have

$$\text{(4.4)}\qquad \overline{X_T(G)}^{\perp}\subset N^\perp=\overline{M}^w\subset X^*_{T^*}(K).$$

If we put $F=C-G$, then $K\supset F$. By the arbitrariness of K, (4.4) implies that

$$\overline{X_T(G)}^{\perp} \subset X_{T^*}^*(F). \tag{4.5}$$

Since the opposite inclusion of (4.5) is clear, one obtains $\overline{X_T(G)}^{\perp} = X_{T^*}^*(F)$. Finally, from

$$\sigma\left(T/\overline{X_T(G)}\right) = \sigma\left(T^*|\overline{X_T(G)}^{\perp}\right) = \sigma(T^*|X_{T^*}^*(F)) \subset F = C - G,$$
$$\sigma\left(T|\overline{X_T(G)}\right) \subset \overline{G},$$

and [3, Theorem 5.17], T is decomposable. The proof of Theorem 2.3 is complete.

References

1. E. Albrecht, *On decomposable operators*, Integral Equations and Operator Theory, vol. 2 (1979), pp. 1–10.
2. E. Bishop, *A duality theory for an arbitrary operator*, Pacific J. Math., vol. 9 (1959), pp. 379–397.
3. I. Erdelyi and S. Wang, *A local spectral theory for closed operators*, London Math. Soc. Lecture Note Series, vol. 105, 1985.
4. C. Foias, *Spectral maximal spaces and decomposable operators in Banach spaces*, Arch. Math. (Basal), pp. Vol. 14 (1963), pp. 341–349.
5. ______, *On the maximal spectral spaces of a decomposable operator*, Rev. Roumaine Math. Pures Appl., vol. 15 (1970), pp. 1199–1606.
6. R. Lange, *On generalization of decomposability*, Glasgow Math. J., vol. 22 (1980), pp. 77–81.
7. ______, *A purely analytic criterion for decomposable operators*, Glasgow Math. J., vol. 21 (1980), pp. 69–70.
8. ______, *Equivalent conditions for decomposable operators*, Proc. Amer. Math. Soc., vol. 82 (1981), pp. 401–406.
9. ______, *Duality and asymptotic spectral decomposition*, Pacific J. Math., vol. 121 (1986), pp. 93–108.
10. B. Nagy, *Operators with the spectral decomposition property are decomposable*, Studia Sci. Math. Hungar., vol. 13 (1978), pp. 429–432.
11. M. Radjabalipour, *Equivalence of decomposable and 2-decomposable operators*, Pacific J. Math., vol. 77 (1978), pp. 243–247.

Central Michigan University
Mt. Pleasant, Michigan
Nanjing University
Nanjing, China