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A GAUSSIAN AVERAGE PROPERTY OF BANACH SPACES

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Introduction

In this paper we introduce a Gaussian average property, abbreviated *GAP*. A Banach space X is said to have *GAP* if there is a constant K so that $\ell(T) \leq K\pi_1(T^*)$ for every finite rank operator from ℓ_2 to X. Here $\ell(T)$ denotes the ℓ-norm defined by Linde and Pietsch [7]; see also N. Tomczak-Jaegermann [13].

We investigate this property in detail and establish that a large class of Banach spaces have it. It turns out that every Banach space which is either of type 2 or is isomorphic to a subspace of a Banach lattice of finite cotype has *GAP* and so does a Banach space of finite cotype which has the Gordon-Lewis property GL_2 with respect to Hilbert spaces.

GAP and GL_2 are closely related, and this enables us to obtain some results on GL_2 by investigating *GAP*. We prove for example, that *GAP* and GL_2 are equivalent properties for cotype 2 spaces and that a K-convex Banach space X has GL_2 if and only if both X and X^* have *GAP*. It also turns out that if a space X is of finite cotype and X^* has *GAP*, then X is K-convex.

We also prove that *GAP* gives rise to some extension theorems of operators with range in a Hilbert space. We prove for example, that if X has *GAP*, then every operator from a subspace of X into a Hilbert space, which factors through L_1, extends to an L_1-factorable operator defined on X. Further, if the dual of a subspace E of a finite cotype Banach space X has *GAP*, then every absolutely summing operator from E to a Hilbert space extends to an absolutely summing operator defined on X. If X^* has *GAP* then the other direction is true for all subspaces E of X. This implies that if X is a Banach space of finite cotype with GL_2 then a subspace E has GL_2 if and only if every 1-summing operator from E to a Hilbert space extends to a 1-summing operator defined on X.

We now wish to discuss the arrangement and contents of the paper in greater detail.

In Section 1 we prove the major results on *GAP* mentioned above. One of the main tools for obtaining these is the duality theorem 1.7 which also relates *GAP* to K-convexity. We provide several examples of Banach spaces with a reasonable structure which fail *GAP*. At the end of the section it is shown that the ℓ_2-sum of

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a sequence of Banach spaces with uniformly bounded *GAP*-constants (respectively uniformly bounded GL_2-constants) has *GAP* (respectively GL_2). This is obtained from an inequality for p-summing operators defined on an ℓ_2-sum of a sequence of Banach spaces with values in a Hilbert space (Theorem 1.18), which turns out to have applications also outside the scope of this paper.

Section 2 is devoted to the extension theorems mentioned above.

0. Notation and preliminaries

In this paper we shall use the notation and terminology commonly used in Banach space theory, as it appears in [8], [9] and [13].

If X and Y are Banach spaces, $B(X, Y)$ ($B(X) = B(X, X)$) denotes the space of bounded linear operators from X to Y. Further, if $1 \leq p < \infty$ we let $\Pi_p(X, Y)$ denote the space of p-summing operators from X to Y equipped with the p-summing norm π_p. We recall that an operator $T \in B(X, Y)$ is said to factor through L_p if it admits a factorization $T = BA$, where $A \in B(X, L_p(\nu))$ and $B \in B(L_p(\nu), Y)$ for some measure ν and we denote the space of all operators which factor through L_p by $\Gamma_p(X, Y)$. If $T \in \Gamma_p(X, Y)$ then we define

$$\gamma_p(T) = \inf\{\|A\|\|B\| \mid T = BA, A \text{ and } B \text{ as above}\}.$$

γ_p is a norm on $\Gamma_p(X, Y)$ turning it into a Banach space.

Throughout the paper we shall identify the tensor product $X \otimes Y$ with the space of ω^*-continuous finite rank operators from X^* to Y in the canonical manner.

We let (g_n) denote a sequence of independent standard Gaussian variables on a fixed probability space $(\Omega, \mathcal{S}, \mu)$ and we let $G(X)$ denote the closure of $\{\sum_{j=1}^n g_j x_j \mid n \in \mathbb{N}$ $x_j \in X, 1 \leq j \leq n\}$ in $L_2(\mu, X)$. Further, we let (e_n) denote the unit vector basis of ℓ_2.

If $n \in \mathbb{N}$ and $T \in B(\ell_2^n, X)$ then, following [13], we define the ℓ-norm of T by

$$\ell(T) = \left(\int \left\|\sum_{j=1}^n g_j(t)Te_j\right\|^2 d\mu(t)\right)^{\frac{1}{2}}.$$

More generally, if $T \in B(\ell_2, X)$, we call T an ℓ-operator if $\sum_{n=1}^\infty g_n T e_n$ converges in $L_2(\mu, X)$ and we put

$$\ell(T) = \left(\int \left\|\sum_{n=1}^\infty g_n(t)Te_n\right\|^2 d\mu(t)\right)^{\frac{1}{2}}.$$

We also need some notation on operators with ranges in a Banach lattice. Recall that if E is a Banach space and X is a Banach lattice, then an operator $T \in B(E, X)$

is called order bounded (e.g., see [12] and [3]), if there exists a $z \in X$, $z \geq 0$ so that

$$|Tx| \leq \|x\| z \quad \text{for all } x \in E \tag{0.1}$$

and the order bounded norm $\|T\|_m$ is defined by

$$\|T\|_m = \inf\{\|z\| \mid z \text{ can be used in (0.1)}\}. \tag{0.2}$$

$\mathcal{B}(E, X)$ denotes the Banach space of all order bounded operators from E to X equipped with the norm $\|\cdot\|_m$.

A Banach space X is said to have the Gordon-Lewis property (abbreviated GL) [2] if every 1-summing operator from X to an arbitrary Banach space Y factors through L_1. It is readily verified that X has *GL*, if and only if there is a constant K so that $\gamma_1(T) \leq K\pi_1(T)$ for every Banach space Y and every $T \in X^* \otimes Y$. In that case $gl(X)$ denotes the smallest constant K with this property.

We shall say that X has GL_2 if it has the property above with $Y = \ell_2$ and we define the constant $gl_2(X)$ correspondingly. An easy trace duality argument yields that *GL* and GL_2 are self dual properties and that $gl(X) = gl(X^*)$ when applicable. It is known [2] that every Banach space with local unconditional structure has the Gordon-Lewis property.

We now present a few theorems which all follow from well-known results and which do not appear in the literature in the form we are going to use them.

The first proposition follows immediately from the contraction principle for independent Gaussian variables; e.g., see [15].

PROPOSITION 0.1. *If X is a Banach space and $(x_j) \subseteq X$ then for all $n \leq m$ and all $1 \leq p < \infty$ we have*

$$\int \left\| \sum_{j=1}^{n} g_j(t) x_j \right\|^p d\mu(t) \leq \int \left\| \sum_{j=1}^{m} g_j(t) x_j \right\|^p d\mu(t).$$

As a corollary to Proposition 0.1 we obtain:

PROPOSITION 0.2. *If X is a Banach space and $f \in G(X)$ then for all $n \in \mathbb{N}$ we have*

$$\left(\int \left\| \sum_{j=1}^{n} g_j(t) \int f\, g_j d\mu \right\|^2 d\mu(t) \right)^{\frac{1}{2}} \leq \|f\|_2, \tag{0.3}$$

and the series $\sum_{j=1}^{\infty} g_j (\int f\, g_j d\mu)$ converges to f in $L_2(\mu, X)$.

Proof. Let $\mathcal{A}$ be the subspace of $G(X)$ consisting of all f of the form $f = \sum_{j=1}^{m} g_j x_j$ for some $m \in \mathbb{N}$ and some sequence $(x_j) \subseteq X$. For every $n \in \mathbb{N}$ we define

$P_n : \mathcal{A} \to G(X)$ by

$$P_n f = \sum_{j=1}^{n} g_j \left(\int f g_j d\mu \right) \text{ for all } f \in \mathcal{A}. \tag{0.4}$$

From the previous proposition it follows that P_n is a bounded linear projection on $\mathcal{A}$ with $\|P_n\| \leq 1$. Hence it can be extended to a norm 1 linear projection on $G(X)$ also denoted P_n. This immediately gives (0.3) and since obviously $P_n f \to f$ for all $f \in \mathcal{A}$ and $\|P_n\| \leq 1$ for all $n \in \mathbb{N}$ we get the same for all $f \in G(X)$. □

PROPOSITION 0.3. *For every $1 \leq p < \infty$ there is a constant K_p so that if X is a Banach space and $T \in B(\ell_2, X)$ is an ℓ-operator then T^* is p-summing with*

$$\pi_p(T^*) \leq K_p \ell(T). \tag{0.5}$$

If $T \in \ell_2 \otimes X$, then T is an ℓ-operator and

$$\ell(T) \leq K_p \pi_p(T). \tag{0.6}$$

Proof. Let $1 \leq p < \infty$. By a result of Kahane [6] there are constants $a_p > 0$ and $b_p > 0$ so that

$$a_p \|f\|_2 \leq \|f\|_p \leq b_p \|f\|_2 \quad \text{for all } f \in G(X). \tag{0.7}$$

To prove (0.5) we let $T \in B(\ell_2, X)$ be an ℓ-operator and define

$$f = \sum_{n=1}^{\infty} g_n T e_n. \tag{0.8}$$

If $I_p \colon \ell_2 \to L_p(\mu)$ denotes the operator defined by $I_p e_n = g_n$ for all $n \in \mathbb{N}$, then I_p is an isomorphism and

$$(I_p T^* x^*)(t) = x^*(f(t)) \quad \text{for almost all } t \in \Omega. \tag{0.9}$$

It follows from [12] that $I_p T^*$ is order bounded and therefore p-summing with

$$\pi_p(T^*) \leq a_p^{-1} \pi_p(I_p T^*) \leq a_p^{-1} \|I_p T^*\|_m = a_p^{-1} \|f\|_p \leq a_p^{-1} b_p \ell(T). \tag{0.10}$$

To prove (0.6) we let $T \in \ell_2 \otimes X$; hence there is a $k \in \mathbb{N}$, $(f_j)_{j=1}^k \subseteq \ell_2$ and $(x_j)_{j=1}^k \subseteq X$ with $T = \sum_{j=1}^k f_j \otimes x_j$. If $g = \sum_{j=1}^k (I_2 f_j) x_j$, then for all $n \in \mathbb{N}$,

$$T e_n = \sum_{j=1}^{k} (e_n, f_j) x_j = \sum_{j=1}^{k} (g_n, I_2 f_j) x_j = \int g(t) g_n(t) d\mu(t), \tag{0.11}$$

and therefore, by Proposition 0.2

$$g = \sum_{n=1}^{\infty} g_n T e_n. \tag{0.12}$$

This shows that T is an ℓ-operator and by [12, Corollary 4.8] we obtain

$$\|I_p T^*\|_m \leq b_p \pi_p(T) \tag{0.13}$$

and

$$\ell(T) \leq a_p^{-1} \|I_p T^*\|_m \leq a_p^{-1} b_p \pi_p(T). \tag{0.14}$$

□

1. The Gaussian average property and related topics

In this section we shall introduce our Gaussian average property and prove our main results, which among other things relates this property to the Gordon-Lewis property. We start with the following definition.

Definition 1.1. Let X be a Banach space. X is said to have the Gaussian average property (*GAP*) if there is a constant K, so that for all $T \in \ell_2 \otimes X$ we have $\ell(T) \leq K\pi_1(T^*)$.

X is said to have property (S_p) $1 \leq p < \infty$ if there is a constant K so that if $T \in B(\ell_2, X)$ with $T^* \in \Pi_1(X^*, \ell_2)$, then $T \in \Pi_p(\ell_2, X)$ with $\pi_p(T) \leq K\pi_1(T^*)$. We shall say that X has (S), if it has (S_p) for some p, $1 \leq p < \infty$.

Recall that a Banach space Y is called a Grothendieck space (abbreviated GT) [15] if $B(Y, \ell_2) = \Pi_1(Y, \ell_2)$. It follows from Grothendieck's inequality that every $\mathcal{L}_1$-space is a GT space. We make the following observation:

PROPOSITION 1.2. *If X is a Banach space so that X^* is a GT-space then X does not have GAP. In particular, L_∞ does not have GAP.*

Proof. Let K be the GT-constant of X and let $n \in \mathbb{N}$ be given. By Dvoretzky's theorem [8] there is an isomorphism $T : \ell_2^n \to X$ so that $\|T\| \leq 2$ and $\|T^{-1}\| = 1$. Clearly $\pi_1(T^*) \leq K\|T\| \leq 2K$ and $\frac{1}{2}\sqrt{n} \leq \ell(T) \leq 2\sqrt{n}$, which shows that X does not have *GAP*. □

It follows easily from the results of the previous section that if X has *GAP*, then the ℓ-norm of an operator $T \in B(\ell_2, X)$ is equivalent to the 1-summing norm of the adjoint. If X has (S_p) then it follows that the p-summing norm of an operator $T \in \Pi_p(\ell_2, X)$ is equivalent to the 1-summing norm of the adjoint.

It is readily seen that both *GAP* and (S) are hereditary properties and from the principle of local reflexivity it is easily seen that X has *GAP*, respectively (S), if and only if X^{**} has *GAP*, respectively (S). Furthermore we have:

THEOREM 1.3. *Let X be a Banach space. Then the following statements hold.*

(*i*) *If X has (S), then it has GAP.*
(*ii*) *If X has (S_p), then it is of cotype* $\max(2, p)$.
(*iii*) *If X has GAP, then it is of finite cotype.*
(*iv*) *If X is of finite cotype and has GL_2, then X has (S) and hence also GAP.*

Proof. (i) and (ii) Let X have (S_p) with constant K for some p, $1 \leq p < \infty$ and put $q = \max(p, 2)$. It follows from Proposition 0.3 that for every $T \in \ell_2 \otimes X$ we have

$$\begin{aligned}\pi_{q,2}(T) &\leq \pi_p(T) \leq K\pi_1(T^*) \leq KK_1\ell(T) \\ &\leq K_pKK_1\pi_p(T) \leq K^2K_pK_1\pi_1(T^*).\end{aligned} \tag{1.1}$$

From (1.1) we obtain directly that X has *GAP*. Furthermore, together with [13, Theorem 12.2], (1.1) gives that X has cotype q.

(iii) Assume that X has *GAP*. If X is not of finite cotype it contains ℓ_∞^n uniformly [11] and since *GAP* is hereditary this implies that ℓ_∞ has *GAP*, which is a contradiction.

(iv) Let X be a Banach space of cotype q with GL_2 and let $p > q$. By self-duality X^* has GL_2 as well and if $T \in B(\ell_2, X)$ with $T^* \in \Pi_1(X^*, \ell_2)$ then $T \in \Gamma_\infty(\ell_2, X^{**})$ and hence by [11] $T \in \Pi_p(\ell_2, X)$. If $q = 2$, we can actually take $p = 2$ as well. □

The next theorem describes some classes of Banach spaces which have *GAP*.

THEOREM 1.4. *Let X be a Banach space.*

(*i*) *If X is of cotype* 2 *then X has GAP if and only if it has GL_2.*
(*ii*) *If X is of type* 2 *then it has GAP.*
(*iii*) *If X is a subspace of a Banach lattice of finite cotype, then X has (S) and hence GAP.*

Proof. (i) If X is of cotype 2 it follows from [13, Theorem 12.2] that there is a constant K so that

$$\pi_2(T) \leq K\ell(T) \quad \text{for all } T \in \ell_2 \otimes X. \tag{1.2}$$

If X has *GAP* with constant C then it follows from (1.2) that for all $T \in \ell_2 \otimes X$

we have

$$\gamma_1(T^*) = \gamma_\infty(T) \le \pi_2(T) \le K\ell(T) \le KC\pi_1(T^*). \tag{1.3}$$

This shows that X^* and hence X has GL_2.

The other direction follows from Theorem 1.3.

(ii) Let X be of type 2 with constant K and let $T \in \ell_2 \otimes X$. Again, by [13, Theorem 12.2], we get

$$\ell(T) \le K\pi_2(T^*) \le K\pi_1(T^*), \tag{1.4}$$

which shows that X has *GAP*.

(iii) Let X be a subspace of a Banach lattice Z of finite cotype. Hence, by [9], Z is q-concave for some q, $1 \le q < \infty$ with constant say K. If $T \in \ell_2 \otimes X$ and $I : X \to Z$ denotes the identity operator, then it follows from [12, Proposition 4.9] that

$$\|IT\|_m \le \pi_1(T^*I^*) \le \pi_1(T^*). \tag{1.5}$$

Since T is of finite rank it follows from [12, Theorem 2.9] that there exists a compact Hausdorff space S and operators $A \in B(\ell_2, C(S))$, $B \in B(C(S), Z)$ so that $\|A\| = 1$, $B \ge 0$, $\|B\| = \|IT\|_m$ and $IT = BA$. Since $B \ge 0$ and Z is q-concave, B is q-summing with $\pi_q(B) \le K\|B\|$ ([9]). Hence T is q-summing as well with

$$\pi_q(T) \le \|A\|\pi_q(B) \le K\|T\|_m \le K\pi_1(T^*). \tag{1.6}$$

This shows that X has (S_q). □

Since *GAP* is a hereditary property, Theorem 1.4 gives the following corollary:

COROLLARY 1.5. *If X of cotype 2 has GL_2 then so does every subspace. In particular, if X is a Banach lattice of cotype 2, then every subspace has GL_2.*

Corollary 1.5 can of course also easily be deduced from the fact that if X is of cotype 2 then $\Pi_1(X, L_2) = \Pi_2(X, L_2)$ and the fact that 2-summing operators extend to 2-summing operators.

The cotype 2 situation is not the only one where *GAP* and GL_2 coincide. We shall return to this after we have proved an important duality theorem. First we need:

PROPOSITION 1.6. *If X is a Banach space of cotype r and $r < q < \infty$, then there is a constant $K_{r,q} \ge 0$ so that*

$$\ell(T) \le K_{r,q}\gamma_\infty(T) \quad \text{for all } T \in \ell_2 \otimes X. \tag{1.7}$$

Proof. Let X be of cotype r and let $q > r$. From [11] it is easily derived that there is a constant $C_{r,q}$ so that

$$K_q^{-1}\ell(T) \le \pi_q(T) \le C_{r,q}\gamma_\infty(T) \quad \text{for all } T \in \ell_2 \otimes X, \tag{1.8}$$

where the first inequality in (1.8) comes from Proposition 0.3. □

We are now able to prove the following duality theorem.

THEOREM 1.7. *If X is a Banach space then the following conditions are equivalent*:

(*i*) *X is K-convex and there is a constant $K \geq 0$ so that $K^{-1}\gamma_\infty(T) \leq \ell(T) \leq K\gamma_\infty(T)$ for all $T \in \ell_2 \otimes X$.*
(*ii*) *X^* has GAP and X is of finite cotype.*

Proof. (i) $\Rightarrow$ (ii). Assume that (i) holds and let C denote the K-convexity constant of X (for the definition of K-convexity we refer to [15]).

If $S \in \ell_2 \otimes X^*$ we get

$$\begin{aligned} \ell(S) &\leq C \sup\{|Tr(T^*S)| \mid T \in \ell_2 \otimes X,\ \ell(T) \leq 1\} \\ &\leq KC \sup\{|Tr(T^*S)| \mid T \in \ell_2 \otimes X,\ \gamma_\infty(T) \leq 1\} \\ &= KC\pi_1(S^*), \end{aligned} \tag{1.9}$$

which shows that X^* has *GAP*. Clearly X is of finite cotype.

(ii) $\Rightarrow$ (i). Since X is of finite cotype it follows from Proposition 1.6 that there is a constant C_1 so that $\ell(T) \leq C_1\gamma_\infty(T)$ for all $T \in \ell_2 \otimes X$. If C_2 denotes the *GAP*-constant of X^* then for every $T \in \ell_2 \otimes X$

$$\begin{aligned} \gamma_\infty(T) &= \sup\{|Tr(S^*T)| \mid S \in \ell_2 \otimes X^*, \pi_1(S^*) \leq 1\} \\ &\leq C_2 \sup\{|Tr(S^*T)| \mid S \in \ell_2 \otimes X^*, \ell(S) \leq 1\} \\ &= C_2\ell^*(T^*) \leq C_2\ell(T) \leq C_1C_2\gamma_\infty(T). \end{aligned} \tag{1.10}$$

This shows that the fourth and fifth entries in (1.10) are equivalent, which clearly implies (see [13]) that X is K-convex. In addition (1.10) shows that $\gamma_\infty(T) \leq C_2\ell(T)$ for all $T \in \ell_2 \otimes X$. Hence we have proved that (ii) $\Rightarrow$ (i). □

Since X has *GAP* if and only if X^{**} has *GAP*, as noted just after Definition 1, it follows that the roles of X and X^* can be interchanged in Theorem 1.7.

Theorem 1.7 has several corollaries.

COROLLARY 1.8. *If X has GAP and X^* is of finite cotype then X is K-convex.*

The next corollary we formulate as a theorem.

THEOREM 1.9. *Let X be a Banach space. The following statements are equivalent*:

(*i*) *X has GL_2 and both X and X^* are of finite cotype.*
(*ii*) *X and X^* have GAP.*

Under these circumstances X is K-convex.

Proof. (i) $\Rightarrow$ (ii). Since GL_2 is a self dual property it follows that both X and X^* have *GAP*.

(ii) $\Rightarrow$ (i). Assume that (ii) holds. It follows from Theorem 1.3 that both X and X^* are of finite cotype.

Since X has *GAP* it follows from Theorem 1.7 that there is a constant $K \geq 0$ so that for all $T \in \ell_2 \otimes X^*$ we have

$$\gamma_\infty(T) \leq K\ell(T). \tag{1.11}$$

If C denotes the *GAP*-constant of X^* we get from (1.11) that if $S \in X \otimes \ell_2$, then

$$\gamma_1(S) = \gamma_\infty(S^*) \leq K\ell(S^*) \leq KC\pi_1(S) \tag{1.12}$$

which shows that X has GL_2. □

It is well known that if X is of cotype 2 then $B(L_\infty, X) = \Pi_2(L_\infty, X)$ or equivalently $\Pi_1(X, \ell_2) = \Pi_2(X, \ell_2)$ and it is an open question whether the converse implication holds. Pisier [14] showed that this is the case if X has GL_2. Here we prove a similar result using *GAP*.

THEOREM 1.10. *Let X be a Banach space and $1 < p \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$. If X has GAP, then $B(L_\infty, X^*) = \Pi_q(L_\infty, X^*)$ if and only if X is of type p-stable.*

In particular, X is of type 2 if and only if it has GAP and $\Pi_1(X^*, \ell_2) = \Pi_2(X^*, \ell_2)$.

Proof. If X is of type p-stable then it follows from [11] that $B(\ell_\infty, X^*) = \Pi_q(\ell_\infty, X^*)$. Next, assume that X has *GAP* with constant M and that $B(L_\infty, X^*) = \Pi_q(L_\infty, X^*)$ with K-equivalence between the norms, hence also $\Pi_1(X^*, \ell_2) = \Pi_p(X^*, \ell_2)$ with K-equivalence between the norms.

If $T = \sum_{j=1}^k e_j \otimes x_j \in \ell_2 \otimes X$, then

$$\begin{aligned}\pi_p(T^*) &\leq \left(\sum_{j=1}^k \|x_j\|^p\right)^{\frac{1}{p}} \sup\left\{\left(\sum_{j=1}^k |(z, e_j)|^q\right)^{\frac{1}{q}} \mid z \in \ell_2 \ \|z\|_2 \leq 1\right\} \\ &\leq \left(\sum_{j=1}^k \|x_j\|^p\right)^{\frac{1}{p}} \end{aligned} \tag{1.13}$$

and therefore

$$\left(\int\left\|\sum_{j=1}^{k}g_j(t)x_j\right\|^2 d\mu(t)\right)^{\frac{1}{2}} = \ell(T) \leq M\pi_1(T^*) \leq MK\pi_p(T^*)$$
$$\leq \left(\sum_{j=1}^{k}\|x_j\|^p\right)^{\frac{1}{p}}. \tag{1.14}$$

which shows that X is of type p.

If $p = 2$ we are done. If $p < 2$ then by [11], $\{p < 2 \mid \Pi_q(L_\infty, X^*) = B(L_\infty, X^*)\}$ is an open interval and therefore X is of type p-stable. □

Let us now look at a few examples.

Example 1.11. Let X be the space constructed by Pisier in [14]. Both X and X^* are of cotype 2, but X is not isomorphic to a Hilbert space. Therefore X is not K-convex and hence cannot have *GAP* nor GL_2 by Corollary 1.8.

There exist K-convex Banach spaces of cotype 2 not having *GAP* (equivalently GL_2), which the following example shows.

Example 1.12. Let $2 < p < \infty$. By Figiel, Kwapień and Pelczyński [1] it follows that there exists a subspace $X \subseteq L_p(0, 1)$ which does not have GL_2 (See also Pisier [16] for the case $p > 4$ and Johnson [4, Lemma 1] for a more general result). X^* is K-convex but does not have *GAP* by Theorem 1.9. Hence it does not embed into a Banach lattice of finite cotype.

Similar arguments as in this example leads to

COROLLARY 1.13. *Let X be a Banach space with GAP. If X^* embeds into a Banach lattice of finite cotype, then X has GL_2.*

From the result of Johnson quoted in Example 1 we can also conclude

COROLLARY 1.14. *Every Banach lattice of finite cotype which is not of weak cotype 2 contains a subspace X, so that X^* does not embed into a Banach lattice of finite cotype.*

We can pose the following problem:

Problem 1.15. Can the above mentioned theorem of Johnson be strengthened. Specifically, is a Banach space of cotype 2, if all subspaces have GL_2?

The convexified Tsirelson space $T^{(2)}$ (see [15]) is of type 2 and weak cotype 2, and one could try to investigate whether there is a subspace X of $T^{(2)}$ failing GL_2. Hence X^* will fail *GAP* and therefore X would be the first example of a weak Hilbert space, which does not embed any Banach lattice of finite cotype.

One of the many results on unconditional structures obtained by Gordon and Lewis in [2] states that the Schatten class c_p, $p \neq 2$, does not have lust, but going through their methods of computing ideal norms for spaces with enough symmetries, in particular those in Chapter 5, it can be derived from their results that in fact c_p does not have (S) for any $p \neq 2$.

Combining this with our Theorems 1.4 and 1.9 we obtain:

Example 1.16. For every q, $2 < q < \infty$, c_q has *GAP*, since it is of type 2, but not (S). c_p does not have *GAP* for $1 \leq p < 2$.

More generally, in [16], Pisier showed among other things, that if λ is a unitarily invariant crossnorm on $\ell_2 \otimes \ell_2$ then $\ell_2 \hat{\otimes}_\lambda \ell_2$ does not embed into a Banach space of finite cotype with lust unless λ is equivalent to the Hilbert Schmidt norm. His argument actually shows that, except for the Hilbert Schmidt case, $\ell_2 \hat{\otimes}_\lambda \ell_2$ does not have (S). Indeed, an inspection of the proof shows that the conclusion of his Proposition 2.1 holds, if the space E is just assumed to have (S) (called (I) there) and this observation together with his Theorem 2.1 show our statement.

The following condition is stronger than (S).

Definition 1.17. A Banach space X is said to have (I), if there is a p, $1 \leq p < \infty$, and a constant K so that

$$i_p(T) \leq K\pi_1(T^*) \quad \text{for all } T \in \ell_2 \otimes X$$

where i_p denotes the p-integral norm [13].

Condition (I) is equivalent to X being of finite cotype and having GL_2. Indeed, if X has (I), then it has (S) and is of finite cotype. (I) immediately implies that $\Pi_1(X^*, \ell_2) \subseteq \Gamma_1(X^*, \ell_2)$ and therefore X^* and hence X has GL_2. On the other hand, if X is of finite cotype and has GL_2, an inspection of the proof of Theorem 1.3, (iv) shows that in fact X has (I) (use $I_p(L_\infty, X) = \Pi_p(L_\infty, X)$ together with the principle of local reflexivity).

This equivalence was also established by Junge [5].

We now wish to show that *GAP* is closed under the formation of ℓ_2-sums of Banach spaces. For this we need the following theorem, which turns out to have some importance in itself.

THEOREM 1.18. *Let (X_n) be a sequence of Banach spaces and put $X = (\sum_{n=1}^{\infty} X_n)_2$. If Y is another Banach space, $1 \leq p < \infty$ and $T \in \Pi_p(X, Y)$*

with $T_n = T_{|X_n}$, *then*

$$\left(\sum_{n=1}^{\infty} \pi_p(T_n)^2\right)^{\frac{1}{2}} \leq \pi_p(T) \quad \text{for } 1 \leq p \leq 2 \tag{1.15}$$

$$\left(\sum_{n=1}^{\infty} \pi_p(T_n)^p\right)^{\frac{1}{p}} \leq \pi_p(T) \quad \text{for } 2 \leq p < \infty. \tag{1.16}$$

If $Y = \ell_2$ *then* (1.15) *holds for all* p, $1 \leq p < \infty$.

Proof. Let $\varepsilon > 0$ be given arbitrarily. For every $n \in \mathbb{N}$ we can find a finite set $\sigma_n \subseteq \mathbb{N}$ and $\{x_i(n) \mid i \in \sigma_n\} \subseteq X_n$ so that

$$\pi_p(T_n)^p \leq \sum_{i \in \sigma_n} \|Tx_i(n)\|^p + \varepsilon 2^{-n}, \tag{1.17}$$

$$\sup\left\{\sum_{i \in \sigma_n} |x^*(x_i(n)|^p \mid x^* \in X_n^*, \|x^*\| \leq 1\right\} \leq 1. \tag{1.18}$$

For every sequence $(\alpha_n) \subseteq \mathbb{R}_+ \cup \{0\}$, from (1.17) and (1.18) we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty} \alpha_n \pi_p(T_n)^p &= \sum_{n=1}^{\infty} \sum_{i \in \sigma_n} \|T(\alpha_n^{1/p} x_i(n))\|^p + \varepsilon \\
&\leq \pi_p(T)^p \sup\Bigg\{\sum_{n=1}^{\infty} \sum_{i \in \sigma_n} |\langle x^*(n), \alpha_n^{1/p} x_i(n)\rangle|^p \mid \\
&\qquad x^*(n) \in X_n^*, \sum_{n=1}^{\infty} \|x^*(n)\|^2 \leq 1\Bigg\} + \varepsilon \\
&\leq \pi_p(T)^p \sup\Bigg\{\sum_{n=1}^{\infty} \|x^*(n)\|^p \alpha_n \sum_{i \in \sigma_n} \left|\left\langle \frac{x^*(n)}{\|x^*(n)\|}, x_i(n)\right\rangle\right|^p \mid \\
&\qquad x^*(n) \in X_n^*, \sum_{n=1}^{\infty} \|x^*(n)\|^2 \leq 1\Bigg\} + \varepsilon \\
&\leq \pi_p(T)^p \sup\left\{\sum_{n=1}^{\infty} \|x^*(n)\|^p \alpha_n \mid x_n^* \in X_n^*, \sum_{n=1}^{\infty} \|x_n^*\|^2 \leq 1\right\} \\
&\qquad + \varepsilon.
\end{aligned} \tag{1.19}$$

If $1 \leq p \leq 2$ we take the supremum in (1.19) over all sequences (α_n) considered with $\sum_{n=1}^{\infty} \alpha_n^{2/(2-p)} = 1$ and let $\varepsilon \to 0$ to obtain (1.15).

For $2 \leq p < \infty$ we put $\alpha_n = 1$ for all $n \in \mathbb{N}$ in (1.19) to obtain (1.16).

Since $\Pi_p(Z, \ell_2) = \Pi_2(Z, \ell_2)$ for every Banach space Z and every $2 \leq p < \infty$ (this follows easily from Maurey's extension theorem [10] and the formula $B(L_\infty, L_2) = \Pi_2(L_\infty, L_2)$) the statement for $Y = \ell_2$ follows from the above. □

This enables us to prove:

THEOREM 1.19. *Let (X_n) be a sequence of Banach spaces, which all have GAP with uniformly bounded constants. Then $X = \left(\sum_{n=1}^{\infty} X_n\right)_2$ has GAP.*

Proof. For every $n \in \mathbb{N}$ we let P_n denote the canonical projection of X onto X_n. If $x_1, x_2, \ldots, x_k \in X$ then it follows immediately from the definition of the norm in X that

$$\int \left\| \sum_{i=1}^{k} g_1(t) x_i \right\|^2 d\mu(t) = \sum_{n=1}^{\infty} \int \left\| \sum_{i=1}^{k} g_i(t) P_n x_i \right\|^2 d\mu(t). \tag{1.20}$$

Therefore if $T \in B(\ell_2, X)$ is an ℓ-operator, then

$$\ell(T) = \left(\sum_{n=1}^{\infty} \ell(P_n T)^2 \right)^{\frac{1}{2}}. \tag{1.21}$$

Let $K \geq 0$ be a constant so that for all $n \in \mathbb{N}$,

$$\ell(S) \leq K \pi_1(S^*) \quad \text{for all } S \in \ell_2 \otimes X. \tag{1.22}$$

Now, if $T \in \ell_2 \otimes X$, then by Theorem 1.18 with $p = 1$, (1.21) and (1.22), we obtain

$$\ell(T) = \left(\sum_{n=1}^{\infty} \ell(P_n T)^2 \right)^{\frac{1}{2}} \leq K \left(\sum_{n=1}^{\infty} \pi_1(T^* P_n^*)^2 \right)^{\frac{1}{2}} \leq K \pi_1(T^*). \tag{1.23}$$

This shows that X has *GAP*. □

Combining Theorems 1.9 and 1.19 we immediately obtain that if (X_n) is a sequence of Banach spaces with uniformly bounded K-convexity constants and GL_2-constants, then $X = \left(\sum_{n=1}^{\infty} X_n\right)_2$ has GL_2. However it was pointed out to us by Junge that this conclusion can be obtained without the K-convexity assumption by combining the inequality in 1.18 with its dual form. We need:

LEMMA 1.20. *Let (X_n) be a sequence of Banach spaces, $X = \left(\sum_{n=1}^{\infty} X_n\right)_2$, $P_n : X \to X_n$ the canonical projection.*

(*i*) *If $T \in B(\ell_2, X)$ with $\sum \gamma_\infty(P_n T)^2 < \infty$ then $T \in \Gamma_\infty(\ell_2, X)$ with*

$$\gamma_\infty(T) \leq \left(\sum_{n=1}^{\infty} \gamma_\infty(P_n T)^2 \right)^{\frac{1}{2}}.$$

(*ii*) *If* $S \in B(X, \ell_2)$ *with* $\sum_{n=1}^{\infty} \gamma_1(SP_n)^2 < \infty$ *then* $S \in \Gamma_1(X, \ell_2)$ *with*

$$\gamma_1(S) \leq \left(\sum_{n=1}^{\infty} \gamma_1(SP_n)^2\right)^{\frac{1}{2}}.$$

Proof. (i) follows immediately from Theorem 1.18 by applying trace duality to the inequality there. Applying (i) to X^* we obtain (ii). □

This leads to:

THEOREM 1.21. *Let* (X_n) *be a sequence of Banach spaces all having* GL_2 *so that* $K = \sup_n gl_2(X_n) < \infty$. *Then* $X = \left(\sum_{n=1}^{\infty} X_n\right)_2$ *has* GL_2.

Proof. Let $T \in \Pi_1(X, \ell_2)$. From Theorem 1.18 and our assumptions we get

$$\sum_{n=1}^{\infty} \gamma_1(TP_n)^2 \leq K^2 \sum_{n=1}^{\infty} \pi_1(TP_n)^2 \leq K^2 \pi_1(T)^2. \tag{1.24}$$

Lemma 1.20 now gives $T \in \Gamma_1(X, \ell_2)$ with

$$\gamma_1(T) \leq K\pi_1(T), \tag{1.25}$$

which shows that X has GL_2. □

Reisner [17] has proved using different methods that the conclusion of Theorem 1.21 holds for more general unconditional sums of Banach spaces.

Let us end this section by discussing the following problem which seems to be important since it has some applications to various areas of Banach space theory.

Problem 1.22. Let (X_n) be a sequence of Banach spaces. Under which assumptions on the X_n's does there exist a constant K so that

$$\pi_1(T) \leq K \left(\sum_{n=1}^{\infty} \pi_1(TP_n)^2\right)^{\frac{1}{2}} \quad \text{for all } T \in X \otimes \ell_2. \tag{1.26}$$

The next theorem gives some conditions for the inequality (1.26) to hold. (iii) was shown to us by Junge.

THEOREM 1.23. *Let* (X_n) *be a sequence of Banach spaces,* $X = (\sum X_n)_2$. *The inequality* (1.26) *holds, if one of the following conditions is satisfied.*

(*i*) X_n^* *has GAP for every* $n \in \mathbb{N}$ *with uniformly bounded GAP-constants.*

(*ii*) X_n *has* GL_2 *for every* $n \in \mathbb{N}$ *and* $\sup gl_2(X_n) < \infty$.
(*iii*) X_n *is of cotype* 2 *for every* $n \in \mathbb{N}$ *with uniformly bounded cotype* 2 *constants.*

Proof. If (i) is satisfied, we choose $K \geq 0$ so that

$$\ell(S) \leq K\pi_1(S^*) \quad \text{for all } S \in \ell_2 \otimes X_n^*.$$

X^* has *GAP* by Theorem 1.19 and by repeating the calculations there with X replaced by X^* combined with Proposition 0.3, for every $T \in X \otimes \ell_2$ we have

$$\pi_1(T) \leq K_1\ell(T^*) = K_1\left(\sum_{n=1}^{\infty}\ell(P_n^*T^*)^2\right)^{\frac{1}{2}} \leq KK_1\left(\sum_{n=1}^{\infty}\pi_1(TP_n)^2\right)^{\frac{1}{2}} \tag{1.27}$$

which gives (1.26).

Next, assume that (ii) holds. Put $K = \sup_n gl_2(X_n)$. If K_G denotes the Grothendieck constant, then by repeating the calculations in the proof of Theorem 1.21, for every $T \in X \otimes \ell_2$ we have

$$\pi_1(T) \leq K_G\gamma_1(T) \leq K_G\left(\sum_{n}\gamma_1(TP_n)^2\right)^{\frac{1}{2}} \leq KK_G\left(\sum_{n=1}^{\infty}\pi_1(TP_n)^2\right)^{\frac{1}{2}} \tag{1.28}$$

which gives (1.26).

Finally, assume that X is of cotype 2 with constant K and let $S \in \ell_2 \otimes X$. By [13, Theorem 12.2] we have

$$\pi_2(P_nS) \leq K\ell(P_nS) \quad \text{for all } n \in \mathbb{N} \tag{1.29}$$

and hence

$$\left(\sum_{n=1}^{\infty}\pi_2(P_nS)^2\right)^{\frac{1}{2}} \leq K\left(\sum_{n=1}^{\infty}\ell(P_nS)^2\right)^{\frac{1}{2}} = K\ell(S) \leq K_2K\pi_2(S), \tag{1.30}$$

where K_2 is the constant from Proposition 0.3.

Dualizing (1.30) and again using the fact that X is of cotype 2, for every $T \in X\otimes\ell_2$ we have

$$\begin{aligned}\pi_1(T) &\leq K\pi_2(T) \leq K^2K_1\left(\sum_{n=1}^{\infty}\pi_2(TP_n)^2\right)^{\frac{1}{2}}\\ &\leq K^2K_1\left(\sum_{n=1}^{\infty}\pi_1(TP_n)^2\right)^{\frac{1}{2}},\end{aligned} \tag{1.31}$$

which gives (1.26). □

2. *GAP* and extension properties of certain classes of operators

In this section we shall prove some results concerning extensions of certain operators defined on a Banach space with *GAP* with values in a Hilbert space. We start with the following:

THEOREM 2.1. *Let X be a Banach space with GAP. Then there is a constant K so that for every subspace $E \subseteq X$ and every $T \in \ell_2 \otimes E$ we have*

$$\pi_1(T^*) \leq K\pi_1(T^*Q) \tag{2.1}$$

where Q is the canonical quotient map of X^* onto E^*.

Consequently, every $S \in \Gamma_1(E, \ell_2)$ admits an extension $\widetilde{S} \in \Gamma_1(X, \ell_2)$ with

$$\gamma_1(\widetilde{S}) \leq K\gamma_1(S). \tag{2.2}$$

Proof. Let C be the *GAP*-constant of X and let $T \in \ell_2 \otimes E$ be arbitrary. It is obvious that $\ell(T\colon \ell_2 \to E) = \ell(T\colon \ell_2 \to X)$ and hence

$$\pi_1(T^*) \leq K_1\ell(T) \leq K_1C\pi_1(T^*Q), \tag{2.3}$$

where K_1 is the constant from Proposition 0.3, (2.3) gives (2.1) with $K = K_1C$.

An easy dualization argument shows that the second statement is equivalent to $\Gamma_1^*(\ell_2, E) \subseteq \Gamma_1^*(\ell_2, X)$ with K-equivalence between the norms. (Γ_1^* denotes the dual operator ideal.)

However, $\Gamma_1^*(\ell_2, E) = \{T \in B(\ell_2, E) \mid T^* \in \Pi_1(E^*, \ell_2)\}$ and similarly for X, and hence the latter statement is exactly (2.1). □

The next theorem gives a characterization of subspaces E of a given Banach space X so that E^* has *GAP* in terms of extensions of 1-summing operators.

THEOREM 2.2. *Let X be a Banach space and E a subspace. Consider the statements*

(*i*) *E^* has GAP.*

(*ii*) *There exists a constant $K \geq 0$ so that every $T \in \Pi_1(E, \ell_2)$ admits an extension $\widetilde{T} \in \Pi_1(X, \ell_2)$ with $\pi_1(\widetilde{T}) \leq K\pi_1(T)$.*

If X is of finite cotype then (*i*) *implies* (*ii*). *If X^* has GAP then* (*ii*) *implies* (*i*).

Proof. By duality, (ii) is equivalent to:

(iii) $\Gamma_\infty(\ell_2, E) \subseteq \Gamma_\infty(\ell_2, X)$ with equivalence between the norms.

Let X be of finite cotype and assume that E^* has *GAP*.

We wish to show that (iii) holds. By Proposition 1.6 and Theorem 1.7 there exist constants $K \geq 0$ and $C \geq 0$ so that if $T \in \ell_2 \otimes E$,

$$\gamma_\infty(T) \leq C\ell(T) \leq KC\gamma_\infty(T : \ell_2 \to X) \tag{2.4}$$

which shows that (iii) holds.

Assume next that X^* has *GAP* with constant M and that (ii) holds. It clearly follows that there is a constant $K \geq 0$ so that every $T \in \Pi_1(E, \ell_2)$ admits an extension $\widetilde{T} \in \Pi_1(X, \ell_2)$ with

$$\pi_1(\widetilde{T}) \leq K\pi_1(T). \tag{2.5}$$

Let now $T = \sum_{j=1}^n f_j^* \otimes e_j \in E^* \otimes \ell_2$ and let $\widetilde{T}$ be an extension of T so that (2.5) holds. Without loss of generality we may assume that the range of $\widetilde{T}$ is contained in $[e_j \mid 1 \leq j \leq n]$ and since X^* has *GAP* we therefore easily obtain

$$\ell(T^*) \leq \ell(\widetilde{T}^*) \leq M\pi_1(\widetilde{T}) \leq KM\pi_1(T), \tag{2.6}$$

which shows that E^* has *GAP*. □

Combining Theorem 2.2 with the results of the previous section we obtain the following corollary.

COROLLARY 2.3. *Let X be a Banach space of finite cotype with GL_2 and let $E \subseteq X$ be a subspace. Then the following statements are equivalent.*

(*i*) *E has GL_2.*
(*ii*) *Every operator $T \in \Pi_1(E, \ell_2)$ admits an extension $\widetilde{T} \in \Pi_1(X, \ell_2)$.*

Proof. Trivially (ii) implies (i) (for this the finite cotype assumption on X is superfluous). Next, assume that E has GL_2 and let $T \in \Pi_1(E, \ell_2)$; hence $T \in \Gamma_1(E, \ell_2)$ as well and since X has *GAP* we get from Theorem 2.1 that T admits an extension $\tilde{T} \in \Gamma_1(X, \ell_2) \subseteq \Pi_1(X, \ell_2)$. □

The assumption that X is of finite cotype cannot be omitted in Corollary 2.3 as the following example shows.

Example 2.4. Let E be a subspace of ℓ_∞ isometric to ℓ_1, and let $T \in B(E; \ell_2)$ be onto. E has GL_2 and T is absolutely summing by Grothendieck's theorem. If T could be extended to a $\tilde{T} \in \Pi_1(\ell_\infty, \ell_2)$, then $\tilde{T}$ and hence also $\tilde{T}^*$ would be nuclear and therefore compact. Since $\tilde{T}^*$ is an isomorphism this is a contradiction.

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