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ABEL SUMMABILITY OF THE AUTOREGRESSIVE SERIES FOR THE BEST LINEAR LEAST SQUARES PREDICTORS

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1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Denote by $L^2(\Omega, \mathcal{F}, P)$ the Hilbert space of complex-valued random variables X, with expectation $E(X) = \int_\Omega X\,dP = 0$ and variance $E(|X|^2) < \infty$, having inner product $\langle X, Y\rangle = E(X\overline{Y})$. A sequence of random variables $\{X_k\}_{k=-\infty}^{\infty}$ in $L^2(\Omega, \mathcal{F}, P)$ is a weakly stationary stochastic process (WSSP) if for all $k,\ l \in Z$, the second moment $E(X_k\overline{X}_{k+l})$ depends only on l. The covariance function $K(l) = E(X_k\overline{X}_{k+l})$ thus defined is nonnegative definite and so

$$K(l) = \int_T e^{-il\theta}\,dF(e^{i\theta}), \qquad l \in Z \tag{1.1}$$

for an essentially unique function F, which is bounded and nondecreasing in θ on $T = [-\pi, \pi)$.

Given $n \geq 1$, the best linear least squares predictor of X_n, based on past and present observations, is defined to be the orthogonal projection of X_n on $M = \overline{sp}\{X_k, k \leq 0\}$, the closed linear span of $\ldots, X_{-2}, X_{-1}, X_0$. The projection is denoted by $\widehat{X}_n$. We assume the WSSP $\{X_k\}_{k=-\infty}^{\infty}$ is purely nondeterministic, in the sense that $\cap_{m=0}^{\infty}\overline{sp}\{X_k, k \leq -m\} = \{0\}$. This guarantees that $X_n \notin M$ for all $n \geq 1$ and that the function F in (1.1) is absolutely continuous with respect to Lebesgue measure on $T, dF(e^{i\theta}) = w(e^{i\theta})d\theta$. Moreover, the function $w(e^{i\theta})$, the spectral density of the WSSP, can be expressed in the form $w(e^{i\theta}) = |\phi(e^{i\theta})|^2$, where the so-called optimal factor $\phi = \phi(z)$ is an outer function in the Hardy space $H^2(D)$ on the unit disk $D = \{z \in \mathbb{C}\colon |z| < 1\}$ and has no zero in D.[1] See [5, pp. 53, 69].

The Spectral Theorem for unitary operators yields a Hilbert space isomorphism between the time domain $L^2(\Omega, \mathcal{F}, P)$ and the spectral domain

$$L^2(w) = \left\{ f\colon \|f\|_2 := \left[\int_T |f(e^{i\theta})|^2 w(e^{i\theta})\,d\theta\right]^{1/2} < \infty \right\},$$

in which $X_k \longleftrightarrow e^{-ik\theta}$, $k = 0, \pm1, \pm2, \ldots$; see [2, p. 241]. Denote by ϕ_n the image of $\widehat{X}_n, n \geq 1$, under the isomorphism, so that ϕ_n is the projection of $e^{-in\theta}$

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[1] Since $\phi(0) \neq 0$, we may assume, without loss of generality, that $\phi(0) = 1$.

on $\overline{sp}\{e^{ik\theta}, k \geq 0\}$. In this context, it is of interest to express ϕ_n in a series of the form $\sum_{k=0}^{\infty} a_k e^{ik\theta}$ which, if valid, leads to the (autoregressive) series representation $\sum_{k=0}^{\infty} a_k X_{-k}$ of $\widehat{X}_n$. (We emphasize that $\{e^{ik\theta}\}_{k=0}^{\infty}$ is not an orthogonal sequence in $L^2(w)$, unless $w \equiv 1$.)

Akutowicz [1] was the first to consider the above question, proposing for the coefficient $a_k = a_k(n)$ the formula

$$a_k = \sum_{j=0}^{k} c_{n+j} d_{k-j}, \quad k = 0, 1, 2, \ldots, \tag{1.2}$$

where the c_k and d_k are, respectively, the Taylor coefficients of the optimal factor ϕ and its reciprocal $1/\phi$. Wiener and Masani [15] proved the autoregressive series converges to $\widehat{X_n}$ in $L^2(\Omega, \mathcal{F}, P)$ for all $n \geq 1$ if w, $1/w \in L^\infty(T)$. Masani [7] later showed $w \in L^\infty(T)$ and $1/w \in L^1(T)$ are enough. More recently, Pourahmadi [10] obtained the sufficiency of the A_2 condition, where w is said to satisfy the A_p condition, $1 < p < \infty$, or to be in the class A_p, if

$$\left[\frac{1}{|I|}\int_I w(e^{i\theta})\, d\theta\right]\left[\frac{1}{|I|}\int_I w(e^{i\theta})^{-\frac{1}{p-1}}\, d\theta\right]^{p-1} \leq C;$$

here the constant $C > 0$ is independent of the interval $I \subset T$ with Lebesgue measure $|I|$. Finally, Pourahmadi used a result of Rosenblum [11] to prove the autoregressive series Abel-summable to $\widehat{X}_n$ in $L^2(\Omega, \mathcal{F}, P)$ provided w satisfies a condition of Helson-Sarason-Szegö [3], [4], which is equivalent to $w(e^{i\theta})/|p(e^{i\theta})|^2$ in $A_2$for some analytic polynomial $p(z)$ with all its roots on T. See [9].

We here characterize, in terms of their optimal factor ϕ, those WSSP's whose autoregressive series are mean-summable in the sense of Abel. The criterion, given in Theorem 2.2, requires, in a certain sense, the (pointwise) invertibility of ϕ in $L^2(T)$. A size condition for this invertibility is proved in Theorem 2.4. The latter condition yields all past results and some significant improvements. Thus, it is shown in Theorem 3.1 that the autoregressive series are mean Abel-summable for all w in $A_\infty = \cup_{p>1} A_p$, which includes densities satisfying the Helson-Sarason-Szegö condition.

2. The basic theorems

Let

$$\chi_n(z) = \frac{\phi(z) - \sum_{j=0}^{n-1} c_j z^j}{z^n \phi(z)}, \quad n = 1, 2, \ldots, \tag{2.1}$$

where $\phi(z) = \sum_{k=0}^{\infty} c_k z^k$ is the optimal factor of the spectral density w. One readily sees that the kth Taylor coefficient of this analytic function on D is $a_k = a_k(n)$ in (1.2). We begin with the following representation of the spectral isomorph of $\widehat{X_n}$ in terms of the boundary values of χ_n.

LEMMA 2.1. *The spectral isomorph, $\phi_n(e^{i\theta})$, of the best linear least squares predictor $\widehat{X_n}$ is given by*

$$\phi_n(e^{i\theta}) = \chi_n(e^{i\theta}), n = 1, 2, \ldots.$$

Proof. Since $e^{-in\theta}$ is the spectral isomorph of the random variable X_n, we have to show $\psi_n(e^{i\theta}) := e^{-in\theta} - \chi_n(e^{i\theta})$ is orthogonal, in $L^2(w)$, to $\overline{s_p}\{e^{ik\theta}, k \geq 0\}$ or, equivalently, to every function $e^{ik\theta}$, $k \geq 0$. But

$$\begin{aligned}
\int_T \psi_n(e^{i\theta})e^{-ik\theta}w(e^{i\theta})\,d\theta &= \int_T \left[\sum_{j=0}^{n-1} c_j e^{ij\theta}\right] \frac{e^{-i(n+k)\theta}}{\phi(e^{i\theta})} |\phi(e^{i\theta})|^2\,d\theta \\
&= \int_T \left[\sum_{j=0}^{n-1} c_j e^{ij\theta}\right] e^{-i(n+k)\theta}\overline{\phi(e^{i\theta})}\,d\theta \\
&= \int_T \left[\sum_{j=0}^{n-1} c_j e^{ij\theta}\right] \sum_{l=0}^{\infty} c_l e^{-i(n+k+l)\theta}\,d\theta \\
&= 0,
\end{aligned}$$

for every $k \geq 0$ and $n \geq 1$. □

Our main result concerning the mean Abel-summability of the autoregressive series for $\widehat{X}_n$ is the following:

THEOREM 2.2. *Suppose $\{X_k\}_{k=-\infty}^{\infty}$ is a purely nondeterministic WSSP, whose spectral density w has optimal factor $\phi(z) = \sum_{j=0}^{\infty} c_j z^j$. Given fixed $n \geq 1$ and r, $0 < r < 1$, define*

$$\widehat{X}_n^{(r)} := \sum_{k=0}^{\infty} r^k a_k X_{-k}, \quad a_k = a_k(n) \text{ as in } (1.2)$$

to be the rth Abel mean of the autoregressive series for the best linear least squares predictor, $\widehat{X}_n$, of X_n. Set

$$\sum_{j=0}^{n-1} c_j z^j = p_n(z) q_n(z),$$

where $p_n(z) = \prod_{l=1}^{l_0}(1 - e^{-i\theta_l}z)^{n_l}$ (with $p_n(z) = 1$, if $l_0 = 0$) and $|q_n(z)| > 0$ on T. Then, in order that

$$\lim_{r\to 1_-} \int_\Omega |\widehat{X}_n^{(r)} - \widehat{X}_n|^2\,dP = 0, \tag{2.2}$$

it is necessary and sufficient that

$$\lim_{r\to 1_-} \int_T \left|\frac{\phi(e^{i\theta})}{\phi(re^{i\theta})} - 1\right|^2 |p_n(e^{i\theta})|^2\,d\theta = 0,$$

or, equivalently, that

$$\lim_{r\to 1_-}\int_T \left|\frac{\phi(e^{i\theta})}{\phi(re^{i\theta})}\right|^2 \left|p_n(e^{i\theta})\right|^2 d\theta = \int_T \left|p_n(e^{i\theta})\right|^2 d\theta. \tag{2.3}$$

Proof. It follows from the isomorphism between the time and spectral domains of the WSSP and Lemma 2.1 that (2.2) holds if and only if

$$\lim_{r\to 1_-}\int_T |\chi_n(re^{i\theta}) - \chi_n(e^{i\theta})|^2 w(e^{i\theta})\, d\theta = 0, \tag{2.4}$$

with $\chi_n(z)$ given by (2.1). From (2.1) again,

$$\chi_n(re^{i\theta}) = \frac{e^{-in\theta}}{r^n} - \frac{e^{-in\theta}}{r^n\phi(re^{i\theta})}\sum_{j=0}^{n-1} r^j c_j e^{ij\theta}.$$

Clearly, $\lim_{r\to 1_-}(e^{-in\theta}/r^n) = e^{-in\theta}$ in $L^2(w)$, which means (2.4) holds if only if

$$\lim_{r\to 1_-}\int_T \left|\chi_n(re^{i\theta}) - \frac{e^{-in\theta}}{r^n} - \left(\chi_n(e^{i\theta}) - e^{-in\theta}\right)\right|^2 w(e^{i\theta})\, d\theta = 0.$$

Again, $\phi \in H^2(D)$ implies $1/\phi$ belongs to the Nevanlinna class, so [13, p. 346]

$$\lim_{r\to 1_-}\frac{1}{\phi(re^{i\theta})} = \frac{1}{\phi(e^{i\theta})}, \quad \text{a.s.}$$

and hence

$$\lim_{r\to 1_-}\left[\chi_n(re^{i\theta}) - \frac{e^{-in\theta}}{r^n}\right] = \chi_n(e^{i\theta}) - e^{-in\theta} \quad \text{a.s.}$$

Therefore, by [12, p. 126, problem 16], (2.4) reduces to

$$\lim_{r\to 1_-}\int_T \left|\chi_n(re^{i\theta}) - \frac{e^{-in\theta}}{r^n}\right|^2 w(e^{i\theta})\, d\theta = \int_T \left|\chi_n(e^{i\theta}) - e^{-in\theta}\right|^2 w(e^{i\theta})\, d\theta,$$

or

$$\lim_{r\to 1_-}\int_T \left|\frac{\phi(e^{i\theta})}{\phi(re^{i\theta})}\right|^2 \left|p_n(re^{i\theta})q_n(re^{i\theta})\right|^2 d\theta = \int_T \left|p_n(e^{i\theta})q_n(e^{i\theta})\right|^2 d\theta = L. \tag{2.5}$$

We first prove the necessity of (2.3) for (2.2). Observe that

$$\left|p_n(re^{i\theta})\right|^2 = \prod_{l=1}^{l_0}\left[(1-r)^2 + 4r\sin^2\left(\frac{\theta-\theta_l}{2}\right)\right]^{n_l}, \tag{2.6}$$

so

$$r^{n_1+\cdots+n_{l_0}} \left|\frac{\phi(e^{i\theta})}{\phi(re^{i\theta})}\right|^2 \left|p_n(e^{i\theta})q_n(re^{i\theta})\right|^2 \leq \left|\frac{\phi(e^{i\theta})}{\phi(re^{i\theta})}\right|^2 \left|p_n(re^{i\theta})q_n(re^{i\theta})\right|^2.$$

This means (2.5), and hence (2.2), implies

$$\lim_{r\to 1_-} \int_T \left|\frac{\phi(e^{i\theta})}{\phi(re^{i\theta})}\right|^2 \left|p_n(e^{i\theta})q_n(re^{i\theta})\right|^2 d\theta = L, \tag{2.7}$$

since, by Fatou's Lemma,

$$\begin{aligned} L &\leq \varliminf_{r\to 1_-} \int_T \left|\frac{\phi(e^{i\theta})}{\phi(re^{i\theta})}\right|^2 \left|p_n(e^{i\theta})q_n(re^{i\theta})\right|^2 d\theta \\ &\leq \varlimsup_{r\to 1_-} \int_T \left|\frac{\phi(e^{i\theta})}{\phi(re^{i\theta})}\right|^2 \left|p_n(e^{i\theta})q_n(re^{i\theta})\right|^2 d\theta \\ &\leq \varlimsup_{r\to 1_-} r^{-(n_1+\cdots+n_{l_0})} \int_T \left|\frac{\phi(e^{i\theta})}{\phi(re^{i\theta})}\right|^2 \left|p_n(re^{i\theta})q_n(re^{i\theta})\right|^2 d\theta = L. \end{aligned}$$

Next, (2.7) is equivalent to

$$\lim_{r\to 1_-} \int_T \left|\frac{\phi(e^{i\theta})}{\phi(re^{i\theta})}\right|^2 \left|p_n(e^{i\theta})q_n(e^{i\theta})\right|^2 d\theta = L. \tag{2.8}$$

Indeed,

$$\varlimsup_{r\to 1_-} \int_T \left|\frac{\phi(e^{i\theta})}{\phi(re^{i\theta})}\right|^2 \left|p_n(re^{i\theta})\right|^2 \left|q_n(re^{i\theta}) - q_n(e^{i\theta})\right|^2 d\theta = 0 \tag{2.9}$$

is implied by either (2.7) or (2.8), as the left side of (2.9) is dominated by both

$$\varlimsup_{r\to 1_-} \left\|1 - \frac{q_n(e^{i\theta})}{q_n(re^{i\theta})}\right\|_\infty^2 \int_T \left|\frac{\phi(e^{i\theta})}{\phi(re^{i\theta})}\right|^2 \left|p_n(e^{i\theta})q_n(re^{i\theta})\right|^2 d\theta = 0$$

and

$$\varlimsup_{r\to 1_-} \left\|1 - \frac{q_n(re^{i\theta})}{q_n(e^{i\theta})}\right\|_\infty^2 \int_T \left|\frac{\phi(e^{i\theta})}{\phi(re^{i\theta})}\right|^2 \left|p_n(e^{i\theta})q_n(e^{i\theta})\right|^2 d\theta = 0.$$

Finally, we claim (2.8) is, in turn, equivalent to (2.3). Thus, [12, p. 89, problem 9] ensures that, given (2.8),

$$\lim_{r\to 1_-} \int_E \left|\frac{\phi(e^{i\theta})}{\phi(re^{i\theta})}\right|^2 \left|p_n(e^{i\theta})q_n(e^{i\theta})\right|^2 d\theta = \int_E \left|p_n(e^{i\theta})q_n(e^{i\theta})\right|^2 d\theta$$

for all measurable $E \subset T$ and so

$$\lim_{r\to 1_-} \int_T \left|\frac{\phi(e^{i\theta})}{\phi(re^{i\theta})}\right|^2 \left|p_n(e^{i\theta})q_n(e^{i\theta})S(e^{i\theta})\right|^2 d\theta = \int_E \left|p_n(e^{i\theta})q_n(e^{i\theta})S(e^{i\theta})\right|^2 d\theta$$

for every simple function S on T. But, $1/q_n(e^{i\theta})$ is the uniform limit on T of such functions, whence (2.3) follows. The same argument yields (2.8) given (2.3).

To obtain the sufficiency of (2.3) for (2.2), we show (2.3) implies (2.5) and so (2.2). To begin, we prove that for each $N \in Z_+$,

$$\lim_{r\to 1_-} (1-r)^{2n_l} \int_{|\theta-\theta_l|\le N(1-r)} \left|\frac{\phi(e^{i\theta})}{\phi(re^{i\theta})}\right|^2 d\theta = 0, \tag{2.10}$$

$l = 1, \ldots, l_0$. Fixing l and $\varepsilon > 0$, we notice, in view of (2.6), that

$$\begin{aligned}(1-r)^{2n_l} \int_{|\theta-\theta_l|\le N(1-r)} \left|\frac{\phi(e^{i\theta})}{\phi(re^{i\theta})}\right|^2 d\theta &\le C \int_{|\theta-\theta_l|\le N(1-r)} \left|\frac{\phi(e^{i\theta})}{\phi(re^{i\theta})}\right|^2 \left|p_n(e^{i\theta})\right|^2 d\theta \\ &\le C \int_{|\theta-\theta_l|\le \varepsilon} \left|\frac{\phi(e^{i\theta})}{\phi(re^{i\theta})}\right|^2 \left|p_n(e^{i\theta})\right|^2 d\theta,\end{aligned}$$

provided r is sufficiently close to 1. By [12, p. 89, problem 9], (2.3) guarantees

$$\overline{\lim_{r\to 1_-}} (1-r)^{2n_l} \int_{|\theta-\theta_l|\le N(1-r)} \left|\frac{\phi(e^{i\theta})}{\phi(re^{i\theta})}\right|^2 d\theta \le C \int_{|\theta-\theta_l|\le \varepsilon} \left|p_n(e^{i\theta})\right|^2 d\theta. \tag{2.11}$$

As the right side of (2.11) goes to 0 with ε, (2.10) follows.

Now, by Fatou's Lemma,

$$\underline{\lim_{r\to 1_-}} \int_T \left|\frac{\phi(e^{i\theta})}{\phi(re^{i\theta})}\right|^2 \left|p_n(re^{i\theta})q_n(re^{i\theta})\right|^2 d\theta \ge L.$$

Therefore, it only remains to prove that (2.3) forces

$$\overline{\lim_{r\to 1_-}} \int_T \left|\frac{\phi(e^{i\theta})}{\phi(re^{i\theta})}\right|^2 \left|p_n(re^{i\theta})q_n(re^{i\theta})\right|^2 d\theta \le L. \tag{2.12}$$

To this end, fix $N \in Z_+$ and let $E(\theta_l) := \{\theta \in T\colon |\theta-\theta_l| \le N(1-r)\}, l = 1, \ldots, l_o$. The left side of (2.5) can be written as

$$\begin{aligned}&\left(\int_{E(\theta_1)} + \cdots + \int_{E(\theta_{l_0})} + \int_{T-\cup E(\theta_l)}\right) \left|\frac{\phi(e^{i\theta})}{\phi(re^{i\theta})}\right|^2 \left|p_n(re^{i\theta})q_n(re^{i\theta})\right|^2 d\theta \\ &= I_1(r) + \cdots + I_{l_0}(r) + J(r).\end{aligned}$$

But,

$$\lim_{r\to 1_-} I_l(r) = 0, \quad l = 1, \dots, l_0,$$

by (2.6) and (2.10). Again, we see, from (2.6), there exists $C > 0$, independent of r and N, such that

$$J(r) \le \left(r + \frac{C}{N^2}\right)^{n_1+\cdots+n_{l_0}} \int_T \left|\frac{\phi(e^{i\theta})}{\phi(re^{i\theta})}\right|^2 \left|p_n(e^{i\theta})q_n(re^{i\theta})\right|^2 \, d\theta.$$

Since (2.7) holds whenever (2.3) does,

$$\overline{\lim_{r\to 1_-}} \int_T \left|\frac{\phi(e^{i\theta})}{\phi(re^{i\theta})}\right|^2 \left|p_n(re^{i\theta})q_n(re^{i\theta})\right|^2 \, d\theta \le \left(1 + \frac{C}{N^2}\right)^{n_1+\cdots+n_{l_0}} L$$

Since $N \in Z_+$ was arbitrary, we have proved (2.12). □

An argument similar to the one used above to show (2.8) implies (2.3) yields:

COROLLARY 2.3. *Let $\widehat{X}_n$, $\widehat{X}_n^{(r)}$ and ϕ be as in Theorem 2.2. Then*

$$\lim_{r\to 1_-} \int_\Omega \left|\widehat{X}_n^{(r)} - \widehat{X}_n\right|^2 dP = 0, n = 1, 2, \dots,$$

provided

(2.13) $$\lim_{r\to 1_-} \int_T \left|\frac{\phi(e^{i\theta})}{\phi(re^{i\theta})}\right|^2 d\theta = \int_T d\theta = 2\pi.$$

A size condition sufficient for (2.2) can be given in terms of the geometric maximal operator, which is defined at a function f, positive a.e. on T, by

$$(Gf)(e^{i\theta}) := \sup_{e^{i\theta}\in I\subset T} \exp\left[\frac{1}{|I|}\int_I \log\left|f(e^{it})\right| dt\right], \quad e^{i\theta} \in T.$$

THEOREM 2.4. *Let $\widehat{X}_n$, $\widehat{X}_n^{(r)}$, w and p_n be as in Theorem 2.2. Then,*

$$\lim_{r\to 1_-} \int_\Omega \left|\widehat{X}_n^{(r)} - \widehat{X}_n\right|^2 dP = 0, n = 1, 2, \dots,$$

whenever

(2.14) $$\int_T w(e^{i\theta})(Gw^{-1})(e^{i\theta}) \left|p_n(e^{i\theta})\right|^2 d\theta < \infty.$$

Proof. To apply Theorem 2.2, we must show that, given (2.14),

$$\int_T |p_n(e^{i\theta})|^2\,d\theta = \lim_{r\to 1_-}\int_T \left|\frac{\phi(e^{i\theta})}{\phi(re^{i\theta})}\right|^2 |p_n(e^{i\theta})|^2\,d\theta$$
$$= \lim_{r\to 1_-}\int_T w(e^{i\theta})\,|\phi(re^{i\theta})|^{-2}\,|p_n(e^{i\theta})|^2\,d\theta.$$

We have

$$\lim_{r\to 1_-}|\phi(re^{i\theta})|^{-2} = |\phi(e^{i\theta})|^{-2} = w(e^{i\theta})^{-1},$$

since $\phi \in H^2(D)$. Thus, it suffices to get

$$|\phi(re^{i\theta})|^{-2} \le (Gw^{-1/2})(e^{i\theta})^2 = (Gw^{-1})(e^{i\theta}), \tag{2.15}$$

for then (2.3), and so (2.2), would follow from (2.14) by the dominated convergence theorem.

From [5, pp. 62–63], the outer function ϕ^{-1} is given by

$$\phi(z)^{-1} = \lambda\exp\left[\frac{1}{2\pi}\int_T \frac{e^{it}+z}{e^{it}-z}\log|\phi(e^{it})^{-1}|\,dt\right],\qquad z = re^{i\theta},$$

with the constant $\lambda \in \mathbb{C}$, $|\lambda| = 1$. A short calculation yields

$$\phi(z)^{-1} = \lambda\alpha\exp\left[\frac{1}{2\pi}\int_T P_r(e^{i(\theta-t)})\log|\phi(e^{it})^{-1}|\,dt\right],$$

where

$$P_r(e^{it}) = \frac{1-r^2}{1-2r\cos t + r^2},$$

and $|\alpha| = 1$, whence

$$|\phi(z)^{-1}| = \exp\left[\frac{1}{2\pi}\int_T P_r(e^{i(\theta-t)})\log|\phi(e^{it})^{-1}|\,dt\right].$$

We need only prove (2.15) for $\theta = 0$, that is, $z = r$. Setting

$$f(e^{it}) = \log|\phi(e^{it})^{-1}|,$$

we have

$$\log|\phi(r)^{-1}| = \frac{1}{2\pi}\int_T P_r(e^{it})f(e^{it})\,dt$$
$$= \frac{1}{2\pi}\int_0^{\pi}\left[P_r(e^{it})f(e^{it}) + P_r(e^{-it})f(e^{-it})\right]dt$$

$$= \frac{1}{2\pi}\int_{|s|\le t} f(e^{is})\,ds\; P_r(e^{it})\Big|_0^{\pi} + \frac{1}{2\pi}\int_0^{\pi}\int_{|s|\le t} f(e^{is})\,ds\,d(-P_r(e^{it}))$$

$$= \frac{1}{2\pi}\int_0^{\pi}\int_{|s|\le t} f(e^{is})\,ds\,d(-P_r(e^{it})),$$

because

$$\int_{|s|\le\pi} f(e^{is})\,ds = \int_T \log\left|\phi(e^{it})^{-1}\right|\,dt = \log\left|\phi(0)^{-1}\right| = -\log|\phi(0)| = 0.$$

So

$$\left|\phi(r)^{-1}\right| = \exp\left[\int_0^{\pi}\left(\frac{1}{2t}\int_{|s|\le t} f(e^{is})\,ds\right)d\nu(t)\right],$$

where

$$d\nu(t) = \frac{t}{\pi}d(-P_r(e^{it}))$$

is a positive measure on $[0,\pi]$ for which

$$\int_0^{\pi} d\nu(t) = \frac{1}{2\pi}\int_T P_r(e^{it})\,dt = 1.$$

Jensen's inequality with the convex function e^x gives

$$\begin{aligned}\left|\phi(r)^{-1}\right| &\le \int_0^{\pi}\exp\left(\frac{1}{2t}\int_{|s|\le t} f(e^{is})\,ds\right)d\nu(t)\\ &\le \int_0^{\pi}\exp\left(\frac{1}{2t}\int_{-t}^{t}\log\left|\phi(e^{is})^{-1}\right|\,ds\right)d\nu(t)\\ &\le (Gw^{-1/2})(0)\int_0^{\pi}d\nu(t) = (Gw^{-1/2})(0)\end{aligned}$$

or

$$|\phi(r)|^{-2} \le (Gw^{-1})(0). \qquad \square$$

3. A_∞ weights

In this section we consider weights $w(e^{i\theta})$ satisfying the A_∞ condition

$$\int_E w(e^{i\theta})\,d\theta \le C\left[\frac{|E|}{|I|}\right]^{\varepsilon}\int_I w(e^{i\theta})\,d\theta,$$

in which $\varepsilon > 0$ is fixed and $C > 0$ is independent of the interval $I \subset T$ and its measurable subsets E. It was shown in [8] that the class, A_∞, of all such weights satisfies

$$A_\infty = \bigcup_{p>1} A_p.$$

Moreover, Hrusčev [6] proved w is in A_∞ if and only if

$$\frac{1}{|I|}\int_I w(e^{i\theta})\,d\theta \exp\left[\frac{1}{|I|}\int_I \log w(e^{i\theta})^{-1}\,d\theta\right] \le C \tag{3.1}$$

for all intervals $I \subset T$.

THEOREM 3.1. *Let $\widehat{X}_n$ and $\widehat{X}_n^{(r)}$ be as in Theorem 2.2. Then*

$$\lim_{r\to 1_-}\int_\Omega \left|\widehat{X}_n^{(r)} - \widehat{X}_n\right|^2 dP = 0, \quad n = 1, 2, \ldots,$$

whenever w is in A_∞.

Proof. According to [14] and [15],

$$\int_T (Gf)(e^{i\theta})w(e^{i\theta})\,d\theta \le C\int_T \left|f(e^{i\theta})\right| w(e^{i\theta})\,d\theta$$

for all f if and only if (3.1) holds. Thus, for w in A_∞,

$$\begin{aligned}\int_T \left|p_n(e^{i\theta})\right|^2 (Gw^{-1})(e^{i\theta})w(e^{i\theta})\,d\theta &\le C\int_T (Gw^{-1})(e^{i\theta})w(e^{i\theta})\,d\theta \\ &\le C\int_T w(e^{i\theta})^{-1}w(e^{i\theta})\,d\theta \\ &\le C < \infty.\end{aligned}$$

That is, when w belongs to A_∞, condition (2.14) of Theorem 2.4, which is sufficient for (2.2), holds. □

It is not difficult to show that a weight satisfying the Helson-Sarason-Szegö condition is in A_∞, so Theorem 3.1 yields the result of Pourahmadi stated in the introduction.

Example 3.1. The outer function

$$\phi_\beta(z) = (1 - z)^\beta$$

is in $H^2(D)$ if and only if $\beta > -1/2$. For such β, the corresponding density

$$w_\beta(e^{i\theta}) = |1 - e^{i\theta}|^{2\beta} = 4\sin^{2\beta}\left(\frac{\theta}{2}\right)$$

is in A_∞; indeed, w_β is in A_p wherever $p > 2\beta + 1$.

Example 3.2. The density determined by the outer function

$$\phi(z) = z(1-z)^{-1/2}\left[\log\frac{1}{1-z}\right]^{-1}$$

in $H^2(D)$ is

$$w(e^{i\theta}) = |\phi(e^{i\theta})|^2 = 4^{-1}\left[\sin\left(\frac{\theta}{2}\right)\right]^{-1}\left[\log\frac{1}{2\sin\left(\frac{\theta}{2}\right)}\right]^{-2}$$

This weight satisfies (2.2) for $n = 1, 2, \ldots$, but is not in A_∞. The first assertion is a consequence of (2.13), which holds in view of the fact that a.e. on T

$$\lim_{r\to 1_-}\left|\frac{\phi(e^{i\theta})}{\phi(re^{i\theta})}\right|^2 = 1$$

and

$$\left|\frac{\phi(e^{i\theta})}{\phi(re^{i\theta})}\right|^2 \le C\left[\sin(\frac{\theta}{2})\right]^{-1}\left[\log\frac{1}{2\sin(\frac{\theta}{2})}\right]^{-2}.$$

We show w is not in A_p for any $p > 1$ and hence not in A_∞. To this end, fix p, $1 < p < \infty$. Then , for $0 < \theta < \pi/2$,

$$\theta^{-1}\int_0^\theta w(e^{it})\,dt \ge C\theta^{-1}\left[\log\frac{1}{\theta}\right]^{-1},$$

while

$$\left[\theta^{-1}\int_0^\theta w(e^{it})^{-\frac{1}{p-1}}\,dt\right]^{p-1} \ge C\theta^{-1}\left[\log\frac{1}{\theta}\right]^2,$$

so their product is unbounded on T.

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