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THE HEREDITARY DUNFORD-PETTIS PROPERTY ON $C(K, E)$

BY

PILAR CEMBRANOS[1]

A Banach space E is said to have the Dunford-Pettis property if for every pair of weakly null sequences $(x_n) \subset E$ and $(x'_n) \subset E'$ one has $\lim\langle x_n, x'_n\rangle = 0$. Following Diestel [1] we shall say that a Banach space E is hereditarily Dunford-Pettis (or also that E has the hereditary Dunford-Pettis property) if all of its closed subspaces have the Dunford-Pettis property. The first known example of a space enjoying this property was c_0 [3]. Besides c_0, the most simple examples of these spaces are $c_0(\Gamma)$ for any set Γ and Schur spaces. Practically the rest of the known examples are among the $C(K)$ spaces (see Theorem 1).

In this paper we characterize when $C(K, E)$, the Banach space of all continuous functions defined on a compact Hausdorff space K with values in a Banach space E, endowed with the supremum norm, has the hereditary Dunford-Pettis property.

The notations and terminology used and not explained here can be found in [1], [5], [7].

Recall that if K is a compact Hausdorff space the ω-th derived set of K is defined by

$$K^{(\omega)} = \bigcap_{n=1}^{\infty} K^{(n)},$$

where $K^{(0)} = K$ and $K^{(n)}$ is the set of all accumulation points of $K^{(n-1)}$ for $n \in \mathbb{N}$; and K is said to be dispersed or scattered if it does not contain any perfect set.

The following characterization of hereditarily Dunford-Pettis $C(K)$ spaces is due essentially to Pelczynski and Szlenk (see [1], [6]).

THEOREM 1. *Let K be a compact Hausdorff space. Then $C(K)$ has the hereditary Dunford-Pettis property if and only if K is dispersed and the ω-th derived set of K is empty.*

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Our first result is a characterization of hereditarily Dunford-Pettis spaces that will be useful in the sequel.

PROPOSITION 2. *A Banach space E has the hereditary Dunford-Pettis property if and only if every normalized weakly null sequence in E has a subsequence that is equivalent to the unit vector basis of c_0.*

Proof. The necessity is a direct consequence of the Bessaga-Pelczynski Selection Principle (for example, see [1], p. 26) and the assertion proved in [1, p. 28]. For the sufficiency let F be a closed subspace of E and let $(x_n) \subset F$ and $(x_n') \subset F'$ be two weakly null sequences. If (x_n) is norm convergent to zero clearly $\langle x_n, x_n' \rangle \to 0$. If this is not the case there exists a subsequence (x_{n_k}) of (x_n) that is equivalent to the unit vector basis of c_0. Then the closed subspace H of F spanned by (x_{n_k}) is isomorphic to c_0. Since (x_{n_k}) and $(x'_{n_k}|_H)$ are weakly null sequences in H and H' respectively, and since H has the Dunford-Pettis property, it follows that $\langle x_{n_k}, x'_{n_k} \rangle \to 0$. Hence F has the Dunford-Pettis property and this concludes the proof.

According to the preceding result we can deduce that the Banach spaces constructed by Hagler in [4] and Talagrand in [8] are hereditarily Dunford-Pettis.

In order to determine when $C(K, E)$ has the hereditary Dunford-Pettis property we will first prove that the problem can be reduced to the study of $c_0(E)$, the Banach space of all null sequences in E endowed with the supremum norm.

THEOREM 3. *$C(K, E)$ has the hereditary Dunford-Pettis property if and only if one of the following two conditions holds*:

(a) *K is finite and E has the hereditary Dunford-Pettis property.*

(b) *$C(K)$ and $c_0(E)$ have the hereditary Dunford-Pettis property.*

Proof. The necessity is clear because $C(K)$ and E are isomorphic to complemented subspaces of $C(K, E)$ and, if K is infinite it is well known that $c_0(E)$ is isomorphic to a subspace of $C(K, E)$.

If (a) holds it is obvious that $C(K, E)$ is hereditarily Dunford-Pettis. Suppose that K is infinite and (b) holds, then by Theorem 1, K is dispersed and $K^{(\omega)} = \varnothing$. Let (f_n) be a normalized weakly null sequence in $C(K, E)$. In view of Proposition 2 we need to prove that (f_n) has a subsequence that is equivalent to the unit vector basis of c_0. To do this we proceed in an analogous way to Diestel in [1, p. 29–30]. If we define the equivalence relationship $\sim$ on K by $t \sim t'$ whenever $f_n(t) = f_n(t')$ holds for all $n \in \mathbf{N}$, then there exist a metrizable quotient space $\tilde{K}$ of K and a sequence $(\tilde{f}_n) \subset$

$C(\tilde{K}, E)$ such that

$$\tilde{f}_n(\pi(t)) = f_n(t) \quad \text{for all } t \in K \text{ and } n \in \mathbf{N}, \tag{1}$$

where π: $K \to \tilde{K}$ is the quotient map. By (1) and Theorem 9 of [2], $(\tilde{f}_n)$ is a normalized weakly null sequence in $C(\tilde{K}, E)$. If $\tilde{K}$ is finite it is clear that $C(\tilde{K}, E)$ is isomorphic to a complemented subspace of $c_0(E)$. If $\tilde{K}$ is infinite, since $\tilde{K}$ is a dispersed compact metric space whose ω-th derived set is empty, then $C(\tilde{K}, E)$ is isomorphic to $c_0(E)$ (see [7]). In any case $C(\tilde{K}, E)$ is hereditarily Dunford-Pettis and, according to Proposition 2, $(\tilde{f}_n)$ has a subsequence $(\tilde{f}_{n_k})$ that is equivalent to the unit vector basis of c_0. Now, by (1), we deduce that (f_{n_k}) is also equivalent to the unit vector basis of c_0. This finishes the proof.

Our aim now is to characterize hereditarily Dunford-Pettis $c_0(E)$ spaces. In order to do this we need to consider Banach spaces E satisfying:

(*) There exists $M > 0$ such that every normalized weakly null sequence $(x_n) \subset E$ has a subsequence (y_n) that is equivalent to the unit vector basis of c_0 and satisfies

$$\left\| \sum_{n=1}^{\infty} a_n y_n \right\| \le M \sup_n |a_n| \quad \text{for all } (a_n) \in c_0.$$

Note that by Proposition 2 every Banach space E verifying (*) is hereditarily Dunford-Pettis. We will prove that (*) is the necessary and sufficient condition for $c_0(E)$ to be hereditarily Dunford-Pettis.

Remark 4. It is easily verified that if E is a Banach space satisfying (*), and if for each $m \in \mathbf{N}$ we consider E^m endowed with the maximum norm, then every weakly null sequence (x^n) in the unit ball of E^m has a subsequence (y^n) such that

$$\left\| \sum_{n=1}^{\infty} a_n y^n \right\| \le M \sup_n |a_n| \quad \text{for all } (a_n) \in c_0;$$

moreover, if (x^n) does not tend to zero in norm we can take the subsequence (y^n) equivalent to the unit vector basis of c_0.

We omit the proof of the following lemma because the vectorial version of the proof of Lemma 9 in [3] works the same here.

LEMMA 5. *Let E be a Banach space and let $(x^n) = ((x_i^n)_i)$ be a sequence in $c_0(E)$, with $0 < \inf\|x^n\| \le \sup\|x^n\| < \infty$, such that $(x_i^n)_n$ is norm convergent* *to zero in E for all $i \in \mathbb{N}$. Then (x^n) has a subsequence that is equivalent to the unit vector basis of c_0.*

THEOREM 6. *Let E be a Banach space satisfying $(*)$. Then $c_0(E)$ has the hereditary Dunford-Pettis property.*

Proof. By Proposition 2 we need to prove that every normalized weakly null sequence in $c_0(E)$ has a subsequence that is equivalent to the unit vector basis of c_0. First we see that it suffices to prove this for sequences (x^n) such that each $x^n = (x_i^n)_i$ is eventually zero. Indeed, let $(y^n) \subset c_0(E)$ be a normalized weakly null sequence. For each $n \in \mathbf{N}$ let $z^n \in c_0(E)$ be such that $z^n = (z_i^n)_i$ is eventually zero and $\|y^n - z^n\| < 1/2^n$. If we put $x^n = z^n/\|z^n\|$ for all $n \in \mathbf{N}$, the sequence (x^n) is a normalized weakly null sequence. Now suppose (x^n) has a subsequence (x^{n_k}) that is equivalent to the unit vector basis of c_0. Thus there exist two positive constants c and C such that

$$c \sup_k |a_k| \le \left\| \sum_{k=1}^{\infty} a_k x^{n_k} \right\| \le C \sup_k |a_k| \quad \text{for } (a_k) \in c_0.$$

Take $k_0 \in \mathbf{N}$ with $\sum_{k \ge k_0} 1/2^k < c/4$; then for each $(a_k) \in c_0$ we have

$$\frac{c}{4} \sup_{k \ge k_0} |a_k| \le \left\| \sum_{k \ge k_0} a_k y^{n_k} \right\| \le \left(\frac{c}{4} + 2C \right) \sup_{k \ge k_0} |a_k|.$$

Therefore $(y^{n_k})_{k \ge k_0}$ is equivalent to the unit vector basis of c_0.

For each $m \in \mathbf{N}$ consider the continuous projection with norm one, P_m: $c_0(E) \to c_0(E)$, defined by

$$P_m(x_1, x_2, \dots) = (x_1, \dots, x_m, 0, 0, \dots)$$

for $x = (x_i) \in c_0(E)$.

Let (x^n) be a normalized weakly null sequence in $c_0(E)$ such that each $x^n = (x_i^n)_i$ is eventually zero. If there exists $m \in \mathbf{N}$ such that $P_m(x^n) = x^n$ for all $n \in \mathbf{N}$, then it follows from Remark 4 that (x^n) has a subsequence that is equivalent to the unit vector basis of c_0. If this is not the case we can extract a subsequence (y^n) of (x^n) such that if r_n is the first positive integer satisfying $P_{r_n}(y^n) = y^n$ then $r_n < r_{n+1}$ for all $n \in \mathbf{N}$. At this step we can find two different situations:

(A) For all $i \in \mathbf{N}$ the sequence

$$(y_i^n)_n \subset E$$

is norm convergent to zero.

(B) There exists $j \in \mathbf{N}$ such that $(y_j^n)_n \subset E$ does not converge to zero in norm.

From Lemma 5 it follows that in case (A) there is nothing more to prove. Suppose now that (B) holds. Since the sequence $(P_j(y^n))_n$ tends to zero weakly but not in norm, according to Remark 4, there is a subsequence (z^n) of (y^n) such that $(P_j(z^n))_n$ is equivalent to the unit vector basis of c_0; moreover, there is $c > 0$ such that

$$\text{(a)} \qquad c \sup_n |a_n| \leq \left\| \sum_{n=1}^{\infty} a_n P_j(z^n) \right\| \leq M \sup_n |a_n| \quad \text{for } (a_n) \in c_0.$$

In addition, we can assume $P_j(z^1) \neq z^1$. If s_1 is the first positive integer satisfying $P_{s_1}(z^1) = z^1$, then $s_1 > j$. By Remark 4, (z^n) has a subsequence $(z^{\sigma_1(n)})_n$, with $\sigma_1(1) > 1$, such that

$$\left\| \sum_{n=1}^{\infty} a_n (P_{s_1} - P_j)(z^{\sigma_1(n)}) \right\| \leq M \sup_n |a_n| \quad \text{for } (a_n) \in c_0.$$

If s_2 is the first positive integer such that $P_{s_2}(z^{\sigma_1(1)}) = z^{\sigma_1(1)}$ then $s_2 > s_1$. Now we can repeat the preceding argument and obtain by induction a family $\{(z^{\sigma_k(n)})_n : k \in \mathbf{N}\}$ of subsequences of (z^n) such that:

(i) $(z^{\sigma_k(n)})_n$ is a subsequence of $(z^{\sigma_{k-1}(n)})_n$ for all $k \in \mathbf{N}$;

(ii) if, for each $k \in \mathbf{N}$, s_{k+1} denotes the first positive integer such that $P_{s_{k+1}}(z^{\sigma_k(k)}) = z^{\sigma_k(k)}$, then (s_k) is an increasing sequence; and

(iii) for each $k \in \mathbf{N}$,

$$\left\| \sum_{n=1}^{\infty} a_n (P_{s_{k+1}} - P_{s_k})(z^{\sigma_{k+1}(n)}) \right\| \leq M \sup_n |a_n| \quad \text{for } (a_n) \in c_0.$$

Let $w^n = z^{\sigma_n(n)}$ for $n \in \mathbf{N}$, and $s_0 = j$. We claim that (w^n) is equivalent to the unit vector basis of c_0. To prove this choose $r \in \mathbf{N}$ and a finite sequence $(a_n)_{n=1}^r$ of scalars. By (a) we have

$$\left\| \sum_{n=1}^{r} a_n w^n \right\| \geq \left\| P_j\left(\sum_{n=1}^{r} a_n w^n \right) \right\| = \left\| \sum_{n=1}^{r} a_n P_j(w^n) \right\| \geq c \max_{1 \leq n \leq r} |a_n|,$$

and

$$\left\| \sum_{n=1}^{r} a_n P_j(w^n) \right\| \leq M \max_{1 \leq n \leq r} |a_n|.$$

From (i), (ii) and (iii) it follows that for each $k \in \{0, 1, \dots, r\}$,

$$\begin{aligned}\left\|(P_{s_{k+1}} - P_{s_k})\left(\sum_{n=1}^{r} a_n w^n\right)\right\| &= \left\|a_k(P_{s_{k+1}} - P_{s_k})(w^k) + \sum_{n=k+1}^{r} a_n(P_{s_{k+1}} - P_{s_k})(w^n)\right\| \\ &\le |a_k|\left\|(P_{s_{k+1}} - P_{s_k})(w^k)\right\| + \left\|\sum_{n=k+1}^{r} a_n(P_{s_{k+1}} - P_{s_k})(w^n)\right\| \\ &\le |a_k|\|w^k\| + M \max_{1\le n\le r} |a_n| \\ &\le (M+1) \max_{1\le n\le r} |a_n|.\end{aligned}$$

Since

$$\left\|\sum_{n=1}^{r} a_n w^n\right\| = \max\left\{\left\|P_j\left(\sum_{n=1}^{r} a_n w^n\right)\right\|, \max_{0\le k\le r}\left\|(P_{s_{k+1}} - P_{s_k})\left(\sum_{n=1}^{r} a_n w^n\right)\right\|\right\}$$

we have

$$c \max_{1\le n\le r} |a_n| \le \left\|\sum_{n=1}^{r} a_n w^n\right\| \le (M+1) \max_{1\le n\le r} |a_n|.$$

Hence (w^n) is equivalent to the unit vector basis of c_0 and this concludes the proof.

THEOREM 7. *If $c_0(E)$ has the hereditary Dunford-Pettis property, then E verifies $(*)$.*

Proof. Since E is isomorphic to a complemented subspace of $c_0(E)$ then E is hereditarily Dunford-Pettis. Suppose that E does not verify $(*)$. Then, according to Proposition 2, for each $n \in \mathbf{N}$ there exists a sequence $(x_i^n)_i \subset E$ such that:

(1) $\|x_i^n\| = 1$ for all $i \in \mathbf{N}$;
(2) $(x_i^n)_i$ is equivalent to the unit vector basis of c_0; and
(3) for every subsequence $(x_{i_k}^n)_k$ of $(x_i^n)_i$ there is a finite sequence $(a_k)_{k=1}^r$ of scalars so that

$$\left\|\sum_{k=1}^{r} a_k x_{i_k}^n\right\| > n \max_{1\le k\le r} |a_k|.$$

Let $y^j = (x_j^1, x_j^2, \ldots, x_j^j, 0, 0, \ldots) \in c_0(E)$ for every $j \in \mathbf{N}$, and let $\pi_n(y)$ denote the n-th coordinate of $y \in c_0(E)$. Since $\pi_n(y^j) = x_j^n$ for all $j \geq n$, from (1) and (2) it follows that (y^j) is a normalized weakly null sequence in $c_0(E)$. We shall prove that no subsequence of (y^j) is equivalent to the unit vector basis of c_0. Let $(y^{j_k})_k$ be a subsequence of (y^j). For each $n \in \mathbf{N}$, $(\pi_n(y^{j_k}))_{k \geq n} = (x_{j_k}^n)_{k \geq n}$ is a subsequence of $(x_i^n)_i$, so by (3) there is a finite sequence $(a_k)_{k=n}^{r_n}$ of scalars such that

$$\left\| \sum_{k=n}^{r_n} a_k \pi_n\left(y^{j_k}\right) \right\| > n \max_{n \leq k \leq r_n} |a_k|;$$

therefore

$$\begin{aligned} \left\| \sum_{k=n}^{r_n} a_k y^{j_k} \right\| &\geq \left\| \pi_n\left(\sum_{k=n}^{r_n} a_k y^{j_k} \right) \right\| \\ &= \left\| \sum_{k=n}^{r_n} a_k \pi_n\left(y^{j_k}\right) \right\| \\ &> n \max_{n \leq k \leq r_n} |a_k|. \end{aligned}$$

Hence $(y^{j_k})_k$ is not equivalent to the unit vector basis of c_0. According to Proposition 2 this contradicts the fact that $c_0(E)$ is hereditarily Dunford-Pettis.

By the preceding results we have the following two corollaries.

COROLLARY 8. *$C(K, E)$ has the hereditary Dunford-Pettis property if and only if one of the two following conditions holds*:

(a) *K is finite and E has the hereditary Dunford-Pettis property.*

(b) *K is dispersed with $K^{(\omega)} = \varnothing$ and E verifies $(*)$.*

COROLLARY 9. *Let E be a Banach space. Then the following assertions are equivalent*:

(a) *$C(K, E)$ is hereditarily Dunford-Pettis for some infinite compact K.*

(b) *$C(K, E)$ is hereditarily Dunford-Pettis for all K such that $C(K)$ is hereditarily Dunford-Pettis.*

(c) *$C(K, E)$ is hereditarily Dunford-Pettis for all dispersed compact K with $K^{(\omega)} = \varnothing$.*

(d) *$c_0(E)$ is hereditarily Dunford-Pettis.*

(e) *E verifies $(*)$.*

The above results allow us to give some examples of Banach spaces E such that $C(K, E)$ is hereditarily Dunford-Pettis whenever $C(K)$ is hereditarily

Dunford-Pettis. In fact, we can deduce that most of the known hereditarily Dunford-Pettis spaces have this property: the spaces with the Schur property, $c_0(\Gamma)$ for all Γ, the Banach spaces constructed by Hagler in [4] and Talagrand in [8], and the hereditarily Dunford-Pettis $C(K)$ spaces. By Corollary 9 it suffices to prove that these spaces satisfy $(*)$. This is clear for the Schur spaces and $c_0(\Gamma)$; for the examples constructed by Hagler and Talagrand it follows from Proposition 5 of [4] and Theorem 1 of [8] respectively. Now we shall prove that it is also true for the hereditarily Dunford-Pettis $C(K)$ spaces.

PROPOSITION 10 *If K is a dispersed compact Hausdorff space with $K^{(\omega)} = \varnothing$ and $\mathbf{N}^*$ is the Alexandroff compactification of $\mathbf{N}$, then $K \times \mathbf{N}^*$ is a dispersed compact space with $(K \times \mathbf{N}^*)^{(\omega)} = \varnothing$.*

Proof. Let $\mathbf{N}^* = \mathbf{N} \cup \{\infty\}$ and let A be a nonempty subset of $K \times \mathbf{N}^*$. We shall prove that A has an isolated point. Since

$$K \times \mathbf{N}^* = \left(\bigcup_{n=1}^{\infty} (K \times \{n\}) \right) \cup (K \times \{\infty\})$$

then

$$A = \left(\bigcup_{n=1}^{\infty} ((K \times \{n\}) \cap A) \right) \cup ((K \times \{\infty\}) \cap A)$$

If there is $m \in \mathbf{N}$ such that $(K \times \{m\}) \cap A \neq \varnothing$, from the fact that $K \times \{m\}$ is homeomorphic to K it follows that there exist $t \in K$ and a neighborhood V of t in K such that

$$(V \times \{m\}) \cap [(K \times \{m\}) \cap A] = \{(t, m)\}.$$

Hence (t, m) is an isolated point of A because $(V \times \{m\}) \cap A$ is a neighborhood of (t, m) in A.

If $(K \times \{n\}) \cap A = \varnothing$ for all $n \in \mathbf{N}$, then $(K \times \{\infty\}) \cap A \neq \varnothing$. Again, from the fact that $K \times \{\infty\}$ is homeomorphic to K it follows that there exist $t \in K$ and a neighborhood V of t in K such that

$$(V \times \{\infty\}) \cap [(K \times \{\infty\}) \cap A] = \{(t, \infty)\}.$$

Then $(V \times \mathbf{N}^*) \cap A = \{(t, \infty)\}$. Since $(V \times \mathbf{N}^*) \cap A$ is a neighborhood of (t, ∞) in A we conclude that (t, ∞) is an isolated point of A. Therefore $K \times \mathbf{N}^*$ is a dispersed compact space.

To prove that the ω-th derived set of $K \times \mathbf{N}^*$ is empty note that it can be verified by induction that

$$(K \times \mathbf{N}^*)^{(n)} \subset (K^{(n)} \times \mathbf{N}) \cup (K^{(n-1)} \times \{\infty\}) \quad \text{for all } n \in \mathbf{N}.$$

If $K^{(\omega)} = \bigcap_{n=1}^{\infty} K^{(n)}$ is empty there exists $m \in \mathbf{N}$ such that $K^{(m)} = \varnothing$; this implies that $(K \times \mathbf{N}^*)^{(m+1)}$, and therefore $(K \times \mathbf{N}^*)^{(\omega)}$, is empty.

PROPOSITION 11. *If $C(K)$ has the hereditary Dunford-Pettis property, then $C(K)$ verifies* $(*)$.

Proof. By Theorem 1, K is dispersed and $K^{(\omega)} = \varnothing$. Then the preceding proposition implies that $K \times \mathbf{N}^*$ is dispersed and its ω-th derived set is empty. Again by Theorem 1 it follows that $C(K \times \mathbf{N}^*)$ is hereditarily Dunford-Pettis. Thus, from Corollary 9 and the fact that $C(K \times \mathbf{N}^*)$ is isomorphic to $C(\mathbf{N}^*, C(K))$, we conclude that $C(K)$ verifies $(*)$.

Finally we note that the following natural question arises:

Problem. Does every hereditarily Dunford-Pettis space satisfy $(*)$?.

Added in Proof. Recently Prof. J. Elton has pointed out to me that, in his Ph.D. dissertation (Yale University 1978-1979), he studied in some detail a topic which is related to this paper: the subsequences of weakly null sequences. In particular Proposition 2 is essentially his Corollary 3.5.

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UNIVERSIDAD COMPLUTENSE DE MADRID
MADRID, SPAIN