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SHARP UPPER BOUNDS FOR THE BETTI NUMBERS OF COHEN-MACAULAY MODULES

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Introduction

At the end of the last century Hilbert proved his celebrated theorem of syzygies for homogeneous ideals of polynomial rings, [Hil90]. This result has been extended and generalized to several different settings. For instance, Serre characterized Noetherian regular local rings as Noetherian local rings with finite global dimension.

This work is concerned with the Betti numbers of finitely generated modules M over Noetherian regular local rings and polynomial rings. In this case Hilbert's and Serre's results imply that the projective dimension of M is finite. Since for such modules projective implies free, a natural problem to consider is to compute the range of Betti numbers, in particular to find sharp lower and upper bounds for the Betti numbers.

For the lower bounds there is the conjecture of Buchsbaum-Eisenbud [BE77]. This conjecture predicts that the i-th Betti number of a finite length module M over a regular local ring R of dimension d is at least $\binom{d}{i}$. This bound has been proved in several cases; see [ChE92] for a reference list of the known results.

On the other hand the problem of finding sharp upper bounds for the Betti numbers has been extensively studied. Macaulay gave examples of height two prime ideals with arbitrarily large number of generators, i.e. first Betti number [Mac27]. Many upper bounds for the number of generators can be found in the literature; see for example [Sal78], [ERV91].

Robbiano, Valla and the author of this paper gave a sharp upper bound for the number of generators of a perfect ideal in terms of its initial degree and multiplicity [ERV91]. The key point of the proof is the result where we maximize some combinatorial functions defined in the set of allowed Hilbert functions. We will refer to this result as the Key Lemma; see Theorem 3.5. Using similar techniques we gave in [EGV94] a sharp upper bound for the last Betti number of a perfect ideal, i.e. Cohen-Macaulay type. As a corollary we obtained sharp upper bounds for all Betti numbers of height three perfect ideals.

Bigatti and Hulett, using the combinatorics of lex-segment ideals, found in the characteristic zero ground field case sharp upper bounds for the Betti numbers of

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perfect ideals with given Hilbert function [Big93], [Hul93]. Hulett extended the result to modules with fixed Hilbert function [HulPh]. Recently Pardue extended to any ground field the result of Bigatti-Hulett [Par94]. From the result of Bigatti and Hulett, the computation of minimal free resolutions for stable ideals [EK90], and the Key Lemma, Valla gave, in the characteristic zero case, sharp upper bounds for the Betti numbers of perfect ideals in terms of their initial degree and multiplicity [Val94]. In [Eli96], we dealt with the problem of finding sharp upper bounds for the Betti numbers of modules. We gave a sharp upper bound for the first Betti number of Cohen-Macaulay modules in terms of the 0-th Betti number and multiplicity; the proof does not extend to higher Betti numbers.

The basic underlying technique of these results is the study of combinatorics of Hilbert functions, in particular the combinatorics of binomial and sub-binomial transforms. It is well known that these transforms play a central role in combinatorial and computational commutative algebra; see [Sta83]. Recall that the Hilbert functions of standard **k**-algebras were characterized by Macaulay in terms of the binomial transforms of their values [Sta78]. Similar results for several types of **k**-algebras can be found in the bibliography: characterizations of the Hilbert functions of reduced **k**-algebras [GMR83], Cohen-Macaulay, Gorenstein [Sta78], and necessary conditions for the Hilbert functions of integral **k**-algebras [Sta91]. In [Gree88], Green gave an upper bound for the dimension of the quotient of a **k**-vector space generated by forms of the same degree by a linear form in terms of the sub-binomial transform of its dimension. From this result and the characterization of Hilbert functions due to Macaulay, we proved the Key Lemma in [ERV91]; see also [Eli91]. The proof of this lemma is too complicated, so an easy proof of this lemma should clarify the main results of [ERV91], [EGV94], and [Val94].

The main purpose of this paper is to give sharp upper bounds for Betti numbers of finitely generated Cohen-Macaulay modules. We compute explicitly upper bounds for all Betti numbers of finitely generated Cohen-Macaulay modules over Noetherian regular local rings (resp. finitely generated graded Cohen-Macaulay modules over polynomial rings) in terms of the 0-th Betti number and multiplicity. The residue field in the local case or the coefficient field in the polynomial case is assumed to be general. We prove that the upper bounds are sharp for modules over polynomial rings or regular Noetherian local rings. This result improves the sharp statement of [Val94] for ideals.

The second purpose of this paper is to obtain a good knowledge of the behavior of the binomial and sub-binomial transforms. The combinatorial techniques developed in this paper have importance by themselves; the main result is Proposition 2.5. From this result we get the sharp statement of this paper, a short an easy proof of the Key Lemma, and some other interesting results about the behavior of the binomial and sub-binomial transforms, see Corollay 2.6. Some of these results were proved in the three codimensional case in [Eli91].

The contents of this paper are the following. The aim of the second section is to study and to develop combinatorial techniques in order to handle binomial and sub-binomial transforms. In the third section we apply the techniques developed in section two to Hilbert functions, we prove the Key Lemma as a corollary and the generalization to any ground field of the main result of [Val94]. In the fourth section we study the combinatorics of the Betti numbers from the point of view of lex-segment ideals. For each $i = 1, \dots, h$ we define a numerical function $\Phi_i(h, b, e)$ that will be the upper bound for the i-th Betti number of modules of codimension h, 0-th Betti number b, and multiplicity e. In the main result of this section we prove that there exists a split Cohen-Macaulay module $M^g_{(h,b,e)}$ with maximal Betti numbers $\beta_i(M^g_{(h,b,e)}) = \Phi_i(h, b, e)$, $i = 0, \dots, h$, i.e. the bounds of the Betti numbers are achieved simultaneously. In section four we prove the main result of this paper, Theorem 4.6: for all $i = 1, \dots, h$ the integer $\Phi_i(h, b, e)$ is a sharp upper bound for the i-th Betti number of codimension h modules with 0-th Betti number b and multiplicity e. We end the paper with some explicit examples and values of the bounds.

1. Notations

Throughout this paper (R, m) will be a Noetherian regular local ring of dimension d with residue field $\mathbf{k}$. We will denote by S the associated graded ring of R, i.e. the polynomial ring $S = \mathbf{k}[X_1, \dots, X_d]$; S_n is the $\mathbf{k}$-vector space of degree n forms.

Let M be a finitely generated R-module, we denote by H_M the Hilbert function of M, and by $e_0(M)$ the multiplicity of M. Let $dp_R(M)$ be the projective dimension of M, and $\beta_i(M)$ will be the i-th Betti number of M, $i = 0, \dots, dp_R(M)$. The codimension of M is the integer $h(M) = \dim(R) - \dim(M)$. Notice that the 0-th Betti number is the minimal number of generators of M. As usual we write $\beta_0(M) = v(M)$. We will use the corresponding notations for graded modules over the polynomial ring S.

We will denote by ω_i the element of S^b with i-th coordinate 1 and 0 otherwise, $i = 1, \dots, b$. A monomial of S^b is an element of the form $a\omega_i$ where a is a monomial of S. Let $<$ be the lex ordering of S, $X_1 > \cdots > X_d$. We can extend this ordering to S^b in the following way: we assume that $\omega_1 < \omega_2 < \cdots < \omega_b$. Given two monomials $a_i\omega_i$, $a_j\omega_j$ of S^b, then $a_i\omega_i > a_j\omega_j$ if and only if $i > j$ or $i = j$ and $a_i > a_j$ with respect to the lex ordering of S. Let F be a submodule of S^b; we say that F is a monomial submodule of S^b if F is generated by monomials. We denote by F_t the piece of degree t of F. We say that F is a lex submodule if F is a monomial submodule and for all t, F_t is the vector space generated by the first $H_F(t)$ monomials with respect to the lex ordering. A lex-vector space $W \subset S_t$ is a $\mathbf{k}$-vector subspace of S_t generated by the first $\dim_{\mathbf{k}}(W)$ monomials with respect the lex ordering. If L is a lex-segment vector space (resp. monomial ideal), we will denote by $\kappa(L)$ the (unique) $\mathbf{k}$-basis (resp. generating system) of L formed by monomials.

2. Combinatorial behavior of the binomial and sub-binomial transforms

Let m, t be positive integers. Let us consider the t-binomial expansion of m:

$$m = \binom{a_t}{t} + \binom{a_{t-1}}{t-1} + \cdots + \binom{a_{j(m)}}{j(m)}$$

where $a_t > a_{t-1} > \cdots > a_{j(m)} \geq j(m) \geq 1$. We define the (t, u)-sub-binomial transform of m by

$$m_{\langle t\rangle(u)} = \binom{a_t - u}{t} + \binom{a_{t-1} - u}{t-1} + \cdots + \binom{a_{j(m)} - u}{j(m)}.$$

We put $0_{\langle t\rangle(u)} = 0$, and $m_{\langle t\rangle(1)} = m_{\langle t\rangle}$. Notice that $m_{\langle t\rangle(0)} = m$. We define the t-binomial transform of m by

$$m^{\langle t\rangle} = \binom{a_t + 1}{t+1} + \binom{a_{t-1} + 1}{t} + \cdots + \binom{a_{j(m)} + 1}{j(m)+1}.$$

For a positive integer t we put $q(t, h) = \binom{t+h-1}{h-1}$, i.e., the dimension of the vector space of forms of degree t in h indeterminates; if $t < 0$ we put $q(t, h) = 0$. We write $p(t, h, s) = q(t, h) - q(t - s, h)$ for all $s \geq 0$; notice that $p(t, h, s) = q(t, h)$ for all $s \geq t + 1$.

The main results about the binomial and sub-binomial transform are due to Macaulay and Green [Sta78], [Gree88]:

PROPOSITION 2.1. *Let W be a vector subspace of S_t. If $\mathbf{k}$ is an infinite field then for a general linear form L then:*

(i) **Macaulay:** $\operatorname{Codim}(S_1 W) \leq \operatorname{Codim}(W)^{\langle t\rangle}$.
(ii) **Green:** $\operatorname{Codim}(W + (L)/(L)) \leq \operatorname{Codim}(W)_{\langle t\rangle}$.

Assume that $\mathbf{k}$ is a general field. Then these bounds are achieved for any lex-segment vector space.

The link between the binomial and sub-binomial transforms is the following equality [ERV91]: given integers $t \geq 1, 0 \leq m \leq q(t, h)$, we have

$$m^{\langle t\rangle} = \sum_{i=0}^{h-1} m_{\langle t\rangle(i)}. \tag{1}$$

Let W be a lex-segment vector space of S_t of codimension m, where $S = \mathbf{k}[X_1, \ldots, X_h]$. Recall that for a lex-segment vector space, X_h is a generic form in the sense of Proposition 2.1, and that $W + (X_h)/(X_h)$ is a lex-segment vector space of $\mathbf{k}[X_1, \ldots, X_{h-1}]$; see [ERV91]. If we denote by an overbar the pass to the quotient by $X_{h-u+1}, \ldots, X_h$, from Proposition 2.1 we get

$$\dim_{\mathbf{k}}(\overline{S_t}/\overline{W}) = m_{\langle t\rangle(u)}. \tag{2}$$

In the following result we will give recursive formulae for the sub-binomial and binomial transforms:

PROPOSITION 2.2. *Let t, u, m be integers, $q(t, h-1) \leq m \leq q(t, h) - 1$, $1 \leq u \leq h$. Then*:

(i) $(m+1)_{\langle t\rangle(u)} - m_{\langle t\rangle(u)} = (m+1-q(t,h-1))_{\langle t-1\rangle(u)} - (m-q(t,h-1))_{\langle t-1\rangle(u)}$,
(ii) $(m+1)^{\langle t\rangle} - m^{\langle t\rangle} = (m+1-q(t,h-1))^{\langle t-1\rangle} - (m-q(t,h-1))^{\langle t-1\rangle}$.

Proof. (i) Let $W \subset S_t$ be the lex-segment vector space such that $\dim_{\mathbf{k}}(S_t/W) = m$, and let $W^+ \subset S_t$ be the lex-segment vector space such that $\dim_{\mathbf{k}}(S_t/W^+) = m+1$. From the equality (2) we deduce

$$\begin{aligned}(m+1)_{\langle t\rangle(u)} - m_{\langle t\rangle(u)} &= \dim_{\mathbf{k}}(\overline{S_t}/\overline{W^+}) - \dim_{\mathbf{k}}(\overline{S_t}/\overline{W}) \\ &= \mathrm{Card}(\kappa(\overline{W}) \setminus \kappa(\overline{W^+})).\end{aligned}$$

Since $m \geq q(t, h-1)$ is the number of monomials without X_1, we have $W^+ \subset W \subset X_1 S_{t-1}$ and hence $\overline{W^+} \subset \overline{W} \subset X_1 \overline{S}_{t-1}$. Let $L \subset X_1 S$ be a lex-segment vector space; we will denote by L/X_1 the lex-segment vector space defined by the monomials X^K/X_1, $X^K \in L$. Notice that $\kappa(\overline{W}/X_1) = \kappa(\overline{W})/X_1$, so dividing the elements of $\kappa(\overline{W})$ and $\kappa(\overline{W^+})$ by X_1, we have

$$\begin{aligned}(m+1)_{\langle t\rangle(u)} - m_{\langle t\rangle(u)} &= \mathrm{Card}(\kappa(\overline{W}/X_1) \setminus \kappa(\overline{W^+}/X_1)) \\ &= \dim_{\mathbf{k}}(\overline{S_{t-1}}/(\overline{W^+}/X_1)) - \dim_{\mathbf{k}}(\overline{S_{t-1}}/(\overline{W}/X_1)) \\ &= \dim_{\mathbf{k}}(\overline{S_{t-1}}/\overline{W^+/X_1}) - \dim_{\mathbf{k}}(\overline{S_{t-1}}/\overline{W/X_1}).\end{aligned}$$

Since W/X_1 and W^+/X_1 are lex-segment subspaces of S_{t-1}, from the equality (2) we obtain (i).

(ii) The proof is similar to the previous one. □

COROLLARY 2.3. *Let r, t, u, a, b be integers such that $1 \leq r \leq t$, $1 \leq u \leq h$, and $p(t, h, r) \leq a \leq b \leq q(t, h)$. Then for all $k = 0, \ldots, r$ we have*

$$b_{\langle t\rangle(u)} - a_{\langle t\rangle(u)} = (b - p(t,h,k))_{\langle t-k\rangle(u)} - (a - p(t,h,k))_{\langle t-k\rangle(u)}.$$

Proof. First of all notice that

$$p(t, h, k) = q(t, h-1) + q(t-1, h-1) + \cdots + q(t-k+1, h-1),$$

for all $k = 0, \dots, r$. Repeatedly applying Proposition 2.2 (i) we get for all $p(t,h,r) \le m \le q(t,h) - 1$,

$$(m+1)_{\langle t\rangle(u)} - m_{\langle t\rangle(u)} = (m+1-p(t,h,k))_{\langle t-k\rangle(u)} - (m-p(t,h,k))_{\langle t-k\rangle(u)}.$$

Hence

$$\begin{aligned} b_{\langle t\rangle(u)} - a_{\langle t\rangle(u)} &= \sum_{m=a}^{b-1}((m+1)_{\langle t\rangle(u)} - m_{\langle t\rangle(u)}) \\ &= (b-p(t,h,k))_{\langle t-k\rangle(u)} - (a-p(t,h,k))_{\langle t-k\rangle(u)}. \end{aligned}$$

□

We will denote the first derivative of the sub-binomial transform by

$$D(m,t,u) = (m+1)_{\langle t\rangle(u)} - m_{\langle t\rangle(u)},$$

and the first derivative of the binomial transform by

$$\Delta(m,t) = (m+1)^{\langle t\rangle} - m^{\langle t\rangle}.$$

Given non-negative integers $u \le h, t$, let $D_*(h,t,u)$ be the list with $D(m,t,u)$ as m-component, $m = 0, \dots, q(t,h) - 1$. We write $D_*(h,t,u)_m = D(m,t,u)$, $D_*(h,0,u) = \{1\}$. In a similar way we define the list $\Delta^*(t,h)$ with $\Delta(m,t)$ as m-component, $m = 0, \dots, q(t,h) - 1$. It is worth to notice that from Proposition 2.2 we get:

(i) $D(m,t,u) = D(m - q(t,h-1), t-1, u)$,
(ii) $\Delta(m,t) = \Delta(m - q(t,h-1), t-1)$.

Given lists $\mathcal{L}_1 = \{a_1, \dots, a_m\}$, $\mathcal{L}_2 = \{b_1, \dots, b_n\}$, we will denote by $\mathcal{L}_1 \sqcup \mathcal{L}_2$ the joint list $\{a_1, \dots, a_m, b_1, \dots, b_n\}$.

PROPOSITION 2.4. *Given integers $h \ge 1$, $t \ge 1$, $1 \le u \le h$ we have:*

(i) $D_*(h,t,u) = D_*(h-1,t,u) \sqcup D_*(h,t-1,u)$.
(ii) $D_*(h,t,u) = D_*(h-1,t,u) \sqcup D_*(h-1,t-1,u) \sqcup \dots \sqcup D_*(h-1,0,u)$,.
(iii) $D_*(h,1,u) = \{0,0,\dots,\overbrace{1,\dots,1}^{h-u}\}$ *if* $h = u$ *then* $D_*(h,t,h) = \{0,0,\dots,0\}$.
(iv) $\Delta^*(n,h) = \Delta^*(n,h-1) \sqcup \Delta^*(n-1,h)$.
(v) $\Delta^*(n,h) = \Delta^*(n,h-1) \sqcup \Delta^*(n-1,h-1) \sqcup \dots \sqcup \Delta^*(1,h-1) \sqcup \{h\}$.
(vi) $\Delta^*(n,2) = \{1,1,\dots,1,2\}$.

Proof. If $m < q(t,h-1)$ then

$$D_*(h,t,u)_m = D_*(h-1,t,u)_m.$$

by the definition of D_*. For $m \geq q(t, h-1)$, by Proposition 2.2 we have

$$D_*(h, t, u)_m = D_*(h, t-1, u)_{m-q(t,h-1)}.$$

hence we get the first equality. From (i) we deduce the second one. The third equality follows from a straightforward computation. The results for Δ^* are proved in a similar way. □

Up to now the behavior of the sub-binomial and binomial transforms seems to be symmetric; the different values of the first derivatives breaks this symmetry. For this we will first give results about the sub-binomial transforms. From these results and the equality (1) we will deduce some properties of binomial transforms.

In the following result we will give the basic properties on the first derivatives of the sub-binomial transform; this is the key result for further developments of this paper. We will apply these results to the values of Hilbert functions, and study the derivatives of $(m, t, u) \longrightarrow m_{\langle t\rangle(u)}$, where $m = H_{S/I}(t)$ and $1 \leq u \leq h$. For this reason we will study these derivatives under the assumptions $0 \leq m \leq q(t, h)$, and $1 \leq u \leq h$.

PROPOSITION 2.5. *Let u be an integer such that $1 \leq u \leq h$.*

(P1) *Given integers t, m such that $0 \leq m \leq q(t, h)$, we have*

$$m_{\langle t+1\rangle(u)} \leq m_{\langle t\rangle(u)}.$$

(P2) *Given integers t, m, a such that $0 \leq m + a \leq q(t, h)$, $0 \leq m, a$, we have*

$$m_{\langle t\rangle(u)} \leq (m+a)_{\langle t\rangle(u)} - a_{\langle t\rangle(u)}.$$

(P3) *Given integers t, m, s, a such that $0 \leq m, a$, and $0 \leq m + a \leq p(t, h, s)$, we have*

$$(m+a)_{\langle t\rangle(u)} - a_{\langle t\rangle(u)} \leq p(t, h, s)_{\langle t\rangle(u)} - (p(t, h, s) - m)_{\langle t\rangle(u)}.$$

(P4) *Given integers t_1, t_2, m such that $t_1 \leq t_2$, $0 \leq m \leq q(t_1, h)$, we have*

$$q(t_1, h)_{\langle t_1\rangle(u)} - (q(t_1, h) - m)_{\langle t_1\rangle(u)} = q(t_2, h)_{\langle t_2\rangle(u)} - (q(t_2, h) - m)_{\langle t_2\rangle(u)}.$$

(P5) *Given integers t_1, t_2, m, s such that $t_1 \leq t_2$, $0 \leq m \leq p(t_1, h, s)$, we have*

$$\begin{aligned} &p(t_2, h, s)_{\langle t_2\rangle(u)} - (p(t_2, h, s) - m)_{\langle t_2\rangle(u)} \\ &\quad \leq p(t_1, h, s)_{\langle t_1\rangle(u)} - (p(t_1, h, s) - m)_{\langle t_1\rangle(u)} \end{aligned}$$

Proof. The equality (P4) follows from Corollary 2.3. Given an integer u, $1 \leq u \leq h$, we will prove by induction on the triplets h, m, t the inequalities (P1), (P2) and (P3) for all a. We will consider in that set of integers the lexicographic ordering, i.e., $\{h, m, t\} > \{h', m', t'\}$ if and only if the first non-zero integer from the left of $\{h - h', m - m', t - t'\}$ is positive.

If $h = u$ then all sub-binomial transforms appearing in (P1), (P2), and (P3) are zero, so we get the inequalities (P1), (P2), and (P3). Hence we may assume $u \leq h-1$.

Assume that (P1), (P2), and (P3) hold for all triplets $\{h', m', t'\} < \{h, m, t\}$; we will prove the inequalities for $\{h, m, t\}$.

First we prove (P1). Assume that $m > p(t+1, h, 1)$. Then

$$\begin{aligned}
m_{\langle t\rangle(u)} &= p(t+1,h,1)_{\langle t\rangle(u)} + m_{\langle t\rangle(u)} - p(t+1,h,1)_{\langle t\rangle(u)}, \\
&\geq p(t+1,h,1)_{\langle t\rangle(u)} + (m - p(t+1,h,1))_{\langle t\rangle(u)} \\
&\qquad \text{by (P2) and induction on } m, \text{ since } 0 \leq m - p(t+1,h,1) < m \\
&\geq p(t+1,h,1)_{\langle t+1\rangle(u)} + (m - p(t+1,h,1))_{\langle t\rangle(u)} \\
&\qquad \text{by (P1) and induction on } m, \text{ since } m > p(t+1,h,1) \\
&= p(t+1,h,1)_{\langle t+1\rangle(u)} + m_{\langle t+1\rangle(u)} - p(t+1,h,1)_{\langle t+1\rangle(u)} \\
&\qquad \text{by Corollary 2.3 applied to the parameters } 1, t+1, u, p(t+1,h,1), m \\
&= m_{\langle t+1\rangle(u)}
\end{aligned}$$

Assume now that $m \leq l = p(t+1, h, 1)$. Let $k \geq 1$ be the biggest integer such that $p(t+1, h, k) \leq m + l$. Notice that for fixed t and h, $p(t, h, s)$ is an increasing function of s. If $k = 1$ then we have $m \leq p(t+1, h, 2) - p(t+1, h, 1) = q(t, h-1)$; by induction on h we get (P1). Let us assume that $k \geq 2$, then we have

$$\begin{aligned}
m_{\langle t\rangle(u)} &= (m+l)_{\langle t+1\rangle(u)} - l_{\langle t+1\rangle(u)} \\
&\qquad \text{by Corollary 2.3 applied to } 1, t+1, u, l, m+l \\
&= (m+l)_{\langle t+1\rangle(u)} - p(t+1,h,k)_{\langle t+1\rangle(u)} + p(t+1,h,k)_{\langle t+1\rangle(u)} - l_{\langle t+1\rangle(u)} \\
&\geq (m+l-p(t+1,h,k))_{\langle t+1\rangle(u)} + p(t+1,h,k)_{\langle t+1\rangle(u)} - l_{\langle t+1\rangle(u)} \\
&\qquad \text{by (P2) applied to } t+1, m+l-p(t+1,h,k), p(t+1,h,k) \\
&\qquad \text{and induction on } m \text{ since } m+l-p(t+1,h,k) < m.
\end{aligned}$$

If $p(t+1, h, k) < m + l$ then by (P3) applied to $t+1, p(t+1, h, k) - l, k, m + l - p(t+1, h, k)$, and induction on m we get (P1) from the last inequality. Assume that $p(t+1, h, k) = m + l$. If $k \geq 3$ then we have

$$\begin{aligned}
m_{\langle t\rangle(u)} &\geq p(t+1,h,k)_{\langle t+1\rangle(u)} - p(t+1,h,k-1)_{\langle t+1\rangle(u)} \\
&\quad + p(t+1,h,k-1)_{\langle t+1\rangle(u)} - l_{\langle t+1\rangle(u)} \\
&\geq (p(t+1,h,k) - p(t+1,h,k-1))_{\langle t+1\rangle(u)} \\
&\quad + p(t+1,h,k-1)_{\langle t+1\rangle(u)} - l_{\langle t+1\rangle(u)} \\
&\qquad \text{by (P2) and induction on } m, \\
&\qquad \text{since } p(t+1,h,k) - p(t+1,h,k-1) < m
\end{aligned}$$

$$\begin{aligned} &\geq m_{\langle t+1\rangle(u)} \\ &\qquad \text{by (P3) applied to } t+1,\ p(t+1,h,k-1)-1,k-1, \\ &\qquad p(t+1,h,k)-p(t+1,h,k-1) \text{ and induction on } m, \\ &\qquad \text{since } 0 \leq p(t+1,h,k-1)-l < m \end{aligned}$$

Let us assume $k = 2$. Then we have $m = p(t+1, h, 2) - p(t+1, h, 1) = q(t, h-1)$, so by induction on h we get (P1).

Now we prove (P2). If $p(t, h, 1) \leq a$ then

$$\begin{aligned} (m+a)_{\langle t\rangle(u)} - a_{\langle t\rangle(u)} &= (m+a-p(t,h,k))_{\langle t-k\rangle(u)} - (a-p(t,h,k))_{\langle t-k\rangle(u)} \\ &\qquad \text{by Corollary 2.3} \\ &\geq m_{\langle t-k\rangle(u)} \\ &\qquad \text{by (P2) and induction on } t \\ &\geq m_{\langle t\rangle(u)} \\ &\qquad \text{by (P1) and induction on } t. \end{aligned}$$

Let us assume that $a < p(t, h, 1)$. If $m + a \leq p(t, h, 1) = q(t, h-1)$ then by induction on h we get (P2).

Let us assume $m + a > p(t, h, 1)$. We put $R = m + a - p(t, h, 1)$, notice that $m > R > 0$. If $R \leq a$ then

$$\begin{aligned} (m+a)_{\langle t\rangle(u)} - a_{\langle t\rangle(u)} &= (m+a)_{\langle t\rangle(u)} - p(t,h,1)_{\langle t\rangle(u)} \\ &\quad + p(t,h,1)_{\langle t\rangle(u)} - a_{\langle t\rangle(u)} \\ &\geq R_{\langle t\rangle(u)} + p(t,h,1)_{\langle t\rangle(u)} - a_{\langle t\rangle(u)} \\ &\qquad \text{by (P2) applied to } t,\ R,\ p(t,h,1) \\ &\qquad \text{and induction on } m \text{ since } R < m \\ &\geq m_{\langle t\rangle(u)} \\ &\qquad \text{by (P3) applied to } t,\ p(t,h,1)-a,\ 1, m+a-p(t,h,1) \\ &\qquad \text{and induction on } m \text{ since } p(t,h,1)-a < m \end{aligned}$$

Assume $R > a$, then we have $m > p(t, h, 1) > a$ and

$$\begin{aligned} (m+a)_{\langle t\rangle(u)} - a_{\langle t\rangle(u)} &= (m+a)_{\langle t\rangle(u)} - m_{\langle t\rangle(u)} + m_{\langle t\rangle(u)} - a_{\langle t\rangle(u)} \\ &\geq a_{\langle t\rangle(u)} + m_{\langle t\rangle(u)} - a_{\langle t\rangle(u)} \\ &\qquad \text{by (P2) and induction on } m, \text{ since } a < m \\ &= m_{\langle t\rangle(u)}. \end{aligned}$$

We will prove (P3). If $s = 1$ then $p(t, h, s) = p(t, h-1, t+1)$, and by induction on h we get (P3), so we can assume $s \geq 2$. Let $r \geq 1$ be the least integer such that $p(t,h,r) \geq m+a$, notice that $r \leq s$. The first step is devoted to prove that we only need to consider the case $r = s$. Assume that $r < s$, and $m > p(t,h,s) - p(t,h,s-1)$. Then we have

$$\begin{aligned}
&p(t,h,s)_{\langle t\rangle(u)} - (p(t,h,s)-m)_{\langle t\rangle(u)} \\
&\quad = p(t,h,s)_{\langle t\rangle(u)} - p(t,h,s-1)_{\langle t\rangle(u)} \\
&\qquad + p(t,h,s-1)_{\langle t\rangle(u)} - (p(t,h,s)-m)_{\langle t\rangle(u)} \\
&\quad \geq (p(t,h,s)-m)_{\langle t\rangle(u)} - (p(t,h,s-1)-m)_{\langle t\rangle(u)} \\
&\qquad + p(t,h,s-1)_{\langle t\rangle(u)} - (p(t,h,s)-m)_{\langle t\rangle(u)} \\
&\qquad\quad \text{by (P3) applied to } t,\ p(t,h,s)-p(t,h,s-1),\ s,\ p(t,h,s-1)-m \\
&\qquad\quad \text{and induction on } m, \text{ since } p(t,h,s)-p(t,h,s-1) < m \\
&\quad = p(t,h,s-1)_{\langle t\rangle(u)} - (p(t,h,s-1)-m)_{\langle t\rangle(u)}.
\end{aligned}$$

We repeat this process until we obtain either

$$p(t,h,s)_{\langle t\rangle(u)} - (p(t,h,s)-m)_{\langle t\rangle(u)} \geq p(t,h,r)_{\langle t\rangle(u)} - (p(t,h,r)-m)_{\langle t\rangle(u)},$$

or there exists ω, $r+1 \leq \omega \leq s$, such that $m \leq p(t,h,\omega) - p(t,h,\omega-1)$. In this case we have

$$\begin{aligned}
&p(t,h,\omega)_{\langle t\rangle(u)} - (p(t,h,\omega)-m)_{\langle t\rangle(u)} \\
&\quad = q(t-\omega+1,h-1)_{\langle t-\omega+1\rangle(u)} - (q(t-\omega+1,h-1)-m)_{\langle t-\omega+1\rangle(u)} \\
&\qquad\quad \text{by Corollary 2.3 applied to } \omega-1,\ t,\ u,\ p(t,h,\omega)-m,\ p(t,h,\omega) \\
&\quad = q(t-r+1,h-1)_{\langle t-r+1\rangle(u)} - (q(t-r+1,h-1)-m)_{\langle t-r+1\rangle(u)} \\
&\qquad\quad \text{by (P4) applied to } t-\omega+1,\ t-r+1,\ m \\
&\quad = p(t,h,r)_{\langle t\rangle(u)} - (p(t,h,r)-m)_{\langle t\rangle(u)} \\
&\qquad\quad \text{by Corollary 2.3 applied to } r-1,\ t,\ u,\ p(t,h,r-m),\ p(t,h,r).
\end{aligned}$$

Hence we may assume that $r = s$, and then $p(t,h,s-1) < m+a \leq p(t,h,s)$. In order to establish (P3) we need to consider two cases

(I) Assume that $a \geq p(t,h,s-1)$. Then we have

$$\begin{aligned}
&p(t,h,s)_{\langle t\rangle(u)} - (p(t,h,s)-m)_{\langle t\rangle(u)} \\
&\quad = q(t-s+1,h-1)_{\langle t-s+1\rangle(u)} - (q(t-s+1,h-1)-m)_{\langle t-s+1\rangle(u)} \\
&\qquad\quad \text{by Corollary 2.3} \\
&\quad \geq (m+a-p(t,h,s-1))_{\langle t-s+1\rangle(u)} - (a-p(t,h,s-1))_{\langle t-s+1\rangle(u)} \\
&\qquad\quad \text{by (P3) applied to } t-s+1,\ m,\ 1,\ a-p(t,h,s-1) \\
&\qquad\quad \text{and induction on } t, \text{ since } t-s+1 < t \\
&\quad = (m+a)_{\langle t\rangle(u)} - a_{\langle t\rangle(u)} \\
&\qquad\quad \text{by Corollary 2.3.}
\end{aligned}$$

(II) Assume that $a < p(t, h, s-1)$. Let $R = p(t, h, s) - (m+a) \geq 0$. If $R = 0$ then we trivially get (P3). Hence we may assume $R > 0$. If $R < m$ then we have

$$\begin{aligned}
&p(t, h, s)_{\langle t\rangle(u)} - (p(t, h, s) - m)_{\langle t\rangle(u)} \\
&\quad = p(t, h, s)_{\langle t\rangle(u)} - (m+a)_{\langle t\rangle(u)} \\
&\qquad + (m+a)_{\langle t\rangle(u)} - (p(t, h, s) - m)_{\langle t\rangle(u)} \\
&\quad \geq (p(t, h, s) - m)_{\langle t\rangle(u)} - a_{\langle t\rangle(u)} \\
&\qquad + (m+a)_{\langle t\rangle(u)} - (p(t, h, s) - m)_{\langle t\rangle(u)} \\
&\qquad\ \text{by (P3) applied to } t,\ R,\ s,\ a, \text{ and induction on } m, \text{ since } R < m \\
&\quad = (m+a)_{\langle t\rangle(u)} - a_{\langle t\rangle(u)}
\end{aligned}$$

Assume that $R \geq m$. In this case we have $m \leq R \leq p(t, h, s) - p(t, h, s-1)$. Let $0 < w = m + a - p(t, h, s-1) < m$; then we have

$$\begin{aligned}
&p(t, h, s)_{\langle t\rangle(u)} - (p(t, h, s) - m)_{\langle t\rangle(u)} \\
&\quad = p(t, h, s)_{\langle t\rangle(u)} - (p(t, h, s) - (m - w))_{\langle t\rangle(u)} \\
&\qquad + (p(t, h, s) - (m - w))_{\langle t\rangle(u)} - (p(t, h, s) - m)_{\langle t\rangle(u)} \\
&\quad \geq p(t, h, s-1)_{\langle t\rangle(u)} - a_{\langle t\rangle(u)} \\
&\qquad + (p(t, h, s) - (m - w))_{\langle t\rangle(u)} - (p(t, h, s) - m)_{\langle t\rangle(u)} \\
&\qquad\ \text{by (P3) applied to } t, m - \omega, s, a \text{ and induction on } m, \text{ since } m - w < m \\
&\quad = p(t, h, s-1)_{\langle t\rangle(u)} - a_{\langle t\rangle(u)} \\
&\qquad + (q(t - s + 1, h - 1) - (m - w))_{\langle t-s+1\rangle(u)} \\
&\qquad - (q(t - s + 1, h - 1) - m)_{\langle t-s+1\rangle(u)} \\
&\qquad\ \text{by Corollary 2.3} \\
&\quad \geq p(t, h, s-1)_{\langle t\rangle(u)} - a_{\langle t\rangle(u)} + w_{\langle t-s+1\rangle(u)} \\
&\qquad\ \text{by (P2) applied to } t - s + 1, \omega, q(t - s + 1, h - 1) - m \\
&\qquad\ \text{and induction on } m, \text{ since } w < m \\
&\quad = p(t, h, s-1)_{\langle t\rangle(u)} - a_{\langle t\rangle(u)} + (w + p(t, h, s-1))_{\langle t\rangle(u)} - p(t, h, s-1)_{\langle t\rangle(u)} \\
&\qquad\ \text{by Corollary 2.3} \\
&\quad = (m+a)_{\langle t\rangle(u)} - a_{\langle t\rangle(u)}.
\end{aligned}$$

Hence (P3) holds.

Let us consider integers t_1, t_2, m, s under the hypothesis of (P5). From Corollary 2.3 we get

$$\begin{aligned}
&p(t_1, h, s)_{\langle t_1\rangle(u)} - (p(t_1, h, s) - m)_{\langle t_1\rangle(u)} \\
&\quad = p(t_2, h, s + t_2 - t_1)_{\langle t_2\rangle(u)} - (p(t_2, h, s + t_2 - t_1) - m)_{\langle t_2\rangle(u)};
\end{aligned}$$

from (P3) applied to $t_2, m, s + t_2 - t_1, p(t_2, h, s) - m$ we get (P5). □

CORLLARY 2.6. *For all positive integers m, t, u, $1 \leq u \leq h$, $m \leq q(t, h)$, we have*

(i) $m_{\langle t\rangle(u)} \geq m_{\langle t+1\rangle(u)}$,
(*ii*) $m^{\langle t\rangle} \geq m^{\langle t+1\rangle}$.

Proof. (i) is the inequality (P1). By (i) and the equality (1) we get (ii):

$$m^{\langle t\rangle} = \sum_{i=0}^{h-1} m_{\langle t\rangle(i)} \geq \sum_{i=0}^{h-1} m_{\langle t+1\rangle(i)} = m^{\langle t+1\rangle}. \qquad \square$$

COROLLARY 2.7. *Let a, b, t_1, t_2 be non-negative integers such that $t_1 \leq t_2$, $1 \leq s \leq t_2 + 1$, $a \leq p(t_1, h, s)$, $b \leq p(t_2, h, s)$, and $1 \leq u \leq h$. Then the following conditions hold*:

(i) *If $a + b \leq p(t_1, h, s)$ then*

$$a_{\langle t_1\rangle(u)} + b_{\langle t_2\rangle(u)} \leq (a+b)_{\langle t_1\rangle(u)}.$$

(ii) *If $a + b > p(t_1, h, s)$ then*

$$a_{\langle t_1\rangle(u)} + b_{\langle t_2\rangle(u)} \leq p(t_1, h, s)_{\langle t_1\rangle(u)} + (a + b - p(t_1, h, s))_{\langle t_2\rangle(u)}.$$

(iii) *If $a + b \leq p(t_1, h, s)$ then*

$$a^{\langle t_1\rangle} + b^{\langle t_2\rangle} \leq (a+b)^{\langle t_1\rangle}.$$

(iv) *If $a + b > p(t_1, h, s)$ then*

$$a^{\langle t_1\rangle} + b^{\langle t_2\rangle} \leq p(t_1, h, s)^{\langle t_1\rangle} + (a + b - p(t_1, h, s))^{\langle t_2\rangle}.$$

Proof. (i)

$$\begin{aligned} a_{\langle t_1\rangle(u)} + b_{\langle t_2\rangle(u)} &\leq a_{\langle t_1\rangle(u)} + b_{\langle t_1\rangle(u)} \\ &\qquad \text{by (P1)} \\ &\leq a_{\langle t_1\rangle(u)} + ((a+b)_{\langle t_1\rangle(u)} - a_{\langle t_1\rangle(u)}) \\ &\qquad \text{by (P2)} \\ &= (a+b)_{\langle t_1\rangle(u)}. \end{aligned}$$

(ii) Assume that $a+b > p(t_1, h, s)$. Let $A = a+b-p(t_1, h, s)$, $B = p(t_1, h, s) - a$, then we have

$$\begin{aligned} a_{\langle t_1\rangle(u)} + b_{\langle t_2\rangle(u)} &= a_{\langle t_1\rangle(u)} + (A+B)_{\langle t_2\rangle(u)} \\ &= a_{\langle t_1\rangle(u)} + A_{\langle t_2\rangle(u)} + ((A+B)_{\langle t_2\rangle(u)} - A_{\langle t_2\rangle(u)}) \\ &\leq a_{\langle t_1\rangle(u)} + (A)_{\langle t_2\rangle(u)} \\ &\quad + p(t_2, h, s)_{\langle t_2\rangle(u)} - (p(t_2, h, s) - (p(t_1, h, s) - a))_{\langle t_2\rangle(u)} \\ &\quad \text{by (P3)} \\ &\leq a_{\langle t_1\rangle(u)} + A_{\langle t_2\rangle(u)} + (p(t_1, h, s)_{\langle t_1\rangle(u)} - a_{\langle t_1\rangle(u)}) \\ &\quad \text{by (P5)} \\ &= p(t_1, h, s)_{\langle t_1\rangle(u)} + (a+b-p(t_1, h, s))_{\langle t_2\rangle(u)}. \end{aligned}$$

(iii) Assume that $a+b \leq p(t_1, h, s)$; then we have

$$\begin{aligned} a^{\langle t_1\rangle} + b^{\langle t_2\rangle} &= \sum_{i=0}^{h-1} (a_{\langle t_1\rangle(u)} + b_{\langle t_2\rangle(u)}) \\ &\leq \sum_{i=0}^{h-1} (a+b)_{\langle t_1\rangle(u)} \\ &\quad \text{by (i)} \\ &= (a+b)^{\langle t_1\rangle}. \end{aligned}$$

(iv) Assume that $a+b > p(t_1, h, s)$; then we have

$$\begin{aligned} a^{\langle t_1\rangle} + b^{\langle t_2\rangle} &= \sum_{i=0}^{h-1} (a_{\langle t_1\rangle(u)} + b_{\langle t_2\rangle(u)}) \\ &\leq \sum_{i=0}^{h-1} (p(t_1, h, s)_{\langle t_1\rangle(u)} + (a+b-p(t_2, h, s))_{\langle t_2\rangle(u)}) \\ &\quad \text{by (ii)} \\ &= p(t_1, h, s)^{\langle t_1\rangle} + (a+b-p(t_2, h, s))^{\langle t_2\rangle}. \end{aligned}$$

□

3. Maximizing combinatorial functions

Let $H\colon \mathbb{N} \longrightarrow \mathbb{N}$ be a numerical function; we say that H is admissible if there exists a graded **k**-algebra $A = S/I$ with Hilbert function $H_A = H$. The following result ([Mac27], [Sta78]) gives a characterization of admissible functions.

PROPOSITION 3.1 (MACAULAY). *Let $H\colon \mathbb{N} \longrightarrow \mathbb{N}$ be a numerical function; H is admissible if and only if $H(i+1) \leq H(i)^{\langle i\rangle}$ for all $i \geq 0$. Moreover, for each numerical function H satisfying the above conditions there exists a lex-segment ideal $I \subset S$ such that $H_{S/I} = H$.*

We will denote by $\mathcal{F}_{h,e,s}$ the set of admissible functions such that

(i) $H(1) = h$,
(ii) $\sum_{i\geq 0} H(i) = e$,
(iii) $s = s(H) = \operatorname{Min}\{n \mid H(n) \neq q(n,h)\}$.

We will denote by $\mathcal{F}_{h,e}$ the set of admissible functions satisfying (i) and (ii). We say that $s(H)$ is the initial degree of H and we define the initial degree of an ideal $I \subset S$ by $s(I) = s(H_{S/I})$. The initial degree of an ideal $I \subset R$ is defined in a similar way.

Remark 3.2. Notice that $\mathcal{F}_{h,e,s}$ (resp. $\mathcal{F}_{h,e}$) is the set of Hilbert functions of graded Artinian **k**-algebras of codimension h, multiplicity e and initial degree s (resp. the set of Hilbert functions of graded Artinian **k**-algebras of codimension h and multiplicity e).

For all u, $1 \leq u \leq h$, let us consider the following function defined on the set of admissible functions:

$$\tau_{\langle\rangle(u)} \colon \mathcal{F}_{h,e} \longrightarrow \mathbb{N},$$

with

$$\tau_{\langle\rangle(u)}(H) := H_{\langle\rangle(u)} = \sum_{t\geq 1} H(t)_{\langle t\rangle(u)}.$$

Given an admissible numerical function H we write

$$H_{\langle\rangle} := \tau_{\langle\rangle(1)}(H) = \sum_{n\geq 1} H(n)_{\langle n\rangle},$$

$$H^{\langle\rangle} := \sum_{u=0}^{h-1} \tau_{\langle\rangle(u)}(H) = \sum_{n\geq 1} H(n)^{\langle n\rangle}.$$

We will denote by $\succeq$ the lexicographic ordering on the set $\mathcal{F}_{h,e}$: given functions $H_1, H_2 \in \mathcal{F}_{h,e}$, $H_1 \neq H_2$, then $H_1 \succeq H_2$ if there exists an integer n_0 such that $H_1(n) = H_2(n)$ for $n = 0, \ldots, n_0$ and $H_1(n_0+1) > H_2(n_0+1)$. It is easy to see that $\tau_{\langle\rangle(u)}$ is not an increasing function on the set $\mathcal{F}_{h,e,s}$ (resp. $\mathcal{F}_{h,e}$) with respect to the lexicographic ordering. Hence in order to maximize $\tau_{\langle\rangle(u)}$ in $\mathcal{F}_{h,e,s}$ and $\mathcal{F}_{h,e}$ we cannot proceed by induction with respect to the lexicographic ordering. We need to jump several admissible functions each step. First we give the candidates to maximize these functions.

Definition 3.3. Given integers h, e, let $t(e)$ be the integer such that $q(t(e)-1, h+1) \leq e < q(t(e), h+1)$. We define the integer $r(e) = e - \sum_{n=0}^{t(e)-1} q(n,h) = e - q(t(e)-1, h+1) \geq 0$ and $r(e) < q(t(e), h)$.

Given integers h, e we consider the numerical function

$$H_{(h,e)}(n) = \begin{cases} q(n,h) & n = 0, \ldots, t(e) - 1 \\ r(e) & n = t(e) \\ 0 & n > t(e). \end{cases}$$

It is easy to see that $H_{(h,e)}$ is an admissible numerical function; i.e., there exists a height h lex-segment ideal $I(e) \subset S = \mathbf{k}[X_1, \ldots, X_h]$ with $H_{S/I(e)} = H_{(h,e)}$. Notice that $H_{(h,e)}$ is maximal with respect to the lexicographic ordering in the set $\mathcal{F}_{h,e}$.

Given integers h, e, s we consider the numerical function

$$H_{(h,e,s)}(n) = \begin{cases} q(n,h) & n = 0, \ldots, s - 1 \\ p(n,h,s) & n = s, \ldots, t - 1 \\ r & n = t \\ 0 & n > t \end{cases}$$

with $r = e - \sum_{n=0}^{s-1} q(n,h) - \sum_{n=s}^{t-1} p(n,h,s) \geq 0$ and $r < p(t,h,s)$. Notice that $H_{(h,e,s)}$ is an admissible numerical function; i.e., there exists a height h lex-segment ideal $I(e,s) \subset S = \mathbf{k}[X_1, \ldots, X_h]$ with $H_{S/I(e,s)} = H_{(h,e,s)}$. Notice that $H_{(h,e,s)}$ is maximal with respect to the lexicographic ordering in the set $\mathcal{F}_{h,e,s}$.

In order to obtain sharp bounds for the Betti numbers in the local case we need to prove:

PROPOSITION 3.4. *Let (R, m) be an h-dimensional regular Noetherian local ring with residue field $\mathbf{k} = A/m$. Then for all $e \geq h + 1$, $s \geq 2$, there exist perfect ideals $J(e) \subset R$, $J(e,s) \subset R$ such that $H_{R/J(e)} = H_{S/I(e)}$ and $\beta_i(R/J(e)) = \beta_i(S/I(e))$ for $i = 1, \ldots, h$; the corresponding results hold for $J(e,s)$.*

Proof. Let $\Lambda \subset \mathbb{N}^h$ be the set of multi-indices appearing in the monomials of $I(e)$. Let U be the monomial ideal of $B = \mathbb{Z}[X_1, \ldots, X_h]$ generated by X^K, $K \in \Lambda$. If $x_1, \ldots, x_h$ is a minimal system of generators of m then we define $J(e)$ as the ideal of R generated by x^K, $K \in \Lambda$.

Let L_* be the free B-resolution of B/U constructed by Eliahou and Kervaire [EK90, Theorem 2.1]. By [EK90], Remark 2 to Proposition 2.6, $L_* \otimes_{\mathbb{Z}} R$ is a minimal resolution of $R/J(e)$ and $\beta_i(R/J(e)) = \operatorname{rank}_{\mathbb{Z}}(L_i)$, $i = 0, \ldots, h$. A similar argument proves that $\beta_i(S/I(e)) = \operatorname{rank}_{\mathbb{Z}}(L_i)$, so $\beta_i(R/J(e)) = \beta_i(S/I(e))$. Since $x_1, \ldots, x_h$ is a system of parameters of R we get $H_{R/J(e)} = H_{S/I(e)}$. For the ideal $J(e,s)$ we proceed in a similar way. □

The following result is the so-called Key Lemma; see [ERV91], Lemma 3.9, 4.1, 4.3.

THEOREM 3.5 (KEY LEMMA). *Given integers h, u, e, s, the function $H_{(h,e)}$ (resp. $H_{(h,e,s)}$) maximizes $H_{()(u)}$ in the set $\mathcal{F}_{h,e}$ (resp. $\mathcal{F}_{h,e,s}$).*

Proof. We prove the result for $\mathcal{F}_{h,e,s}$. Given $H \in \mathcal{F}_{h,e,s}$ we define the discrepancy between H and $H_{(h,e,s)}$ as follows: if $H \neq H_{(h,e,s)}$ then

$$d(H) = \mathrm{Min}\{i \mid H(i) \neq H_{(h,e,s)}(i)\}.$$

Notice that $d = d(H) \geq s$. If $H = H_{(h,e,s)}$ then we define $d = \infty$. Assume that $d < \infty$. Let $j = \mathrm{Max}\{n \mid H(n) \neq 0\}$; we put $a = H(d)$ and $b = H(j)$.

We consider the following two cases.

(I) If $a + b \leq p(d, h, s)$ then we consider the admissible numerical function

$$F(n) = \begin{cases} H(n) & n = 0, \dots, d-1 \\ H(n) + b & n = d \\ H(n) & n = d+1, \dots, j-1 \\ 0 & n \geq j. \end{cases}$$

Notice that $F \succeq H$, and from Corollary 2.7 (i) we obtain $H_{\langle\rangle(u)} \leq F_{\langle\rangle(u)}$.

(II) If $a + b > p(d, h, s)$ then we consider the admissible numerical function

$$F(n) = \begin{cases} H(n) & n = 0, \dots, d-1 \\ p(d, h, s) & n = d \\ H(n) & n = d+1, \dots, j-1 \\ a + b - p(d, h, s) & n = j \\ 0 & n \geq j+1. \end{cases}$$

Notice that $F \succeq H$, and from Corollary 2.7 (ii) we obtain $H_{\langle\rangle(u)} \leq F_{\langle\rangle(u)}$.

Hence we have proved that if $H \neq H_{(h,e,s)}$ then there exists an admissible numerical function $F \succeq H$ in $\mathcal{F}_{h,e,s}$ with $H_{\langle\rangle(u)} < F_{\langle\rangle(u)}$. Since $H_{(h,e,s)}$ is the maximum of $\mathcal{F}_{h,e,s}$ with respect to the lexicographic ordering we deduce the result. □

COROLLARY 3.6. *Let h, e, s be integers.*

(i) *The function $H_{(h,e)}$ maximizes $H_{\langle\rangle}$ and $H^{\langle\rangle}$ in the set $\mathcal{F}_{h,e}$,*
(ii) *The function $H_{(h,e,s)}$ maximizes $H_{\langle\rangle}$ and $H^{\langle\rangle}$ in the set $\mathcal{F}_{h,e,s}$.*

We end this section by generalizing the main result of [Val94] to any ground field; recall that the results of [Val94] are valid in characteristic zero. We prove this result as a corollary of the Key Lemma. First we need to translate the Key Lemma in terms of the combinatorics of lex-segment ideals. The following two statements are implicit in [Val94] and were suggested to the author by G. Valla; the second one is equivalent to the Key Lemma.

Let $m = X_1^{a_1} \dots X_h^{a_h}$ be a monomial of S; we denote $\max(m)$ for the integer $\max\{i \mid a_i > 0\}$. If I is a lex-segment ideal we write $\max_u(I)$ for the cardinal of the set of monomials $m \in \kappa(I)$ such that $\max(m) = u$. Notice that $v(I) = \sum_{u=1}^{h} \max_u(I)$.

PROPOSITION 3.7. *Let I be a height h lex-segment ideal of S. Then*

$$\max_u(I) = \tau_{\langle\rangle(h-u+1)}(H_{S/I})$$

for all $u = 1, \ldots, h$.

Proof. We denote by I_n the ideal $I_n = I + (X_{n+1}, \ldots, X_h)/(X_{n+1}, \ldots, X_h)$ of the polynomial ring $T_n = \mathbf{k}[X_1, \ldots, X_n]$, $n = 1, \ldots, h$. It is easy to prove that

$$\max_u(I) = v(I_u) - v(I_{u-1}).$$

From [ERV91], Theorem 2.9, we deduce

$$\begin{aligned} \max_u(I) &= e_0(T_{u-1}/I_{u-1}) \\ &= \sum_{t\geq 0} H_{T_{u-1}/I_{u-1}}(t) \\ &= \sum_{t\geq 0} H_{S/I}(t)_{\langle t\rangle(h-u+1)} \\ &\qquad \text{by equality (2)} \\ &= \tau_{\langle\rangle(h-u+1)}(H_{S/I}). \end{aligned}$$

□

From this result and Theorem 3.5 it is straightforward to prove:

THEOREM 3.8 (LEX VERSION OF THE KEY LEMMA). *Let e, s be integers.*

(i) *For all height h lex-segment ideals $I \subset S$, $e = e_0(S/I)$, we have*

$$\max_u(I) \leq \max_u(I(e)).$$

(ii) *For all height h lex-segment ideals $I \subset S$, $e = e_0(S/I)$ $s = s(I)$, we have*

$$\max_u(I) \leq \max_u(I(e, s)).$$

Let us recall that Eliahou and Kervaire computed the Betti numbers of stable ideals [EK90]. They proved that if I is an stable ideal, then

$$\beta_i(S/I) = \sum_{m\in\kappa(I)} \binom{\max(m) - 1}{i - 1}.$$

From this and the Lex Version of the Key Lemma we can maximize the Betti numbers on the set of lex-segment ideals. This is the purpose of the next result.

PROPOSITION 3.9. *Let e, s be integers.*

(i) *For all height h lex-segment ideals $I \subset S$, $e = e_0(S/I)$, we have*

$$\beta_i(S/I) \leq \beta_i(S/I(e))$$

for $i = 0, \ldots, h$.

(ii) *For all height h lex-segment ideals $I \subset S$, $e = e_0(S/I)$, $s = s(I)$, we have*

$$\beta_i(S/I) \leq \beta_i(S/I(e, s))$$

for $i = 0, \ldots, h$.

The corresponding results hold for a regular local ring.

The proofs of the upper bounds of Betti numbers of ideals are based on several reductions ([ERV91], [EGV94], [Val94], [Eli96]). We summarize and generalize to modules without any restriction on the ground field these reductions in the Lex Reduction Step below. The allowed transformations conserve the multiplicity, codimension, and the 0-th Betti number and do not decrease the higher Betti numbers.

Lex Reduction Step

1. Let M be a Cohen-Macaulay R-module of codimension $h = \dim(R) - \dim(M)$, multiplicity e and depth s. Without loss of generality we may assume that $\mathbf{k}$ is infinite, so a generic set of elements $\alpha_1, \ldots, \alpha_s \in m \setminus m^2$ form a regular sequence on M. If we denote by $(\ \ )^*$ the tensoring by $R/(\alpha_1, \ldots, \alpha_s)R$ then we have $\dim(R^*) = h$, $\beta_i(M^*) = \beta_i(M)$, $e_0(M) = e_0(M^*)$, and M^* is an Artinian R^*-module. Hence we may assume that M is an Artinian R-module of codimension h.
2. We denote by $gr_R(M)$ the associated graded module of M. From [HRV86] it follows that $N = gr_m(M)$ is an $S = gr_m(R)$-module with $\beta_i(N) \geq \beta_i(M)$ for all $i = 0, \ldots, h$. On the other hand it is well known that $e_0(N) = e_0(M)$ and it is easy to see that $\beta_0 = \beta_0(N) = \beta_0(M)$. Hence we may assume that M is an Artinian graded quotient of S^{β_0} with multiplicity e and codimension h.
3, Given a homogeneous submodule F of S^{β_0} we will denote by $Lt(F)$ the monomial submodule of S^{β_0} generated by the leading terms of the elements of F with respect to the lexicographic ordering of S^{β_0}. Let us consider a minimal resolution of M:

$$0 \longrightarrow S^{\beta_h} \xrightarrow{\varphi_h} S^{\beta_{h-1}} \xrightarrow{\varphi_{h-1}} \cdots \xrightarrow{\varphi_2} S^{\beta_1} \xrightarrow{\varphi_1} S^{\beta_0} \xrightarrow{\varphi_0} M \longrightarrow 0,$$

with $\beta_i = \beta_i(M)$. Notice that $\ker(\varphi_0) \subset S^{\beta_0}$ is a homogeneous submodule of S^{β_0}, so $Lt(\ker(\varphi_0))$ is a direct sum of monomial ideals of S:

$$Lt(\ker(\varphi_0)) \cong \bigoplus_{j=1}^{\beta_0} I_j.$$

By [HRV86], $L = \bigoplus_{j=1}^{\beta_0} S/I_j$ is an Artinian S-module of multiplicity e, $\beta_0(L) = \beta_0(M)$, and $\beta_i(L) \geq \beta_i(M)$.

4. Let J_j be the lex-segment ideal of S with the same Hilbert function as I_j. Note that $\beta_i(L) = \sum_{j=1}^{\beta_0} \beta_i(S/I_j)$, $i = 1, \ldots, h$, and from the Bigatti-Hulett-Pardue result we get

$$\beta_i(S/J_j) \geq \beta_i(S/I_j),$$

$i = 1, \ldots, h$ ([Big93], [Hul93] and [Par94]). Hence $L' = \bigoplus_{j=1}^{\beta_0} S/J_j$, is an Artinian graded S-module of multiplicity e, splitting as a direct sum of quotients of S by lex-segment ideals and satisfies $\beta_0(L') = \beta_0(L)$, $\beta_i(L') \geq \beta_i(L)$, $i = 1, \ldots, h$. We will say that L' is a lex reduction of M.

Notice that $h = h(M) = h(L')$, $e_0(M) = e_0(L')$, $\beta_0(M) = \beta_0(L')$ and $\beta_i(M) \leq \beta_i(L')$ for $i = 1, \ldots, h$.

For graded modules over S we proceed in a similar way. In the case of ideals, i.e., $M = R/I$ (resp. $M = S/I$), we may assume that $s(I) = s(I')$ where $L' = S/I'$ is a lex reduction of M.

Let $I(e)$ be the lex-segment ideal defined at the beginning of this section. Recall that $I(e)$ is the lex-segment ideal of $S = \mathbf{k}[X_1, \ldots, X_h]$ such that $H_{R/I(e)} = H_{(h,e)}$, $e \geq h + 1$. Then we also denote by $I(e)$ the extension of $I(e) \subset S = \mathbf{k}[X_1, \ldots, X_h]$ to the ring $\mathbf{k}[X_1, \ldots, X_d]$, $d \geq h$. Notice that the multiplicity and Betti numbers of $I(e)$ and its extension are the same. We proceed in a similar way with $I(e, s)$.

Let R be a Noetherian regular local ring of dimension d, $d \geq h$, and let $x_1, \ldots, x_d$ be a system of generators of m. Let $\Lambda \subset \mathbb{N}^h$ be the set of multi-indices appearing in the monomials of $I(e)$. We define $J(e)$ as the ideal of R generated by x^K, $K \in \Lambda$. Since the cosets of $x_{h+1}, \ldots, x_d$ in $R/J(e)$ form a regular sequence, from Proposition 3.4 we deduce that the multiplicity and Betti numbers of $I(e)$ and $J(e)$ are the same. For the ideal $J(e, s)$ we proceed in a similar way.

From the Lex version of the Key Lemma and the Lex Reduction Step we get the main result of [Val94] in the graded case. From the Key Lemma and Proposition 3.4 we get the corresponding result for local rings, completing the result of Valla:

THEOREM 3.10. *Let R be a Noetherian regular local ring of dimension d.*

(i) *Let I be a height h perfect ideal of R, $e = e_0(R/I)$. Then for all $i = 1, \ldots, h$,*

$$\beta_i(R/I) \leq \beta_i(S/I(e)).$$

For all integers e, h such that $e \geq h + 1$, there exists a perfect height h ideal $J(e) \subset R$ with $e(R/J(e)) = e$ and $\beta_i(R/J(e)) = \beta_i(S/I(e))$, $i = 0, \ldots, h$.

(ii) *Let I be a height h perfect ideal of R with initial degree s and $e = e_0(R/I)$.*

Then for all $i = 1, \dots, h$,

$$\beta_i(R/I) \leq \beta_i(S/I(e,s)).$$

For all integers e, h, s *such that* $e \geq h+1$, $s \geq 2$, *there exists a perfect height* h *ideal* $J(e,s) \subset R$ *with initial degree* s, $e(R/J(e,s)) = e$ *and* $\beta_i(R/J(e,s)) = \beta_i(S/I(e,s))$, $i = 0, \dots, h$.

The corresponding results hold for graded modules over S.

4. Combinatorics of Betti numbers

The purpose of this section is to define and compute explicitly the upper bounds for the Betti numbers of Cohen-Macaulay modules. We will construct split modules with maximal Betti numbers; i.e., we prove the sharp statement of the main result of this paper.

Recall that Valla [Val94, Theorem 3.10] explicitly computed the upper bounds for the Betti numbers of perfect ideals: given integers h, e, $e \geq h+1$, the upper bound for the i-th Betti number of a perfect height h ideal of multiplicity e is

$$\beta_i(S/I(e)) = \sum_{l=i-1}^{h-1} \binom{l}{i-1} (q(t(e)-1, l+1) + r(e)_{\langle t(e)\rangle(h-l)}),$$

$i = 1, \dots, h$. For the definitions of $r(e)$ and $t(e)$ see Definition 3.3.

Let i be an integer, $1 \leq i \leq h$. We write $\Phi_i(h, b, e)$ for the maximum of

$$\sum_{j=1}^{b} \beta_i(S/I(e_j))$$

with $e_1 + \cdots + e_b = e$ and $e_j \geq h+1$, for $j \in \{1, \dots, b\}$. The next section is devoted to proving that Φ_i is the sharp bound for the i-th Betti number. Notice that Φ_1 is the function Φ of [Eli96].

In the main result of this section we compute $\Phi_i(h, b, e)$ explicitly: we give a set of integers $\varepsilon_1(e), \dots, \varepsilon_b(e)$, $\varepsilon_1(e) + \cdots + \varepsilon_b(e) = e$, $\varepsilon_j(e) \geq h+1$, maximizing $\sum_{j=1}^{b} \beta_i(S/I(e_j))$ for all $i = 1, \dots, h$. Notice that the maximum is achieved for a set of integers independent of i. We define the integers $\varepsilon_j(e)$ as follows: given integers $h, b, e \geq b(h+1)$, we consider

$$j^*(h, e, b) = \mathrm{Max}\{j \mid e \geq bq(j, h+1)\} \geq 1.$$

Notice that if $b = 1$ then $j^*(h, e, 1) = t(e) - 1$. We put $j^* = j^*(h, e, b)$, and we define p and W by

$$e - bq(j^*, h+1) = q(j^*+1, h)p + W,$$

with $W < q(j^*+1, h)$. We define the integers $\varepsilon_j(e)$ by

$$\varepsilon_j(e) = \begin{cases} q(j^*+1, h+1) & i = 1, \dots, p \\ q(j^*, h+1) + W & i = p+1 \\ q(j^*, h+1) & i = p+2, \dots, b. \end{cases}$$

The main result of this section is the following:

THEOREM 4.1. *For all* $i = 1, \dots, h$,

$$\Phi_i(h, b, e) = \sum_{j=1}^{b} \beta_i(S/I(\varepsilon_j(e))).$$

Proof. Given integers h, l, e, we write

$$F(h, l, e) = q(t(e) - 1, l + 1) + r(e)_{\langle t(e)\rangle(h-l)},$$

so $\Phi_i(h, b, e)$ is the maximum of

$$\sum_{l=i-1}^{h-1} \binom{l}{i-1} \sum_{j=1}^{b} F(h, l, e_j),$$

with $e_1 + \cdots + e_b = e$ and $e_j \geq h+1$, for $j \in \{1, \dots, b\}$. Hence in order to prove the result we only need to prove:

Claim. For all $l = 0, \dots, h-1$ the set of integers $\varepsilon_1(e), \dots, \varepsilon_b(e)$ maximizes the function $\sum_{j=1}^{b} F(h, l, e_j)$.

LEMMA 4.2. *Let a, b be integers such that $t(a) \geq t(b) + 1$.*

(i) *Assume $r(a) + r(b) < q(t(b), h)$. Let us consider the transformation $(a, b) \longrightarrow (a - r(a), b + r(a))$. Then $t(a - r(a)) = t(a)$, $t(b + r(a)) = t(b)$, $r(a - r(a)) = 0$, $r(b + r(a)) = r(a) + r(b)$, and*

$$F(h, l, a) + F(h, l, b) \leq F(h, l, a - r(a)) + F(h, l, b + r(a)).$$

(ii) *Assume $r(a) + r(b) = q(t(b), h)$. Let us consider the transformation $(a, b) \longrightarrow (a - r(a), b + r(a))$. Then $t(a - r(a)) = t(a)$, $t(b + r(a)) = t(b) + 1$, $r(a - r(a)) = r(b + r(a)) = 0$, and*

$$F(h, l, a) + F(h, l, b) \leq F(h, l, a - r(a)) + F(h, l, b + r(a)).$$

(iii) *Assume* $r(a) + r(b) > q(t(b), h)$. *If* $\gamma = q(t(b), h) - r(b)$ *let us consider the transformation* $(a, b) \longrightarrow (a - \gamma, b + \gamma)$. *Then* $t(a - \gamma) = t(a)$, $t(b + \gamma) = t(b) + 1$, $r(a - \gamma) = r(a) - \gamma$, $r(b + \gamma) = 0$, *and*

$$F(h, l, a) + F(h, l, b) \le F(h, l, a - \gamma) + F(h, l, b + \gamma).$$

(iv) *Assume* $r(a) = 0$. *If* $\gamma = q(t(b), h) - r(b)$ *let us consider the transformation* $(a, b) \longrightarrow (a - \gamma, b + \gamma)$. *Then* $t(a - \gamma) = t(a) - 1$, $t(b + \gamma) = t(b) + 1$, $r(a - \gamma) = q(t(a) - 1, h) - \gamma$, $r(b + \gamma) = 0$, *and*

$$F(h, l, a) + F(h, l, b) = F(h, l, a - \gamma) + F(h, l, b + \gamma).$$

Proof. We will denote by $\Delta(x)$ the integer

$$\Delta(x) = F(h, l, a - x) + F(h, l, b + x) - F(h, l, a) - F(h, l, b).$$

The aim of the lemma is to prove $\Delta(x) \ge 0$ for some values of x.

(i) If $r(a) + r(b) < q(t(b), h)$ then it is easy to prove that $t(a - r(a)) = t(a)$, $t(b + r(a)) = t(b)$, $r(a - r(a)) = 0$, and $r(b + r(a)) = r(a) + r(b)$.

Notice that $\Delta(r(a)) = (r(a) + r(b))_{\langle t(b)\rangle(h-l)} - r(a)_{\langle t(a)\rangle(h-l)} - r(b)_{\langle t(b)\rangle(h-l)}$. From Corollary 2.7 (i) we get $\Delta(r(a)) \ge 0$.

(ii) If $r(a) + r(b) = q(t(b), h)$ then $t(a - r(a)) = t(a)$, $t(b + r(a)) = t(b) + 1$, $r(a - r(a)) = r(b + r(a)) = 0$.

$$\begin{aligned}
\Delta(r(a)) &= q(t(b), l + 1) - q(t(b) - 1, l + 1) - r(a)_{\langle t(a)\rangle(h-l)} - r(b)_{\langle t(b)\rangle(h-l)} \\
&= q(t(b), h)_{\langle t(b)\rangle(h-l)} - r(b)_{\langle t(b)\rangle(h-l)} - r(a)_{\langle t(a)\rangle(h-l)} \\
&\ge 0 \quad \text{by Corollary 2.7}(i) \text{ with } s = t(a) + 1
\end{aligned}$$

(iii) Assume $r(a) + r(b) > q(t(b), h)$. We write $\gamma = q(t(b), h) - r(b)$. Then $t(a - \gamma) = t(a)$, $t(b + \gamma) = t(b) + 1$, $r(a - \gamma) = r(a) - \gamma$, $r(b + \gamma) = 0$.

$$\begin{aligned}
\Delta(\gamma) &= (r(a) + r(b) - q(t(b), h))_{\langle t(a)\rangle(h-l)} + q(t(b), l) \\
&\quad - r(a)_{\langle t(a)\rangle(h-l)} - r(b)_{\langle t(b)\rangle(h-l)} \\
&\ge 0 \quad \text{by Corollary 2.7 } (ii) \text{ with } s = t(a) + 1.
\end{aligned}$$

(iv) Assume $r(a) = 0$. If $\gamma = q(t(b), h) - r(b)$ then $t(a - \gamma) = t(a) - 1$, $t(b + \gamma) = t(b) + 1$, $r(a - \gamma) = q(t(a) - 1, h) - \gamma$, and $r(b + \gamma) = 0$. Again we can compute $\Delta(\gamma)$:

$$\begin{aligned}
\Delta(\gamma) &= q(t(a) - 2, l + 1) + (q(t(a) - 1, h) - \gamma)_{\langle t(a)-1\rangle(h-l)} + q(t(b), l + 1) \\
&\quad - q(t(a) - 1, l + 1) - q(t(b) - 1, l + 1) - r(b)_{\langle t(b)\rangle(h-l)}
\end{aligned}$$

$$
\begin{aligned}
&= -q(t(a)-1,h)_{\langle t(a)-1\rangle(h-l)} + (q(t(a)-1,h)-\gamma)_{\langle t(a)-1\rangle(h-l)} \\
&\quad + q(t(b),h)_{\langle t(b)\rangle(h-l)} - r(b)_{\langle t(b)\rangle(h-l)} \\
&= -q(t(b),h)_{\langle t(b)\rangle(h-l)} + (q(t(b),h)-\gamma)_{\langle t(b)\rangle(h-l)} \\
&\quad + q(t(b),h)_{\langle t(b)\rangle(h-l)} - r(b)_{\langle t(b)\rangle(h-l)} = 0 \quad \text{by (P4)}
\end{aligned}
$$
□

COROLLARY 4.3. *Given integers* $e_1, \ldots, e_b, e_1 + \cdots + e_b = e$, *there exist integers* $t, e'_1, \ldots, e'_b, e'_1 + \cdots + e'_b = e$, *such that*

(i) $\sum_{j=1}^b F(h,l,e_j) \le \sum_{j=1}^b F(h,l,e'_j)$,
(ii) $q(t,h+1) \le e'_1 \le \cdots \le e'_b \le q(t+1,h+1)$.

Proof. First step is to prove that we can assume $| t(e_j) - t(e_l) | \le 1$ for all $1 \le j,l \le b$. Let us assume that there exist j,l such that $| t(e_j) - t(e_l) | \ge 2$. Notice that the integer $| t(e_j) - t(e_l) |$ drops one under the transformations (ii) and (iii) of the last result. The output of (i) satisfies the hypothesis of (iv), so (i) followed by (iv) drops $| t(e_j) - t(e_l) |$ at least one. Notice that under the transformations (i),. . . ,(iv) the integer $F(h,l,e_j) + F(h,l,e_l)$ does not decrease. Hence, repeatedly applying the transformations (i),. . . ,(iv) we obtain a new set of integers $\{e''_1, \ldots, e''_b\}$ such that $e''_1 + \cdots + e''_b = e$ and

$$
\begin{aligned}
&\textstyle\sum_{j=1}^b F(h,l,e_j) \le \sum_{j=1}^b F(h,l,e''_j), \\
&v \le t(e''_1) \le \cdots \le t(e''_b) \le v+1, \\
&q(v-1,h+1) \le e''_1 \le \cdots \le e''_s \\
&\qquad\qquad < q(v,h+1) \le e''_{s+1} \le \cdots \le e''_b < q(v+1,h+1)
\end{aligned}
$$

for some integers v, $1 \le s \le b$. Hence there exist integers $1 \le s_1 \le s \le s_2 \le b$ such that

$$0 = r(e''_1) = \cdots = r(e''_{s_1}) < r(e''_{s_1+1}) \le \cdots \le r(e''_s) < q(v,h)$$

and

$$0 = r(e''_{s+1}) = \cdots = r(e''_{s_2}) < r(e''_{s_2+1}) \le \cdots \le r(e''_b) < q(v+1,h).$$

According to the sign of $q(v,h) - r(e''_i) - r(e''_j)$ we will apply Corollary 2.7 (i) or (ii) to the pairs $\{e''_i, e''_j\}$, $i = 1, \ldots s$, $j = s_2+1, \ldots b$. If $q(v,h) \ge r(e''_i) + r(e''_j)$ then we replace e''_i by $e''_i + r(e''_j)$ and e''_j by $q(v,h+1)$. From Corollary 2.7 (i) we get

$$F(h,l,e''_i) + F(h,l,e''_j) \le F(h,l,e''_i + r(e''_j)) + F(h,l,q(v,h)).$$

If $q(v,h) < r(e''_i) + r(e''_j)$ then we replace e''_i by $q(v,h+1)$ and e''_j by $e''_i + e''_j - q(v,h+1)$. From Corollary 2.7 (ii) we get

$$F(h,l,e''_i) + F(h,l,e''_j) \le F(h,l,q(v,h+1)) + F(h,l,e''_i + e''_j - q(v,h+1)).$$

We can repeatedly apply this procedure until we get integers $e'_1, \dots, e'_b$ such that $e'_1 + \cdots + e'_b = e$ and

$$\sum_{j=1}^{b} F(h, l, e_j) \le \sum_{j=1}^{b} F(h, l, e'_j),$$

and one of the following conditions holds:

(1) $q(v, h+1) \le e'_1 \le \cdots \le e'_b \le q(v+1, h+1)$.
(2) $q(v-1, h+1) \le e'_1 \le \cdots \le e'_b \le q(v, h+1) = e'_{s+1} = \cdots = e'_b$.

Notice that (ii) is (1) (resp. (2)) for $t = v$ (resp. $t = v - 1$); i.e., we get (ii). □

Now we are able to prove the claim. From the last corollary we may assume that there exists an integer $0 \le s \le b$ such that

$$q(t, h+1) \le e_1 \le \cdots \le e_s < q(t+1, h+1) = e_{s+1} = \cdots = e_b.$$

If $s = 0$ then we are done; otherwise we have $t = j^*(h, e, b)$ and

$$0 \le r(e_1) \le \cdots \le r(e_s) < q(t+1, h).$$

From Corollary 2.7 (i), (ii), see proof of the last result, there exists a new set of integers $e'_1, \dots, e'_s$ such that $e'_1 + \cdots + e'_s = e_1 + \cdots + e_s$ and

$$\begin{aligned}
&\textstyle\sum_{j=1}^{s} F(h, l, e_j) \le \sum_{j=1}^{s} F(h, l, e'_j),\\
&e'_j = q(t+1, h+1), \text{ for } j = 1, \dots, p,\\
&e'_{p+1} = q(t, h+1) + W \le q(t+1, h+1),\\
&e'_j = q(t, h+1), \text{ for } j = p+2, \dots, s,
\end{aligned}$$

for some integer $p \le s$. Notice that by definition $e'_j = \varepsilon_j(e)$, $j = 1, \dots, s$, $e_j = \varepsilon_j(e)$, $j = s+1, \dots, b$, so the claim is proved. □

Definition 4.4. Let $h, b, e \ge b(h+1)$ be integers; we will denote by $M^g_{(h,b,e)}$ the split S-module

$$M^g_{(h,b,e)} = \bigoplus_{i=1}^{b} \frac{S}{I(\varepsilon_i)}.$$

In the local case we define

$$M^l_{(h,b,e)} = \bigoplus_{i=1}^{b} \frac{R}{J(\varepsilon_i)}.$$

From the last result we get:

THEOREM 4.5. *Given integers $h, b, e \geq b(h+1)$ the module $M^{g}_{(h,b,e)}$ has multiplicity e, codimension h, b as 0-th Betti number and*

$$\beta_i(M^{g}_{(h,b,e)}) = \Phi_i(h, b, e)$$

for all $i = 1, \dots, h$. The corresponding properties holds for $M^{l}_{(h,b,e)}$.

Next result is the main result of this paper. We will prove it in the local case; the graded case is done in a similar way.

THOEREM 4.6. *Let R be a Noetherian regular local ring of dimension d. Let M be a finitely generated Cohen-Macaulay R-module, then for all $i = 1, \dots, h$,*

$$\beta_i(M) \leq \Phi_i(h(M), \beta_0(M), e_0(M)).$$

This bound is sharp: for all $h, b \geq 1$ and $e \geq b(h+1)$ the module $M = M^{l}_{(h,b,e)}$ satisfies $h(M) = h$, $\beta_0(M) = b$, $e_0(M) = e$, and $\beta_i(M) = \Phi_i(h, b, e)$ for all $i = 1, \dots, h$.

Proof. From now on we let $b = \beta_0(M)$, $h = h(M)$, $e = e_0(M)$. Let $N = \bigoplus_{j=1}^{b} S/I_j$ be a lex reduction of M, so $e = e_0(N)$, $e_j = e_0(S/I_j)$, $e_1 + \cdots + e_b = e$, $h(N) = h$, $\beta_0(N) = b$, and $\beta_i(N) \geq \beta_i(M)$, $i = 1, \dots, b$. Hence from Theorem 3.10 we have

$$\beta_i(M) \leq \beta_i(N) = \sum_{j=1}^{b} \beta_i\left(\frac{S}{I_j}\right) \leq \sum_{j=1}^{b} \beta_i\left(\frac{S}{I(e_j)}\right)$$

From Theorem 4.1 and the definition of Φ_i we deduce

$$\beta_i(M) \leq \Phi_i(h, b, e) = \Phi_i(h(M), \beta_0(M), e_0(M))$$

for $i = 1, \dots, h$. The sharp statement follows from Theorem 4.5. □

Remark 4.7. Recall that for non-Cohen-Macaulay modules there is not an upper bound for the number of generators; see [ERV91].

Remark 4.8. Assume that $\mathbf{k}$ is a characteristic zero field. Notice that $M^{g}_{(h,b,e)}$ is itself a ring. From [GGR86] we can lift this $S = \mathbf{k}[X_1, \dots, X_h]$-module to a $S[Z] = \mathbf{k}[X_1, \dots, X_h, Z]$-module $M^{g,lift}_{(h,b,e)}$ that is also a reduced ring. See also [ERV91], [EGV94], [Val94].

We end the paper giving some explicit computations.

Example 1. We compute explicitly the module $M^g_{(3,3,19)}$. In this case we have $\varepsilon_1 = 10$, $\varepsilon_2 = 5$, $\varepsilon_3 = 4$, and $M^g_{(3,3,19)} = S/I(10) \oplus S/I(5) \oplus S/I(4)$, where $I(10)$ is the ideal generated by $X^3, X^2Y, XY^2, XYZ, XZ^2, X^2Z, Y^3, Y^2Z, YZ^2, Z^3$, $I(5)$ is the ideal generated by $X^2, XY, Y^2, YZ, XZ, Z^3$, $I(4)$ is the ideal generated by X^2, XY, Y^2, YZ, XZ, Z^2. The module $M^g_{(3,3,19)} = M$ has Betti numbers $\beta_0(M) = 3$, $\beta_1(M) = v(M) = 22$, $\beta_2(M) = 31$, $\beta_3(M) = 12$. The Hilbert function of M is $H_M = \{3, 9, 7, 0, \ldots\}$.

Example 2. In the following table we compute some values of the upper bounds of the Betti numbers for $h = 3$ and $\beta_0 = 2$:

e_0	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
β_1	12	12	12	13	13	14	16	16	16	17	17	18	20	20	20
β_2	16	16	16	18	18	20	23	23	23	25	25	27	30	30	30
β_3	6	6	6	7	7	8	9	9	9	10	10	11	12	12	12

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