REGULARITY THEOREMS FOR $[F, d_n]$-TRANSFORMATIONS

BY

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1. Introduction

The $[F, d_n]$-method of summation was introduced by the first author in [2] as follows: Let $\{d_n\}$ $(n \geq 1)$ $(d_n \neq -1)$ be a real or complex sequence. The transformation-matrix $\{c_{nm}\}$ corresponding to this sequence is defined by $c_{00} = 1$, by the identity

$$\sum_{m=0}^{n} c_{nm}\, x^m = \prod_{j=1}^{n} (d_j + x)(d_j + 1)^{-1}, \qquad n \geq 1 \tag{1.1}$$

for $0 \leq m \leq n$, and by $c_{nm} = 0$ for $m > n$.

In [2] it was proved that if $d_n > 0$ for $n \geq n_0$ and $\sum d_n^{-1}$ is divergent, then the corresponding $[F, d_n]$-transformation is regular.

In a recent paper C. L. Miracle [4] obtained a family of regular $[F, d_n]$-transformation-matrices with complex elements defining the sequences $\{d_n\}$ on the following way. Suppose $\{\lambda_n\}$ is a positive sequence with

$$\sum \lambda_n^{-1} = +\infty.$$

The sequences $\{d_n\}$ are defined by taking successively the square roots of $-\lambda_n$, the cube roots of λ_n or the fourth roots of $-\lambda_n$, (see Theorems 2.1, 2.2, and 2.3 of [4]). In the conclusion of his paper C. L. Miracle asks whether the method used would be continuable to higher roots of positive sequences $\{\lambda_n\}$ yielding regular transformation-matrices. Our Theorem 1 answers this question and improves his results, namely, instead of the positiveness of $\{\lambda_n\}$ we assume only (2.1) and (2.2) which are weaker conditions. In Theorems 2 and 3 of the paper we prove the corrected and extended forms of some results stated in [1]. Theorems 4 and 5 show how further regular transformation-matrices with complex terms can be obtained from known ones. In §4 we deal with analytic continuation by these methods.

2. Regularity theorems

THEOREM 1. *Let $\{\lambda_n\}$ $(n \geq 1)$ $(\lambda_n \neq -1)$ be a sequence of real or complex numbers satisfying the following:*

$$\text{the } [F, \lambda_n]\text{-transformation is regular,} \tag{2.1}$$

$$(1 + |\lambda_n|)|1 + \lambda_n|^{-1} \leqq K < +\infty, \qquad n = 1, 2, \cdots. \tag{2.2}$$

Let r be a fixed positive integer. Denote by $-\lambda_p^{(1)}, -\lambda_p^{(2)}, \cdots, -\lambda_p^{(r)}$ $(p \geq 1)$ the r roots of

$$x^r + \lambda_p = 0,$$

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i.e., let

$$(2.3) \qquad (x+\lambda_p^{(1)})(x+\lambda_p^{(2)})\cdots(x+\lambda_p^{(r)}) = x^r+\lambda_p\,,\ p=1,2,\cdots$$

and define for $\nu = (p-1)r+q\ (0<q\le r)$

$$(2.4) \qquad d_\nu = \lambda_p^{(q)}, \qquad \nu = 1, 2, \cdots .$$

Then the $[F, d_n]$*-transformation is regular.*

THEOREM 2. *Let the* $[F, d_n]$*-transformation be regular. Then*

$$(2.5) \qquad \limsup_{n\to\infty} \operatorname{Re}(d_n) \ge 0.$$

(2.5) *is the best possible statement in the sense that there exists a sequence* $\{d_n^*\}$ *with* Re $(d_n^*) < 0$ *for all* n *and the* $[F, d_n^*]$*-transformation is regular.*

Theorem 2 improves Corollary 2.1 of [1].

THEOREM 3. *Let* $\{d_n\}$ $(n \ge 1)$ $(d_n \ne -1)$ *be a fixed sequence. Denote*

$$(2.6) \qquad 1 + d_n = r_n e^{i\phi_n} \qquad (0 \le \phi_n < 2\pi)$$

and suppose there exist α, β *such that*

$$(2.7) \qquad 0 < \alpha \le \liminf_{n\to\infty} \phi_n \le \limsup_{n\to\infty} \phi_n \le \beta < 2\pi$$

and

$$(2.8) \qquad \beta - \alpha < \pi.$$

Then the $[F, d_n]$*-transformation is not regular. The statement is best possible in the sense that there exists a sequence* $\{d_n^*\}$ *for which there exist* α, β *satisfying* (2.7) *with* $\beta - \alpha = \pi$ *and the* $[F, d_n^*]$*-transformation is regular.*

THEOREM 3 corrects and improves Theorem 2.2 and 2.3 of [1].

THEOREM 4. *Let the* $[F, \lambda_n]$*-transformation be regular, and* $q \ge 1$ *fixed. Let* $1 + d_n = q(1 + \lambda_n)$ *for* $n \ge 1$. *Then the* $[F, d_n]$*-transformation is regular. If* $q < 1$ *the statement in general is false.*

THEOREM 5. *Let* $\{a_n\}$ $(n \ge 1)$ $(a_n \ne -1)$ *and* $\{b_n\}$ $(n \ge 1)$ $(b_n \ne -1)$ *be two sequences for which the corresponding* $[F, a_n]$ *and* $[F, b_n]$*-transformations are regular. Let the sequence* $\{d_n\}$ $(n \ge 1)$ *be merged from the sequences* $\{a_n\}$ *and* $\{b_n\}$ *preserving the original order of the* a_n *and* b_n *respectively in the new sequence* $\{d_n\}$. *Then the* $[F, d_n]$*-transformation is regular.*

3. Proofs

Proof of Theorem 1. First we observe that by (2.3) and (2.4), $d_\nu \ne -1$ $(\nu \ge 1)$ since $\lambda_p \ne -1$ $(p \ge 1)$. The case $r = 1$ is trivial. We may assume $r > 1$. Let k be any positive integer; then by (2.3) and (2.4)

$$(3.1) \qquad \prod_{\nu=1}^{kr} (x + d_\nu) = \prod_{p=1}^{k} \prod_{q=1}^{r} (x + \lambda_p^{(q)}) = \prod_{p=1}^{k} (x^r + \lambda_p).$$

Denote the matrix of the $[F, d_n]$-transformation by $\{c_{n,m}\}$ and that of the $[F, \lambda_n]$-transformation by $\{a_{n,m}\}$.

Let n be any positive integer. Then if $n = kr + s$ with $0 \leq s < r$, we have by (2.3), (2.4) and (3.1)

$$\prod_{\nu=1}^{n} \frac{x + d_\nu}{1 + d_\nu} = \prod_{p=1}^{k} \frac{x^r + \lambda_p}{1 + \lambda_p} \cdot \prod_{q=1}^{s} \frac{x + \lambda_{k+1}^{(q)}}{1 + \lambda_{k+1}^{(q)}} \tag{3.2}$$

and thus by (11) it is clear that

$$\sum_{m=0}^{n} | c_{nm} | \leqq \sum_{m=0}^{k} | a_{km} | \cdot \prod_{q=1}^{s} \frac{1 + | \lambda_{k+1}^{(q)} |}{| 1 + \lambda_{k+1}^{(q)} |}. \tag{3.3}$$

Now, since the $[F, \lambda_n]$-transformation is regular, by the well known theorem of Toeplitz-Schur the first factor of the right-hand side $\leq H < +\infty$. By (2.3) clearly $| \lambda_{k+1}^{(q)} | = | \lambda_{k+1} |^{1/r}$ and since $s < r$ we have from (3.3)

$$\sum_{m=0}^{n} | c_{nm} | \leqq H \cdot \prod_{q=1}^{r} \frac{1 + | \lambda_{k+1} |^{1/r}}{| 1 + \lambda_{k+1}^{(q)} |}$$

which by (2.3)

$$= H \frac{(1 + | \lambda_{k+1} |^{1/r})^r}{| 1 + \lambda_{k+1} |}$$

and further by (2.2)

$$\leqq H \cdot K \cdot \frac{(1 + | \lambda_{k+1} |^{1/r})^r}{1 + | \lambda_{k+1} |}$$

and by an easy estimate

$$\leqq H \cdot K \cdot 2^r.$$

So

$$\textstyle\sum_{m=0}^{n} | c_{nm} | < C < +\infty, \qquad n = 0, 1, \cdots. \tag{3.4}$$

Also, if $n = kr + s$ $(0 \leq s < r)$ and $m = jr + t$ $(0 \leq t < r)$ we have for c_{nm}, the coefficient of x^m in the left-hand side of (3.2)

$$| c_{nm} | \leqq | a_{kj} | \cdot \prod_{q=1}^{s} \frac{1 + | \lambda_{k+1}^{(q)} |}{| 1 + \lambda_{k+1}^{(q)} |}$$

and using the same arguments as above for the second factor on the right-hand side

$$| c_{nm} | \leq | a_{kj} | \cdot K \cdot 2^r. \tag{3.5}$$

Now, if $n \to \infty$ also $k \to \infty$, and thus by the Toeplitz-Schur theorem, since $[F, \lambda_n]$ is regular

$$\lim_{k\to\infty} a_{kj} = 0, \qquad j = 0, 1, \cdots.$$

Therefore by (3.5)

$$\lim c_{nm} = 0, \qquad m = 0, 1, \cdots. \tag{3.6}$$

By (1.1) obviously

$$\textstyle\sum_{m=0}^{n} c_{nm} = 1, \qquad n = 0, 1, \cdots, \tag{3.7}$$

(3.4), (3.6) and (3.7) show that the conditions of the Toeplitz-Schur-theorem for regularity are satisfied, Q.E.D.

Remarks. (i) From the proof it is clear that instead of the *fixed* integer r we could allow r to take a bounded sequence of integer values $\{r_k\}$ and define $\{d_\nu\}$ successively by the r_k-*th* roots of the λ_k's.

(ii) The assumptions of the theorem are clearly satisfied if

$$\textstyle\prod_{n=1}^{\infty} (1 + |\lambda_n|) |1 + \lambda_n|^{-1} < +\infty \quad \text{and} \quad \sum_{n=1}^{\infty} |\lambda_n + 1|^{-1} = +\infty$$

(see [3, Theorem 3.c]); and especially if $\lambda_n > 0$ $(n \geq n_0)$ and $\sum \lambda_n^{-1} = +\infty$.

Proof of Theorem 2. Suppose, contrariwise, that

$$\limsup_{n\to\infty} \operatorname{Re}(d_n) < 0.$$

Then for a suitable δ, $0 < \delta < 1$,

$$\operatorname{Re}(d_k) \leq -\delta, \qquad k \geq k_0. \tag{3.8}$$

Clearly we may assume that for $1 \leq k < k_0$

$$\delta \neq 1 - d_k. \tag{3.9}$$

From (3.8) by elementary geometric considerations

$$|d_k - 1 + \delta| > |d_k + 1|, \qquad k \geq k_0. \tag{3.10}$$

Denote by $\{t_n\}$ the $[F, d_n]$-transform of the sequence $\{(-1 + \delta)^n$. By (1.1)

$$t_n = \textstyle\prod_{k=1}^{n} (d_k - 1 + \delta)(d_k + 1)^{-1} = \prod_{k=1}^{k_0-1} \cdot \prod_{k=k_0}^{n}.$$

The first factor on the right-hand side is $\neq 0$ by (3.9) and the absolute value of the second is > 1 by (3.10). Thus t_n does not tend to zero as $n \to \infty$ although $(-1 + \delta)^n$ does. This contradicts the regularity of $[F, d_n]$, and the theorem follows.

For showing that (2.5) is the best possible result of this type we choose

$$d_k^* = -(k + 1)^{-2}, \qquad k = 1, 2, \cdots.$$

The regularity of the $[F, d_n^*]$-transformation follows by [3, Theorem 3.c].

Proof of Theorem 3. First, it is obvious that we may assume

$$\alpha \leq \pi \leq \beta. \tag{3.11}$$

By (2.7) and (2.8) we can choose $\varepsilon > 0$ such that

$$0 < \alpha - \varepsilon < \phi_k < \beta + \varepsilon < 2\pi, \qquad k \geq k_0 \tag{3.12}$$

and

$$\beta - \alpha < \pi - 4\varepsilon. \tag{3.13}$$

Denote

$$\gamma = 2^{-1}(\alpha + \beta - \pi) \tag{3.14}$$

and let $z = e^{2i\gamma}$. It is not hard to see that we may assume

$$d_k \neq -z, \qquad 1 \leq k < k_0 \tag{3.15}$$

since if it would not be true, we might increase β such that (3.11), (3.12), (3.13) remain still satisfied with the same value of ε, and such that (3.15) holds too.

Denote by $\{t_n\}$ the $[F, d_n]$-transform of $\{z^n\}$. By (1.1)

$$| t_n |^2 = \prod_{k=1}^{k_0-1} \left| \frac{d_k + z}{d_k + 1} \right|^2 \cdot \prod_{k=k_0}^{n} \frac{| d_k + e^{2i\gamma} |^2}{| d_k + 1 |^2} \tag{3.16}$$

which by (3.15) and simple computation

$$= A \cdot \prod_{k=k_0}^{n} \{1 + 4r_k^{-2} \sin \gamma(r_k \sin (\phi_k - \gamma) + \sin \gamma)\}$$

where $A > 0$.

Now, by (3.12), (3.13) and (3.14)

$$\varepsilon < \phi_k - \gamma < \pi - \varepsilon, \qquad k \geq k_0 ;$$

thus

$$\sin (\phi_k - \gamma) = \delta > 0. \tag{3.17}$$

Also by (2.7) and (3.11)

$$0 < \gamma < \pi$$

and so

$$\sin \gamma > 0. \tag{3.18}$$

By (3.16), (3.17) and (3.18)

$$| t_n |^2 > A \cdot \textstyle\sum_{k=k_0}^{n} 4\delta \sin \gamma \cdot r_k^{-1}. \tag{3.19}$$

Now suppose the $[F, d_n]$-transformation is regular. By the first part of the proof of Theorem 3.c of [3]

$$\textstyle\sum_{k=1}^{\infty} r_k^{-1} = \sum_{k=1}^{\infty} | 1 + d_k |^{-1} = +\infty$$

is a necessary condition for regularity. Thus by (3.19)

$$\lim_{n\to\infty} | t_n | = +\infty. \tag{3.20}$$

From the other side by (1.1)

$$| t_n | = | \textstyle\sum_{m=0}^{n} c_{nm} z^m | \leq \sum_{m=0}^{n} | c_{nm} || z^m |$$

and since $|z| = 1$

$$\leq \sum_{m=0}^{n} |c_{nm}|$$

which by the Toeplitz-Schur theorem if the transformation is regular

$$\leq H < +\infty .$$

This contradicts (3.20) and so proves the theorem.

For proving that the statement is best possible of this type, we choose

$$d_{2k-1}^{*} = i\sqrt{k}, \qquad d_{2k}^{*} = -i\sqrt{k}, \qquad k = 1, 2, \cdots .$$

Clearly $\alpha = \pi/2$, $\beta = 3\pi/2$ satisfy (2.7) for this sequence $\{d_n^*\}$. Here $\beta - \alpha = \pi$. The regularity of the $[F, d_n^*]$-transformation follows from Theorem 1 by taking $r = 2$, $\lambda_k = k$.

Proof of Theorem 4. It is easy to see (compare [2, Lemma 5.1]) that the $[F, q(\lambda_n + 1) - 1]$ transformation is the $[F, \lambda_n]$-transform of the $[F, q - 1]$-transform. Since the $[F, \lambda_n]$-transformation is supposed to be regular and the $[F, q - 1]$-transformation is regular for $q \geq 1$ by Theorem 3.1 of [2], the regularity of $[F, q(\lambda_n + 1) - 1]$ follows. For proving that for $q < 1$ the theorem is not true in general, we choose $\lambda_n = 0$ for all n. Then $d_n = q - 1 < 0$ and so by Theorem 2 the $[F, d_n]$-transformation is not regular.

Proof of Theorem 5. Denote by $\{A_{nm}\}$ and $\{B_{nm}\}$ the matrices of the $[F, a_n]$- and $[F, b_n]$-transformations respectively, and as usual, by $\{c_{nm}\}$ the matrix of the $[F, d_n]$-transformation. Let n be any integer and suppose the set $\{d_1, d_2, \cdots, d_n\}$ contains the $r = r(n)$ terms $a_1, a_2, \cdots, a_r$ and the $(n - r)$ terms $b_1, b_2, \cdots, b_{n-r}$. Then

$$\prod_{\nu=1}^{n} \frac{x + d_\nu}{1 + d_\nu} = \left(\prod_{\nu=1}^{r} \frac{x + a_\nu}{1 + a_\nu}\right) \cdot \left(\prod_{\nu=1}^{n-r} \frac{x + b_\nu}{1 + b_\nu}\right).$$

Comparing the coefficients of x^m on both sides, we get by (1.1)

$$c_{nm} = \sum_{\nu=0}^{m} A_{r\nu} B_{n-r,m-\nu}, \qquad m = 0, 1, \cdots . \tag{3.21}$$

(Note that $A_{ij} = B_{ij} = 0$ if $i < j$.)

From (3.21)

$$\sum_{m=0}^{n} |c_{nm}| \leq \sum_{m=0}^{n} \sum_{\nu=0}^{m} |A_{r\nu}| |B_{n-r,m-\nu}| \tag{3.22}$$

which clearly

$$\leq \left(\sum_{\nu=0}^{r} |A_{r\nu}|\right) \cdot \left(\sum_{\nu=0}^{n-r} |B_{n-r,\nu}|\right)$$

and since the $[F, a_n]$- and $[F, b_n]$-transformations are regular, by the Toeplitz-Schur theorem

$$\leq H_1 \cdot H_2 < +\infty .$$

If $n \to \infty$ either r or $(n - r)$ or both tend to ∞. Without loss of generality we may assume $r \to \infty$, because the assumptions for the sequences $\{a_n\}$ and $\{b_n\}$ are symmetric.

From (3.21)

(3.23) $$|c_{nm}| \leq (\max_{0 \leq \nu \leq m} |A_{r\nu}|) \cdot (\textstyle\sum_{\nu=0}^{n-r} |B_{n-r,\nu}|)$$

which by the regularity of $[F, b_n]$

$$\leq H_2 \cdot (\max_{0 \leq \nu \leq m} |A_{r\nu}|).$$

Now, since the $[F, a_n]$-transformation is regular, by the Toeplitz-Schur theorem

$$\lim_{r \to \infty} A_{r\nu} = 0, \qquad \nu = 0, 1, \cdots.$$

Thus

(3.24) $$\lim_{n \to \infty} c_{nm} = 0, \qquad m = 0, 1, \cdots.$$

Since by (1.1)

(3.25) $$\textstyle\sum_{m=0}^{n} c_{nm} = 1$$

for all n, by (3.22), (3.24) and (3.25) the regularity of the $[F, d_n]$-transformation follows.

4. Analytic continuation of the geometric series

It is known that the $[F, \lambda_n]$-transform, say $\{\sigma_n(z)\}$ of the sequence $\{s_n(z)\}$ ($s_n(z) = 1 + z + \cdots + z^n$) tends to the value $(1 - z)^{-1}$ for $z \neq 0$, if and only if

(4.1) $$\lim_{n \to \infty} \prod_{\nu=1}^{n} \frac{\lambda_\nu + z}{\lambda_\nu + 1} = 0.$$

Combining this fact with our Theorem 1 we improve Theorems (3.1)–(3.5) and (3.7)–(3.11) of [4] by

THEOREM 6. *Suppose* $\{\lambda_n\}$ *satisfy the conditions of Theorem* 1 *and denote by* D *the set of* z *for which* (4.1) *holds and by* E *the set of* z *for which* (4.1) *does not hold. Let* $\{d_n\}$ *be defined as in Theorem* 1 *by* (2.3) *and* (2.4). *Then the* $[F, d_n]$-*transformation sums the geometric series to the value* $(1 - z)^{-1}$ *for every* z *for which* $z^r \in D$, *and does not sum it to* $(1 - z)^{-1}$ *for* z ($z \neq 0$) *for which* $z^r \in E$.

Proof. As in (3.2) if $n = kr + s$ $(0 \leq s < r)$

$$\prod_{\nu=1}^{n} \frac{d_\nu + z}{d_\nu + 1} = \prod_{p=1}^{k} \frac{\lambda_p + z^r}{\lambda_p + 1} \cdot \prod_{q=1}^{s} \frac{z + \lambda_{k+1}^{(q)}}{1 + \lambda_{k+1}^{(q)}}.$$

Now, since $1 + |z| + |\lambda_{k+1}^{(q)}|$ is greater than $|1 + \lambda_{k+1}^{(q)}|$ and also than $|z + \lambda_{k+1}^{(q)}|$ we obtain easily

$$\left| \prod_{\nu=1}^{n} \frac{d_\nu + z}{d_\nu + 1} \right| < \left| \prod_{p=1}^{k} \frac{\lambda_p + z^r}{\lambda_p + 1} \right| \cdot \prod_{q=1}^{r} \frac{1 + |z| + |\lambda_{k+1}^{(q)}|}{|1 + \lambda_{k+1}^{(q)}|}$$

which by (2.3) and (2.2)

$$< K \cdot \frac{(1 + |z| + |\lambda_{k+1}|^{1/r})^r}{1 + |\lambda_{k+1}|} \cdot \left| \prod_{p=1}^{k} \frac{\lambda_p + z^r}{\lambda_p + 1} \right|$$

and by an easy estimate

$$\leqq 2^r(1 + |z|)^r \cdot K \cdot \left| \prod_{p=1}^{k} \frac{\lambda_p + z^r}{\lambda_p + 1} \right| .$$

If $z^r \in D$, by (4.1) the last expression tends to zero if $k \to \infty$; thus also

$$\lim_{n \to \infty} \prod_{\nu=1}^{n} \frac{d_\nu + z}{d_\nu + 1} = 0.$$

Therefore the $[F, d_n]$-transformation sums the geometric series to $(1 - z)^{-1}$ if $z^r \in D$. On the other hand, if $z^r \in E$ the expressions

$$\prod_{\nu=1}^{kr} \frac{d_\nu + z}{d_\nu + 1} = \prod_{p=1}^{k} \frac{\lambda_k + z^r}{\lambda_k + 1}$$

do not tend to a finite limit as $k \to \infty$; thus the $[F, d_n]$-transform does not sum the geometric series to $(1 - z)^{-1}$ if $z \neq 0$ and $z^r \in E$. By Theorems (4.1)–(4.4) of [1] and by our Theorem 6 the results stated in Theorems (3.1)–(3.5) and (3.7)–(3.11) follow as special cases.

REFERENCES

1. V. F. COWLING AND C. L. MIRACLE, *Some results on the generalized Lototsky transform*, Canad. J. Math., vol. 14 (1962), pp. 418–435.
2. A. JAKIMOVSKI, *A generaliation of the Lototsky method of summability*, Michigan Math. J., vol. 6 (1959), pp. 277–290.
3. A. MEIR, *On the* $[F, d_n]$-transformations of A. Jakimovski, Bull. Res. Council Israel, vol. 10F4 (1962), pp. 165–187.
4. C. L. MIRACLE, *Some regular* $[F, d_n]$*-matrices with complex elements*, Canad. J. Math., vol. 15 (1963), pp. 503–525.

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