ON THE SOLVABILITY OF SOME FACTORIZED LINEAR GROUPS

BY

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Let G be a group with subgroups A and B such that $G = AB = BA$. To what extent does the structure of A and B determine the structure of the "factorized" group G?

For finite groups some results in this direction are known (cf. e.g. [21], [7], [9], [8], [22], [23], [11]). Except for a few trivialities, there seems to be only one general result on this subject in the literature, viz. Itô's theorem [10] stating that the product of two abelian groups is solvable of derived length at most two. –In view of this result one might feel tempted to make the following conjecture: $G = AB$ is solvable of derived length at most $a + b$ if A and B are nilpotent of class a and b, respectively.

Even for finite groups this conjecture still is open; although solvability has been obtained here [23], [11]. If A and B are finite of coprime order, then the statement of the conjecture is contained in [6]. To get near such a result in general, it seems necessary to develop new commutator techniques generalizing the trick Itô used in the proof of his theorem. Devoid of these tools one may only hope to extend the solvability results from the class of finite groups to some more general classes of groups suitably restricted by finiteness conditions. One simple example of such an extension is indicated in [16].

In this paper two such extensions are given. Under suitable conditions the solvability results on finite groups may be carried over to some "large" subgroups of factorized locally finite groups. In particular the solvability results carry over immediately to locally finite groups satisfying the minimum condition for subgroups. For these groups a characterisation of solvability may be given in terms of factorizations.

As solvable locally finite groups that satisfy the minimum condition for subgroups admit a faithful linear representation, one may ask whether the solvability results also extend to linear groups. In fact, one can prove: the linear group $G = AB$ is almost solvable if A and B both are almost nilpotent. –The methods of proof for this theorem are quite different from the ones used before; they make strong use of the theory of algebraic linear groups.

Notation. If the group G operates by automorphisms on the group H and if $U \subseteq H$, then $\mathbf{C}_G\, U$ denotes the centralizer of U in G, i.e. the set of all those elements of G leaving every element of U fixed; $\mathbf{N}_G\, U$ is the set of all those elements of G that transform every element of U into an element of U.

$\mathbf{F}G = \mathbf{F}_1 G$ is the product of all finitely generated normal subgroups of G. The terms of the *upper* $\mathbf{F}$-*chain* of G are defined inductively:

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$$\mathbf{F}_j G = \bigcup_{i<j} \mathbf{F}_i G \qquad \text{for limit ordinals } j, \text{ and}$$

$$\mathbf{F}_{i+1} G/\mathbf{F}_i G = \mathbf{F}(G/\mathbf{F}_i G) \quad \text{otherwise.}$$

As in similar contexts, it is easily seen that the following two statements about the group G are equivalent.

(a) G is a [terminal] member of its upper $\mathbf{F}$-chain;
(b) every epimorphic image $H \neq 1$ of G has $\mathbf{F}H \neq 1$.

It is sometimes covenient to denote by $\mathbf{F}_\wedge G$ the terminal member of the upper $\mathbf{F}$-chain of G.

$\mathbf{J}G$ = intersection of all subgroups X of finite index $[G:X]$ in G.

A group is almost abelian, if it possesses an abelian subgroup of finite index.

The element $g \in GL(n, K)$ is *semi-simple* if the matrix g may be transformed to diagonal form by an element of $GL(n, K^*)$, where K^* is the algebraic closure of the field K.

$g \in GL(n, K)$ is *unipotent*, if all its eigenvalues are 1.

A *Borel subgroup* of the linear algebraic group G is a maximal closed connected solvable subgroup of G.

A *torus* of G is a closed connected abelian subgroup consisting of semi-simple elements.

It should be observed that in this paper 1 denotes the unit element of any group as well as the unit group and the number "one".

1. Locally finite groups

Before taking up the problem of the structure of factorized locally finite groups, we shall give a few very simple results on factorized groups in general.

LEMMA 1.1. *If in the factorized group $G = AB$ the subgroups A and B have normal subgroups A_0 and B_0 of finite index in A and B, respectively, then the normal closure C of $\{A_0, B_0\}$ has finite index in G.*

Remark. It would be of interest to know whether in general the subgroup $\{A_0, B_0\}$ has already finite index in G, or not.

Proof. As $G = AB$, it is clear that the sets

$$a_i A_0 B_0 b_j \subseteq a_i C b_j = C a_i b_j$$

cover all of G, if a_i and b_j run independently over a set of representatives of A over A_0 and of B over B_0, respectively. As these sets are finite, $| G:C |$ is finite.

Evidently, every factor group G/N of the factorized group $G = AB$ inherits the factorization $G/N = (AN/N)(BN/N)$. For certain types of subgroups of G one can prove a similar statement.

LEMMA 1.2 (Wielandt [23, Hilfssatz 7]). *Let A_0, B_0 be subgroups of the factorized group $G = AB$ with $A_0 \trianglelefteq A$, $B_0 \trianglelefteq B$, $H = \{A_0, B_0\}$. If for every* *pair a, b of elements, $a \in A$, $b \in B$, the statement $H^a = H^b$ implies $H^a = H^b = H$, then the normalizer $N = \mathbf{N}_G H$ admits the factorization*

$$N = (N \cap A)(N \cap B).$$

Remark. The conditions of this lemma are certainly met if either $H = \{A_0, B_0\}$ is not contained in any proper conjugate of itself in G or if at least one of A/A_0 and B/B_0 happens to be a torsion group.

Proof. If $ab^{-1} \in N$, then $H^a = H^b$. Hence $H^a = H^b = H$. This implies $a, b \in N$, and $N = (N \cap A)(N \cap B)$.

LEMMA 1.3. *If in $G = AB$ the normal subgroups A_0, B_0 of A, B, respectively, generate a finite subgroup $H = \{A_0, B_0\}$ of order h, then there exists a subgroup F of G, generated by at most $2h$ elements, and such that*

$$H \subseteq F \subseteq N_G H \quad \text{and} \quad F = (F \cap A)(F \cap B).$$

Proof. Being finite, H is not contained in any proper conjugate of itself in G; hence Lemma 1.2 applies: $N = \mathbf{N}_G H = (N \cap A)(N \cap B)$. Take $A_1 = N \cap A$, $B_1 = N \cap B$, and consider the subgroup $A_1 H$ of N. By Dedekind's law, one gets $A_1 H = A_1 B_2$ with $B_2 = A_1 H \cap B_1$. But since H is of finite order h, we may replace B_2 in this factorization by a subgroup B^* of B_2 generated by at most h elements with $A_1 H = A_1 B^*$. Now consider the subgroup HB^*; another application of Dedekind's law yields a subgroup A_2 with $A_2 B^* = HB^*$. In the same way as before, A_2 may be replaced by a subgroup A^* of A_2 generated by at most h elements with $A_2 B^* = A^* B^* = F$. This subgroup F has the desired properties.

DEFINITION 1.4. The factorization $G = AB$ is an *(S)-factorization* if every finite factor group U/N of any subgroup U of G with $U = (U \cap A)(U \cap B)N$ is solvable.

An example of how this notion of (S)-factorization may be used to establish some kind of solvability for certain subgroups of G is given by the following.

PROPOSITION 1.5. *If the locally finite group G has an (S)-factorization $G = AB$, then the subgroup $\{\mathbf{F}A, \mathbf{F}B\}$ is locally solvable. –If furthermore A is a member of its upper $\mathbf{F}$-chain and if $B = \mathbf{F}B$, then G is locally solvable.*

Proof. Let X and Y be arbitrary finitely generated normal subgroups of A and B, respectively. By the local finiteness of G, the union $M = \{X, Y\}$ is finite. So by Lemma 1.3, there is a finite subgroup F containing M such that $F = (F \cap A)(F \cap B)$. Since $G = AB$ is an (S)-factorization of G, the subgroup F is solvable, and hence also M. But then $\{\mathbf{F}A, \mathbf{F}B\}$ is locally solvable, since the subgroups formed like M make up a local system of $\{\mathbf{F}A, \mathbf{F}B\}$.

For the second statement of the proposition, we may assume that G has no locally solvable normal subgroup $\neq 1$, as in locally finite groups local solvability is preserved in extensions. Then the locally solvable subgroup $F = \{\mathbf{F}A, \mathbf{F}B\}$ contains the normal subgroup $\mathbf{F}A$ of A, and the normal closure of $\mathbf{F}A$ in G is contained in F; it thus is locally solvable. This fact is contrary to the assumption that G has no locally solvable normal subgroup $\neq 1$, unless $FA = 1$. Hence $A = 1$ and $F = B = 1$. –This proves the proposition.

Remarks. (1) Noting that for locally finite groups extensions of locally solvable groups by locally solvable groups are locally solvable, one sees easily that the same procedure of proof yielding the first part of Proposition 1.5 gives that $\mathbf{F}_\wedge G$, the terminal member of the ascending $\mathbf{F}$-chain of G, also is locally solvable, and so is the group

$$\{\mathbf{F}(AS/S),\ \mathbf{F}(BS/S)\}\mathbf{F}_\wedge(G/S)$$

for any locally solvable normal subgroup S of G.

(2) If in the situation of Proposition 1.5 one had $|A:\mathbf{F}A|\,|B:\mathbf{F}B| < \infty$, and if one somehow knew that $\{\mathbf{F}A, \mathbf{F}B\}$ has finite index in G, then one could easily infer that G itself is locally solvable.

In order to get information on the structure of G and not only of "large" subgroups, we have to restrict the possible structure of G severely, concentrating on locally finite groups satisfying the minimum condition for subgroups.

PROPOSITION 1.6. *For the factorized group $G = AB$ the following properties are equivalent*:
(a) *G satisfies the minimum condition for subgroups and is almost abelian.*
(b) *A and B both satisfy the minimum condition for subgroups, and G is almost abelian.*
(c) *G satisfies the minimum condition for subgroups, A and B both are almost abelian, and $\mathbf{J}A\mathbf{J}B = \mathbf{J}B\mathbf{J}A = \mathbf{J}G$.*

The author is indebted to Professor Reinhold Baer for this proposition in its present form.

Remark. There is the conjecture that a locally finite group is almost abelian if it satisfies the minimum condition for subgroups. –The connection to Proposition 1.5 is given by the remark that for any almost abelian group G one has the equality $G = \mathbf{F}_\wedge G = \mathbf{F}_2 G$.

Proof. Obviously, (b) is a consequence of (a). –Assume (b): with G also the subgroups A and B are almost abelian. As they satisfy the minimum condition, it is clear that $\mathbf{J}A$ and $\mathbf{J}B$ are abelian groups without proper subgroups of finite index. By assumption, there is an abelian normal subgroup V with finite index in G. As $V \cap \mathbf{J}A$ is of finite index in $\mathbf{J}A$, and as the latter does not admit any proper subgroup of finite index, one has $\mathbf{J}A = V \cap \mathbf{J}A$; and similarly, $\mathbf{J}B = V \cap \mathbf{J}B$. So the subgroup $S = \mathbf{J}A\mathbf{J}B = \mathbf{J}B\mathbf{J}A$ of V is abelian and satisfies the minimum condition for subgroups. Since the abelian normal subgroup V has finite index in G, there are only finitely many conjugates of S in G; let T be their product. Since T still is a subgroup of V, it is abelian and satisfies the minimum condition for subgroups. The characteristic subgroup $\mathbf{J}G$ of G contains $\mathbf{J}A$ and $\mathbf{J}B$, hence also T. But since T is (because of the finiteness of AT/T and BT/T) of finite index in G, one has $T = \mathbf{J}G$. Now consider $N = \mathbf{N}_G S$. Evidently $T \subseteq N$. Since G is a torsion group, Lemma 1.2 applies: $N = (N \cap A)(N \cap B)$. But this shows that $| N:S |$ is finite, hence also $| T:S |$. As T does not have any proper subgroup of finite index, this means $T = S$, and $\mathbf{J}G = \mathbf{J}A\mathbf{J}B$. Thus (c) is a consequence of (b).

Now assume (c): $\mathbf{J}A$ and $\mathbf{J}B$ being abelian, Itô's theorem [10] states that $\mathbf{J}G$ is metabelian. As a solvable group satisfying the minimum condition for subgroups is almost abelian (cf. e.g. [3] or [1, Lemma 3.3]) and since $\mathbf{J}G$ does not admit any proper subgroup of finite index, $\mathbf{J}G$ is itself abelian. Thus G is almost abelian; and (a) is a consequence of (c).

THEOREM 1.7. *If the locally finite group G satisfies the minimum condition for subgroups and if it admits an (S)-factorization $G = AB$, then G is solvable.*

Proof. By Proposition 1.5, the subgroup $F = \{\mathbf{F}A, \mathbf{F}B\}$ is locally solvable. Evidently, $\mathbf{F}A \supseteq \mathbf{J}A$, $\mathbf{F}B \supseteq \mathbf{J}B$; and so the subgroup $C = \{\mathbf{J}A, \mathbf{J}B\}$ of G is locally solvable. But the minimum condition implies that C is almost abelian (cf. e.g. [1, Satz 3.4]). As in any group X, a subgroup without proper subgroups of finite index is contained in $\mathbf{J}X$, one obtains $C = \mathbf{J}C$, and thus C is a divisible abelian group.

By induction on $| A:\mathbf{J}A || B:\mathbf{J}B |$ we may assume that the group

$$N = \mathbf{N}_G(\mathbf{J}A) = A(N \cap B) \supseteq C$$

is solvable, except if $N = G$. In case $N \neq G$, the index $| G:N |$ is finite; consequently there is a solvable normal subgroup M of G contained in N with $| G:M | < \infty$. But since $G = AB$ is an (S)-factorization of G, the finite factor group G/M of G is solvable, hence so is G.

Thus one may assume that $\mathbf{J}A$ and—for the same reason—$\mathbf{J}B$ are normal in G. But then, by Lemma 1.1, the normal subgroup $C = \mathbf{J}A\mathbf{J}B$ has finite index in G; and again G is solvable.

The following theorem complements some known results on solvable groups with minimum condition for subgroups (cf. e.g. [3], [1]).

THEOREM 1.8. *For a locally finite group G satisfying the minimum condition for subgroups the following properties are equivalent:*

(a) *G is solvable.*

(b) *G is a product of finitely many pairwise permutable nilpotent subgroups.*

(c) *G is a product of pairwise permutable locally nilpotent subgroups.*

Proof. Let G be solvable. Then the intersection $\mathbf{J}G$ of all subgroups of finite index in G is an abelian subgroup of finite index in G, (see [3] or [1, Lemma 3.3]). Thus there is a finite subgroup H of G with $H\mathbf{J}G = G$. Since H is a product of pairwise permutable nilpotent subgroups and $\mathbf{J}G$ is a normal abelian subgroup, one obtains a factorization of G into finitely many pairwise permutable nilpotent subgroups. Thus (a) implies (b); and (c) is a weakened form of (b).

To prove that in turn (c) implies (a), it is sufficient to show that G is locally solvable; thus it suffices to show the local solvability of any product of finitely many locally nilpotent subgroups of G that are pairwise permutable. So consider the group $G^* = G_1 \cdots G_n$ with $G_i G_j = G_j G_i$. The subgroups G_i of G^* being locally nilpotent, it is clear by the result of [23] and [11] that the group $G_{ij} = G_i G_j$ has the (S)-factorization $G_i G_j = G_j G_i$. Theorem 1.7 together with property (c) of Proposition 1.6 yields that the subgroups $\mathbf{J}G_i$ and $\mathbf{J}G_j$ centralize each other; thus the subgroup $J = \{\mathbf{J}G_i ; i = 1, \cdots, n\}$ of G^* is abelian and divisible. By induction, we may assume that the subgroup $H = G_1 \cdots G_{n-1}$ is solvable and $\mathbf{J}H = \{\mathbf{J}G_i ; i = 1, \cdots, n-1\}$. Consider now $N = \mathbf{N}_{G^*}(\mathbf{J}H)$; by Dedekind's law it has the form $N = H(N \cap G_n)$. As furthermore $N \supseteq J$, it is evident that $| G^*:N |$ divides $| G_n:\mathbf{J}G_n |$ and hence is finite. Moreover, J is of finite index in N: for in the factor group

$$N/\mathbf{J}H = H(N \cap G_n)/\mathbf{J}H = [H/\mathbf{J}H][(N \cap G_n)\mathbf{J}H/\mathbf{J}H]$$

the subgroup $(N \cap G_n)\mathbf{J}H/\mathbf{J}H$ has a finite index, and so has

$$\mathbf{J}(N \cap G_n)\mathbf{J}H/\mathbf{J}H = \mathbf{J}G_n\,\mathbf{J}H/\mathbf{J}H = J/\mathbf{J}H.$$

Consequently, J also has finite index in G^*. Hence J contains an abelian normal subgroup A of finite index in G^*. In particular, A is of finite index in J. But since J does not have any proper subgroup of finite index, $J = A$ is normal in G^*. The factor group G^*/J is finite and a product of pairwise permutable nilpotent subgroups; hence G^*/J is solvable (cf. [23] and [11]), and so is G^*. –Thus G is locally solvable, and hence—by the minimum condition—solvable.

Remark. Essentially, the proofs of 1.7 and 1.8 just yield that the group G is almost abelian; and then the assumption of the existence of an (S)-factorization (in 1.7) or the result of [23] and [11] (in 1.8) establishes solvability.

Added in Proof (May 21, 1965). Since in a locally finite group G satisfying the minimum condition for subgroups for any prime p the Sylow p-subgroups of G form a single class of conjugate subgroups (cf. [1]), it is not difficult to show the following fact, which, for finite groups, is proved in [22]: *If the locally finite group G satisfies the minimum condition for subgroups and admits a factorization $G = AB$ with subgroups A and B, then there is a Sylow p-subgroup G_p of G which admits a factorisation $G_p = A_pB_p$, where A_p and B_p are Sylow p-subgroups of A and B, respectively.* – By means of this observation, one may extend part of Satz 1 of the author's recent paper, *Zur Struktur Mehrfach Faktorisierter Endlicher Gruppen*, Math. Zeitschrift, vol. 87 (1965), pp. 42–48, to locally finite groups with minimum condition.

THEOREM 1.9. *The locally finite group G which satisfies the minimum condition for subgroups possesses a normal Sylow p-subgroup for the prime p if and* *only if there are three subgroups* A, B, C *of* G *with normal Sylow* p*-subgroups* A_p, B_p, C_p, *respectively, such that* G *has the form*

$$G = AB = BC = CA.$$

For the proof of this statement, one just has to show that the Sylow p-subgroups A_pB_p and B_pC_p of G commute and hence coincide. This is done by the same trick as in the finite case.

2. Linear groups

It is well known [12] that an abelian group A admits a faithful representation as linear group over some field of characteristic 0 if and only if elementary abelian subgroups of A have bounded dimensions considered as vector spaces over the corresponding prime fields. Since an abelian group satisfying the minimum condition for subgroups evidently meets this condition, and since furthermore finite extensions of linear groups admit a faithful linear representation, it seems quite natural to ask whether and how Theorems 1.7 and 1.8 extend to linear groups. The simplest result in this direction—and the only one on (S)-factorizations—is the following.

PROPOSITION 2.1. *If the linear group* $G \subseteq GL(n, K)$ *is finitely generated and admits an* (S)*-factorization* $G = AB$, *then* G *is solvable.*

Proof. An important theorem of Mal'cev states that the finitely generated subgroup G of $GL(n, K)$ has a system of normal subgroups N_i such that $\bigcap_i N_i = 1$ and $G_i = G/N_i \simeq U_i \subseteq GL(n, K_i)$ where K_i ranges over suitable finite fields. Since $G = AB$ is an (S)-factorization, each of the finite factor groups G_i is solvable. But there is a function $f(n)$ of the degree n bounding the derived length of a solvable linear group of degree n (cf. [24, p. 295]); thus the $f(n)$-*th* term of the derived series of G lies in each of the N_i and hence equals 1. This shows that G is solvable.

For arbitrary linear groups no such approximation theorem holds, and one must look for other tools in order to extend the solvability results. These tools are readily found in the theory of linear algebraic groups; the following digression is intended to present these auxiliary results.

Let G be a linear group, i.e. G is a subgroup of some $GL(n, K)$ with a given embedding, where K is a commutative field which we assume algebraically closed. If the group $GL(n, K)$ is endowed with the Zariski topology (cf. [2] or [18] for this and the following notions) then the closure $\bar{G}$ of G in $GL(n, K)$ is the smallest algebraic linear subgroup of $GL(n, K)$ containing G.

Let $(\bar{G})_0$ be the component of the identity of $\bar{G}$, then define

$$G_0 = G \cap (\bar{G})_0$$

to be the component of the identity of G. Evidently, it is of finite index in G. Furthermore, one has $\bar{G_0} = (\bar{G})_0$, i.e. G_0 is dense in $(\bar{G})_0$.

The notion of dimension is defined for algebraic groups, it may be carried over to arbitrary linear groups by setting

$$\dim G = \dim \bar{G} = \dim \bar{G}_0 .$$

The property $\mathcal{P}$ of linear groups will be called *algebraic*, if for a linear group G the relation $G \in \mathcal{P}$ implies $\bar{G} \in \mathcal{P}$, and if furthermore every normal subgroup of a $\mathcal{P}$-group is also a $\mathcal{P}$-group. Solvability of finite length and nilpotency of given class are the best known examples of algebraic properties. Another example is the property of the matrix group G to be of (upper) triangular form. –It should be noted that the algebraic property $\mathcal{P}$ of the linear group G (e.g. $\dim G \leq d$) in general depends on the particular embedding of G into $GL(n, K)$, and need not be shared by another subgroup H of $GL(n, K)$ isomorphic to G, i.e. $\mathcal{P}$ is not an abstract property, in general.

LEMMA 2.2. *If the linear group $G \subseteq GL(n, K)$ has a $\mathcal{P}$-subgroup S of finite index, with $\mathcal{P}$ an algebraic property of linear groups, then $\bar{G}_0$—and hence G_0—is a $\mathcal{P}$-group, and one has* $\dim G = \dim S$.

Proof. The intersection J of all conjugates of S in G is of finite index in G. Since G normalizes J, it also normalizes $\bar{J}$, and hence also $\bar{J}_0$. Being a finite union of algebraic sets, $G\bar{J}_0$ is algebraic, i.e. $\bar{G} = G\bar{J}_0$. Since $\bar{G}_0$ is the smallest closed subgroup of finite index of $\bar{G}$, one has $\bar{G}_0 = \bar{J}_0$. –The statement about the dimensions follows directly from the definitions.

An example of the situation described in Lemma 2.2 is given by the following result which is probably well known, but which we were unable to find in print.

PROPOSITION 2.3. *If L is a locally nilpotent subgroup of $GL(n, K)$, then L has a normal subgroup N with $| L:N | \leq f(n)$ that is nilpotent of class at most* $\max(1, n - 1)$.

Remark. It is an easy consequence of this proposition that a locally nilpotent linear group is a ZA-group—a result given by Garaščuk [4].

Proof. Being locally nilpotent, L is a fortiori locally solvable, and thus even solvable ([24, p. 295]). By the theorem[2] of Lie-Kolchin-Mal'cev (cf. [2], [19, Theorem 21], [13, Theorem 1]) the solvable group L has a normal subgroup T with $| L:T | \leq f(n)$ that may be assumed to have trigular form. Take any finite set of elements of T and let E be the subgroup generated by this set. Let $\bar{E}$ be the closure of E in $GL(n, K)$; then $\bar{E}$ contains with every element g also its unipotent component g_u and its semi-simple component g_s $(g = g_u g_s = g_s g_u)$. As T is locally nilpotent, its subgroup E is nilpotent, and so is $\bar{E}$. The set of all unipotent elements of $\bar{E}$ is a normal subgroup of $\bar{E}$ since $\bar{E}$ has triangular form whenever E has, and by [20, Cor. 2] the semi-simple elements form a normal subgroup of $\bar{E}$, too. So the unipotent components of the elements of T generate a nilpotent subgroup U of $GL(n, K)$ of class at most $n - 1$ which is centralized by the set of all semi-simple com-

[2] This theorem usually is given in the form that the index $| L:T |$ is finite. The existence of a bound for this index in terms of the degree n is easily obtained by the methods of [19, Theorems 15 and 21].

ponents of elements of T. But this set generates a subgroup S consisting exclusively of semi-simple matrices (cf. [20]). The group S is abelian, and so $U \times S$ is nilpotent of class at most max $(1, n - 1)$. As T is a subgroup of $U \times S$, its class cannot be larger.

Remark. Dr. K. W. Gruenberg pointed out that the above proposition may also be proved by rather direct calculations.

The following lemma is taken over from [17]; we quote it in the form it is stated in [18, exposé 3] from where the main idea of the proofs of Lemma 2.5 and Proposition 2.6 are taken. –This is the only point in this paper where algebraic geometry appears explicitly.

LEMMA 2.4. *Let f be a regular mapping of the algebraic variety V into the algebraic variety W such that $W = \overline{f(V)}$, and let $d = \dim V - \dim W$. For any subvariety S of W, let R be a component of $f^{-1}(S)$ such that $S = \overline{f(R)}$; then* $\dim R \geq \dim S + d$. *–If T is the union of all the subvarieties R of V such that* $\dim R > \dim \overline{f(R)} + d$, *then U is a closed non-dense subset of V; in particular,* $\dim U < \dim V$.

In many cases the notion of dimension may play a similar role for linear groups as the notion of order does for finite groups.

LEMMA 2.5. *If A and B are closed subgroups of the algebraic linear group G, then for the complex AB one has*

$$\dim AB = \dim \overline{AB} = \dim A + \dim B - \dim (A \cap B).$$

Proof. At first we show that any component of $\overline{AB}$ has the same dimension as $A_0 B_0$, thus reducing the proof of the lemma to its proof for the connected subgroups A_0 and B_0. –Every component of the affine algebraic set $A \times B$ has the form $aA_0 \times bB_0$. The restriction ρ to $A \times B$ of the rational map of $G \times G$ onto G defined by $(g, h) \rightarrow gh^{-1}$ maps $A \times B$ onto AB and $A_0 \times B_0$ onto $A_0 B_0$; the component $aA_0 \times bB_0$ is mapped onto $aA_0 B_0 b^{-1}$; hence the image of every component of $A \times B$ has the same dimension as $A_0 B_0$. –Every component of $\overline{AB}$ has a dense subset which is the union of images under ρ of finitely many components of $A \times B$. Thus this dense subset and also the component of $\overline{AB}$ has the same dimension as $A_0 B_0$. –Now let ρ denote the restriction of the above mapping to $A_0 \times B_0$. We apply Lemma 2.4, replacing f by ρ, V by $A_0 \times B_0$, W by $\overline{A_0 B_0}$. Now take for S the subvariety 1 of $\overline{A_0 B_0}$. Any component R of $\rho^{-1}(1) = \{(d, d); d \in A_0 \cap B_0\}$ has the same dimension as $A_0 \cap B_0$. Lemma 2.4 yields

$$\dim A_0 \cap B_0 \geq \dim A_0 + \dim B_0 - \dim A_0 B_0 .$$

As furthermore $\dim a(A_0 \cap B_0)b = \dim A_0 \cap B_0$ for any pair $a \in A_0$, $b \in B_0$, the second part of Lemma 2.4 yields

$$\dim A_0 \cap B_0 \leq \dim A_0 + \dim B_0 - \dim A_0 B_0 .$$

Both inequalities together yield the statement of the lemma.

LEMMA 2.6[3]. *Let S be a solvable linear algebraic group, S_0 its connected component with $S_0 = UT$, where U is the subgroup of S_0 consisting of all the unipotent elements of S_0 and T is a maximal torus of S_0. Then for every almost nilpotent subgroup N of S one has*

$$(*) \qquad \dim N \leq \dim U + \dim \mathbf{C}_T U.$$

For the following short proof the author is greatly indebted to Professor T. A. Springer.

Proof. Since N has a nilpotent subgroup of the same dimension (Lemma 2.2), one may assume that N is nilpotent; and as the smallest closed subgroup of S that contains N is also nilpotent, one may assume N is algebraic and even connected. But if N is a connected closed subgroup of S it lies in S_0, and we may assume S to be connected, and $S = UT$.

By [18, exposé 6, p. 4, Th. 2], N is the direct product of its unipotent part V and its unique maximal torus L. Evidently it is sufficient to prove the assertion of the lemma for the subgroup $S_1 = UL$, which is again connected, i.e. one may assume (and we shall do this) $S = S_1$ and $T = L$.

It follows from [18, exposé 9, p. 1, Lemma 1] that there is a chain of closed connected subgroups V_i, $i = 0, \cdots, m$ of N such that

(a) $V = V_0 \subset V_1 \subset \cdots \subset V_m = U$;
(b) the normalizer of V_i contains V_{i+1} and T;
(c) there exists a morphism θ_i of V_i onto the additive group K^+, which is a group homomorphism with kernel V_{i-1} defining an isomorphism of V_i/V_{i-1} onto K^+; moreover, if $t \in T$, $g \in V_i$ one has

$$\theta_i(tgt^{-1}) = \chi_i(t)\theta_i(g)$$

where χ_i is a rational character of T.

If we define $Z_i = \mathbf{C}_T V_i$, then $Z_0 = T$. We have to prove the inequality

$$\dim V_0 + \dim Z_0 \leq \dim V_m + \dim Z_m .$$

For this it suffices to prove

$$\dim V_{i-1} + \dim Z_{i-1} \leq \dim V_i + \dim Z_i , \qquad 1 \leq i \leq m,$$

or that in fact

$$\dim Z_{i-1} \leq \dim Z_i + 1, \qquad 1 \leq i \leq m.$$

But an element t of Z_{i-1} lies also in Z_i if and only if $\chi_i(t) = 1$, (cf. [18, exposé

[3] A result similar to Lemma 2.6 lies behind the argument of M. Goto, *Note on a characterization of solvable Lie algebras*, J. Sci. Hiroshima Univ., Ser. A. I., vol. 26 (1962), pp. 1–2.

4, p. 13, Cor. 1]). Hence Z_i has codimension 1 at most in Z_{i-1}, which establishes the last inequality and hence the lemma.

Now we have all the auxiliary results and shall take up again the question of the structure of factorized groups.

THEOREM 2.7. *Let the linear group G be the product of (any number of) pairwise permutable locally nilpotent subgroups G_λ, $\lambda \in \Lambda$. The product of any two of the G_λ is solvable. –G itself is solvable if and only if the product of any three of the G_λ is solvable.*

To prove the second statement, it is sufficient to show that the product of any finite number of the G is solvable, if the product of any three of them is, i.e. that G is locally solvable; for a locally solvable linear group is well known to be solvable [24, p. 295].

The statement of the solvability of the product F of finitely many of the G_λ will be clear if there is a solvable normal subgroup S of finite index in $F = G_{\lambda_1} \cdots G_{\lambda_n}$: for then the finite group F/S is a product of the pairwise permutable nilpotent subgroups SG_{λ_i}/S, $i = 1, \cdots, n$, and such a group is known to be solvable [23], [11].

The conditional existence of such a normal subgroup S for finite products F as well as its existence in the case of two factors is contained in the following.

THEOREM 2.8. *If the linear group G is a product of n pairwise permutable almost nilpotent subgroups G_i, $i = 1, \cdots, n$, then every subgroup of the form $G_i G_j = G_j G_i$ has a solvable normal subgroup of finite index. –G has a solvable normal subgroup of finite index if and only if every product of any three of the G_i has such a subgroup.*

Proof. The necessity is obvious. –Suppose the statement of the theorem is false, and let G be a counterexample such that n has a minimal value (hence either $n = 2$ or $n \geq 4$). Assume the field K is algebraically closed and investigate the algebraic group $\bar{G}$. One may assume $\bar{G}$ is semi-simple, i.e. 1 is the only connected solvable normal subgroup of $\bar{G}$; for otherwise let S be the largest solvable connected normal subgroup of $\bar{G}$. Then $\bar{G}/S$ is also linear, and its dense subgroup GS/S meets all our assumptions.

Let G^i be the product of the $n - 1$ subgroups $G_j \neq G_i$. By the minimality of n, the subgroup G^i has a solvable normal subgroup of finite index. Thus by Lemma 2.2,

$$\dim G^i = \dim \overline{G^i} \leq \dim B,$$

where B is a Borel subgroup of the semi-simple group $\bar{G}$.

As G is dense in $\bar{G}$, we have by Lemma 2.5,

$$\dim \bar{G} = \dim G \leq \dim G^i + \dim G_i, \qquad i = 1, \cdots, n.$$

But with $B = UT$, where T is a maximal torus of G and U is the subgroup consisting of all the unipotent elements of B, one has also [18]

$$\dim G = 2 \dim U + \dim T.$$

As for Borel subgroups of semi-simple groups one has $\dim \mathbf{C}_T U = 0$; one obtains $\dim G_i \leq \dim U$, by Lemma 2.6. But this means (by Lemma 2.5)

$$\dim G^i = \dim B \quad \text{and} \quad \dim G_i = \dim U, \qquad i = 1, \cdots, n;$$

$$\dim (\overline{G^i} \cap \overline{G_i}) = 0.$$

With this knowledge apply Lemma 2.5 again to obtain

$$\dim G^i = (n-1) \dim U = \dim B = \dim U + \dim T.$$

This is possible in a semi-simple algebraic group only if $n = 3$ (and $\dim T = \dim U$). But for this case the existence of a solvable subgroup of finite index in G had been postulated; Lemma 2.2 yields even the existence of such a subgroup in $\bar{G}$. –Thus, such a counterexample cannot exist, and the theorem is proved.

Remarks. (1) Apparently no linear groups are known that are a product of three pairwise permutable almost nilpotent subgroups and do not have a solvable subgroup of finite index. The above argument could be refined to show that if such a group exists, then there is a group of this type contained in a finite extension of $PGL(2, K)$ for a suitable algebraically closed field K.

(2) Such an example cannot exist in the situation of Theorem 2.7 if one assumes furthermore that G is finitely generated; the same argument yielding Proposition 2.1 proves then that G is solvable.

Finally, a simple and well known observation should be stated.

PROPOSITION 2.9. *If the linear algebraic group G has the form $G = XY$, where X and Y are almost solvable subgroups of G, then G_0 is solvable.*

Proof. $G = XY$ evidently entails $G = \bar{X}\bar{Y}$, and $\bar{X}_0$, $\bar{Y}_0$ are solvable by Lemma 2.2. But then $G_0 = \bar{X}_0 \bar{Y}_0$ (cf. [14]). But $\bar{X}_0$ is contained in a Borel subgroup B of G_0 and $\bar{Y}_0$ in some conjugate B^g. Hence $G_0 = BB^g$, which means $G_0 = B = B^g$; and G_0 is solvable.

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