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BOUNDEDNESS OF COMMUTATORS OF FRACTIONAL AND SINGULAR INTEGRALS FOR THE EXTREME VALUES OF p

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1. Introduction

It is well known that commutators of singular integrals with multiplication by a measurable function $b(x)$ are bounded operators on L^p, $1 < p < \infty$, as long as b is a *BMO* function [C-R-W].

Moreover if the commutator of all Riesz transforms are bounded for some p, $1 < p < \infty$, the function b must necessarily belong to *BMO*. Similar results are also known for the fractional integral operators I_α in connection with the boundedness from L^p into L^q, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$ [Ch].

Later on, Segovia and Torrea [S-T] have considered this problem in the more general context of vector valued operators including in this approach, commutators associated for example to maximal functions.

It this paper we find sufficient conditions on the function b in order to obtain $H^1 \to L^1$ and $L^{n/\alpha} \to BMO$ boundedness of such commutators. In most of the cases the given conditions will be also necessary. See [P] for a discussion in the case b is a *BMO* function. We have chosen to work in the general context of vector valued operators of singular integral type as to include a larger class of commutators. Following this line, we first prove two general theorems (Theorems A and B in Section 2) expressing the conditions on b in terms of the kernel of the given operator. Afterwards, in Section 3, we apply our theorems to some particular cases like the Hilbert transform, fractional integrals and maximal operators of smooth approximations to the identity.

As an example, commutators with the Hilbert transform are bounded from H^1 into L^1 only in the trivial case that b equals a constant; this is also the case in the other extreme, L^∞ into *BMO*. Similar results are proven for the fractional integral, therefore since a constant function corresponds to the zero function in *BMO* we have that for non-zero *BMO* functions the commutator with the Hilbert transform or fractional integral is *not* bounded in the extreme cases; see Theorems (3.1) and (3.10). The picture improves in the periodic case. In fact we prove that commutators with the conjugate function are bounded from L^∞ into *BMO* if and only if b belongs to a class a little bit more restricted than *BMO*, the space BMO_φ for $\varphi(t) = |\log t|^{-1}$.

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These classes, which generalize the *BMO* and Lipschitz spaces, were first introduced by Spanne [S].

From the examples presented here we may conclude that even though all the operators involved can be seen as vector valued singular integrals, those arising from approximations to the identity behave better on $p = \infty$ than classical singular integrals, while the situation is reversed on the extreme $p = 1$. Also, our two last results show how maximal operators acting on ℓ^r-valued functions become "more singular" in this sense, when we approach $r = 1$.

2. Main results

We are going to consider vector valued *operators of Calderón-Zygmund type*; i.e., if E and F are two Banach spaces and if we denote by $\mathcal{L}(E, F)$ the space of all linear bounded operators from E into F the operator T must satisfy:
(a) T: $L^p(\mathbf{R}^n, E) \to L^q(\mathbf{R}^n, F)$ boundedly for some pair p, q, $1 < p \leq q < \infty$.
(b) For every E-valued bounded and compactly supported function f, Tf can be represented by

$$Tf(x) = \int K(x, y) f(y) dy$$

for $x \notin \operatorname{supp} f$ and where the kernel K is a locally integrable function from $\mathbf{R}^n \times \mathbf{R}^n \backslash \{(x, x)\}$, taking values on $\mathcal{L}(E, F)$.

We will say that the kernel K satisfies also the (D^α_∞) condition if for some constant C and α such that $1/q = 1/p - \alpha/n$,

$$\|K(x, y) - K(x, z)\|_{\mathcal{L}(E,F)} \leq C \frac{|y - z|}{|x - z|^{n+1-\alpha}}$$

as long as $|x - z| \geq 2|y - z|$.

Similarly, we will say that K satisfies the $(D^\alpha_\infty)'$ condition if the kernel $\widetilde{K}(x, y) = K(y, x)$ satisfies (D^α_∞).

For this class of operators we shall consider the commutator T_b with a locally integrable function b: $\mathbf{R}^n \to \mathbf{R}$, formally defined by

$$T_b f(x) = b(x)(Tf)(x) - T(bf)(x) \tag{2.1}$$

where the product of $b(x)$ by elements in E or F must be understood as the multiplication of vectors by scalars.

Before stating our main results we remind the definitions and some properties of the vectorial versions of H^1 and BMO spaces.

Let E be a Banach space and f a locally integrable function $f\colon \mathbf{R}^n \to E$ (in the Bochner sense). We define the sharp function of f by

$$f^{\#}(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q \|f(y) - f_Q\|_E dy$$

where Q denotes an arbitrary cube with sides parallel to the axes and f_Q the average of f over Q; i.e., $f_Q = |Q|^{-1} \int_Q f(x)\,dx$.

The space $BMO(\mathbf{R}^n, E)$ is defined as the space of functions f such that $\|f\|_{BMO} = \|f^{\#}\|_\infty < \infty$.

More generally we may introduce the space $BMO_\varphi(\mathbf{R}^n, E)$ for φ a positive non-decreasing function defined on $\mathbf{R}^+$, as the space of locally integrable functions f such that

$$\|f\|_{BMO_\varphi} = \sup_Q \frac{1}{|Q|\,\varphi(|Q|)} \int_Q \|f(y) - f_Q\|_E dy < \infty. \tag{2.2}$$

Similarly the space $BMO_\varphi(T^n, E)$ can be defined (T^n being the n-dimensional thorus) as the corresponding subspace of $L^1(T^n, E)$ but clearly this time it suffices to know the values of φ near zero.

Now we give the definition of the space $H^1(\mathbf{R}^n, E)$. Let a be a function belonging to $L^1(\mathbf{R}^n, E)$. We will say that a is an E-atom if its support is contained in a cube Q in such a way that $\|a(x)\|_E \le 1/|Q|$ and $\int_Q a(x)\,dx = 0$. The space $H^1(\mathbf{R}^n, E)$ can be defined then in the usual way in terms of these atoms. If the Banach space E is U.M.D. the Riesz transforms R_j can be defined for integrable function and moreover [B] we have

$$H^1(\mathbf{R}^n, E) = \left\{ f \in L^1(\mathbf{R}^n, E) \,/ R_j f \in L^1(\mathbf{R}^n, E)\ 1 \le j \le n \right\}$$

Finally we will denote by $L_c^{n/\alpha}(\mathbf{R}^n, E)$ the subspace of $L^{n/\alpha}(\mathbf{R}^n, E)$ of compactly supported functions.

Now we are in the position to state our main results.

THEOREM A. *Let T be an operator as above with a kernel satisfying $(D_\infty^\alpha)'$ and let b be a $BMO(\mathbf{R}^n)$ function. Then the following statements are equivalent.*

(2.3) *The commutator T_b is bounded from $L_c^{n/\alpha}(\mathbf{R}^n, E)$ into $BMO(\mathbf{R}^n, F)$.*

(2.4) *The function b satisfies the following condition:*

For any cube Q and $u \in Q$,

$$\left(\frac{1}{|Q|} \int_Q |b(x) - b_Q| dx \right) \left\| \int_{(2Q)^c} K(u, y) f(y)\, dy \right\|_F \le C \|f\|_{n/\alpha}$$

for every $f \in L_c^{n/\alpha}(\mathbf{R}^n, E)$.

THEOREM B. *Let T be an operator as above with a kernel K satisfying (D_∞^α) and let b be a $BMO(\mathbf{R}^n)$ function. Then the following statements are equivalent.*

(2.5) *The commutator T_b is bounded from $H^1(\mathbf{R}^n, E)$ into $L^{n/(n-\alpha)}(\mathbf{R}^n, E)$.*

(2.6) *The function b satisfies the following condition:*
For any E-atom a supported in a cube Q and $u \in Q$,

$$\int_{(2Q)^c} \left\| K(x, u) \int_Q b(y)a(y)dy \right\|_F^{n/(n-\alpha)} dx \leq C.$$

Before beginning the proofs of Theorems A and B, in the next lemma we give an estimate for the operator T which will be very useful.

(2.7) LEMMA. *Let T be an operator as above with K satisfying $(D_\infty^\alpha)'$ and let f be a function in $L^{n/\alpha}(\mathbf{R}^n, E)$ with compact support. For a cube Q we decompose f into $f = f_1 + f_2$ with $f_1 = f\chi_{2Q}$. Then if $x, u \in Q$ we have*

$$\|Tf_2(x) - Tf_2(u)\|_F \leq C\|f\|_{n/\alpha}. \tag{2.8}$$

Proof. We first observe that for $x,\ u \in Q$ we have

$$\|Tf_2(x) - Tf_2(u)\|_F \leq \int \|K(x, y) - K(u, y)\|_{\mathcal{L}(E,F)} \|f_2(y)\|_E \, dy.$$

We decompose $\mathbf{R}^n$ as the union of $2Q$ and the sets $2^jQ \setminus 2^{j-1}Q$, $j \in \mathbf{N}$, $j \geq 2$ and, by shortness, we omit the subscripts on the norms.

By using $(D_\infty^\alpha)'$ and Hölder's inequality, we obtain

$$\begin{aligned}
\|Tf_2(x) - Tf_2(u)\| &\leq \sum_{j=2}^{\infty} \int_{2^jQ\setminus 2^{j-1}Q} \|K(x, y) - K(u, y)\| \, \|f_2(y)\| \, dy \\
&\quad + \int_{2Q} \|K(x, y) - K(u, y)\| \, \|f_2(y)\| \, dy \\
&\leq C\sum_{j=1}^{\infty} 2^{-j} \frac{1}{|2^jQ|^{1-\alpha/n}} \int_{2^jQ} \|f_2(y)\| \, dy \\
&\leq C\sum_{j=1}^{\infty} 2^{-j} \|f\|_{n/\alpha} = C\|f\|_{n/\alpha}
\end{aligned}$$

□

Proof of Theorem A. Let f be a bounded function with compact support. We observe that since $b \in BMO$ and $f \in L^p(\mathbf{R}^n, E)$ for any $p > 1$, then the commutator $T_b(f)$ makes sense and moreover is a $L^p(\mathbf{R}^n, F)$-function and hence it is a locally integrable function. In order to estimate the *BMO* norm of $T_b f$ we take a cube Q and make a decomposition as in Lemma (2.7):

$$f = f_1 + f_2 \qquad \text{with} f_1 = f\chi_{2Q}.$$

Then for $x \in Q$,

$$\begin{aligned}
T_b f(x) - (T_b f)_Q &= T_b f(x) - \frac{1}{|Q|}\int_Q T_b f(z)\,dz \\
&= T_b f_1(x) - \frac{1}{|Q|}\int_Q T_b f_1(z)\,dz \ + T_b f_2(x) \\
&\quad - \frac{1}{|Q|}\int_Q T_b f_2(z)\,dz \\
&= T_b f_1(x) - (T_b f_1)_Q + (b(x) - b_Q) T f_2(x) \\
&\quad - T((b - b_Q) f_2)(x) \\
&\quad - \frac{1}{|Q|}\int_Q (b(z) - b_Q) T f_2(z)\,dz \\
&\quad - \frac{1}{|Q|}\int_Q T((b - b_Q) f_2)(z)\,dz.
\end{aligned}$$

Let $u \in Q$. Then

$$\begin{aligned}
T_b f(x) - (T_b f)_Q &= T_b f_1(x) - (T_b f_1)_Q + (b(x) - b_Q)(T f_2(x) - T f_2(u)) \\
&\quad + (b(x) - b_Q) T f_2(u) \\
&\quad - \frac{1}{|Q|}\int_Q (b(z) - b_Q)(T f_2(z) - T f_2(u))\,dz \\
&\quad + \frac{1}{|Q|}\int_Q [T((b - b_Q) f_2)(z) - T((b - b_Q) f_2)(x)]\,dz.
\end{aligned}$$

Now, if we let

$$\begin{aligned}
\sigma_1(x) &= T_b f_1(x), \\
\sigma_2(x,u) &= (b(x) - b_Q)(T f_2(x) - T f_2(u)), \\
\sigma_3(x,u) &= T((b - b_Q) f_2)(u) \ - T((b - b_Q) f_2)(x)
\end{aligned}$$

and

$$\sigma_4(x,u) = (b(x) - b_Q) T f_2(u),$$

we have

$$\begin{aligned}
T_b f(x) - (T_b f)_Q &= \sigma_1(x) - (\sigma_1)_Q + \sigma_2(x,u) + \sigma_4(x,u) \\
&\quad - (\sigma_2(\ ,u))_Q + (\sigma_3(\ ,u))_Q. \qquad (2.9)
\end{aligned}$$

We claim that the following estimates hold:

$$\frac{1}{|Q|}\int_Q \|\sigma_1(x)\|\ dx \le C\,\|b\|_{BMO}\,\|f\|_{n/\alpha}\ , \tag{2.10}$$

$$\frac{1}{|Q|}\int_Q \|\sigma_2(x,u)\|\ dx \le C\,\|b\|_{BMO}\,\|f\|_{n/\alpha} \quad \text{for any } u \in Q \tag{2.11}$$

and

$$\frac{1}{|Q|}\int_Q \|\sigma_3(x,z)\|\ dz \le C\,\|b\|_{BMO}\,\|f\|_{n/\alpha} \quad \text{for any } x \in Q. \tag{2.12}$$

To prove (2.10) we note that $b \in BMO$ and the assumptions on T allow us to apply the results of [S-T] concluding that T_b is bounded from $L^p(\mathbf{R}^n, E)$ into $L^q(\mathbf{R}^n, F)$, $1/q = 1/p - \alpha/n$. Therefore, using Hölder's inequality and taking into account that $p \le n/\alpha$, we get

$$\begin{aligned}
\frac{1}{|Q|}\int_Q \|\sigma_1(x)\|\ dx &\le \left(\frac{1}{|Q|}\int_Q \|T_b f_1(x)\|^q\ dx\right)^{1/q} \\
&\le \frac{C\,\|b\|_{BMO}}{|Q|^{1/q}}\left(\int_{2Q} \|f(x)\|^p\, dx\right)^{1/p} \\
&\le \frac{C\,\|b\|_{BMO}}{|Q|^{1/q}}\,|Q|^{1/p-\alpha/n}\,\|f\|_{n/\alpha} \\
&= C\,\|b\|_{BMO}\,\|f\|_{n/\alpha}\,.
\end{aligned}$$

For (2.11) we use Lemma(2.7). In fact, since $u \in Q$ we have

$$\begin{aligned}
\frac{1}{|Q|}\int_Q \|\sigma_2(x,u)\|\ dx &\le \frac{1}{|Q|}\int_Q |b(x) - b_Q|\ \|Tf_2(x) - Tf_2(u)\|\ dx \\
&\le C\|f\|_{n/\alpha}\frac{1}{|Q|}\int_Q |b(x) - b_Q|\ dx \\
&\le C\ \|b\|_{BMO}\|f\|_{n/\alpha}.
\end{aligned}$$

Finally, in proving (2.12), we again use the fact that the kernel of T satisfies $(D^\alpha_\infty)'$. First for $x,\ z \in Q$ we have

$$\|\sigma_3(x,z)\| \le \int_{(2Q)^c} \|K(x,y) - K(z,y)\|\ |b(y) - b_Q|\ \|f(y)\|\ dy.$$

Decomposing $(2Q)^c$ as the union of $2^j Q - 2^{j-1}Q,\ j \geq 2$ and using the $(D^\alpha_\infty)'$ condition in each term and Holder's inequality we get

$$\begin{aligned}
\|\sigma_3(x,z)\| &\leq \sum_{j=2}^{\infty} \frac{2^{-j}}{(|2^jQ|)^{1-\alpha/n}} \int\limits_{2^jQ} |b(y) - b_Q|\ \|f(y)\|\ dy \\
&\leq C\sum_{j=2}^{\infty} 2^{-j}(|2^jQ|)^{\alpha/n}\left(\frac{1}{|2^jQ|}\int\limits_{2^jQ}|b(y)-b_Q|^{n/(n-\alpha)}dy\right)^{1-\alpha/n} \\
&\quad\cdot\left(\frac{1}{|2^jQ|}\int\limits_{2^jQ}\|f(y)\|^{n/\alpha}dy\right)^{\alpha/n} \\
&\leq C\|f\|_{n/\alpha}\sum_{j=2}^{\infty}2^{-j}\left(\frac{1}{|2^jQ|}\int\limits_{2^jQ}|b(y)-b_Q|^{n/(n-\alpha)}dy\right)^{1-\alpha/n}.
\end{aligned}$$

But it is well known that for a function in BMO the last factor in the above sum can be estimated by a fixed constant times $j\,\|b\|_{BMO}$, which leads to the desired conclusion.

Now the equivalence between (2.3) and (2.4) follows easily. Assume first that T_b is bounded from $L^{n/\alpha}_c(\mathbf{R}^n, E)$ into $BMO(\mathbf{R}^n, F)$. By (2.9), for $x,\ u \in Q$ we have

$$\begin{aligned}
\sigma_4(x,u) &= T_bf(x) - (T_bf)_Q - \sigma_1(x) + (\sigma_1)_Q - \sigma_2(x,u) \\
&\quad + (\sigma_2(\ ,u))_Q - (\sigma_3(x,\ ))_Q.
\end{aligned}$$

Integrating in x over Q and using the boundedness of T_b and the estimates (2.10), (2.11) and (2.12) we get

$$\frac{1}{|Q|}\int\limits_{Q}\|\sigma_4(x,u)\|\ dx \leq C\|f\|_{n/\alpha}. \tag{2.13}$$

That means that for any cube Q and $u \in Q$,

$$\frac{1}{|Q|}\int\limits_{Q}|b(x)-b_Q|\ dx\ \left\|\ \int\limits_{(2Q)^c}K(u,y)f(y)\ dy\right\| \leq C\|f\|_{n/\alpha},$$

giving (2.4). Conversely, let us assume (2.4) holds. As we have just seen, this is equivalent to (2.13). Therefore coming back to (2.9) and inserting the estimates (2.10), (2.11), (2.12) and (2.13) we get

$$\frac{1}{|Q|}\int_Q \|T_bf(x) - (T_bf)_Q\|\ dx \leq C\|f\|_{n/\alpha}.$$

Taking the supremum over all cubes Q we obtain the boundedness of T_b from $L_c^{n/\alpha}$ into BMO. $\square$

Proof of Theorem B. Let a be an atom supported in some cube Q. Since a is bounded and with compact support, a belongs to L^p and hence T_b can be applied to a and gives a function in L^q, according to [S-T]. Moreover, for any x, we may write

$$\begin{aligned}
T_b a(x) &= \chi_{2Q}(x) T_b a(x) + \chi_{(2Q)^c}(x) T_b a(x) \\
&= \chi_{2Q}(x) T_b a(x) + \chi_{(2Q)^c}(x)(b(x) - b_Q) T a(x) \\
&\quad + \chi_{(2Q)^c}(x) T((b - b_Q)a)(x) \\
&= \chi_{2Q}(x) T_b a(x) + \chi_{(2Q)^c}(x)(b(x) - b_Q) T a(x) \\
&\quad + \chi_{(2Q)^c}(x) \int_Q [K(x,y) - K(x,u)](b(y) - b_Q) a(y) dy \\
&\quad + \chi_{(2Q)^c}(x) \int_Q K(x,u)(b(y) - b_Q) a(y) dy \\
&= \mu_1(x) + \mu_2(x) + \mu_3(x,u) + \mu_4(x,u)
\end{aligned}$$

where u is any point in the cube Q.

We claim that the following estimates hold:

$$\|\mu_i\|_{\frac{n}{n-\alpha}} \leq C \qquad \text{for } i = 1, 2, \tag{2.14}$$

and

$$\|\mu_3(\ , u)\|_{\frac{n}{n-\alpha}} \leq C \qquad \text{for } u \in Q. \tag{2.15}$$

In order to prove (2.14) for μ_1 we observe that since $q > n/(n-\alpha)$, applying Hölder's inequality we get

$$\begin{aligned}
\left(\int_{\mathbf{R}^n} \|\mu_1(x)\|^{\frac{n}{n-\alpha}} dx \right)^{\frac{n-\alpha}{n}} &= \left(\int_{2Q} \|T_b a(x)\|^{\frac{n}{n-\alpha}} dx \right)^{\frac{n-\alpha}{n}} \\
&\leq C |Q|^{\frac{n-\alpha}{n} - \frac{1}{q}} \|T_b a\|_q \\
&\leq C |Q|^{\frac{1}{p'}} \|a\|_p \\
&\leq C |Q|^{\frac{1}{p'}} |Q|^{\frac{1}{p}} |Q|^{-1} = C,
\end{aligned}$$

where we have also used the fact that for $b \in BMO$, T_b is bounded from L^p into L^q and the function a is an E-atom.

On the other hand, using the zero average of a and the (D_α^∞) property of the kernel, we have

$$\begin{aligned}
\int_{\mathbf{R}^n} \|\mu_2(x)\|^{\frac{n}{n-\alpha}} dx &\le \int_{(2Q)^c} |b(x) - b_Q|^{\frac{n}{n-\alpha}} \|Ta(x)\|^{\frac{n}{n-\alpha}}\, dx \\
&\le \int_{(2Q)^c} |b(x) - b_Q|^{\frac{n}{n-\alpha}} \left\| \int_Q (K(x,y) - K(x,u))a(y)dy \right\|^{\frac{n}{n-\alpha}} dx \\
&\le \sum_{j=1}^{\infty} \frac{1}{2^{jn/(n-\alpha)}} \left(\frac{1}{|2^j Q|} \int_{2^j Q} |b(x) - b_Q|^{\frac{n}{n-\alpha}} dx \right) \\
&\quad \times \left(\int \|a(y)\| dy \right)^{\frac{n}{n-\alpha}} \\
&\le C \|b\|_{BMO}^{\frac{n}{n-\alpha}} \sum_{j=2}^{\infty} 2^{-jn/(n-\alpha)} j^{n/(n-\alpha)} = C,
\end{aligned}$$

where in the last inequality we used again the property of BMO functions mentioned in the proof of (2.12).

Finally, in order to prove (2.15), we make use again of the (D_α^∞) condition on the kernel, splitting the integral over the complement of $2Q$. In this way, for $u \in Q$ we have

$$\begin{aligned}
\int_{\mathbf{R}^n} \|\mu_3(x,u)\|^{\frac{n}{n-\alpha}} dx &\le \int_{(2Q)^c} \left\| \int_Q (K(x,y) - K(x,u))(b(y) - b_Q)a(y)dy \right\|^{\frac{n}{n-\alpha}} dx \\
&\le C \sum_{j=2}^{\infty} \int_{2^j Q} \left(\frac{1}{2^j |2^j Q|^{1-\alpha/n}} \int_Q |b(y) - b_Q|\, \|a(y)\| dy \right)^{\frac{n}{n-\alpha}} dx \\
&\le C \sum_{j=2}^{\infty} \frac{2^{-jn/(n-\alpha)}}{|2^j Q|} \int_{2^j Q} \left(\frac{1}{|Q|} \int_Q |b(y) - b_Q| dy \right)^{\frac{n}{n-\alpha}} dx \\
&\le C \|b\|_{BMO}^{n/(n-\alpha)} \sum_{j=2}^{\infty} 2^{-jn/(n-\alpha)} \le C.
\end{aligned}$$

Now the proof of the equivalence between (2.5) and (2.6) follows easily. In fact the inequalities (2.14) and (2.15) show that $\|T_b a\|_{n/n-\alpha} \le C$ for any atom a if and only if for any atom a supported in Q and any $u \in Q$, $\|\mu_4(x,u)\|_{n/n-\alpha} \le C$. Using the zero average of a, we get precisely condition (2.6). $\square$

(2.16) *Remark.* We conclude this section by observing that both Theorems A and B remain valid if we work with operators acting on functions defined on the n-dimensional torus T^n, with obvious modifications. For theorem A we ask T_b to be bounded from $L^\infty(\mathrm{T}^n, E)$ into $BMO(\mathbf{R}^n, F)$ and in (2.4) we take cubes small enough so $2Q$ is included in T^n.

Theorem B remains the same; the only care is to have in mind that functions which are constantly equal to a unit vector e are also atoms.

3. Applications

In this section we shall apply our general theorems to some particular operators such as the Hilbert transform, the conjugate function, the fractional integral and the maximal operators associated to some approximations of the identity.

Application 1. *The Hilbert transform and the conjugate function.* Let us denote by H the Hilbert transform on $\mathbf{R}$; i.e.,

$$Hf(x) = \text{p.v.} \int_{\mathbf{R}} \frac{f(y)}{x-y} dy.$$

Then, for $b \in L^1_{\mathrm{loc}}$and $f \in L^\infty_c(\mathbf{R})$ the commutator H_b is given by

$$H_b f(x) = b(x)Hf(x) - H(bf)(x) = \text{p.v.} \int_{\mathbf{R}} \frac{b(x)-b(y)}{x-y} f(y)\, dy.$$

That this operator is well defined even if b is just locally integrable follows by observing that the first term is a product of a L^1_{loc} function times a function in L^p, while the second is the Hilbert transform of an integrable function and hence the principal value exists almost everywhere.

Similarly, $\widetilde{H}$ shall denote the conjugate function operator defined by

$$\widetilde{H} f(x) = \text{p.v.} \frac{1}{2\pi} \int_{-\pi}^{\pi} \cot\left(\frac{x-y}{2}\right) f(y)\, dy, \qquad x \in [-\pi, \pi]$$

and $\widetilde{H}_b$ will be the corresponding commutator with $b \in L^1(T)$.
An application of Theorems A and B lead to the following results.

(3.1) THEOREM. *Let b be a locally integrable function on* $\mathbf{R}$. *Then the following statements are equivalent*:

(a) *The operator H_b is bounded from $L_c^\infty(\mathbf{R})$ into $BMO(\mathbf{R})$.*
(b) *The operator H_b is bounded from $H^1(\mathbf{R})$ into $L^1(\mathbf{R})$.*
(c) *The function b equals to a constant almost everywhere.*

Proof. We first observe that both conditions, (2.4) of Theorem A and (2.6) of Theorem B, are equivalent to (c). In fact, if we write (2.4) in our particular case, then for any interval I and $u \in I$, the inequality

$$\left(\frac{1}{|I|}\int_I |b(x) - b_I|\,dx\right)\left|\int_{(2I)^c} \frac{1}{u-y} f(y)\,dy\right| \leq C\|f\|_\infty,$$

must hold for any $f \in L_c^\infty$. Now for I and $u \in I$ fixed, we take $f_n(y) = \chi_{[-n,n]}(u - y)\, sg(u-y)$, $n \in \mathbf{N}$, obtaining

$$\left(\frac{1}{|I|}\int_I |b(x) - b_I|\,dx\right) \int_{|I|\leq|u-y|<n} |u-y|^{-1}\,dy \leq C,$$

Letting n go to infinity we have $b(x) = b_I$ a.e. in I, and hence b must be constant almost everywhere.

Similarly, condition (2.6) of Theorem B can be written as

$$\left(\int_{(2I)^c} \frac{dy}{|u-y|}\right)\left|\int_I b(y)a(y)\,dy\right| \leq C,$$

where a is any atom supported in I and $u \in I$. Clearly this is impossible unless $\int_I b(y)a(y)dy = 0$ for every interval I and any atom a supported in I; that is, b must be equal to a constant almost everywhere.

Next observe that if we assume $b \in BMO$ we may apply Theorems A and B proving our theorem. Therefore we only need to show that if either (a) or (b) holds then b must be a BMO function. To do this, we first show that (a) and (b) are equivalent and then, by interpolation, we may conclude that H_b is bounded on $L^p(\mathbf{R})$, $1 < p < \infty$, and therefore, by the results in [C-R-W], b must belong to BMO.

We shall prove that under appropiate conditions on f and g,

$$\int H_b f(x) g(x)dx = \int f(x) H_b g(x)dx \tag{3.2}$$

holds. First, assume that f and g are smooth functions with compact support and let K be the union of the supports of f and g. For almost every x, we have

$$\begin{aligned} H_b f(x)g(x) &= g(x)\int_K \frac{b(x)-b(y)}{x-y}(f(y)-f(x))dy \\ &\quad + g(x)f(x)\text{p.v.}\int_K \frac{b(x)-b(y)}{x-y}dy \\ &= G_1(x) + g(x)f(x)H_b(\chi_K)(x). \end{aligned}$$

Since $|G_1(x)| \leq \|f'\|_\infty \|g\|_\infty (|b(x)||K| + \int_K |b|)\chi_K(x)$, then $G_1 \in L^1$. Interchanging the roles of f and g and substracting we get

$$H_b f(x)g(x) - f(x)H_b g(x) = \int_K \frac{b(x)-b(y)}{x-y}(f(y)g(x) - f(x)g(y))dy, \quad (3.3)$$

where the second member is integrable in x. Then, since the integral is an antisymmetric function of x and y, we have

$$\int (H_b f(x)g(x) - f(x)H_b g(x))dx = 0 \quad (3.4)$$

Let us now assume that statement (a) in the theorem holds. Then for f and g smooth functions with compact support, both $H_b f$ and $H_b g$ are *BMO* functions. Thus $H_b fg$ and $fH_b g$ are integrable functions and by (3.4) we have (3.2).

Now, in order to prove (b) it is enough to show that for any smooth atom f, $\|H_b f\|_1 \leq C$. Since $H_b f$ is *BMO*, and therefore locally integrable, then

$$\|H_b f\|_{L^1} = \sup\left|\int H_b f(x)g(x)dx\right| \quad (3.5)$$

where the supremum is taken over all smooth functions g with compact support and $\|g\|_\infty \leq 1$. Using (3.2) and (a) we get

$$\begin{aligned} \|H_b f\|_{L^1} &\leq \sup\left|\int f(x)H_b g(x)dx\right| \\ &\leq \sup \|f\|_{H_1}\|H_b g\|_{BMO} \leq C\|g\|_\infty \|f\|_{H_1} \leq C', \end{aligned}$$

which proves (b).

Next, we shall show that (b) implies (a).

Let f and g smooth functions with compact support and assume $\int g(x)dx = 0$. Then g is a multiple of an atom and therefore $fH_b g$ belongs to L^1, which implies (3.2). Let $f \in L_c^\infty$ (not necessarely smooth) with $\int f(x)dx = 0$.

If φ is a nonnegative smooth function, with compact support and $\int \varphi(x)dx = 1$, then for $f_n(x) = (f * n\varphi(n\cdot))(x)$ we have

$$\int H_b f_n(x) g(x) dx = \int f_n(x) H_b g(x) dx. \tag{3.6}$$

Since the functions f_n belong to H^1 and satisfy

$$\|f_n\|_\infty \leq \|f\|_\infty,$$

$$\lim_{n\to\infty} f_n(x) = f(x) \text{ a.e.},$$

and

$$\lim_{n\to\infty} \|f_n - f\|_{H^1} = 0$$

then by (b) and the Lebesgue bounded convergence theorem, taking the limit in (3.6), we get (3.2) under our assumptions on f and g.

Finally we observe that (3.2) also holds if we assume that $f \in L_c^\infty$ and g is a smooth atom. In fact, if φ is as before we can write

$$f = \left[f - \left(\int f\right)\varphi\right] + \left(\int f\right)\varphi = h + s.$$

The function h belongs to L_c^∞ and $\int h dx = 0$ and, on the other hand, s is smooth and with compact support, so that (3.2) holds substituting h and s for f. Thus, (3.2) is also true for $f = h + s$. In order to estimate the *BMO* norm of $H_b f$ we write

$$\|H_b f\|_{BMO} = \sup \left| \int H_b f(x) g(x) dx \right|$$

where the supremum is taken over all smooth atoms g. By (3.2) and (b) we get

$$\begin{aligned} \|H_b f\|_{BMO} &= \sup \left| \int f(x) H_b g(x) dx \right| \\ &\leq \sup \|f\|_\infty \|H_b g\|_{L^1} \leq C \|f\|_\infty, \end{aligned}$$

ending the proof of the theorem. □

Similarly for the commutators of the conjugate function we can prove the following result.

(3.7) THEOREM. *Let b be an integrable function on T. The following statements are equivalent:*

(a) *The operator $\widetilde{H}_b$ maps $L^\infty(T)$ into $BMO(T)$.*

(b) $\widetilde{H}_b$ *maps* $H^1(T)$ *into* $L^1(T)$.
(c) $b \in BMO_{|\log t|^{-1}}(T)$.

Proof. As in the preceding theorem, duality arguments prove that (a) and (b) are equivalent and by interpolation and the results in [C-R-W], b belongs to $BMO(T)$. Therefore by Theorem A we only need to show that (2.4) is equivalent to (c). In fact if $b \in BMO_{|\log t|^{-1}}(T)$ there exists a constant A such that for any interval $I \subset T$,

$$\frac{|\log |I||}{|I|} \int_I |b(x) - b_I| dx \leq A. \tag{3.8}$$

Then for I small enough, $u \in I$ and $f \in L^\infty(T)$,

$$\begin{aligned}\left| \int_{T \setminus 2I} \cot\left(\frac{u-y}{2}\right) f(y)\, dy \right| &\leq C \|f\|_\infty \int_{|I|<|u-y|<2\pi} \frac{dy}{|u-y|} \\ &\leq C |\log 2\pi - \log |I|| \, \|f\|_\infty \\ &\leq C |\log |I|| \, \|f\|_\infty .\end{aligned}$$

This estimate together with (3.6) gives that for any small interval I and $u \in I$,

$$\frac{1}{|I|} \int_I |b(x) - b_I| dx \left| \int_{T-2I} \cot\left(\frac{u-y}{2}\right) f(y) dy \right| \leq C \|f\|_\infty \tag{3.9}$$

Conversely, if (b) satisfies (3.9), choosing $f(y) = sg(\cot(\frac{u_0-y}{2})) \in L^\infty(T)$ for u_0 the center of I, we get

$$\begin{aligned}\left| \int_{T \setminus 2I} \cot\left(\frac{u_0-y}{2}\right) f(y) dy \right| &= \int_{T \setminus 2I} \left| \cot\left(\frac{u_0-y}{2}\right) \right| dy \\ &\geq C \int_{2\pi > |u_0-y| > |I|} \frac{dy}{|u_0-y|} \\ &\geq C(|\log 2\pi - \log |I||) \\ &\geq C |\log |I||\end{aligned}$$

and hence (3.9) implies $b \in BMO_{|\log t|^{-1}}(T)$. $\square$

Application 2. *Fractional integral operators.* For $0 < \alpha < n$, let I_α be defined by

$$I_\alpha f(x) = \int_{\mathbf{R}} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

Then $I_{\alpha,b}$ will denote its commutator which, even if b is a locally integrable function, is defined for $f \in L_c^\infty$ by the integral

$$I_{\alpha,b} f(x) = \int_{\mathbf{R}} \frac{b(x) - b(y)}{|x - y|^{n-\alpha}} f(y) dy.$$

This follows from the fact that bf is an integrable function and hence $I_\alpha(bf)$ is finite almost everywhere. Similarly we shall denote by $\widetilde{I}_\alpha$ and $\widetilde{I}_{\alpha,b}$ the finite measure versions of these operators.

By applying Theorems A and B we can get the following results

(3.10) THEOREM. *Let b be a locally integrable function on $\mathbf{R}^n$. Then the following statements are equivalent*:

(a) *$I_{\alpha,b}$ maps $L_c^{n/\alpha}$ into $BMO(\mathbf{R}^n)$.*
(b) *$I_{\alpha,b}$ maps $H^1(\mathbf{R}^n)$ into $L^{\frac{n}{n-\alpha}}(\mathbf{R}^n)$.*
(c) *b is constant almost everywhere.*

Proof. By duality arguments, it is easy to see that (a) is equivalent to (b). Again the main observation is that the equality

$$\int_{\mathbf{R}^n} I_{\alpha,b}\, f(x)\, g(x)\, dx = - \int_{\mathbf{R}^n} f(x)\, I_{\alpha,b}\, g(x)\, dx \tag{3.11}$$

holds if we assume either (a) or (b). In fact if f and g are bounded functions with compact supports, Fubini's theorem can be applied to the left hand side of (3.11) giving the right one. For, on one side $I_a(|bf|)$ is an absolutely convergent integral a.e., giving a function in weak $L^{\frac{n}{n-\alpha}}$ and hence locally integrable, while $I_a(|f|)$ is locally bounded and therefore $I_a(|f|)|bg|$ belongs to $L^1(\mathbf{R}^n)$. This is enough to prove the equivalence between (a) and (b). Therefore, by interpolation, $I_{\alpha,b}$ maps $L^p(\mathbf{R}^n)$ into $L^q(\mathbf{R}^n)$, $1/q = 1/p - \alpha/n$ for any $1 < p < n/\alpha$, and hence b must belong to BMO (see [Ch]). Now we are in the hypothesis of Theorem A and we may conclude that if either (a) or (b) is assumed, b must satisfy the inequality (2.4). We pick the functions

$$f_N(y) = |u - y|^{-\alpha} \chi_{B(o,N)}(u - y) \chi_{(2Q)^c}(y) \qquad \text{for } N \in \mathbf{N}.$$

By inserting them in (2.4) we obtain

$$\left(\frac{1}{|Q|} \int_Q |b(x) - b_Q| dx \right) \left(\int_{|Q|^{1/n} < |u-y| < N} \frac{dy}{|u - y|^n} \right)$$
$$\leq C \left(\int_{|Q|^{1/n} < |u-y| < N} \frac{dy}{|u - y|^n} \right)^{\alpha/n}$$

and therefore b must satisfy

$$(\log N - \log |Q|^{1/n})^{1-\alpha/n} \left(\frac{1}{|Q|} \int_Q |b(x) - b_Q| dx \right) \leq C$$

for any cube Q and N large enough. Letting $N \to \infty$, it follows that b must be constant almost everywhere. □

As in the previous case the situation changes when we consider the finite measure version $\widetilde{I}_\alpha$ of this operator. In this case we obtain the following result.

(3.12) THEOREM. *Let b be an integrable function on T. Then the following statements are equivalent*:

(a) $\widetilde{I}_{\alpha,b}$ *maps* $L^{n/\alpha}(T)$ *into* $BMO(T)$ *boundedly.*
(b) $\widetilde{I}_{\alpha,b}$ *maps* $H^1(T)$ *into* $L^{n/(n-\alpha)}(T)$ *boundedly.*
(c) $b \in BMO_{|\log t|^{\alpha/n-1}}(T)$.

The proof is omitted since it follows the same lines as Theorem (3.7), choosing the functions f_N as in the preceding theorem.

Application 3. *Maximal functions.* We shall consider now some maximal operators associated to smooth approximations to the identity. More precisely, let $\varphi(t)$, $t > 0$, be a non negative differentiable function such that for some $0 \leq \alpha < n$,

$$\begin{aligned} &\text{(i)} && \varphi(t) \text{ is non-increasing, } \varphi \not\equiv 0; \\ &\text{(ii)} && |\varphi'(t)| \leq C t^{-n-1+\alpha}; \\ &\text{(iii)} && \int_0^\infty \varphi(t) t^{n-1-\alpha} < \infty; \\ &\text{(iv)} && \varphi(t) \leq C t^{-n+\alpha}. \end{aligned} \tag{3.13}$$

Observe that condition (iv) is implied by the others three together.
We define the maximal function $M_{\varphi,\alpha}$ as

$$M_{\varphi,\alpha} f(x) = \sup_{\epsilon>0} \epsilon^{-n+\alpha} \left| \int_{\mathbb{R}^n} \varphi(|x-y|/\epsilon) f(y)\, dy \right| .$$

Under the assumptions (3.10) it is easy to obtain the pointwise inequality

$$M_{\varphi,\alpha}\, f(x) \leq C_{n,\alpha} \left(\int_0^\infty \varphi(t) t^{n-1-\alpha} dt \right) M_\alpha f(x) \tag{3.14}$$

where M_α is the fractional maximal function

$$M_\alpha f(x) = \sup_{\epsilon>0} \left\{ \epsilon^{-n+\alpha} \int_{|x-y|<\epsilon} |f(y)|\, dy \right\}.$$

Therefore, for $f \in L^{n/\alpha}$, Hölder's inequality implies

$$M_{\varphi,\alpha}\, f(x) \le C_1 M_\alpha f(x) \le C_2 \|f\|_{n/\alpha} \tag{3.15}$$

showing that $M_{\varphi,\alpha}$ maps $L^{n/\alpha}(\mathbf{R}^n)$ into $L^\infty(\mathbf{R}^n)$.

Now, given a function $b \in BMO(\mathbf{R}^n)$ we define the commutator type maximal operator by

$$M_{\varphi,\alpha,b} f(x) = \sup_{\epsilon>0} \epsilon^{-n+\alpha} \left| \int (b(x) - b(y))\varphi(|x-y|/\epsilon) f(y) dy \right|.$$

In order to show that this operator is well defined for $f \in L_c^{n/\alpha}$ observe that from the previous inequalities we get

$$|M_{\varphi,\alpha,b} f(x)| \le c_2 |b(x)|\ \|f\|_{n/\alpha} + M_\alpha(bf)(x).$$

Since $b \in BMO$, b belongs locally to L^r for any $r > 1$. Choosing $r > n/(n-\alpha)$ and p such that $1/p = 1/r + \alpha/n$ it follows, by Hölder's inequality, that $bf \in L^p$, $1 < p < n/\alpha$ and consequently $M_\alpha(bf) \in L^q$, $1/q = 1/p - \alpha/n$ and hence it is finite almost everywhere.

Moreover for this commutator we have the following result.

THEOREM. *Let b belong to BMO. Then*:

(3.16) *If b is a non-zero BMO function then $M_{\varphi,\alpha,b}$ maps $L_c^{n/\alpha}(\mathbf{R}^n)$ into $BMO(\mathbf{R}^n)$.*

(3.17) *$M_{\varphi,\alpha,b}$ maps $H^1(\mathbf{R}^n)$ into $L^{n/(n-\alpha)}(\mathbf{R}^n)$ if and only if b is a zero function in BMO.*

Proof. We shall apply the theory developed in Section 2, choosing $E = \mathbf{C}$, $F = L^\infty(0,\infty)$ the operator defined by

$$T_{\varphi,\alpha} f(x) = \left\{ \epsilon^{-n+\alpha} \int \varphi(|x-y|/\epsilon) f(y)\, dy \right\}_{\epsilon>0}.$$

Clearly $M_{\varphi,\alpha} f(x) = \|T_{\varphi,\alpha} f(x)\|_F$ and from (3.14) we may conclude that $T_{\varphi,\alpha}$ is a bounded operator from L^p into $L^q(\mathbf{R}^n, F)$, $1 < p \le n/\alpha$, $1/q = 1/p - \alpha/n$, in view of the $L^p - L^q$ boundedness of the fractional maximal operator M_α. Moreover it is given by $K_{\varphi,\alpha}(x,y) \in \mathcal{L}(\mathbf{C}, F)$, $x \ne y$,

$$K_{\varphi,\alpha}(x,y)\xi = \left\{ \epsilon^{-n+\alpha} \varphi(|x-y|/\epsilon)\xi \right\}_{\epsilon>0}, \qquad \xi \in \mathbf{C},$$

since by property (iv) of (3.13) we have

$$\|K_{\varphi,\alpha}(x,y)\|_{\mathcal{L}(\mathbf{C},F)} = \sup_{\epsilon} \epsilon^{-n+\alpha}\varphi(|x-y|/\epsilon) \leq C|x-y|^{\alpha-n}.$$

Therefore $K_{\varphi,\alpha}$ is locally integrable in $\mathbf{R}^n \times \mathbf{R}^n \backslash \{(x,x)\}$. Also this kernel satisfies both D_∞^α and $(D_\infty^\alpha)'$. In fact, let $|x-y| > 2|x-x'|$; then by the mean value theorem and property (ii) in (3.13) we have

$$\begin{aligned}\|K_{\varphi,\alpha}(x,y) - K_{\varphi,\alpha}(x',y)\|_{\mathcal{L}(\mathbf{C},F)} &= \sup_{\epsilon>0} \epsilon^{-n+\alpha}|\varphi(|x-y|/\epsilon) - \varphi(|x'-y|/\epsilon)| \\ &\leq C \sup_{\epsilon>0} \epsilon^{-n+\alpha} \frac{|x-x'|/\epsilon}{(|x-y|/2\epsilon)^{n+1-\alpha}} \\ &\leq C \frac{|x-x'|}{|x-y|^{n+1-\alpha}}.\end{aligned}$$

Finally we observe that if $b \in BMO$, the commutator of $T_{\varphi,\alpha,b}$ is given by

$$\begin{aligned}(T_{\varphi,\alpha,b}f)(x) &= b(x)T_{\varphi,\alpha}f(x) - T_{\varphi,\alpha}(bf)(x) \\ &= \left\{\epsilon^{-n+\alpha}\int (b(x)-b(y))\varphi(|x-y|/\epsilon)f(y)dy\right\}_{\epsilon>0}\end{aligned}$$

and consequently

$$\|T_{\varphi,\alpha,b}f(x)\|_F = M_{\varphi,\alpha,b}f(x).$$

Since for any F-valued function g is true that $g \in BMO(\mathbf{R}^n, F)$ implies that $\|g\|_F$ belongs to $BMO(\mathbf{R}^n)$ with $\|\ \|g\|_F\|_{BMO} \leq \|g\|_{BMO(\mathbf{R}^n,F)}$, in order to prove (3.16) it will be enough to show that

$$\|T_{\varphi,\alpha,b}f\|_{BMO(\mathbf{R}^n,F)} \leq C\|f\|_{L^{n/\alpha}}, \quad f \in L_c^{n/\alpha}.$$

By Theorem A of Section 2 this is equivalent to the following condition: For any cube Q and $u \in Q$ the inequality

$$\left(\frac{1}{|Q|}\int_Q |b(x)-b_Q|dx\right)\left\|\int_{(2Q)^c} K_{\varphi,\alpha}(u,y)f(y)dy\right\|_F \leq C\|f\|_{n/\alpha}$$

holds with C independent of Q, u, and f. But this is equivalent to saying

$$\left(\frac{1}{|Q|}\int_Q |b(x)-b_Q|dx\right) M_{\varphi,\alpha}f(\chi_{(2Q)^c})(u) \leq C\|f\|_{n/\alpha}.$$

Since $b \in BMO$ and, by (3.15) $M_{\varphi,\alpha}$ is bounded from $L^{n/\alpha}$ into L^∞, the latter inequality holds proving (3.16).

In order to prove (3.17) we observe that $M_{\varphi,\alpha,b}$ maps $H^1(\mathbf{R}^n)$ into $L^{n/(n-\alpha)}(\mathbf{R}^n)$ if and only if $T_{\varphi,\alpha,b}$ maps $H^1(\mathbf{R}^n)$ into $L^{n/(n-\alpha)}(\mathbf{R}^n, F)$. By Theorem B this is equivalent to the following condition:

For any atom a with support in a cube Q and $u \in Q$ the inequality

$$\int\limits_{(2Q)^c} \left\| K_{\varphi,\alpha}(x,u) \int\limits_Q b(y)\, a(y)\, dy \right\|_F^{n/(n-\alpha)} dx \leq C$$

holds for a constant C independent of a, Q and u. But that is to say

$$\left| \int\limits_Q b(y)a(y)dy \right|^{n/(n-\alpha)} \int\limits_{(2Q)^c} [\sup_\epsilon \epsilon^{\alpha-n} \varphi(|x-y|/\epsilon)]^{n/(n-\alpha)} dx \leq C. \tag{3.18}$$

Now by (i) of (3.13) there is $t_0 > 0$ such that $\varphi(t_0) = \gamma > 0$. Then, by taking $\epsilon = |x-u|/t_0$on one hand, and using (iv) again on the other hand we get

$$C_1|x-u|^{\alpha-n} \leq \sup_\epsilon \epsilon^{\alpha-n}\varphi(|x-u|/\epsilon) \leq C_2|x-u|^{\alpha-n}.$$

Therefore (3.18) is equivalent to

$$\left| \int\limits_Q b(y)\, a(y)dy \right|^{n/(n-\alpha)} \int\limits_{(2Q)^c} |x-u|^{-n} dx \leq C.$$

This implies that $\int b(y)a(y)\, dy$ must be zero for any atom and therefore b must be constant almost everywhere. □

As in the previous examples we may consider the periodic version of the operator $M_{\varphi,\alpha}$ acting on functions defined on the n-dimensional torus. In this case a wider class of functions b gives rise to bounded commutators from H^1 into $L^{n/(n-\alpha)}$. More precisely:

THEOREM. *Let $b \in BMO(T^n)$. Then*

(3.19) *$M_{\varphi,\alpha,b}$ maps $L^{n/\alpha}(T^n)$ into $BMO(T^n)$ and*

(3.20) *$M_{\varphi,\alpha,b}$ maps $H^1(T^n)$ into $L^{n/(n-\alpha)}(T^n)$ if only if $b \in BMO_{|\log t|^{\alpha/n-1}}(T^n)$.*

The proof of (3.19) is the same as in the previous case. For (3.20) we just observe that arguing as above, the inequality (3.18) is equivalent to asking that for any atom a supported in a cube Q,

$$\left(\int\limits_Q b(y)\, a(y)\, dy \right) |\log\, |Q||^{1-\alpha/n} \leq C. \tag{3.21}$$

Now, if we let $\rho(t) = 1/|\log t|^{1-\alpha/n}$, the function $\widetilde{a}(y) = a(y)/\rho(|Q|)$ is a ρ-atom in the sense that it has zero average, is supported in Q and

$$\|\widetilde{a}\|_\infty \leq (|Q|\rho(|Q|))^{-1}.$$

Therefore (3.21) is equivalent to say that b belongs to the dual of the atomic space generated by the ρ-atoms which can be identified with BMO_ρ (see [V]).

Application 4. *Maximal operators acting on vector valued functions.* From the previous examples we may conclude that commutators for integral and maximal operators behave similarly on H^1. However, the picture is completely different on the other edge $L^{n/\alpha}$.

In the sequel we will try to show how the gap narrows when we allow the operator $M_{\varphi,\alpha}$ to act on vector valued functions.

To this end we introduce the operators $\mathbf{M}_{\varphi,\alpha}$ and $\mathbf{M}_{\varphi,\alpha,b}$ acting on sequence valued functions $f(x) = (f_j(x))_{j\in\mathbf{N}}$:

$$\mathbf{M}_{\varphi,\alpha} f(x) = (M_{\varphi,\alpha} f_j(x))_{j\in\mathbf{N}}$$

$$\mathbf{M}_{\varphi,\alpha,b} f(x) = (M_{\varphi,\alpha,b} f_j(x))_{j\in\mathbf{N}}$$

With this notation we have the following results.

(3.22) THEOREM. *Let* $0 \leq \alpha < n$, φ *a function satisfying* (3.13) *and* b *a locally integrable function. For the operator* $\mathbf{M}_{\varphi,\alpha,b}$ *we have*:

(3.23) $\mathbf{M}_{\varphi,\alpha,b}$ *maps* $L^{n/\alpha}(\mathbf{R}^n, \ell^r)$ *into* $BMO(\mathbf{R}^n, \ell^r)$ *provided* $b \in BMO(\mathbf{R}^n)$ *and* $n/\alpha \leq r \leq \infty$.

(3.24) $\mathbf{M}_{\varphi,\alpha,b}$ *maps* $L^{n/\alpha}(T^n, \ell^r)$ *into* $BMO(T^n, \ell^r)$ *provided* $b \in BMO_{|\log t|^{\alpha/n-1/r}}(T^n)$ *and* $1 \leq r \leq n/\alpha$.

Proof. We consider the operator $T_{\varphi,\alpha}$ introduced in the proof of the previous theorem acting now on sequences of functions; that is, for $f = (f_j)_{j\in\mathbf{N}}$ we set

$$\widetilde{T}_{\varphi,\alpha} f(x) = (T_{\varphi,\alpha} f_j(x))_{j\in\mathbf{N}}.$$

It follows inmediately that the associated kernel $\widetilde{K}_{\varphi,\alpha}$ is given by

$$\widetilde{K}_{\varphi,\alpha}(x, y)\ (\alpha_j)_{j\in\mathbf{N}} = \left(\left\{\epsilon^{\alpha-n}\varphi\left(\frac{|x-y|}{\epsilon}\right)\alpha_j\right\}_{\epsilon>0}\right)_{j\in\mathbf{N}}$$

and that it satisfies both D^α_∞ and $(D^\alpha_\infty)'$ conditions.

Moreover it follows from (3.14) and from the vector valued singular integrals theory in [RdeF-R-T] that the operator $\widetilde{T}_{\varphi,\alpha}$ is bounded from $L^p(\mathbf{R}^n, \ell^r)$ into $L^q(\mathbf{R}^n$,

$\ell^r(L^\infty(0,\infty))$ for $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$. Therefore given a function $b \in BMO$ we may apply the theorem of Section 2 for $E = \ell^r$ and $F = \ell^r(L^\infty(0,\infty))$, $1 < r \le \infty$, to the commutator

$$\begin{aligned}\widetilde{T}_{\varphi,\alpha,b} f(x) &= b(x)(\widetilde{T}_{\varphi,\alpha} f)(x) - \widetilde{T}_{\varphi,\alpha}(b(.)f(.))(x) \\ &= (T_{\varphi,\alpha,b}(f_j)(x))_{j\in\mathbf{N}}.\end{aligned}$$

As in the proof of the previous theorem we observe that (3.23) will be true as soon as we can show that $\widetilde{T}_{\varphi,\alpha,b}$ maps $L^{n/\alpha}(\mathbf{R}^n, \ell^r)$ into $BMO(\mathbf{R}^n, \ell^r(L^\infty(0,\infty))$, $n/\alpha \le r \le \infty$. According to Theorem A it will be enough to check that for any cube Q and any $u \in Q$,

$$\left(\frac{1}{|Q|}\int_Q |b(x) - b_Q|\, dx\right) \left\| \left(\int_{(2Q)^C} K_{\varphi,\alpha}(u,y)\, f_j(y)\, dy \right)_j \right\|_{\ell^r(L^\infty)} \le C\|(f_j)\|_{L^{n/\alpha}(\mathbf{R}^n,\ell^r)} \tag{3.25}$$

Now it is easy to see from (3.14) and the boundedness $L^{n/\alpha} \to L^\infty$ for the fractional maximal function that the operator $\mathsf{M}_{\varphi,\alpha}$ maps $L^{n/\alpha}(\mathbf{R}^n, \ell^{n/\alpha})$ into $L^\infty(\mathbf{R}^n, \ell^{n/\alpha})$, also $L^{n/\alpha}(\mathbf{R}^n, \ell^\infty)$ into $L^\infty(\mathbf{R}^n, \ell^\infty)$. Therefore, by interpolation, $\mathsf{M}_{\varphi,\alpha}$ is bounded from $L^{n/\alpha}(\mathbf{R}^n, \ell^r)$ into $L^\infty(\mathbf{R}^n, \ell^r)$ if $n/\alpha \le r \le \infty$ and hence $\widetilde{T}_{\varphi,\alpha}$ maps $L^{n/\alpha}(\mathbf{R}^n, \ell^r)$ into $L^\infty(\mathbf{R}^n, \ell^r(L^\infty(0,\infty)))$. This implies that (3.25) holds in this case for any $b \in BMO$.

For the periodic case let us take r such that $1 < r < n/\alpha$. Now, with the previous notation, statement (3.24) would hold if we are able to prove the inequality

$$\sup_{u\in Q}\left(\frac{1}{|Q|}\int_Q |b(x) - b_Q|\, dx\right) \left\| \int_{T^n\setminus 2Q} K_{\varphi,\alpha}(u,y)\, f(y)\, dy) \right\|_{\ell^r(L^\infty)} \le C\|f\|_{L^{n/\alpha}(\ell^r)}. \tag{3.26}$$

But for $u \in Q$ we have

$$\begin{aligned}&\left\| \int_{T^n\setminus 2Q} K_{\varphi,\alpha}(u,y)\, f(y)\, dy \right\|_{\ell^r(L^\infty)} \\ &= \left\| \left(\sup_{\epsilon>0} \left| \int_{T^n\setminus 2Q} \epsilon^{\alpha-n}\varphi\left(\frac{|u-y|}{\epsilon}\right) f_j(y)\, dy \right| \right)_j \right\|_{\ell^r} \\ &\le C \left\| (\sup_{\epsilon>0} \epsilon^\alpha \left(\int_{T^n\setminus 2Q} \epsilon^{-n}\varphi\left(\frac{|u-y|}{\epsilon}\right) |f_j(y)|^r\, dy \right)^{1/r}_j \right\|_{\ell^r}\end{aligned}$$

$$\leq C \left\| \left(\int_{T^n\backslash 2Q} |u-y|^{\alpha r-n} \, |f_j|^r(y)\, dy \right)_j^{1/r} \right\|_{\ell^r}$$

$$= C \left(\int_{T^n\backslash 2Q} |u-y|^{\alpha r-n} \left(\sum_j |f_j|^r(y) \right) dy \right)^{1/r}$$

where we have used the size condition on φ.

Now since $s = \frac{1}{r}\,\frac{n}{\alpha} > 1$ we may use Hölder's inequality in the last expresion to bound it by

$$C \left(\int_{T^n\backslash 2Q} |u-y|^{(\alpha r-n)s'} \right)^{1/rs'} \left(\int \left(\sum_j |f_j|^r \right)^{\frac{1}{r}\frac{n}{\alpha}} \right)^{\alpha/n}$$

$$= C \left(\int_{T^n\backslash 2Q} |u-y|^{-n}\, dy \right)^{1/r-\alpha/n} \|(f_j)_j\|_{L^{n/\alpha}(\ell^r)}$$

$$\leq C(\log\, |Q|)^{1/r-\alpha/n}\, \|(f_j)_j\|_{L^{n/\alpha}(\ell^r)}.$$

Therefore (3.26) will be true as long as

$$(\log\, |Q|)^{1/r-\alpha/n} \frac{1}{|Q|} \int_Q |b(x) - b_Q|\, dx \leq C$$

but this is exactly our assumption on b.

Comment. It should be observed that for the "vectorial" commutator $\widetilde{T}_{\varphi,\alpha,b}$, used in the proof of the above theorem, we could get a complete characterization of the functions b; that means we could get "if and only if" results in the whole range of r's. The main obstacle to transfer the "only if" part to the maximal operator lies in the fact that for g a measurable F-valued function it is no longer true that

$$\| \, \|g(x)\|_F \|_{BMO(\mathbf{R}^n)} \simeq \|g\|_{BMO(\mathbf{R}^n,F)}$$

as in the L^p case. Indeed only the less than or equal part is valid and that is the clue in proving the BMO-boundedness of the commutators appearing in the last two theorems.

It might be of some interest to explicit the full class of functions b that can be obtained for the "vectorial commutator $\widetilde{T}_{\varphi,\alpha,b}$".

THEOREM. *Let $b \in BMO$ and $\widetilde{T}_{\varphi,\alpha,b}$ the commutator defined above. Then:*

(3.27) *For $1 < r < n/\alpha$, $\widetilde{T}_{\varphi,\alpha,b}$ maps $L^{n/\alpha}(\mathbf{R}^n, \ell^r(0,\infty))$ into $BMO(\mathbf{R}^n, \ell^\infty(0,\infty))$ if and only if b equals to a constant.*

(3.28) *For $1 < r < n/\alpha$, $\widetilde{T}_{\varphi,\alpha,b}$ maps $L^{n/\alpha}(T^n, \ell^r(L^\infty(0,\infty)))$ into $BMO(T^n, \ell^\infty(0, \infty))$ if and only if $b \in BMO_{|\log t|^{\alpha/n-1/r}}(T^n)$.*

(3.29) *For $n/\alpha \le r \le \infty$, $\widetilde{T}_{\varphi,\alpha,b}$ is bounded as above for any $b \in BMO$ in the periodic and non periodic case.*

Proof. We only need to prove the "only if" parts of (3.27) and (3.28) since the others are either trivial or they are already contained in the previous theorems.

Let us start by (3.28). For a given k we consider the cube $Q = Q(0, 2^{-k})$ and we define a sequence of functions by

$$
\begin{aligned}
f_j(x) &= 2^{j\alpha}\chi_{Q(0,2^{-j})\setminus Q(0,2^{-j-1})}(x) \qquad && \text{if } 0 \le j \le k-1\\
f_j(x) &\equiv 0 && \text{if } j \ge k.
\end{aligned}
$$

Easy calculations show that

$$\|(f_j)\|_{L^{n/\alpha}(T^n,\ell^r)} = k^{\alpha/n}.$$

Furthermore for $u \in Q$, $M_\alpha f_j(u) \ge C > 0$, $0 \le j \le k-1$, and hence

$$\left(\sum_j |M_\alpha f_j(u)|^r\right)^{1/r} \ge Ck^{1/r}.$$

Therefore noticing that $\log|Q| = -kC_n$, we have

$$
\begin{aligned}
\frac{1}{|Q|\,|\log|Q||^{\alpha/n-1/r}}\int_Q |b(x)-b_Q|\,dx &= C_n k^{-\alpha/n+1/r}\frac{1}{|Q|}\int_Q |b(x)-b_Q|\,dx\\
&\le Ck^{-\alpha/n}\|b\|_{BMO}\left(\sum |M_\alpha f_j(u)|^r\right)^{1/r}
\end{aligned}
$$

for any $u \in Q$. Now, by Theorem A, our assumption on T_b implies that the last expresion must be bounded by

$$C\|b\|_{BMO}\, k^{-\alpha/n}\|(f_j)\|_{L^{n/\alpha}(T^n,\ell^r)} \le C\ \|b\|_{BMO}.$$

Since this argument may be repeated for any cube $Q \subset T^n$, b must belong to $BMO_{|\log t|^{\alpha/n-1/r}}(T^n)$.

Finally for (3.27) we consider a cube $Q = Q(0,r)$ and a positive integer N. Let us denote by Q_r the cube $Q(0, 2^{kr})$ and choose the sequence f_k defined by

$$
\begin{aligned}
f_k(x) &= r^{-\alpha}2^{-k\alpha}\chi_{Q_r\setminus Q_{r-1}}(x) \quad && \text{if } 1 \le k \le N\\
f_k(x) &\equiv 0 && \text{if } k > N.
\end{aligned}
$$

The same computations as above show that

$$\|(f_k)\|_{L^{n/\alpha}(\mathbf{R}^n,\ell^r)} = N^{\alpha/n}.$$

while for any $u \in Q$,

$$\left\{\sum_k (M_\alpha f_k(u))^r\right\}^{1/r} \geq CN^{1/r}.$$

Therefore Theorem A applied to our situation implies that for any cube $\overline{Q} \subset Q$,

$$\left(\frac{1}{|\overline{Q}|}\int_{\overline{Q}} |b(x) - b_Q|\, dx\right) \leq CN^{\alpha/n-1/r}.$$

Letting N go to infinity and using $1 < r < n/\alpha$ we get

$$\left(\frac{1}{|\overline{Q}|}\int_{\overline{Q}} |b(x) - b_Q|\, dx\right) = 0.$$

Since this conclusion holds for any cube it follows that b must be constant almost everywhere.

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