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ON THE ZEROS OF POLYNOMIALS WITH RESTRICTED COEFFICIENTS

PETER BORWEIN AND TAMÁS ERDÉLYI

1. Introduction

The study of the location of zeros of polynomials from

$$\mathcal{F}_n := \left\{ p\colon\ p(x) = \sum_{i=0}^{n} a_i x^i\,, \quad a_i \in \{-1, 0, 1\} \right\}$$

begins with Bloch and Pólya [2]. They prove that the average number of real zeros of a polynomial from $\mathcal{F}_n$ is at most $c\sqrt{n}$. They also prove that a polynomial from $\mathcal{F}_n$ cannot have more than

$$\frac{cn \log\log n}{\log n}$$

real zeros. This result, which appears to be the first on this subject, shows that polynomials from $\mathcal{F}_n$ do not behave like unrestricted polynomials. Schur [11] and by different methods Szegő [12] and Erdős and Turán [7] improve the above bound to $c\sqrt{n \log n}$ (see also [5]).

In [6] we give the right upper bound of $c\sqrt{n}$ for the number of real zeros of polynomials from a large class, namely for all polynomials of the form

$$p(x) = \sum_{j=0}^{n} a_j x^j\,, \qquad |a_j| \le 1\,, |a_0| = |a_n| = 1\,, a_j \in \mathbb{C}\,.$$

In this paper we extend this result by proving that a polynomial of the form

$$p(x) = \sum_{j=0}^{n} a_j x^j\,, \qquad |a_j| \le 1\,, |a_0| = 1\,, a_j \in \mathbb{C}\,,$$

cannot have more than $c\sqrt{n}$ zeros inside any polygon with vertices on the unit circle, where c depends only on the polygon.

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We also prove another essentially sharp result stating that a polynomial of the form

$$p(x) = \sum_{j=0}^{n} a_j x^j \,, \qquad |a_0| = |a_n| = 1 \,, |a_j| \le 1 \,, a_j \in \mathbb{C} \,,$$

has at most $c(n\alpha + \sqrt{n})$ zeros in the strip $\{z \in \mathbb{C}\colon \ |\mathrm{Im}(z)| \le \alpha\}$, where c is an absolute constant.

Theorems 2.1–2.3, our main results, have self contained proofs distinct from those in [6]. They sharpen and generalize some results of Amoroso [1], Bombieri and Vaaler [4], and Hua [8], who gave upper bounds for the number of zeros of polynomials with integer coefficients at 1.

The class $\mathcal{F}_n$ and various related classes have been studied from a number of points of view. Littlewood's monograph [9] contains a number of interesting, challenging, and still open problems about polynomials with coefficients from $\{-1, 1\}$. The distribution of zeros of polynomials with coefficients from $\{0, 1\}$ is studied in [10] by Odlyzko and Poonen.

2. New results

Throughout the paper,

$$D(z_0, r) := \{z \in \mathbb{C}\colon \ |z - z_0| < r\}$$

denotes the open disk of the complex plane centered at $z_0 \in \mathbb{C}$ with radius $r > 0$.

THEOREM 2.1. *Every polynomial p of the form*

$$p(x) = \sum_{j=0}^{n} a_j x^j \,, \qquad |a_0| = 1 \,, |a_j| \le 1 \,, a_j \in \mathbb{C} \,,$$

has at most $c\sqrt{n}$ zeros inside any polygon with vertices on the unit circle, where the constant c depends only on the polygon.

THEOREM 2.2. *There is an absolute constant c such that*

$$p(x) = \sum_{j=0}^{n} a_j x^j \,, \qquad |a_0| = |a_n| = 1 \,, |a_j| \le 1 \,, a_j \in \mathbb{C} \,,$$

has at most $c(n\alpha + \sqrt{n})$ zeros in the strip

$$\{z \in \mathbb{C}\colon \ |\mathrm{Im}(z)| \le \alpha\} \,,$$

and in the sector

$$\{z \in \mathbb{C}\colon \ |\arg(z)| \le \alpha\} \,.$$

THEOREM 2.3. *Let $\alpha \in (0, 1)$. Every polynomial p of the form*

$$p(x) = \sum_{j=0}^{n} a_j x^j, \qquad |a_0| = 1, |a_j| \le 1, a_j \in \mathbb{C},$$

has at most c/α zeros inside any polygon with vertices on the circle

$$\{z \in \mathbb{C}: \ |z| = 1 - \alpha\},$$

where the constant c depends only on the number of the vertices of the polygon.

The sharpness of Theorem 2.1 can be seen by the theorem below proved in [6].

THEOREM 2.A. *For every $n \in \mathbb{N}$, there exists a polynomial p_n of the form given in Theorem 2.1 with real coefficients so that p_n has a zero at 1 with multiplicity at least $\lfloor\sqrt{n}\rfloor - 1$.*

When $0 < \alpha \le n^{-1/2}$, the sharpness of Theorem 2.2 is shown by the polynomials

$$q_n(z) := p_n(z) + z^{2n+1} p_n(z^{-1}),$$

where p_n are the polynomials in Theorem 2.A. Namely the polynomials q_n are of the required form with $\lfloor\sqrt{n}\rfloor - 1 \ge c(n\alpha + \sqrt{n})$ zeros at 1. When $n^{-1/2} \le \alpha \le 1$, the sharpness of Theorem 2.2 is shown by the polynomials $q_n(z) := z^n - 1$.

The next theorem proved in [3] shows the sharpness of Theorem 2.3.

THEOREM 2.B. *For every $\alpha \in (0, 1)$, there exists a polynomial p_n of the form given in Theorem 2.3 with real coefficients so that p_n has a zero at $1 - \alpha$ with multiplicity at least $\lfloor 1/\alpha \rfloor - 1$. (It can also be arranged that $n \le 1/\alpha^2 + 2$.)*

As a remark to Theorem 2.3 we point out that a more or less straightforward application of Jensen's formula gives that there is an absolute constant $c > 0$ such that every polynomial p of the form

$$p(x) = \sum_{j=0}^{n} a_j x^j, \qquad |a_0| = 1, |a_j| \le 1, a_j \in \mathbb{C},$$

has at most $(c/\alpha)\log(1/\alpha)$ zeros in the disk $D(0, 1 - \alpha)$, $\alpha \in (0, 1)$. (An interested reader may view this remark as an exercise. We will not use it, and hence will not present its proof.) A very recent (unpublished) example, suggested by F. Nazarov, shows that this upper bound for the number of zeros in the disk $D(0, 1 - \alpha)$ is, up to the absolute constant $c > 0$, best possible. So, in particular, the constant in Theorem 2.3 cannot be made independent of the number of vertices of the polygon.

3. Lemmas

To prove our theorems we need some lemmas. Our first lemma states Jensen's formula. Its proof may be found in most of the complex analysis textbooks.

LEMMA 3.1. *Suppose h is a nonnegative integer and*

$$f(z)=\sum_{k=h}^{\infty} c_k(z-z_0)^k\,, \qquad c_h \neq 0,$$

is analytic on the closure of the disk $D(z_0,r)$ and suppose that the zeros of f in $D(z_0,r)\setminus\{z_0\}$ are $a_1,a_2,\dots,a_m$, where each zero is listed as many times as its multiplicity. Then

$$\log|c_h| + h\log r + \sum_{k=1}^{m}\log\frac{r}{|a_k-z_0|} = \frac{1}{2\pi}\int_0^{2\pi}\log|f(z_0+re^{i\theta})|\,d\theta\,.$$

A straightforward calculation gives the next lemma.

LEMMA 3.2. *Suppose f is an analytic function on the open unit disk $D(0,1)$ that satisfies the growth condition*

$$|f(z)| \le \frac{1}{1-|z|}\,, \qquad z\in D(0,1)\,. \tag{1}$$

Let $D(z_0,r)\subset D(0,1)$. Then there is a constant $c(r)$ depending only on r such that

$$\frac{1}{2\pi}\int_0^{2\pi}\log|f(z_0+re^{i\theta})|\,d\theta \le c(r)\,.$$

The next lemma is used in the proof of both Theorems 2.1 and 2.2.

LEMMA 3.3. *There is an absolute constant c_1 such that every polynomial of the form*

$$p(x)=\sum_{j=0}^{n} a_j x^j\,, \qquad |a_0|=1\,, |a_j|\le 1\,, a_j\in\mathbb{C},$$

has at most $c_1(nr+\sqrt{n})$ zeros in any open disk $D(z_0,r)$ with $|z_0|=1$, $z_0\in\mathbb{C}$.

To prove Lemma 3.3 we need the lemma below. Lemma 3.4 also plays a central role in [6] where its reasonably simple proof may be found.

LEMMA 3.4. *There are absolute constants $c_2 > 0$ and $c_3 > 0$ such that*

$$|f(0)|^{c_2/a} \le \exp\left(\frac{c_3}{a}\right) \|f\|_{[1-a,1]}, \qquad a \in (0, 1],$$

for every f analytic on the open unit disk $D(0, 1)$ satisfying the growth condition (1) *in Lemma* 3.2.

For the sake of completeness we present the proof of Lemma 3.4 in Section 4.

Proof of Lemma 3.3. Without loss of generality we may assume that $z_0 := 1$ and $n^{-1/2} \le r \le 1$ (the case $0 < r < n^{-1/2}$ follows from the case $r = n^{-1/2}$, and the case $r > 1$ is obvious.) Let p be a polynomial of the form given in the lemma. Observe that such a polynomial satisfies the growth condition (1) in Lemma 3.4. Choose a point $z_1 \in [1 - r, 1]$ such that

$$|p(z_1)| \ge \exp\left(\frac{-c_3}{r}\right).$$

There is such a point by Lemma 3.4. Using the bounds for the coefficients of p, we have

$$\log|p(z)| \le \log((n+1)(1+4r)^n) \le \log(n+1) + 4nr, \qquad |z| \le 1 + 4r.$$

Let m denote the number of zeros of p in the open disk $D(z_1, 2r)$. Applying Jensen's formula on the disk $D(z_1, 4r)$, then using the above inequality, we obtain

$$-\frac{c_3}{r} + m\log 2 \le \log|p(z_1)| + m\log 2 \le \frac{1}{2\pi} 2\pi(\log(n+1) + 4nr).$$

This, together with $n^{-1/2} \le r \le 1$, implies $m \le c(nr + \sqrt{n})$. Now observe that $D(1, r)$ is a subset of $D(z_1, 2r)$, and the result follows. □

LEMMA 3.5. Suppose that p is a polynomial of the form

$$p(x) = \sum_{j=0}^{n} a_j x^j, \qquad |a_0| = 1, |a_j| \le 1, a_j \in \mathbb{C}.$$

Let $D(z_0, r) \subset D(0, 1)$. Let $r \in \left[\frac{3}{4}, 1\right)$. Let $0 < \delta < r$. Then there is a constant $c(r)$ depending only on r such that p has at most $c(r)\delta^{-1}$ zeros in the open disk $D(z_0, r - \delta)$.

Proof. Let p be a polynomial of the form given in the lemma. Observe that $r \in \left[\frac{3}{4}, 1\right)$ implies that $|z_0| < \frac{1}{4}$. Hence

$$|p(z_0)| \ge \frac{2}{3}.$$

Let m denote the number of zeros of p in the open disk $D(z_0, r-\delta)$. Applying Jensen's formula on $D(z_0, r)$ and Lemma 3.2, we obtain

$$\begin{aligned}\log\frac{2}{3} + m\log\frac{r}{r-\delta} &\le \log|p(z_0)| + m\log\frac{r}{r-\delta}\\ &\le \frac{1}{2\pi}\int_0^{2\pi}\log|p(z_0+re^{i\theta})|\,d\theta \le c(r)\,.\end{aligned}$$

As $0<r<1$ and $0<\delta<r$, we have

$$\log\frac{r}{r-\delta} = \log\frac{1}{1-(\delta/r)} \ge \frac{\delta}{r} \ge \delta\,,$$

and with the previous inequality this implies that $m \le (c(r)+1)\delta^{-1}$. □

4. Proof of Lemma 3.4

In this section, for the sake of completeness, we present the proof of Lemma 3.4 given in [6]. We need some lemmas.

HADAMARD THREE CIRCLES THEOREM. *Suppose f is regular in*

$$\{z \in \mathbb{C}\colon r_1 \le |z| \le r_2\}\,.$$

For $r \in [r_1, r_2]$, let

$$M(r) := \max_{|z|=r}|f(z)|\,.$$

Then

$$M(r)^{\log(r_2/r_1)} \le M(r_1)^{\log(r_2/r)}M(r_2)^{\log(r/r_1)}\,.$$

COROLLARY 4.1. *Let $a \in (0,1]$. Suppose f is regular inside and on the ellipse E_a with foci at $1-a$ and $1-a+\frac{1}{4}a$ and with major axis*

$$\left[1-a-\frac{13a}{32},\, 1-a+\frac{21a}{32}\right].$$

Let $\widetilde{E}_a$ be the ellipse with foci at $1-a$ and $1-a+\frac{1}{4}a$ and with major axis

$$\left[1-a-\frac{6a}{32},\, 1-a+\frac{14a}{32}\right].$$

Then

$$\max_{z\in\widetilde{E}_a}|f(z)| \le \left(\max_{z\in[1-a,1-a+\frac{1}{4}a]}|f(z)|\right)^{1/2}\left(\max_{z\in E_a}|f(z)|\right)^{1/2}.$$

Proof. This follows from the Hadamard Three Circles Theorem with the substitution

$$w = \frac{a}{8}\left(\frac{z+z^{-1}}{2}\right) + \left(1 - a + \frac{a}{8}\right).$$

The Hadamard Three Circles Theorem is applied with $r_1 := 1$, $r := 2$, and $r_2 := 4$. □

COROLLARY 4.2. *Let $\widetilde{E}_a$ be as in Corollary* 4.1. *Then*

$$\max_{z \in \widetilde{E}_a} |f(z)| \le \left(\frac{32}{11a}\right)^{1/2} \left(\max_{z \in [1-a,1]} |f(z)|\right)^{1/2}$$

for every f analytic in the open unit disk $D(0, 1)$ that satisfies the growth condition (1) (*see Lemma* 3.2) *and for every $a \in (0, 1]$.*

Proof. This follows from Corollary 4.1 and the Maximum Principle. □

Proof of Lemma 3.4. Let $h(z) = \frac{1}{2}(1-a)(z+z^2)$. Observe that $h(0) = 0$, and there are absolute constants $c_4 > 0$ and $c_5 > 0$ such that

$$|h(e^{it})| \le 1 - c_4 t^2, \qquad -\pi \le t \le \pi,$$

and for $t \in [-c_5 a, c_5 a]$, $h(e^{it})$ lies inside the ellipse $\widetilde{E}_a$. Now let $m := \lfloor 2\pi c_5/a \rfloor + 1$. Let $\xi := \exp(2\pi i/(2m))$ be the first $2m$th root of unity, and let

$$g(z) := \prod_{j=0}^{2m-1} f(h(\xi^j z)).$$

Using the Maximum Principle and the properties of h, we obtain

$$\begin{aligned} |f(0)|^{2m} &= |g(0)| \le \max_{|z|=1} |g(z)| \le \left(\max_{z \in \widetilde{E}_a} |f(z)|\right)^2 \prod_{k=1}^{m-1} \left(\frac{1}{c_4(k/m)^2}\right)^2 \\ &= \left(\max_{z \in \widetilde{E}_a} |f(z)|\right)^2 e^{c_6(m-1)} \left(\frac{m^{m-1}}{(m-1)!}\right)^4 < \left(\max_{z \in \widetilde{E}_a} |f(z)|\right)^2 e^{c_7(m-1)}, \end{aligned}$$

and the lemma follows by Corollary 4.2. □

5. Proof of the Theorems

Proof of Theorem 2.1. It is sufficient to prove the upper bound of the theorem for the number of zeros in a triangle with vertices 0, w, and w^{-1}, where $|w| = 1$

and $\mathrm{Re}(w) \geq \frac{1}{2}$. Let $S_1 := D(z_0, r - \delta)$, where $z_0 := \frac{1}{4}$, $r := \frac{3}{4}$, and $\delta := n^{-1/2}$. Let $S_2 := D(z_0, r)$, where $z_0 := w$ and $r := cn^{-1/2}$. Let $S_3 := D(z_0, r)$, where $z_0 := w^{-1}$ and $r := cn^{-1/2}$. Note that if $c = c(w)$ is sufficiently large, then the triangle with vertices 0, w, and w^{-1} is covered by the union of S_1, S_2, and S_3. Hence the theorem follows from Lemmas 3.3 and 3.5. □

Proof of Theorem 2.2. Without loss of generality we may assume that $n^{-1/2} \leq \alpha \leq \frac{1}{2}$. it is sufficient to prove that every polynomial p of the form

$$p(x) = \sum_{j=0}^{n} a_j x^j , \qquad |a_0| = 1 , |a_j| \leq 1 , a_j \in \mathbb{C} ,$$

has at most $c(n\alpha + \sqrt{n})$ zeros in

$$\{z \in \mathbb{C}: \ |\mathrm{Im}(z)| \leq \alpha , \ \ |z| \leq 1\} .$$

The remaining part of the theorem follows from this by studying the polynomials $q_n(z) := z^n p_n(z^{-1})$.

Let $S_1 := D(z_0, r - \delta)$, where $z_0 := \frac{1}{4}$, $r := \frac{3}{4}$, and $\delta := \alpha$. Let $S_2 := D(z_0, r - \delta)$, where $z_0 := -\frac{1}{4}$, $r := \frac{3}{4}$, and $\delta := \alpha$. Let $S_3 := D(z_0, r)$, where $z_0 := 1$ and $r := 4\alpha$. Let $S_4 := D(z_0, r)$, where $z_0 := -1$ and $r := 4\alpha$. Note that S_1, S_2, S_3, and S_4 cover

$$\{z \in \mathbb{C}: \ |\mathrm{Im}(z)| \leq \alpha , \ \ |z| \leq 1\} .$$

Hence the theorem follows from Lemmas 3.4 and 3.5 (note that $n^{-1/2} \leq \alpha \leq \frac{1}{2}$ implies $\alpha^{-1} \leq n\alpha$). □

Proof of Theorem 2.3. Without loss of generality we may assume that $\alpha \in (0, \frac{1}{2}]$, otherwise the statement of the theorem is trivial. It is sufficient to prove the upper bound of the theorem for the number of zeros in a triangle with vertices 0, w, and $\overline{w}$, where $|w| = 1 - \alpha$ and $\mathrm{Re}(w) \geq \frac{1}{2}$. Let $w =: |w|e^{i\theta}$. Let $S_1 := D(z_0, r - \delta)$, where $z_0 := \frac{1}{4}e^{i\theta}$, $r := \frac{3}{4}$, $\delta := \alpha$. Let $S_2 := D(z_0, r - \delta)$, where $z_0 := \frac{1}{4}e^{-i\theta}$, $r := \frac{3}{4}$, $\delta := \alpha$. Note that the triangle with vertices 0, w, and $\overline{w}$ is covered by the union of S_1 and S_2. Hence the theorem follows from Lemma 3.5. □

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Peter Borwein, Department of Mathematics and Statistics, Simon Fraser University, Burnaby, B.C., Canada V5A 1S6
pborwein@cecm.sfu.edu

Tamás Erdélyi, Department of Mathematics, Texas A&M University, College Station, Texas 77843