MARKOV PROCESSES AND MARTIN BOUNDARIES PART I

BY

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The results of this paper were announced in [9]: it is shown how the theory of Martin boundaries can be carried out for certain general types of Markov processes including Brownian motions and Markov chains for which the theory was established by several authors [4], [7], [11], [13]. The method employed here is, essentially, Martin's method translated into probability languages. See also [11], [13] for the ideas involved.

The class of continuous parameter Markov processes for which there is a potential kernel (of function type) will be discussed in Part I and certain classes of continuous parameter processes proceeding in simple jumps and discrete parameter processes, in Part II.

1. Outline of Part I

Basic notions on Markov processes are defined in Section 2. Terminology and notation are taken mainly from [1] and [5]. Several known facts on excessive functions as well as several new results on superharmonic functions and harmonic functions are collected in Section 3 and basic facts on resolvent kernels, in Section 4. These sections constitute the preliminary part.

In Sections 5–8 we generalize results of Hunt [6, Part III, Sections 17, 18] by a method different from Hunt's.

Let X be a transient Hunt process (see Section 2 for the definition) taking values in a locally compact separable space S such that

$$G_0(x, A; X) = \int_0^\infty P_x\{x_t \in A\}\, dt$$

is bounded in $x \in S$ if A is compact. Brownian motion Z on a Green space R (for instance, any bounded domain of Euclidean n-space) is such a process. The Newtonian potential kernel (or the Green function) of R, denoted by $G(x, y; Z)$, is associated with Z in the following way;

$$G_0(x, A; Z) = \int_A G(x, y; Z)\, dy, \tag{1.1}$$

where dy is the volume element of R. The problem of determining a potential kernel (of function type) $G(x, y; X)$ associated with a general process X is discussed in Section 5. It will be natural that the first requirement is to keep the relation (1.1) with X and a measure m over S in place of Z and dy;

$$G_0(x, A; X) = \int_A G(x, y; X) m\, (dy). \tag{1.2}$$

Received March 9, 1964.

This condition determines $G(x, y; X)$ uniquely up to a set of (m) measure 0 for each x. The crucial point is to eliminate arbitrariness of measure 0. We will denote by $G_\alpha(x, A; X)$ the resolvent kernel of X and assume the existence of the co-resolvent kernel $\hat{G}_\alpha(x, A; X)$ defined by the relation

$$\int_B G_\alpha(x, A; X)m\,(dx) = \int_A \hat{G}_\alpha(y, B; X)m\,(dy),$$

where α is any positive number and A and B are any Borel sets of S with compact closure. The definition of the potential kernel involves the co-resolvent kernel as well as the resolvent kernel and the measure m. $G(x, y; X)$ is said to be a potential kernel (of function type) if it is excessive in x for each y and co-excessive in y for each x and if (1.2) and

$$\hat{G}_0(y, A; X) = \int_A G(x, y; X)m\,(dx)$$

are satisfied, where $\hat{G}_0(y, A; X) = \lim_{\alpha\to 0} \hat{G}_\alpha(y, A; X)$. Such a kernel, if it exists, is unique (with arbitrariness of (m) measure 0 eliminated). An obvious necessary condition for the existence of such kernel $G(x, y; X)$ is that $G_0(x, A; X)$ and $\hat{G}_0(x, A; X)$ are absolutely continuous with respect to $m(A)$ for each x. The first key result (Theorem 1) is that this condition is also sufficient.

Sections 6–8 are devoted to a potential theory based on the kernel $G(x, y; X)$ under hypothesis (B) involving some regularity properties of the co-resolvent kernel. But since the potential theory itself is not our purpose, we will only give results enough to cover the application to Martin boundaries. A new phenomenon in the potential theory based on $G(x, y; X)$ is that $G(x, y; X)$ may not be a potential as a function of x for some y. We will denote by S_P the set of y's for which $G(\cdot, y; X)$ is a potential. Conditions for $S = S_P$ are studied in Section 13. In the Brownian motion case, the resolvent kernel of Z is also the unique co-resolvent kernel (relative to the Lebesgue measure of R) satisfying hypothesis (B), and the corresponding potential kernel turns out to be the Newtonian potential kernel.

The Martin boundary is introduced in Section 9. Let r be a measure defined over S such that $\int r(dx)G(x, y; X)$ is continuous in y, taking values in $(0, \infty]$. The (generalized) Martin potential kernel associated with $G(x, y; X)$ and with r is defined by

$$\kappa(x, y; X) = \frac{G(x, y; X)}{\int r\,(dx)G(x, y; X)} \quad \text{if} \quad \int r\,(dx)G(x, y; X) < \infty$$

$$= 0 \qquad\qquad \text{if} \quad \int r\,(dx)G(x, y; X) = \infty.$$

Obviously this generalizes the (usual) Martin potential kernel associated with the Newtonian potential kernel $G(x, y; Z)$ and with a reference point x_0

which is defined by

$$\kappa(x, y; Z) = \frac{G(x, y; Z)}{G(x_0, y; Z)} \quad \text{if} \quad y \neq x_0$$
$$= 0 \quad\quad \text{if} \quad y = x_0.^{1}$$

$\kappa(x, y; X)$ induces a boundary S' of S in a way similar to the way that $\kappa(x, y; Z)$ induces the Martin boundary R' of R. The second key result of this paper (Theorem 3) is that $\kappa(x, y; X)$ can be extended uniquely in a certain sense from $S \times S$ to $S \times (S + S')$. The method of Martin [11] for $\kappa(x, y; Z)$ can not be applied to the general case.

Let A be a Borel set of $S + S'$. The reduced function $\bar{H}_A\, u$ of an excessive function u is defined in Section 10. We will denote by S_1' the set of points η of S' for which $\bar{H}_{\{\eta\}}\, \kappa(x, \eta; X)$ is not identically zero and by S_1 the set of points y of S_P for which $\int r(dx)G(x, y; X)$ is finite. The (generalized) Martin representation theorm of excessive functions is proved in Section 11 which asserts that each (r) integrable excessive function u has the unique integral representation

$$u = \int_{S_1+S_1'} \kappa(\cdot, \eta; X)\mu(d\eta).$$

This is the analogue of the (usual) Martin representation theorem of positive superharmonic functions on R;

$$(1.3) \qquad h = \int_{R-\{x_0\}+R_1'} \kappa(\cdot, \eta; Z)\mu^h(d\eta),$$

where h is a positive superharmonic function finite at x_0 and R_1' is the set of minimal boundary points.

Doob [3] introduced Brownian h-path process Z^h for each positive superharmonic function h of R and proved a theorem which gives the measure μ^h of (1.3) a probabilistic interpretation in terms of Z^h. In Section 12 we will generalize the theorem to a general process X as follows: Doob's theorem is true if and only if every point y such that $\int r(dx)G(x, y; X)$ is finite belongs to S_P. This conclusion is closely connected with the following result of Section 13: $S = S_P$ if and only if any h-path process of X is a transient Hunt process.

Throughout Sections 2–13 we assume that the basic process X is a transient Hunt process, namely, a standard process satisfying hypotheses (A_6) and (A_7) of Section 2. But hypotheses (A_6) and (A_7) can be removed if X is a standard process satisfying hypothesis (B). We have chosen to prove this fact in the final section rather than to do without these hypotheses from the beginning, for the following two reasons. One is that we did not like to make too complicated the organization of the paper. The second reason is that the

[1] Actually, Martin defined $\kappa(x, y; Z) = 1$ if $x = y = x_0$. See footnote 18.

argument employed to remove hypotheses (A_6) and (A_7) is more naturally connected with a quite different approach to the whole material of this paper which we will discuss in another place (see footnote 29).

2. Markov processes

Let S be a locally compact, noncompact, separable Hausdorff space and Δ, a point adjoined to S as the point at infinity. We will denote by $\mathfrak{B}$ the σ-field of all Borel subsets of S and by $\mathfrak{A}$ the σ-field consisting of all the subsets of S which, for each finite measure μ defined on $(S, \mathfrak{B})$, are in the completed σ-field of $\mathfrak{B}$ relative to μ. Let w denote a function from $[0, \infty)$ to $S \cup \{\Delta\}$, $x_t = x_t(w) = w(t)$ the value at t and $\zeta(w) = \inf \{t \geqq 0, x_t(w) = \Delta\}$.[2] The sample space for our process will be taken as the set W of all w's which are right continuous, have the left-hand limits in S for $t \in [0, \zeta)$ and satisfy $w(t) = \Delta$ for $t \geqq \zeta$. The set W is closed under the shift operation θ_t defined by $(\theta_t w)(s) = w(t + s)$, $s, t \geqq 0$. $\mathfrak{F}$ is the σ-field in W generated by sets $\{x_t \in A\}$ for each $t \geqq 0$ and for each A of $\mathfrak{B}$. Obviously ζ is $\mathfrak{F}$-measurable. $\mathfrak{F}_t$ is the σ-field in $W_t = \{\zeta > t\}$ generated by sets $\{x_s \in A, \zeta > t\}$ for each s $(0 \leqq s \leqq t)$ and for each A of $\mathfrak{B}$. For each x of S, let $P_x(\cdot)$ denote a probability measure over $(W, \mathfrak{F})$. For an $\mathfrak{F}$-measurable function φ and $\mathfrak{F}$-measurable set Λ, we write $E_x(\varphi; \Lambda)$ for $\int_\Lambda \varphi(w) P_x(dw)$.

The system $X = (W, x_t, \zeta, \mathfrak{F}_t, \mathfrak{F}, P_x, x \in S)$ is called a *right continuous* (*stationary*) *Markov process* if it satisfies the following conditions: (A_1) For each Λ of $\mathfrak{F}$, $P_x(\Lambda)$ is $\mathfrak{A}$-measurable; (A_2) for each x of S, $x_0(w) = x$ a. e.[3] (P_x); (A_3) for each x of S, $t \geqq 0$, Λ in $\mathfrak{F}_t$ and bounded $\mathfrak{F}$-measurable function φ

$$E_x\{\varphi(\theta_t w); \Lambda\} = E_x\{E_{x_t}(\varphi); \Lambda\}.$$

A random time (= nonnegative function defined on W allowing the value infinity) τ is said to be an $(\mathfrak{F})$ *stopping time* if $\{\tau < t < \zeta\} \in \mathfrak{F}_t$ for all $t \geqq 0$. For an $(\mathfrak{F})$ stopping time τ, let $\mathfrak{F}_{\tau+}$ denote the σ-field formed by all the sets Λ in $W_\tau = \{\zeta > \tau\}$ such that $\Lambda \in \mathfrak{F}$ and $\Lambda \cap \{\tau < t < \zeta\} \in \mathfrak{F}_t$ for all $t \geqq 0$.

A right continuous Markov process X is said to be *standard* if the following conditions are satisfied: (A_4) For each x of S, $(\mathfrak{F})$ stopping time τ, Λ in $\mathfrak{F}_{\tau+}$ and bounded $\mathfrak{F}$-measurable function φ

$$E_x\{\varphi(\theta_\tau w); \Lambda\} = E_x\{E_{x_\tau}(\varphi); \Lambda\};$$

and (A_5) if $\{\tau_n\}$ is any increasing sequence of $(\mathfrak{F})$ stopping times and if $\tau = \lim_{n\to\infty} \tau_n$, then for each x, $x_{\tau_n} \to x_\tau$ a.e. (P_x) on W_τ.

Let A be a subset of S. The *nonnegative hitting time* $\tau(A)$ and *positive hitting time* $\tau^+(A)$ are defined as follows;

$$\tau(A) = \inf \{t \geqq 0, x_t \in A\} \quad \text{and} \quad \tau^+(A) = \inf \{t > 0, x_t \in A\}.$$

If there is no t satisfying the condition in the parentheses, we set

[2] If there are no such t, then we set $\zeta(w) = \infty$.

[3] Almost everywhere relative to P_x.

$\tau(A)$ (resp. $\tau^+(A)$) $= \zeta$. Hereafter such convention will be made without special mention whenever a random time is defined by some specified condition. If A is open, $\tau(A)$ and $\tau^+(A)$ are equal and they define an $(\mathfrak{F})$ stopping time. But in order to treat hitting times of more general sets, we must introduce a new system of σ-fields which is slightly larger than the system $(\mathfrak{F}, \mathfrak{F}_t)$. For each finite measure μ on $(S, \mathcal{B})$, define

$$P_\mu(\cdot) = \int_S P_x(\cdot)\mu(dx),$$

which gives again a measure over $(W, \mathfrak{F})$. Let $\mathcal{G}$ be the intersection, over all μ, of the (P_μ) completed σ-fields of $\mathfrak{F}$ and let $\mathcal{G}_t$ be the σ-field formed by all subsets of $\mathcal{G} \cap W_t$, which, for each μ, differ by at most a set of (P_μ) measure 0 from a set of $\mathfrak{F}_t$. For each bounded $\mathcal{G}$-measurable function φ, $E_x(\varphi)$ is well defined for all x and $\mathcal{A}$-measurable as a function of x. Also $\mathcal{G}$ (resp. $\mathcal{G}_t$) includes all sets $\{x_s \in A\}$ (resp. $\{x_s \in A, \zeta > t \geqq s\}$) for each $s \geqq 0$ and for each A of $\mathcal{A}$. A random time τ is called a $(\mathcal{G})$ *stopping time if* $\{\tau < t < \zeta\} \in \mathcal{G}_t$ for all $t \geqq 0$. For a $(\mathcal{G})$ stopping time τ, $\mathcal{G}_{\tau+}$ denotes the σ-field formed by the sets Λ in $W_\tau = \{\zeta > \tau\}$ such that $\Lambda \in \mathcal{G}$ and $\Lambda \cap \{\tau < t < \zeta\} \in \mathcal{G}_t$ for all $t \geqq 0$. Suppose X is a standard process. Then it is known [14] that *the properties* (A_4) *and* (A_5) *remain still valid if "*$\mathfrak{F}$*" is replaced by "*$\mathcal{G}$*" in each statement.* Moreover, *if A is an analytic set of S, $\tau(A)$ and $\tau^+(A)$ are* $(\mathcal{G})$ *stopping times* (see [5], [6]).

We will say a standard process X is a *Hunt process* if, (A_6) for each x of S, the left hand limits $x_{\zeta-}$ at $t = \zeta$ exist in $S \cup \{\Delta\}$ a.e. (P_x) on the set $\{\zeta < \infty\}$. A process X is said to be *transient* if (A_7), for each x of S and for each compact set A of S, x_t is not in A for all sufficiently large t, a.e. (P_x). If X is a transient Hunt process, then (A_6) is true even if we remove the phrase *on the set* $\{\zeta < \infty\}$ in the statement. Let A be an open set of S with compact closure and B, a closed neighborhood of A. Define

$$\tau_1 = \text{the hitting time for } A,$$

$$\tau_2 = \text{the hitting time for } S - B \text{ after } \tau_1,$$

$$\tau_3 = \text{the hitting time for } A \text{ after } \tau_2.$$

$\tau_4, \tau_5, \cdots$ are defined successively. Hypotheses (A_6) and (A_7) can be restated as follows: For any A and B as above

$$P_x\{\tau_n < \zeta \text{ for every } n\} = 0 \qquad \text{for all } x$$

and for each compact set C

$$P_x\{\tau(S - C) < \infty\} = 1 \qquad \text{for all } x.$$

This kind of fact will be used repeatedly in later sections.

From now on we will assume the basic process X is a transient Hunt process.

3. Excessive functions and some other related functions

Let f be an $\mathcal{A}$-measurable function defined on S, τ a $\mathcal{G}$-measurable random time, $\tau(A)$ and $\tau^+(A)$ the hitting times of an analytic set A, and B a set of $\mathcal{A}$. Define $f(x_\tau) = 0$ if $\tau \geqq \zeta$. With this conevntion, one may write $E_x\{f(x_\tau)\}$ for $E_x\{f(x_\tau); \tau < \zeta\}$. We use the following system of notation:

$$H_\tau(x, B) = P_x\{x_\tau \in B\},$$

$$H_\tau f(x) = E_x\{f(x_\tau)\},$$

$$H_A(x, B) = H_{\tau(A)}(x, B),$$

$$H_A^+(x, B) = H_{\tau^+(A)}(x, B),$$

$$G_\alpha(x, B) = \int_0^\infty e^{-\alpha t} H_t(x, B)\, dt \qquad \text{for } \alpha \geqq 0.$$

$H_A f$, $H_A^+ f$ and $G_\alpha f$ are defined similarly to $H_\tau f$. $\{H_t(x, B), t \geqq 0\}$, $\{H_A(x, B)$ (or $H_A^+(x, B)$), $A \in \mathcal{A}\}$ and $\{G_\alpha(x, B), \alpha > 0\}$ are called, respectively, the *transition function, system of harmonic measures* and *resolvent kernel of the process* X.

Let $\mathbf{A}^+$ be the space of functions defined on S, $\mathcal{A}$-measurable and nonnegative (allowing the value infinity). A function u of $\mathbf{A}^+$ is said to be *excessive* if $H_t u \leqq u$ for all $t \geqq 0$ and if $H_t u \to u$ $(t \to 0)$. Basic results on excessive functions are found in [6, Part I]. Here we will list some of them.

PROPOSITION 3.1. *The following three statements are equivalent to each other.* (i) *u is excessive.* (ii) *$\alpha G_\alpha u \leqq u$ for all $\alpha > 0$ and $\alpha G_\alpha u \to u$ $(\alpha \to \infty)$.* (iii) *$H_\tau u \leqq u$ for each $(\mathcal{G})$ stopping time τ and $H_{\tau_n} u \to u$ $(n \to \infty)$ if $\{\tau_n\}$ is a sequence of $(\mathcal{G})$ stopping times and if τ_n decreases to* 0 a.e. (P_x) *for all x.*

PROPOSITION 3.2. *Let u be an excessive function, $\{\tau_n\}$ a decreasing sequence of $(\mathcal{G})$ stopping times, $\tau = \lim_{n\to\infty} \tau_n$ and Λ a set of $\mathfrak{F}_{\tau+}$. Then* (i) *$u(x_t)$ is right continuous in t,* a.e. (P_x) *for each x.* (ii) *$E_x\{u(x_\tau); \Lambda\} \geqq E_x\{u(x_{\tau_n}); \Lambda\}$ and the right side increases to the left side as $n \to \infty$. Therefore* (iii) *$u(x_{\tau_n})$ is uniformly integrable relative to (P_x).*

A function u of $\mathbf{A}^+$ is said to be *quasi-excessive* if it satisfies

$$\alpha G_\alpha u \leqq u \qquad \text{for all } \alpha > 0. \tag{3.1}$$

Then, as will be shown in the next section, the left side increases with α and the function $\lim_{\alpha\to\infty} \alpha G_\alpha u$ is excessive. This limit function is called the *regularization* or *smoothed version* of u and denoted by reg. u. If a function u of $\mathbf{A}^+$ satisfies

$$H_t u \leqq u \qquad \text{for all } t \geqq 0, \tag{3.2}$$

the left side increases as $t \downarrow 0$ and $\lim_{t\to 0} H_t u$ defines an excessive function. But since (3.2) implies (3.1), such a function u is also quasi-excessive. We have

$$\lim_{t\to 0} H_t u = \lim_{\alpha\to\infty} \alpha G_\alpha u = \text{reg. } u.$$

Let u be an excessive function, A an analytic set of S and $\tau(t, A, w) = t + \tau(A, \theta_t w)$. It follows that $\tau(t, A) \geqq \tau^+(A) \geqq \tau(A)$ and that $\tau(t, A)$ decreases to $\tau^+(A)$ as $t \to 0$. Therefore by Proposition 3.2,

$$H_{\tau(t,A)} u = H_t H_A u \leqq H_A^+ u \leqq H_A u$$

and $H_t H_A u$ increases to $H_A^+ u$ as $t \to 0$, which implies that $H_A u$ is quasi-excessive and that $H_A^+ u = \text{reg. } H_A u$. In particular, if A is open (more generally, nearly open [6]), then $H_A u = H_A^+ u$, so that $H_A u$ is also excessive. The following proposition comes from [6, Part I, Proposition 6.1].

PROPOSITION 3.3. *Let u be an excessive function and $\{A_n\}$, a monotone sequence of open subsets of S increasing to A. Then $H_{A_n} u$ increases to $H_A u$ as $n \to \infty$.*

Let u be a function of $\mathbf{A}^+$ and G, an open subset of S. The function u is is said to be *superharmonic* if, for each open subset A of S with compact closure

$$u \geqq H_{\tilde{A}} u,$$

where we write $\tilde{A}$ or $A^\sim$ for $S - A$. (Hereafter this notation will be used without mention.) The function u is said to be *harmonic on G* (just *harmonic* if $G = S$) if, for each open subset A of G whose closure is compact in G,

$$u = H_{\tilde{A}} u.$$

PROPOSITION 3.4. *If u is superharmonic, then it is quasi-excessive. In particular, if u is harmonic, then it is excessive.*

Let K be any compact set of S and A, an open set with compact closure. If u is superharmonic,

$$\begin{aligned} u &\geqq H_{K \cup \tilde{A}}\, u \\ &\geqq E_\bullet\{u(x_{\tau(K)});\ \tau(K) \leqq \tau(\tilde{A})\}. \end{aligned}$$

Letting $A \uparrow S$,[4] we have $u \geqq H_K u$, so that u is quasi-excessive according to a theorem of Dynkin [1, Theorem 4.1].[5] Next suppose u is harmonic. Let x be any fixed point of S and A, an open neighborhood of x with compact closure. By definition of a harmonic function, $u = H_{\tilde{A}} u$, so that

$$H_t u(x) = H_t H_{\tilde{A}} u(x) = E_x\{u(x_{\tau(t,\tilde{A})})\},$$

[4] In general, by $A \uparrow S$ we understand a sequence of sets $\{A_n\}$ to be chosen as follows: A_n increases to S, each A_n has compact closure in S and A_{n+1} is a neighborhood of A_n. This convention is repeatedly used in Sections 7, 8.

[5] In [1], the proof is carried out under the condition that $u \geqq H_K^+ u$. But it is applicable for our case,with no change.

where $\tau(t, \tilde{A}, w) = t + \tau(\tilde{A}, \theta_t w)$. Take w such that $\tau(\tilde{A}, w) > 0$. Then $\tau(t, \tilde{A}, w) = \tau(\tilde{A}, w)$ for every $t \leqq \tau(\tilde{A}, w)$. Since $P_x\{w; \tau(\tilde{A}, w) > 0\} = 1$, we have

$$\begin{aligned} \liminf_{t\to 0} H_t u(x) &\geqq E_x\{\liminf_{t\to 0} u(x_{\tau(t,\tilde{A})})\} \\ &= E_x\{u(x_{\tau(\tilde{A})})\} \\ &= H_{\tilde{A}} u(x) \\ &= u(x), \end{aligned}$$

so that

$$\begin{aligned} \lim_{\alpha\to\infty} \alpha G_\alpha u(x) &= \lim_{\alpha\to\infty} \int_0^\infty e^{-t} H_{t/\alpha} u(x)\, dt \\ &\geqq \int_0^\infty e^{-t} \liminf_{\alpha\to\infty} H_{t/\alpha} u(x)\, dt \\ &\geqq u(x). \end{aligned}$$

PROPOSITION 3.5. *If u is harmonic and if A is an analytic set with compact closure,*

$$u = H_{\tilde{A}} u = H_{\tilde{A}}^{+} u.$$

Let $\bar{A}$ be the closure of the set A, $\bar{A}^{\sim}$ the complement of $\bar{A}$, and A_0 an open neighborhood of $\bar{A}$ with compact closure. Noting that u is excessive, we have

$$\begin{aligned} u &\geqq H_{\tilde{A}} u \geqq H_{\tilde{A}}^{+} u \\ &\geqq H_{\bar{A}^\sim}^{+} u = H_{\bar{A}^\sim} u \\ &\geqq H_{\tilde{A}_0} u = u. \end{aligned}$$

PROPOSITION 3.6. *Let G_1 and G_2 be open subsets of S, and u an excessive function. If u is harmonic on each G_i $(i = 1, 2)$ and if u is bounded on each compact set of $G = G_1 \cup G_2$, then u is harmonic on G.*

It is enough to show that $H_{\tilde{K}} u \geqq u$ for each compact set K of G. There are subsets A_i and B_i of G_i $(i = 1, 2)$ such that A_i is open, B_i is a compact neighborhood of A_i and $A_1 \cup A_2 \supset K$. Define

$$\begin{aligned} \tau_1(w) &= \tau(\tilde{B}_1, w) \quad \text{if} \quad x_0(w) \in A_1 \\ &= \tau(\tilde{B}_2, w) \quad \text{if} \quad x_0(w) \in A_2 \cap \tilde{A}_1 \\ &= 0 \qquad\qquad \text{otherwise} \end{aligned}$$

and

$$\tau_n(w) = \tau_{n-1}(w) + \tau_1(\theta_{\tau_{n-1}} w).$$

Then $H_{\tau_1} u = u$ and hence $H_{\tau_n} u = H_{\tau_{n-1}} H_{\tau_1} u = H_{\tau_{n-1}} u = \cdots = u$ for every n. Moreover hypotheses (A_6) and (A_7) imply that

$$\lim_{n\to\infty} P_x\{\tau_n < \tau(\tilde{K})\} = 0$$

for each x of S. But we have

$$\begin{aligned} H_{\tau_n} u &= E_\bullet\{u(x_{\tau_n}); \tau_n < \tau(\tilde{K})\} + E_\bullet\{u(x_{\tau_n}); \tau_n \geqq \tau(\tilde{K})\} \\ &\leqq E_\bullet\{u(x_{\tau_n}); \tau_n < \tau(\tilde{K})\} + E_\bullet\{u(x_{\tau(\tilde{K})}); \tau_n \geqq \tau(\tilde{K})\} \\ &\leqq [\sup_{y \in K} u(y)] P_\bullet\{\tau_n < \tau(\tilde{K})\} + H_{\tilde{K}} u, \end{aligned}$$

where we use the fact that $\{\zeta > \tau_n \geqq \tau(\tilde{K})\} \in \mathcal{G}_{\tau(\tilde{K})+}$. When $n \to \infty$ the last side of this inequality goes to $H_{\tilde{K}} u$, so that

$$u = \lim_{n \to \infty} H_{\tau_n} u \leqq H_{\tilde{K}} u.$$

4. Resolvent kernels

We introduce several spaces of functions defined over S; $\mathbf{B}$ = the space of bounded and $\mathcal{A}$-measurable functions, $\mathbf{B}_0$ = the subspace of $\mathbf{B}$ formed by functions of compact support,[6] $\mathbf{C}$ = the space of bounded and continuous functions, $\mathbf{C}_0$ = the subspace of $\mathbf{C}$ formed by functions of compact support.

A function $R_\alpha(x, A)$, defined for $\alpha > 0$, x of S and A of $\mathcal{B}$, is said to be a *resolvent kernel* if it satisfies the following conditions (a)–(d). (a) For each $\alpha > 0$ and x of S, $R_\alpha(x, \cdot)$ is a measure finite for compact sets. Let f be a bounded and Borel measurable (= $\mathcal{B}$-measurable) function of compact support. We will write $R_\alpha f$ for $\int f(y) R_\alpha(\cdot, dy)$. (b) $R_\alpha f$ is $\mathcal{A}$-measurable and bounded on every compact set; (c) the resolvent equation

$$R_\alpha f - R_\beta f + (\alpha - \beta) R_\alpha R_\beta f = 0. \tag{4.1}$$

is satisfied and (d) $\lim_{\alpha \to \infty} R_\alpha f(x) = 0$ for each x.

We will now list several elementary properties of $R_\alpha(x, A)$. Let f be a function of $\mathbf{B}_0$. Clearly $R_\alpha f(x)$ is well defined for such f.

(i) (b)–(d) are satisfied for such f.

(ii) $R_\alpha R_\beta f = R_\beta R_\alpha f$.

(iii) If $f \geqq 0$ and if $\alpha \leqq \beta$, $R_\alpha f \geqq R_\beta f$.

(iv) $R_\alpha f$ is continuous in α, so that $R_\alpha f(x)$ is jointly measurable in (α, x).

(v) $R_\alpha f = \lim_{\beta \to \infty} R_\alpha(\beta R_\beta f)$.

Given a number $\alpha \geqq 0$, a function u of $\mathbf{A}^+$ is said to be $(\mathbf{R}, \alpha)$ *excessive* if $\beta R_{\alpha+\beta} u \leqq u$ for all $\beta > 0$ and if $\lim_{\beta \to \infty} \beta R_{\alpha+\beta} u = u$. By (iii),

$$R_0(x, A) = \lim_{\alpha \to 0} R_\alpha(x, A)$$

exists for each A of $\mathcal{A}$ and defines a measure on $\mathcal{A}$. Moreover, for each A of $\mathcal{A}$, there are Borel sets (= sets in $\mathcal{B}$) B and C such that $B \subset A \subset C$ and such that $R_0(x, B) = R_0(x, A) = R_0(x, C)$.

(vi) A function u of $\mathbf{A}^+$ is $(\mathbf{R}, \alpha_0)$ excessive if and only if it is $(\mathbf{R}, \alpha)$ excessive for all $\alpha > \alpha_0$.

[6] A function is said to be of compact support if it vanishes outside of a compact set.

(vii) If f is in $\mathbf{A}^+$, $R_\alpha f$ is $(\mathbf{R}, \alpha)$ excessive.

PROPOSITION 4.1. *Let u be a function of $\mathbf{A}^+$ and let $\beta R_{\alpha+\beta} u$ increase with β. Then $v = \lim_{\beta\to\infty} \beta R_{\alpha+\beta} u$ is $(\mathbf{R}, \alpha)$ excessive.*

First we note that the condition in the proposition that

$$\beta R_{\alpha+\beta} u \leqq \gamma R_{\alpha+\gamma} u \qquad \text{for} \quad \beta \leqq \gamma \tag{4.2}$$

is equivalent to the condition that

$$\beta R_{\alpha+\beta} R_{\alpha+\gamma} u \leqq R_{\alpha+\gamma} u \qquad \text{for} \quad \beta \leqq \gamma. \tag{4.3}$$

Also (4.2) or (4.3) implies that

$$\gamma R_{\alpha+\beta} R_{\alpha+\gamma} u \leqq R_{\alpha+\beta} u \qquad \text{for} \quad \beta \leqq \gamma. \tag{4.4}$$

These assertions follow from the fact that, if $\beta \leqq \gamma$, the equation

$$R_{\alpha+\beta} u = R_{\alpha+\gamma} u + (\gamma - \beta) R_{\alpha+\beta} R_{\alpha+\gamma} u$$

is true for any function u of $\mathbf{A}^+$. When $\gamma \to \infty$ in (4.4) the left side goes to $R_{\alpha+\beta} v$, so that

$$R_{\alpha+\beta} v \leqq R_{\alpha+\beta} u.$$

On the other hand, using the property (v) of $R_\alpha(x, A)$, we have

$$R_{\alpha+\beta} u \leqq \liminf_{\gamma\to\infty} R_{\alpha+\beta}(\gamma R_{\alpha+\gamma} u) = R_{\alpha+\beta} v.$$

Therefore we have shown $R_{\alpha+\beta} u = R_{\alpha+\beta} v$, which implies that v is $(\mathbf{R}, \alpha)$ excessive.

The function v in the above proposition is said to be the $(\mathbf{R}, \alpha)$ *regularization of* u.

An obvious sufficient condition that (4.3) is satisfied is that

$$\beta R_{\alpha+\beta} u \leqq u \qquad \text{for all} \quad \beta > 0. \tag{4.5}$$

Hence Proposition 4.1 can be applied to functions of $\mathbf{A}^+$ satisfying (4.5).

Let $\mathbf{R} = \{R_\alpha(x, A)\}$ be a resolvent kernel and m, a measure defined over $(S, \mathcal{B})$.

$\mathbf{R}$ is said to be *dominated by* m if, for each $\alpha > 0$ and for each x of S, $R_\alpha(x, \cdot)$ is absolutely continuous with respect to m.

$\mathbf{R}$ is said to be *integrable* if $R_0(\cdot, A)$ is bounded on every compact set when A is compact.

$\mathbf{R}$ is said to be *substochastic* if $\alpha R_\alpha(x, S) \leqq 1$ for every $\alpha > 0$ and $x \in S$.

$\mathbf{R}$ is said to be *regular* if, for each function f of $\mathbf{C}_0$, $\alpha R_\alpha f$ converges boundedly on every compact set to f as $\alpha \to \infty$.

The resolvent kernel $\mathbf{G} = \{G_\alpha(x, A)\}$ of the process X, defined in Section 3, is substochastic and regular. It is known [6] that this kernel $G_\alpha(x, A)$ has the following properties.

(i) The constant function is $(\mathbf{G}, 0)$ excessive.
(ii) $\lim_{\alpha\to\infty} \alpha R_\alpha f = f$ for each function f of $\mathbf{C}$.
(iii) Each $(\mathbf{G}, \alpha)$ excessive function can be approximated by an increasing sequence of bounded $(\mathbf{G}, \alpha)$ excessive functions.
(iv) The minimum of two $(\mathbf{G}, \alpha)$ excessive functions is also $(\mathbf{G}, \alpha)$ excessive.

The properties (i)–(iii) can be proved for any substochastic and regular resolvent kernel $\mathbf{R}$. (For (iii) it is sufficient that $\mathbf{R}$ is substochastic.) But it remains an open question whether or not (iv) is true for such a general $\mathbf{R}$.[7]

5. Determining the potential kernel

Hunt [6, Part III] gave a method of determining a potential kernel (of function type). There are, however, some interesting cases to which Hunt's method is not applicable. For instance, one-dimensional diffusions with an entrance boundary do not satisfy the condition, one of those conditions assumed by Hunt, that the transition function of the process under consideration leaves invariant the space of functions continuous on S and vanishing at the point at infinity of S. Also his approach does not explain why, in the case of (continuous parameter) Markov chains, $G_0(x, \{y\})$ acts as a potential kernel (of function type). To cover such cases we present a different method.

Let m be a measure defined over $(S, \mathfrak{B})$ and finite for compact sets and let $\langle \cdot, \cdot \rangle$ denote the inner product with respect to m. A resolvent kernel[8] $\hat{\mathbf{G}} = \{\hat{G}_\alpha(x, A)\}$ is called the *co-resolvent kernel of* $\mathbf{G} = \{G_\alpha(x, A)\}$ *with respect to* m if, for each f, g of $\mathbf{C}_0$ and for each $\alpha > 0$,

$$(5.1) \qquad \langle f, G_\alpha g \rangle = \langle \hat{G}_\alpha f, g \rangle$$

It is easy to show that (5.1) is true for each f, g of $\mathbf{B}_0$. We will use the word α-excessive (or just *excessive*[9] when $\alpha = 0$) for $(\mathbf{G}, \alpha)$ excessive. Also we will use the word *α-co-excessive* (or *co-excessive* when $\alpha = 0$) for $(\hat{\mathbf{G}}, \alpha)$ excessive.

Given a number $\alpha \geqq 0$, a jointly $(= \mathfrak{A} \times \mathfrak{A})$ measurable function $G_\alpha(x, y)$ is said to be the *potential kernel of exponent* α (*associated with* $\mathbf{G}$, m *and* $\hat{\mathbf{G}}$) if the following conditions are satisfied: (a) $G_\alpha(x, dy) = G_\alpha(x, y)m(dy)$; (b) $\hat{G}_\alpha(y, dx) = G_\alpha(x, y)m(dx)$; (c) $G_\alpha(\cdot, y)$ is α-excessive for each fixed y and (d) $G_\alpha(x, \cdot)$ is α-co-excessive for each fixed x.

Suppose there is the potential kernel of exponent α_0 for some $\alpha_0 > 0$. Then it follows from the resolvent equation that $\mathbf{G}$ and $\hat{\mathbf{G}}$ are dominated by m. Therefore, as will be proved soon, there is the potential kernel of exponent α for all $\alpha \geqq 0$. The fact that $\mathbf{G}$ is dominated by m implies that $m(A)$ must be positive if A is a non-void open set. Moreover the uniqueness of $G_\alpha(x, y)$ is

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[7] An argument of Ray [12] may be useful for this question.
[8] In general, such $\hat{\mathbf{G}}$ is not necessarily determined uniquely by $\mathbf{G}$ and m.
[9] This phraseology amounts to that in Section 2.

shown as follows. Given a number $\alpha \geqq 0$, let $G_\alpha^{(i)}(x, y)(i = 1, 2)$ be two potential kernels of exponent α. When y is fixed, by (b),

$$G_\alpha^{(1)}(\cdot, y) = G_\alpha^{(2)}(\cdot, y) \tag{5.2}$$

a. e. (m) (= almost everywhere relative to the measure m). But since $\mathbf{G}$ is dominated by m,

$$\beta \int G_\alpha^{(1)}(z, y) G_{\alpha+\beta}(x, dz) = \beta \int G_\alpha^{(2)}(z, y) G_{\alpha+\beta}(x, dz)$$

for all x of S. Letting $\beta \to \infty$ we have (5.2) everywhere on S, because both $G_\alpha^{(1)}(\cdot, y)$ and $G_\alpha^{(2)}(\cdot, y)$ are α-excessive for each y.

THEOREM 1. *Assume that* $\mathbf{G}$ *and* $\hat{\mathbf{G}}$ *are dominated by* m. *Then there is the unique potential kernel of exponent* α *for all* $\alpha \geqq 0$. *Moreover we have for all* $\beta > \alpha \geqq 0$

$$\begin{aligned} G_\alpha(x, y) &= G_\beta(x, y) + (\beta - \alpha) \int G_\alpha(x, z) G_\beta(z, y) m\,(dz) \\ &= G_\beta(x, y) + (\beta - \alpha) \int G_\beta(x, z) G_\alpha(z, y) m\,(dz). \end{aligned} \tag{5.3}$$

Generally if $f(x, y)$ is a jointly measurable function, then $\mathbf{G}$ (resp. $\hat{\mathbf{G}}$) operates on f with respect to the variable x (resp. y), while the other variable is fixed. In other words,

$$G_\alpha f(x, y) = \int f(z, y) G_\alpha(x, dz), \qquad \hat{G}_\alpha f(x, y) = \int (x, z) \hat{G}_\alpha(y, dz).$$

Such convention will be used throughout the rest of this paper.

Let $\alpha > 0$ and let $g_\alpha(x, y)$ be any jointly measurable version of $G_\alpha(x, dy)/m(dy)$. For a fixed x and for a positive function f of $\mathbf{B}_0$, it follows from (5.1) that

$$\begin{aligned} \int \hat{G}_{\alpha+\beta}\, g_\alpha(x, y) f(y) m\,(dy) &= G_\alpha G_{\alpha+\beta} f(x) \\ &\leqq \frac{1}{\beta} G_\alpha f(x) = \frac{1}{\beta} \int g_\alpha(x, y) f(y) m(dy), \end{aligned} \tag{5.4}$$

so that

$$\beta \hat{G}_{\alpha+\beta}\, g_\alpha(x, \cdot) \leqq g_\alpha(x, \cdot) \qquad \text{a.e. } (m).$$

Since $\hat{\mathbf{G}}$ is dominated by m,

$$\beta \hat{G}_{\alpha+\gamma} \hat{G}_{\alpha+\beta}\, g_\alpha(x, y) \leqq \hat{G}_{\alpha+\gamma}\, g_\alpha(x, y) \qquad \text{for every } y.$$

Therefore, according to Prop. 4.1 and its remark, $\beta \hat{G}_{\alpha+\beta}\, g_\alpha(x, y)$ increases with $\beta \to \infty$ and its limit $G_\alpha(x, y)$ is jointly measurable and α-co-excessive. Multiplying β to the first half of (5.4) and letting $\beta \to \infty$, we have

$$(5.5) \qquad G_\alpha f(x) = \int G_\alpha(x, y) f(y) m(dy).$$

(5.1) and (5.5) imply that, for each positive function f of $\mathbf{B}_0$

$$\hat{G}_\alpha f(\cdot) = \int f(x) G_\alpha(x, \cdot) m(dx) \qquad \text{a.e. } (m).$$

Again using the fact that $\hat{\mathbf{G}}$ is dominated by m,

$$\beta \hat{G}_{\alpha+\beta} \hat{G}_\alpha f(y) = \int f(x) \beta \hat{G}_{\alpha+\beta} G_\alpha(x, y) m(dx) \qquad \text{for all } y.$$

When $\beta \to \infty$ we get

$$\hat{G}_\alpha f(y) = \int f(x) G_\alpha(x, y) m(dx).$$

From Proposition 4.1, $\hat{G}_{\alpha+\beta}\, g_\alpha(x, y) = \hat{G}_{\alpha+\beta}\, G_\alpha(x, y)$, so that

$$\beta \hat{G}_{\alpha+\beta} g_\alpha(x, y) = \beta \int G_\alpha(x, z) G_{\alpha+\beta}(z, y) m(dz)$$
$$= \beta G_\alpha G_{\alpha+\beta}(x, y) \uparrow G_\alpha(x, y).$$

Hence $G_\alpha(\cdot, y)$ is α-excessive for each y, for it is the increasing limit of α-excessive functions $\beta G_\alpha\, G_{\alpha+\beta}(x, y)$.

For a fixed x, it follows from the resolvent equation of $\mathbf{G}$ that (5.3) is true for a.a. y (m) ($=$ almost all y relative to the measure m). Operating $\gamma \hat{G}_{\beta+\gamma}$ and letting $\gamma \to \infty$, one gets (5.3) from the observation that the both sides of (5.3) are β-co-excessive.

Finally, noting that $G_\alpha(x, y)$ is a decreasing function of α by (5.3), one can define

$$G_0(x, y) = \lim_{\alpha \to 0} G_\alpha(x, y),$$

which is easily proved to be the potential kernel of exponent 0, Q. E. D.

Fix a point x of S. Since $G_\alpha(x, \cdot)$, $\alpha > 0$, is finite a.e. (m), the equation

$$(5.6) \qquad \int G_\alpha(x, z) G_\beta(z, y) m(dz) = \int G_\beta(x, z) G_\alpha(z, y) m(dz)$$

is true for a.a. $y(m)$. But the both sides are β-co-excessive ($\beta \geqq \alpha$), so that (5.6) is true for all y. The equation (5.6) can be written as follows;

$$G_\alpha\, G_\beta(x, y) = G_\beta\, G_\alpha(x, y) = \hat{G}_\alpha\, G_\beta(x, y) = \hat{G}_\beta\, G_\alpha(x, y).$$

Suppose $\mathbf{G}$ is dominated by m. Then if there is a symmetric (jointly measurable) version of $G_\alpha(x, dy)/m(dy)$ for each $\alpha > 0$, it is obvious that $\mathbf{G}$ itself is a co-resolvent kernel. Conversely, if $\hat{\mathbf{G}} = \mathbf{G}$, the corresponding kernel $G_\alpha(x, y)$ must be symmetric for each $\alpha \geqq 0$. Indeed, by the assumption,

$G_\alpha(\cdot, y)$ and $G_\alpha(y, \cdot)$ are α-excessive and equal a.e. (m) for each fixed y, so that they are identical everywhere.

PROPOSITION 5.1. *The following four statements are equivalent to each other.* (i) $\mathbf{G}$ (*or* $\hat{\mathbf{G}}$) *is dominated by* m. (ii) *If* u *and* v *are* α-*excessive* (*or* α-*co-excessive*) *for some* $\alpha > 0$ *and if* $u = v$ a.e. (m), *then* $u = v$ *everywhere.* (iii) *Let* f *be a function of* $\mathbf{B}_0$. (*) *There is some positive number* α_0 *such that, whenever* $G_{\alpha_0} f$ (*or* $\hat{G}_{\alpha_0} f$) *vanishes* a.e. (m), *it vanishes everywhere.* (iv) *The statement* (*) *is true only if* f *is a nonnegative function of* $\mathbf{B}_0$.

It is enough to show that (iv) implies (i) in the case of $\mathbf{G}$: the rest is verified easily. Let A be a set of $\mathfrak{A}$ with compact closure such that $m(A) = 0$. By (5.1)

$$\langle f, G_{\alpha_0}(\cdot, A)\rangle = \int_A \hat{G}_{\alpha_0} f(y) m(dy) = 0,$$

so that $G_{\alpha_0}(\cdot, A) = 0$ a.e. (m). Therefore $G_{\alpha_0}(x, A) = 0$ for all x. By the resolvent equation, $G_\alpha(x, A) = 0$ for all $\alpha \geqq 0$ and for all x.

6. Some properties of the potential kernel

From now on the potential kernel of exponent 0 will be called simply the *potential kernel.* Also we will write $G(x, y)$ for $G_0(x, y)$.

The following hypothesis on the triple $(\mathbf{G}, m, \hat{\mathbf{G}})$ is always assumed in the rest of this paper.

HYPOTHESIS (B). $\mathbf{G}$ *is integrable and dominated by* m. $\hat{\mathbf{G}}$ *is regular and* $\hat{G}_\alpha f$, $\alpha \geqq 0$, *is continuous and finite everywhere for each* f *of* $\mathbf{B}_0$.

Obviously this hypothesis implies that $\hat{\mathbf{G}}$ is also integrable. In the previous section we noted that if $\mathbf{G}$ is dominated by m, then $m(A)$ is positive for each non-void open set. From this remark and Proposition 5.1 it follows that, given $\mathbf{G}$ and m, the co-resolvent kernel $\hat{\mathbf{G}}$ satisfying (B), if it exists, is unique and that $\hat{\mathbf{G}}$ is dominated by m. Therefore there is the potential kernel of exponent α for each number $\alpha \geqq 0$. The kernel $G_\alpha(x, y)$ is lower semicontinuous relative to y for each fixed x, because any α-co-excessive function is so under hypothesis (B).

Hypothesis (B) is weaker[10] and, sometimes, easier to be verified than Hunt's hypotheses (F) and (G). For instance, in two examples cited in the beginning of Section 5, it is easy to find the measure m and the co-resolvent kernel for which (B) is satisfied. Under hypothesis (B) we will derive some plausible properties of the potential kernel and establish potential theoretic results like those obtained by Hunt.

Suppose f is in $\mathbf{C}_0$, nonnegative and $f(x_0) > 0$. Since $\hat{\mathbf{G}}$ is regular,

$$\hat{G}_\alpha f(x_0) \geqq \frac{1}{\beta} \cdot \beta \hat{G}_\beta f(x_0) \geqq \frac{1}{2\beta} f(x_0) > 0$$

[10] For the proof of this fact, see [6, Part III, Sections 17 and 18].

for each $\alpha \geqq 0$ and for all sufficiently large β. Since $\hat{G}_\alpha f$ is continuous by (B), we get

PROPOSITION 6.1. *Let A be a compact set and B, a neighborhood of A. Then*

$$\inf_{x \in A} \hat{G}_\alpha(x, B) > 0 \qquad \textit{for each} \quad \alpha \geqq 0.$$

PROPOSITION 6.2. (i) *If A is open and if y is in A, then $H_A G(x, y) = G(x, y)$ for all x, where $H_A G(x, y) = \int G(z, y) H_A(x, dz)$.* (ii) *For each fixed y, $G(\cdot, y)$ is harmonic on $S - \{y\}$.*

In order to show (i) it is enough to prove

$$H_A G(x, y) \geqq G(x, y) \qquad \text{for all } x \text{ of } S \text{ and } y \text{ of } A.$$

Let f be a function of $\mathbf{B}_0$ vanishing outside of A and x, any point of S. Then

$$\begin{aligned} \langle H_A G(x, \cdot), f \rangle &= H_A(G_0 f)(x) \\ &= E_x\left[E_{x_{\tau(A)}}\left\{\int_0^\infty f(x_t)\, dt\right\}\right] = E_x\left\{\int_{\tau(A)}^\infty f(x_t)\, dt\right\} \\ &= E_x\left\{\int_0^\infty f(x_t)\, dt\right\} = G_0 f(x) \\ &= \langle G(x, \cdot), f \rangle, \end{aligned}$$

so that $H_A G(x, \cdot) = G(x, \cdot)$ a.e. (m) on A. Define $G^A(x, \cdot) = G(x, \cdot)$ on A and $= 0$ on $\tilde{A}$. Then

$$\alpha \hat{G}_\alpha(H_A G)(x, y) \geqq \alpha \hat{G}_\alpha G^A(x, y) \qquad \text{for all } y.$$

Noting that $H_A G(x, \cdot)$ is co-excessive, when $\alpha \to \infty$ we get

$$H_A G(x, y) \geqq \liminf_{\alpha \to \infty} \alpha \hat{G}_\alpha G^A(x, y) \geqq G^A(x, y),$$

where the second inequality follows from the facts that $\hat{\mathbf{G}}$ is regular and that $G^A(x, \cdot)$ is lower semicontinuous.

The second statement is immediate from (i).

7. Potentials

Let $\mathbf{L}_0(m)$ be the space of functions defined over S, $\mathfrak{a}$-measurable and (m) integrable over each compact set.

PROPOSITION 7.1. *An excessive (or more generally, quasi-excessive) function u is in $\mathbf{L}_0(m)$ if and only if it is finite* a.e. (m).

Suppose u is quasi-excessive and finite a.e. (m). Fix any point y_0 of S and $\alpha > 0$. Since $\hat{\mathbf{G}}$ is regular, there is a point x_0 such that $G_\alpha(x_0, y_0) > 0$ and $u(x_0) < \infty$. Therefore, for some small $\varepsilon > 0$, the set $A = \{y; G_\alpha(x_0, y) > \varepsilon\}$ is an open neighborhood of y_0. Then

$$\infty > u(x_0) \geqq \alpha G_\alpha u(x_0) \geqq \alpha\varepsilon \int_A u(y) m(dy),$$

which proves the *if* part, while the other half is obvious.

PROPOSITION 7.2. *If* u *is quasi-excessive and finite* a.e. (m), *then* $u = \text{reg. } u$ a.e. (m).

It is enough to show that $u \leqq \text{reg. } u$ a.e. (m). Let f be a positive function of $\mathbf{C}_0$. Then

$$\langle f, u \rangle \leqq \lim_{\alpha\to\infty} \langle \alpha\hat{G}_\alpha f, u \rangle = \lim_{\alpha\to\infty} \langle f, \alpha G_\alpha u \rangle$$
$$= \langle f, \text{reg. } u \rangle.$$

Hence $u \leqq \text{reg. } u$ a.e. (m), for u is in $\mathbf{L}_0(m)$.

Let A be a compact set of S. An excessive function u of $\mathbf{L}_0(m)$ is said to be a *potential* if $\lim_{A \uparrow S} H_{\tilde{A}}\, u = 0$ a.e. (m).[11]

With this definition of a potential, $G(\cdot, y)$ may not be a potential. We will denote by S_P the set of points y for which $G(\cdot, y)$ is a potential. It is important to know when $S = S_P$. This problem will be discussed in Section 13.

We will soon prove that each potential has an integral representation

$$G\mu = \int_S G(\cdot, y)\mu\,(dy),$$

where μ is a measure defined over $(S, \mathcal{B})$. But its converse is false unless $S = S_P$. Keeping this fact in mind we will call a function $G\mu$ of $\mathbf{L}_0(m)$ a *potential in the wide sense*.

PROPOSITION 7.3. *The measure of a potential in the wide sense is finite for compact sets of* S.

Fix any point y_0 of S. Similarly to the argument in Proposition 7.1 there is a point x_0 such that $G(x_0, y_0) > 0$ and such that $G\mu(x_0) < \infty$. Hence the set $A = \{y; G(x_0, y) > \varepsilon\}$ is an open neighborhood of y_0 for a small $\varepsilon > 0$. Therefore

$$\infty > G\mu(x_0) \geqq \int_A G(x_0, y)\mu\,(dy) \geqq \varepsilon\cdot\mu(A).$$

A similar argument gives us

PROPOSITION 7.4. *If* $G\mu = 0$, *then* $\mu = 0$.

We need one more auxiliary result.

PROPOSITION 7.5. *Let* $\{G\mu_n\}$ *be a sequence of potentials in the wide sense which are dominated by a function* v *of* $\mathbf{L}_0(m)$. *Then* (i) *there is at least one weak limit of* $\{\mu_n\}$.[12] (ii) *If* μ *is a weak limit of* $\{\mu_n\}$ *and if* $G\mu_n$ *converges to a function* u a.e. (m), $\alpha G_\alpha u$ *increases with* α *and*

$$\text{reg. } u \geqq G\mu. \tag{7.1}$$

[11] For the meaning of $A \uparrow S$, see footnote 4.

[12] We say μ is a weak limit of $\{\mu_n\}$ if an infinite subsequence $\{\mu_{n(k)}\}$ converges weakly to μ, that is, if $\int f\, d\mu_{n(k)} \to \int f\, d\mu$ $(k \to \infty)$ for each f of $\mathbf{C}_0$.

(iii) *Equality holds in* (7.1) *if, for each positive function f of* $\mathbf{B}_0$ *and for any number* $\varepsilon > 0$, *there is a compact set A such that*

$$\int_{\tilde{A}} \hat{G}_0 f(y)\mu_n(dy) < \varepsilon \qquad \textit{for all } n. \tag{7.2}$$

(Note that the condition of (iii) is satisfied if every μ_n vanishes outside of a compact set independent of n.)

For the assertion (i) it is enough to show that, for each compact set A, $\mu_n(A)$ is bounded. Let B be a compact neighborhood of A. By Proposition 6.1, $c = \inf_{y \in A} \hat{G}_0(y, B) > 0$. Hence

$$\infty > \int_B v(x)m(dx) \geqq \int_B G\mu_n(x)m(dx)$$

$$\geqq \int_A \hat{G}_0(y, B)\mu_n(dy) \geqq c \cdot \mu_n(A).$$

For (ii), let $\mu_{n(k)}$ converge weakly to μ and let f be a positive function of $\mathbf{B}_0$. Since $v \geqq G\mu_{n(k)}$ and since $\hat{G}_0 f$ is positive and continuous,

$$\langle f, u\rangle = \lim_{k\to\infty} \langle f, G\mu_{n(k)}\rangle$$

$$= \lim_{k\to\infty} \int \hat{G}_0 f(y)\mu_{n(k)}(dy)$$

$$\geqq \int \hat{G}_0 f(y)\mu(dy) = \langle f, G\mu\rangle,$$

so that $u \geqq G\mu$ a.e. (m). But since

$$u = \lim_{n\to\infty} G\mu_n \geqq \liminf_{n\to\infty} \alpha G_\alpha(G\mu_n) \geqq \alpha G_\alpha(\lim_{n\to\infty} G\mu_n) = \alpha G_\alpha u \quad \text{a.e. } (m),$$

$\alpha G_\alpha u$ increases with α everywhere on S according to the remark below Proposition 4.1. Hence reg. $\mu \geqq G\mu$.

Finally we will prove (iii). By a simple evaluation we have

$$\lim_{k\to\infty} \int \hat{G}_0 f(y)\mu_{n(k)}(dy) \leqq \int \hat{G}_0 f(y)\mu(dy) + \lim_{k\to\infty} \int_{\tilde{A}} \hat{G}_0 f(y)\mu_{n(k)}(dy)$$

for each compact set A. Therefore if (7.2) is satisfied, then $\langle f, u\rangle \leqq \langle f, G\mu\rangle$, which implies that reg. $u \leqq G\mu$.

We will now give results on the integral representation of potentials.

PROPOSITION 7.6. *A function is a potential if and only if it is a potential in the wide sense of a measure* μ *vanishing outside of* S_P.

Since the proof of the *if* part is quite easy, it is omitted. We will prove the *only if* part. Let u be a potential and let $u_n = nG_n(u \wedge n)$, where $u \wedge n = \min\{u(\cdot), n\}$. The function u_n is also a potential and increases to u when

$n \to \infty$. If A is compact, then $E_x\{\tau(\tilde{A})\} \leqq G_0(x, A) < \infty$. By a formula of Dynkin[13]

$$u_n = E_\bullet\left\{\int_0^{\tau(\tilde{A})} f_n(x_t)dt\right\} + H_{\tilde{A}} u_n ,$$

where $f_n = n[(u \wedge n) - u_n] \geqq 0$. Letting $A \uparrow S$ we have

$$u_n = E_\bullet\left\{\int_0^{\zeta} f_n(x_t)dt\right\} = G_0 f_n \qquad \text{a.e. } (m) \tag{7.3}$$

Since both sides are excessive, (7.3) is true everywhere on S. Therefore u_n is the potential in the wide sense of the measure $\mu_n(dy) = f_n(y)m(dy)$. Using the fact, which follows from Proposition 6.2, that

$$H_{\tilde{A}} u_n = \int H_{\tilde{A}} G(\cdot, y)\mu_n(dy)$$
$$\geqq \int_{\tilde{A}} G(\cdot, y)\mu_n(dy),$$

we get

$$\int_{\tilde{A}} \hat{G}_0 f(y)\mu_n(dy) \leqq \langle f, H_{\tilde{A}} u_n\rangle \leqq \langle f, H_{\tilde{A}} u\rangle. \tag{7.4}$$

When $A \uparrow S$, $\langle f, H_{\tilde{A}} u\rangle \to 0$. Hence according to the preceding proposition u is the potential in the wide sense of μ, a weak limit of $\{\mu_n\}$. It follows from the Fubini theorem that μ has no mass outside of S_P.

PROPOSITION 7.7. *There is a potential, bounded and strictly positive everywhere on S.*

First we note that if f is a function of $\mathbf{A}^+$ and if $G_0 f$ is finite everywhere, then $G_0 f$ is a potential. Indeed

$$\infty > G_0 f \geqq H_{\tilde{A}} G_0 f = E_\bullet\left\{\int_{\tau(\tilde{A})}^{\zeta} f(x_t)\, dt\right\}$$
$$\to 0 \qquad \text{as } A \uparrow S.$$

Also we note that if f is strictly positive on a neighborhood of a point x, then $G_0 f(x) > 0$. This follows from the right continuity of paths.

Let $\{A_n\}$ be an open cover of S and let each A_n be of compact closure. Let f_n be a function of $\mathbf{B}_0$, strictly positive on A_n. Denote by k_n an upper bound of $G_0 f_n$. Then it is clear that $G_0(\sum 2^{-n}\cdot k_n^{-1}\cdot f_n)$ is what we want.

As a result of the above proposition we have

[13] For instance, see p. 632 of Loève's book, *Probability theory*, 3rd ed., Princeton, Van Nostrand, 1963.

PROPOSITION 7.8. *Any excessive function can be approximated by an increasing sequence of bounded potentials.*

PROPOSITION 7.9. *Let u be a potential and A, an open set. Then $H_A u$ is a potential in the wide sense of a measure vanishing outside of $\bar{A}$ (= closure of A).*

First suppose u is bounded. $H_A u$ is a potential in the wide sense of a measure μ, for u is a potential and hence $H_A u$ is so. We have to show that $\mu(\bar{A}^\sim) = 0$, where $\bar{A}^\sim$ is the complement of $\bar{A}$. Take any compact set C of $\bar{A}^\sim$. It is enough to show that $\mu(C) = 0$. Define

$$u_1 = \int_C G(\cdot, y)\mu(dy).$$

Since $H_A u$ is harmonic on $\bar{A}^\sim$ and since both u_1 and $H_A u - u_1$ are excessive, u_1 is also harmonic on $\bar{A}^\sim$. By Proposition 6.2, u_1 is harmonic on $\tilde{C}$. Therefore, according to Proposition 3.6, u_1 is harmonic on the whole space S. Also u_1 is a potential, for $u_1 \leqq H_A u$. Hence $u_1 = 0$, which implies that $\mu(C) = 0$ by Proposition 7.4.

If u is unbounded, take an increasing sequence of bounded potentials $\{u_n\}$ which approximates u. By what we proved above, $H_A u_n = G\mu_n$ with $\mu_n(\bar{A}^\sim) = 0$. Since u is a potential, an evaluation like (7.4) yields that $H_A u$ is the potential in the wide sense of a weak limit of $\{\mu_n\}$.

By the remark to Proposition 7.5, the argument of the preceding paragraph remains still valid for any excessive function of $\mathbf{L}_0(m)$ if A is of compact closure. That is,

PROPOSITION 7.10. *Let u be an excessive function of $\mathbf{L}_0(m)$ and A, an open set with compact closure. Then the conclusion of Proposition 7.9 is valid.*

PROPOSITION 7.11. *If μ_1 and μ_2 define the same potential in the wide sense, then $\mu_1 = \mu_2$.*[14]

Set $u = G\mu_i$ $(i = 1, 2)$. First suppose that each μ_i has no mass outside of a compact set. Then, for each f of $\mathbf{C}_0$

$$\begin{aligned}\int f(y)\mu_i(dy) &= \lim_{\alpha\to\infty} \int \alpha\hat{G}_\alpha f(y)\mu_i(dy)\\ &= \lim_{\alpha\to\infty} \alpha \int (\hat{G}_0 f - \alpha\hat{G}_0\hat{G}_\alpha f)(y)\mu_i(dy)\\ &= \lim\nolimits_{\alpha\to\infty} \alpha\{\langle f, G\mu_i\rangle - \langle \alpha\hat{G}_\alpha f, G\mu_i\rangle\}\\ &= \lim\nolimits_{\alpha\to\infty} \alpha\{\langle f, u\rangle - \langle \alpha\hat{G}_\alpha f, u\rangle\},\end{aligned}$$

so that $\mu_1 = \mu_2$. Next we will consider the general case. Let A be an open

[14] This proposition implies that the family of functions $G(x, \cdot)$ with the index $x \in S$ separates points of S.

set with compact closure and B, a compact set contained in A. Then

$$H_A\{G\mu_1\} = H_A\left\{\int_B G(\cdot, y)\mu_1(dy)\right\} + H_A\left\{\int_{\tilde{B}} G(\cdot, y)\mu_1(dy)\right\}$$

$$= \int_B G(\cdot, y)\mu_1(dy) + H_A\left\{\int_{\tilde{B}} G(\cdot, y)\mu_1(dy)\right\}.$$

By the previous proposition there are measures μ_A and μ'_A such that $H_A\{G\mu_1\} = G\mu_A$, $H_A\{\int_{\tilde{B}} G(\cdot, y)\mu_1(dy)\} = G\mu'_A$ and $\mu_A(\bar{A}^\sim) = \mu'_A(\bar{A}^\sim) = 0$. The restriction of μ_1 to the set B has no mass on on $\bar{A}^\sim$. Hence appealing to the first displayed special case we can conclude that $\mu_A \geqq \mu_1$ over A.[15] Let μ_∞ be a weak limit of μ_A as $A \uparrow S$. Obviously $\mu_\infty \geqq \mu_1$ and $G\mu_\infty \geqq G\mu_1 = u$. But by Proposition 7.5, $u \geqq G\mu_\infty$. Therefore it follows from Proposition 7.4 that $\mu_\infty = \mu_1$. Since μ_A and μ_∞ depend just on u, the above argument is applied to μ_2 with the same $\{\mu_A, \mu_\infty\}$ to show that $\mu_\infty = \mu_2$.

PROPOSITION 7.12. *If μ is a signed measure on S and if $u = G\mu$ is an excessive function of $\mathbf{L}_0(m)$, then μ must be a measure and hence u is a potential in the wide sense.*

Let $\mu = \mu^+ - \mu^-$ be the Jordan decomposition of μ and A, an open set with compact closure. Let $H_A\{G\mu^+\} = G\mu_A^+$ and $H_A\{G\mu^-\} = G\mu_A^-$. The preceding proposition and its proof imply that $\mu_A^+ \geqq \mu_A^-$ and that μ^+ (resp. μ^-) is the unique weak limit of μ_A^+ (resp. μ_A^-) as $A \uparrow S$. Hence $\mu^+ \geqq \mu^-$, i.e., $\mu^- = 0$.

8. Riesz decomposition of excessive functions

PROPOSITION 8.1. *Each excessive function of $\mathbf{L}_0(m)$ can be written uniquely as the sum of a potential and a harmonic function.*

Let A be a Borel set with compact closure and let $h' = \lim_{A\uparrow S} H_{\tilde{A}}\, u$. h' is quasi-excessive and $h' = H_{\tilde{A}}\, h'$ on $S_{h'} = \{x; h'(x) < \infty\}$. Define

$$p' = u - h' \quad \text{on } S_{h'}$$

$$= \infty \qquad \text{otherwise.}$$

Obviously $u = p' + h'$. It follows that $\lim_{A\uparrow S} H_{\tilde{A}}\, p' = 0$ on $S_{h'}$ and that $p' \geqq H_{\tilde{A}}\, p'$ everywhere on S. Hence p' is quasi-excessive. Set

$$p = \text{reg. } p', \qquad h = \text{reg. } h'.$$

p is a potential, for $\lim_{A\uparrow S} H_{\tilde{A}}\, p \leqq \lim_{A\uparrow S} H_{\tilde{A}}\, p' = 0$ on $S_{h'}$. Also $u = p + h$ everywhere on S. It remains to show that h is harmonic, i.e.

$$h = \lim_{A\uparrow S} H_{\tilde{A}}\, h. \tag{8.1}$$

But on $S_{h'}$

$$h \leqq h' = \lim_{A\uparrow S} H_{\tilde{A}}\,(p + h) = \lim_{A\uparrow S} H_{\tilde{A}}\, h \leqq h. \tag{8.2}$$

[15] This means that $\mu_A(B) \geqq \mu_1(B)$ for each Borel subset B of A.

Hence, everywhere on S

$$h \leqq \text{reg. } (\lim_{A \uparrow S} H_{\tilde{A}}\, h) \leqq \lim_{A \uparrow S} H_{\tilde{A}}\, h \leqq h,$$

which proves (8.1).

To prove the uniqueness, suppose that

$$u = p_1 + h_1 = p_2 + h_2 ,$$

where each p_i is a potential and each h_i is harmonic. Let B be the set of points in S such that u is finite and such that $\lim_{A \uparrow S} H_{\tilde{A}}\, p_i = 0$ $(i = 1, 2)$. Then

$$h_1 = h_2 , \qquad p_1 = p_2 \tag{8.3}$$

on B. However since $m(\tilde{B}) = 0$, (8.3) holds everywhere on S.

PROPOSITION 8.2. *If u is a potential, then*

$$\lim_{A \uparrow S} H_{\tilde{A}}\, u = 0 \quad or \quad \infty . \tag{8.4}$$

In particular the above limit is 0 *on the set* $\{x; u(x) < \infty\}$.

By the display (8.2), $h = h'$ on $S_{h'}$. But $h = 0$, for u is a potential.

In a way similar to the proof of Proposition 8.1 we can obtain

PROPOSITION 8.3. *Let $\{A_n\}$ be a decreasing sequence of Borel sets such that the complement of each compact set contains some A_n. Then* reg. $(\lim_{n \to \infty} H_{A_n} u)$ *is harmonic. Moreover*

$$\text{reg. } (\lim_{n \to \infty} H_{A_n} u) = \lim_{n \to \infty} H_{A_n} u$$

over the set $\{x; \lim_{n \to \infty} H_{A_n} u(x) < \infty\}$.

Summing up the results in this section and the preceding section we have the Riesz theorem cited in the section title, as follows.

THEOREM 2. *Each excessive function u of* $\mathbf{L}_0(m)$ *is decomposed uniquely in the form*

$$u = \int_{S_P} G(\cdot, y)\mu(dy) + (\text{a harmonic function}).$$

Note. In Section 7 we defined a potential as an excessive function of $\mathbf{L}_0(m)$ such that $\lim_{A \uparrow S} H_{\tilde{A}}\, u = 0$ a.e. (m). An alternative definition of a potential, not involving the measure m, is given as follows. Let $\mathbf{1}$ be the indicator function of the whole space S. A subset A of S is said to be a *polar set* if there is an $\mathfrak{A}$-measurable set B including A such that $H_B^+ \mathbf{1} = 0$. An excessive function u is called a *potential* if, except on a polar set, u is finite and $\lim_{A \uparrow S} H_{\tilde{A}}\, u = 0$. With this alternative definition, Proposition 8.1 and 8.2 remain still valid. We will show that these two definitions are equivalent.

Making use of the fact that $\hat{\mathbf{G}}$ is regular we can see, without difficulty, that a polar set is of (m) measure 0. Moreover an excessive function u is finite except on a polar set if and only if it is so a.e. (m). To prove this fact it is enough to show the *if* part, for the *only if* part is immediate from the first asserted statement. Let u be an excessive function finite a.e. (m) and let $A = \{x; u(x) = \infty\}$. Then $H_A^+ \mathbf{1}$ is an excessive function vanishing outside of A. Since $m(A) = 0$, $H_A^+ \mathbf{1} = 0$.

Hence it is clear that the first definition is weaker than the second one. Conversely, suppose u is a potential in the first sense. Then, by Proposition 8.2, $\lim_{A \uparrow S} H_{\tilde{A}} u = 0$ on the set $\{x; u(x) < \infty\}$. Therefore it is a potential in the second sense.

9. The Martin boundary

Following the terminology of Hunt [7], we will say a measure r, defined over $(S, \mathcal{B})$, is a *reference measure* if the function

$$rG = \int r\ (dx)\ G(x, \cdot)$$

is strictly positive and continuous on S, allowing the value infinity. Let $\mathbf{L}(r)$ be the space of (r) integrable functions and let $S_r = \{y; rG(y) < \infty\}$. Set

$$(9.1)\qquad \begin{aligned} \kappa(x, y) &= \frac{G(x, y)}{rG(y)} \quad \text{if} \quad y \in S_r \\ &= 0 \qquad\qquad \text{if} \quad y \in \tilde{S}_r. \end{aligned}$$

We will use the notation $f\kappa$ for $\int m(dx) f(x) \kappa(x, \cdot)$.

Similarly to the proof of Proposition 7.7 it is shown that, given any excessive function u of $\mathbf{L}_0(m)$, there is a reference measure having the form $r(dx) = f(x)m(dx)$ such that u is in $\mathbf{L}(r)$ and such that rG is finite everywhere.

Suppose there is a measure r such that

$$m(A) = \int r\ (dx) G_0(x, A).$$

Then $rG = 1$ everywhere on S and hence r is a reference measure. Also $\kappa(x, y) = G(x, y)$ for every x and y.[16] First we will show that $\hat{\mathbf{G}}$ is substochastic. Denote by $\mathbf{1}$ the indicator function of S. By (5.1), for each positive function g of $\mathbf{B}_0$

$$\begin{aligned} \langle \alpha \hat{G}_\alpha \mathbf{1}, g \rangle &= \langle \mathbf{1}, \alpha G_\alpha g \rangle \\ &= \int m(dx) \cdot \alpha G_\alpha g(x) \end{aligned}$$

[16] This means that the definition of the function κ under hypothesis (D) in the previous paper [9] is only a special case of the definition under hypothesis (C) there.

$$= \int r(dy) \cdot \alpha G_0 G_\alpha g(y)$$

$$\leqq \int r(dy) \cdot G_0 g(y) = \langle \mathbf{1}, g \rangle.$$

Hence $\alpha \hat{G}_\alpha \mathbf{1} \leqq 1$ a.e. (m). But since $\alpha \hat{G}_\alpha \mathbf{1}$ is lower semi-continuous by hypothesis (B), $\alpha \hat{G}_\alpha \mathbf{1} \leqq 1$ everywhere on S. Since $\hat{\mathbf{G}}$ is substochastic and regular, $\mathbf{1}$ is co-excessive (see the final paragraph of Section 4). On the other hand

$$m(A) = \int r(dx) G_0(x, A) = \int_A rG(y) m(dy),$$

so that $rG = 1$ a.e. (m). Moreover rG is co-excessive. Hence we have

$$rG = \lim_{\alpha \to \infty} \alpha \hat{G}_\alpha (rG) = \lim_{\alpha \to \infty} \alpha \hat{G}_\alpha \mathbf{1} = \mathbf{1}.$$

PROPOSITION 9.1. *If u is an excessive function of $\mathbf{L}(r)$, then it is in $\mathbf{L}_0(m)$. In particular, if $\int r(dx) u(x) = 0$, then u is identically zero.*

The proof is similar to that of Proposition 7.1. For each y_0, since $\int r(dx) G(x, y_0) > 0$ and $\int r(dx) u(x) < \infty$, there is an x_0 such that $u(x_0) < \infty$ and $G(x_0, y_0) > 0$. Hence there is some $\alpha > 0$ such that $G_\alpha(x_0, y_0) > 0$. Therefore u is (m) integrable on a neighborbood of y_0. The latter half is proved similarly.

PROPOSITION 9.2. (i) *If f is in $\mathbf{B}_0$, then $f\kappa$ is bounded and continuous.* (ii) *If f and g are in $\mathbf{B}_0$ and vanish outside of a compact set A, then*

$$(9.2) \qquad | f\kappa(y) - g\kappa(y) | \leqq c \times \sup_{x \in S} | f(x) - g(x) | \qquad \textit{for all } y \textit{ of } S,$$

where c is a constant depending only on A.

The continuity of $f\kappa$ is evident from $f\kappa(y) = \hat{G}_0 f(y)/rG(y)$. The boundedness is proved as follows. Let B be an open set with compact closure $\bar{B}$ and let f vanish outside of B. Since $\kappa(\cdot, y)$ is (r) integrable and therefore since it is in $\mathbf{L}_0(m)$, we have

$$(9.3) \qquad \begin{aligned} \kappa(\cdot, y) \geqq H_B \kappa(\cdot, y) &= \int_{\bar{B} \cap S_r} G(\cdot, z) \mu_y(dz) \\ &= \int_{\bar{B} \cap S_r} \kappa(\cdot, z) \{ rG(z) \} \mu_y(dz) \end{aligned}$$

for each y. Moreover the equality holds on the set B in the above display. Hence

$$\begin{aligned} f\kappa(y) &= \int_{\bar{B} \cap S_r} f\kappa(z) \{ rG(z) \} \mu_y(dz) \\ &\leqq \{ \sup_{z \in \bar{B} \cap S_r} | f\kappa(z) | \} \times \int_{\bar{B} \cap S_r} rG(z) \mu_y(dz). \end{aligned}$$

Again using (9.3)

$$1 \geqq \int r\ (dx)\kappa(x, y) \geqq \int_{\bar{B}\cap S_r} \left\{\int r\ (dx)\kappa(x, z)\right\} rG(z)\mu_y(dz)$$

$$= \int_{\bar{B}\cap S_r} rG(z)\mu_y(dz),$$

so that $f\kappa$ is bounded.

The second statement follows from the inequality

$$| f\kappa(y) - g\kappa(y) | \leqq \left\{\int_A m(dx)\kappa(x, y)\right\} \times \sup_{x\in S} | f(x) - g(x) | ,$$

for the first term of the right side is bounded relative to y by the first statement.

Before introducing the Martin boundary relevant to the kernel $\kappa(x, y)$, we will refer to the *classical cases* well-studied already.

I. *Brownian motion case* [2], [11]. X is Brownian motion on a Green space. The measure m is chosen as the Lebesgue measure on the space. Hypothesis (B) is satisfied by the unique co-resolvent kernel $\hat{\mathbf{G}} = \mathbf{G}$. The corresponding potential kernel is the Newtonian potential kernel. The reference measure is taken as the unit distribution at any fixed (reference) point x_0. Then, according to the symmetry of the Newtonian potential kernel, our κ-function is nothing but the K-function of Martin except with the position of variables reversed (that is, $\kappa(x, y) = K(y, x)$)[17] and except the definition of the value at $x = y = x_0$.[18]

II. *Markov chain case* [4], [7], [13]. By a Markov chain we here mean a Hunt process taking values in a denumerable space with discrete topology. X is a transient Markov chain and $m(A)$ is the number of points in A. $\hat{G}_\alpha(x, \{y\}) = G_\alpha(y, \{x\})$. (B) is satisfied and the potential kernel of exponent α is given by $G_\alpha(x, \{y\})$. r is any measure such that $rG(y) > 0$ for all y.

In the above two cases the Martin boundary S' could be characterized[19] by the following properties (a)–(d). (a) $S + S'$ is a compact metric space. (b) S is dense and open in $S + S'$ and its relative topology coincides with its original topology. (c) To each η of S' corresponds an excessive function $\kappa(x, \eta)$ and if $\eta \neq \eta'$, then $\kappa(x, \eta) \neq \kappa(x, \eta')$ for some x. (d) For each η of

[17] As to the K-function, potentialists usually have followed the original notation of Martin. Probabilists have also used the same letter K to denote the function κ defined here. The difference looks very simple in the present stage and one will have no trouble. But if he wants to consider more complicated kernels associated with the function K like the Θ-kernel of Naïm, he will have to be more careful about the position of variables. This is the reason why we employed the new symbol κ.

[18] By definition, $\kappa(x_0, x_0) = 0$ and $K(x_0, x_0) = 1$. Such difference is irrelevant to the boundary theory of this case.

[19] This means that $S + S'$ is uniquely determined up to homeomorphism.

S' and for each sequence y of S converging to η with the topology of $S + S'$, $\kappa(x, y)$ converges to $\kappa(x, \eta)$ for all x.

So far the Ascoli-Arzelà theorem has been applied to the proof of the existence of the boundary, based on the fact that $\kappa(x, y)$ is uniformly bounded and equicontinuous as a family of functions of x if x varies on a compast set and if y varies on a set the distance of which with the compact set is positive. But in our general case such condition may not be satisfied and hence we need some new device to define a boundary of Martin type. Indeed we will show there is a unique boundary having the above-stated properties (a), (b), (c) and having a property which is a little weaker than the property (d) (see Theorem 3). This boundary coincides with the previous one in the classical cases.

Let $\{f_n\}$ be a countable subspace of $\mathbf{C}_0$ such that each function f of $\mathbf{C}_0$ can be uniformly approximated by a linear combination of functions in $\{f_n\}$ each of which vanishes outside of a compact set (depending only on f). Moreover let ρ_1 be the metric of one-point compactification of S and let

$$\rho_2(y, y') = \sum \frac{1}{2^n} \frac{| f_n \kappa(y) - f_n \kappa(y') |}{1 + | f_n \kappa(y) - f_n \kappa(y') |}.$$

The set of points which are adjoined by the completion of S relative to $\rho_1 + \rho_2$, denoted by S', is said to be the *Martin boundary* (*relative to the kernel* $\kappa(x, y)$). It is easy to show that S' has the properties (a), (b) in the classical cases.

By definition, each $f_n \kappa$ can be extended continuously to $S + S'$. Hence, for each f of $\mathbf{C}_0$, $f\kappa$ can be so also by the preceding proposition. In other words the pseudometric ρ_2 generates the uniformity relative to which $f\kappa$ is uniformly continuous whenever f is in $\mathbf{C}_0$. Therefore $S + S'$ does not depend on the choice of $\{f_n\}$. The extension of $f\kappa$ ($f \in \mathbf{C}_0$) will be denoted by the same symbol $f\kappa$. Then, for each η of S', $f\kappa(\eta)$ is a positive linear functional on $\mathbf{C}_0$, so that it defines uniquely a measure on S, say $\kappa(dx, \eta)$.

THEOREM 3. *$\kappa(x, y)$ can be extended uniquely to $S \times (S + S')$ in such a way that, for each η fo S', $\kappa(x, \eta)$ is excessive and $\kappa(dx, \eta) = m(dx)\kappa(x, \eta)$. Hence S' can be characterized by the properties* (a), (b), (c) *in the classical case and, in place of* (d), *by the property*: (d), *if $\eta \in S'$ and if y (in S) $\to \eta$ with the topology of $S + S'$, then for each f of $\mathbf{C}_0$*

$$\int m\ (dx) f(x)\kappa(x, y) \to \int m\ (dx) f(x)\kappa(x, \eta).$$

The proof of the latter half is a routine work, so it is omitted. Also the uniqueness part of the former half is trivial.

We will prove the existence of $\kappa(x, \eta)$, $\eta \in S'$, satisfying the asserted condition. For $\eta \in S'$, define

$$\kappa(x, \eta) = \sup_{\alpha > 0} \left\{ \alpha \int G_\alpha(x, z)\kappa(dz, \eta) \right\}. \tag{9.4}$$

Since the integral of the right side is right continuous with respect to α, $\kappa(\cdot, \cdot)$ is jointly measurable over $S \times (S + S')$. We want to show $\kappa(x, \cdot)$ is lower semicontinuous on $S + S'$. Indeed, if f is positive and lower semicontinuous, then $f\kappa$ is lower semicontinuous on $S + S'$. Hence if we set

$$\kappa_\alpha(x, \cdot) = \alpha \int m\,(dz) G_\alpha(x, z)\kappa(z, \cdot) \qquad (\cdot \in S)$$

$$= \alpha \int G_\alpha(x, z)\kappa(dz, \cdot) \qquad (\cdot \in S')$$

$\kappa_\alpha(x, \cdot)$ is lower semicontinuous on $S + S'$. However $\kappa(x, \cdot)$ is the upper envelope of $\kappa_\alpha(x, \cdot)$ at each point of $S + S'$ because of (9.4) on S' and because of the fact that $\kappa(x, \cdot)$ is excessive relative to x on S. Hence it was proved that $\kappa(x, \cdot)$ is lower semicontinuous on $S + S'$. Next we prove $\kappa(dx, \eta) = m(dx)\kappa(x, \eta)$, $\eta \in S'$. It is enough to show that, for each positive function f of $\mathbf{C}_0$

$$\int m\,(dx) f(x)\kappa(x, \cdot) = f\kappa \qquad \text{on } S + S', \tag{9.5}$$

This is true on S by definition. Since the left side is lower semicontinuous and since the right side is continuous, the inequality "$\leqq$" is true. On the other hand, if $\eta \in S'$, then

$$\int f(x)\kappa(x, \eta) m(dx)$$

$$\geqq \limsup_{\alpha \to \infty} \int f(x) \left\{ \alpha \int G_\alpha(x, z)\kappa\,(dz, \eta) \right\} m(dx)$$

$$= \limsup_{\alpha \to \infty} \int \alpha \hat{G}_\alpha f(z)\kappa(dz, \eta)$$

$$\geqq \int \lim_{\alpha \to \infty} \alpha \hat{G}_\alpha f(z)\kappa(dz, \eta) = f\kappa(\eta),$$

so that (9.5) was proved. Hence we have

$$\kappa(x, \eta) \geqq \alpha \int G_\alpha(x, z)\kappa(z, \eta) m(dz) = \alpha \int \kappa(z, \eta) G_\alpha(x, dz),$$

which shows that $\kappa(\cdot, \eta)$ is quasi-excessive. Therefore the right side in the last display increases with α and its limit ($\alpha \to \infty$) is equal to its upper envelope ($\alpha > 0$), that is, to $\kappa(x, \eta)$, Q. E. D.

Proposition 9.3. *Let* $\mathbf{\Lambda}$ *be the collection of functions* $\lambda(x, \eta)$ *defined on* $S \times (S + S')$ *such that* (i) *if* $y \in S$, *then* $\lambda(x, y) = \kappa(x, y)$, (ii) *for each* η *of* S', $\lambda(\cdot, \eta)$ *is excessive and* (iii) *for each* x *of* S, $\lambda(x, \cdot)$ *is lower semicontinuous on* $S + S'$. *Then the extended kernel* $\kappa(x, \eta)$ *of the previous theorem is the upper* *envelope of* $\mathbf{A}$. *In particular, if* $\kappa(x, y)$ *has the continuous extension* $\kappa'(x, \eta)$ *to* S' *for each* x, *then* $\kappa(x, \eta)$, $\eta \in S'$, *is the regularization of* $\kappa'(x, \eta)$.

The proof is easy, so it is omitted.

10. Reduced functions

PROPOSITION 10.1. *Let* A *be a Borel (or more generally, analytic) set of* $S + S'$ *and let* u *be an excessive function of* $\mathbf{L}_0(m)$. *Then there is a unique excessive function* $\bar{H}_A u$ *such that, if* A *is open,* $\bar{H}_A u$ *is defined by* $H_{A \cap S} u$ *and if* A *is a general Borel set,* $\bar{H}_A u$ *coincides with* $\inf \bar{H}_{A'} u$ (A'; *open and including* A) *except on a set of* (m) *measure* 0.

$\bar{H}_A u$ is said to be the *reduced function of* u *to the set* A.

The uniqueness of $\bar{H}_A u$ is obvious. The proof of the existence is analogous to the proof of Hunt [6, Part I] for the measurability of hitting times. Define $H_A^n u = \min(n, H_{A\cap S} u)$ for each open set A. Next define $H_A^n u = \inf H_{A'}^n$ (A'; open and including A) for a compact set A. Then it follows that $H_A^n u$ is a Choquet capacity (i.e., right continuous and alternating of order 2 on all compact sets) and therefore it can be extended to all analytic sets. According to Proposition 3.3, such extension gives the original $H_A^n u$ for an open set A. Hence we have for each analytic set A

$$\text{(10.1)} \qquad \sup_{A' \subset A} H_{A'}^n u = H_A^n u = \inf_{A'' \supset A} H_{A''}^n u,$$

where A' is compact and A'', open. If A is open, $H_A^n u$ is $\mathfrak{a}$-measurable clearly. Suppose A is compact and A_k' is a sequence of open sets decreasing to A. Then $H_A^n u = \lim_{k\to\infty} H_{A_k'}^n u$, which proves the $\mathfrak{a}$-measurability of $H_A^n u$. From this, the $\mathfrak{a}$-measurability of $H_A^n u$ for a general analytic set A is proved by a standard tenchnique (for example, see [5, p. 34]). Therefore according to the second equality of (10.1), $H_A^n u$ is quasi-excessive. Set $H_A^\infty u = \lim_{n\to\infty} H_A^n u$. We will show that the regularization of the quasi-excessive function $H_A^\infty u$ is the desired function $\bar{H}_A u$. Let B be any open set including A. It is evident that reg. $(H_A^\infty u) \leqq \inf_B \{\text{reg. } (H_B^\infty u)\}$. Now choose a function f, strictly positive everywhere on S, such that $\int f(x)u(x)m(dx) < \infty$. Then an argument on capacitability similar to that for $H_A^n u$ yields that

$$\int \{H_A^\infty u(x)\} f(x) m\,(dx) = \inf_B \left[\int \{H_B^\infty\, u(x)\} f(x) m(dx)\right].$$

Hence there is a decreasing sequence $\{B_k\}$ of open sets such that

$$\lim_{k\to\infty} \int \{H_{B_k}^\infty u(x)\} f(x) m(dx)$$

is equal to the right side of the above equation. From this it follows that

$$\text{reg. } (H_A^\infty u) = \lim_{k\to\infty} H_{B_k}^\infty u \qquad \text{a.e. } (m),$$

which completes the proof of Proposition 10.1.

PROPOSITION 10.2. *$\bar{H}_A u = \sup \bar{H}_{A'} u$ (A'; compact and included in A) and there is an increasing sequence of compact sets $\{A'_k\}$ such that $A'_k \subset A$ and such that $\bar{H}_{A_k'} u \to \bar{H}_A u$.*

In a way similar to the proof of the preceding proposition one can find a sequence $\{A'_k\}$ for which the latter half of the proposition is true a.e. (m). However since $\lim_{k\to\infty} \bar{H}_{A_k'} u$ is excessive, it must be equal to $\bar{H}_A u$ everywhere on S. The former half is immediate from the latter half.

We will study some properties of the reduced functions to subsets of the boundary.

PROPOSITION 10.3. (i) *If A is a subset of S', then $\bar{H}_A u$ is harmonic.* (ii) *If A and B are subsets of S' and if B includes A, then*

$$\bar{H}_A \bar{H}_B u = \bar{H}_B \bar{H}_A u = \bar{H}_A u.$$

If A is compact, the first assertion follows from Proposition 8.3. For a general (analytic) set A, it is enough to apply the latter half of the preceding proposition. As for the assertion (ii), the general case is again reduced to the case of compact sets. Moreover it is enough to prove that $\bar{H}_B \bar{H}_A u = \bar{H}_A u$, because then $\bar{H}_A \bar{H}_A u = \bar{H}_A u$ and hence

$$\bar{H}_A \bar{H}_B u \geqq \bar{H}_A \bar{H}_A u = \bar{H}_A u \geqq \bar{H}_A \bar{H}_B u.$$

Let A and B be compact subsets of S' and $\hat{A}$ and $\hat{B}$, their open neighborhoods in $S + S'$. Assuming $\hat{A} \subset \hat{B}$, we have $H_{\hat{B}\cap S} H_{\hat{A}\cap S}\, u = H_{\hat{A}\cap S}\, u$. Set $S^u = \{x; u(x) < \infty\}$. Since $\tilde{S}^u$ is a polar set and since $H_{\hat{A}\cap S}\, u \downarrow H_A^\infty u$ $(\hat{A} \downarrow A)$ on S^u, we have

$$H_{\hat{B}\cap S} H_A^\infty u = H_A^\infty u$$

on S^u. By Proposition 8.3, $H_A^\infty u = \text{reg.}\ (H_A^\infty u) = \bar{H}_A u$ on S^u, so that $H_{\hat{B}\cap S} H_A^\infty u = H_{\hat{B}\cap S} \bar{H}_A u = \bar{H}_A u$. Letting $\hat{B} \downarrow B$, we have $H_B^\infty \bar{H}_A u = \bar{H}_A u$ on S^u. Hence $\bar{H}_B \bar{H}_A u = \bar{H}_A u$.

PROPOSITION 10.4. *Let A be a Borel set of S'. Then $\bar{H}_A \kappa(x, \eta)$ is jointly measurable on $S \times (S + S')$ in the following sense: For each pair of finite measures ν on S and μ on $S + S'$, there is a jointly Borel measurable function on $S \times (S + S')$ which coincides with $\bar{H}_A \kappa(x, \eta)$ except on a set of the product $(\nu \times \mu)$ measure 0. Moreover*

$$\bar{H}_A \left\{ \int_{S+S'} \kappa(\cdot\,, \eta) \mu\,(d\eta) \right\} = \int_{S+S'} \{\bar{H}_A \kappa(\cdot\,, \eta)\} \mu\,(d\eta). \tag{10.2}$$

The first statement is reduced to showing the joint measurability of $H_A^n \kappa(x, \eta)$. $H_A^n \kappa(x, \eta)$ is jointly measurable for an open set A and hence so for a compact set A. From this, the usual argument on capacitability yields the joint measurability of $H_A^n \kappa(x, \eta)$ for each analytic set. The proof of the latter half is also a routine.

11. The Martin representation of excessive functions

Let S_1' be the set of points η of S' for which there is an (r) integrable excessive function u such that $\bar{H}_{\{\eta\}}\, u$ is not identically zero and let $S_1 = S_P \cap S_r$.

This section is devoted to the proof of the following Martin representation theorem.

THEOREM 4. *The class of excessive functions u of $\mathbf{L}(r)$ is in one-one correspondence with the class of finite (Radon) measures μ on $S_1 + S_1'$ through the integral formula*

$$u = \int_{S_1+S_1'} \kappa(\cdot\,, \eta)\mu\,(d\eta). \tag{11.1}$$

The total mass of μ lies on S_1' if and only if u is harmonic.

The formula (11.1) is said to be the *canonical representation of* u and the measure μ, the *canonical measure.*

To prove the theorem we will prepare a series of propositions like those in the classical cases.

PROPOSITION 11.1. *Let $\{\mu_n\}$ be a sequence of measures on $S + S'$ such that $\mu_n(S + S')$ is bounded and let μ be a weak limit of $\{\mu_n\}$. If*

$$u_n = \int_{S+S'} \kappa(\cdot\,, \eta)\mu_n\,(d\eta)$$

is dominated by a function of $\mathbf{L}_0(m)$ and if u_n converges a.e. (m) *to a function u, then $\alpha G_\alpha\, u$ increases with α and*

$$\text{reg. } u = \lim_{\alpha\to\infty} \alpha G_\alpha\, u = \int_{S+S'} \kappa(\cdot\,, \eta)\mu\,(d\eta).$$

The proof is easy, recalling that $\langle f, \kappa(\cdot, \eta)\rangle = f\kappa(\eta)$ is continuous on $S + S'$ for each f of $\mathbf{C}_0$.

PROPOSITION 11.2. *If A is a Borel set of S', then there is a measure μ on A such that*

$$\bar{H}_A\, u = \int_A \kappa(\cdot\,, \eta)\mu\,(d\eta)$$

$$\mu(A) = \int r\,(dx)\bar{H}_A\, u(x).$$

A measure μ on a Borel subset A of $S + S'$ is identified with the measure on the whole space which coincides with μ on A and vanishes outside of A. Hence, for example, the expression that μ_n on A_n converges weakly to a measure on A means that the sequence of the corresponding measures on the whole space converges weakly to a measure on the whole space vanishing outside of A.

If A is an open set with compact closure $\bar{A}$ in S, then

$$\bar{H}_A u = H_A u = \int_{\bar{A}} G(\cdot, \eta)\mu'(d\eta),$$

by Proposition 7.10. Since u is supposed to be (r) integrable, μ' has no mass outside of S_r. Therefore we have

$$\bar{H}_A u = \int_{\bar{A}} \kappa(\cdot, \eta)\mu(d\eta), \qquad \mu(d\eta) = \{rG(\eta)\}\mu'(d\eta)$$

$$\mu(\bar{A}) = \int_{\bar{A}} \{rG(\eta)\}\mu'(d\eta) = \int r(dx)\bar{H}_A u(x).$$

If A is any open set of $S + S'$, choose a sequence $\{A_n\}$ of open sets with compact closure in S increasing to $A \cap S$. Let μ_n be the measure on $\bar{A}_n$ corresponding to $\bar{H}_{A_n} u$ in the above-stated way. It is clear that $\{\mu_n\}$ has a weak limit μ and that μ is a measure on $\bar{A}$, the closure of A in $S + S'$. Since $\bar{H}_{A_n} u \to \bar{H}_A u$, by Proposition 11.1 we have

$$\bar{H}_A u = \int_{\bar{A}} \kappa(\cdot, \eta)\mu(d\eta).$$

Moreover since $\int r(dx)\bar{H}_{A_n} u(x)$ is the total mass of μ_n, its limit is equal to the total mass of μ. Hence

$$\mu(\bar{A}) = \int r(dx)\bar{H}_A u(x).$$

If A is a compact subset of S', choose a sequence $\{A_n\}$ of open sets in $S + S'$ decreasing to A. Let μ_n be the measure on $\bar{A}_n$ corresponding to $\bar{H}_{A_n} u$ and let μ be a weak limit of $\{\mu_n\}$. Similarly to the preceding case, μ is a measure on A and

$$\bar{H}_A u = \int_A \kappa(\cdot, \eta)\mu(d\eta),$$

$$\mu(A) = \int r(dx)\{\lim_{n\to\infty} \bar{H}_{A_n} u(x)\}.$$

But we have already shown (see the proof of Proposition 10.3) that

$$\lim_{n\to\infty} \bar{H}_{A_n} u = H_A^\infty u = \bar{H}_A u$$

on the set $S^u = \{x; u(x) < \infty\}$. Since $r(\tilde{S}^u) = 0$, we get

$$\mu(A) = \int r(dx)\bar{H}_A u(x).$$

Finally if A is any Borel set of S', choose an increasing sequence $\{A_n\}$ of compact sets of S' such that $A_n \subset A$ and such that $\bar{H}_{A_n} u \to \bar{H}_A u$. For convenience, let A_0 denote the empty set. Define, for $n \geqq 1$,

$$u_n' = \bar{H}_{A_n} u - \bar{H}_{A_{n-1}} u \quad \text{on } S^u$$
$$= \infty \quad \text{on } \tilde{S}^u.$$

In the same way as in Proposition 8.1 it follows that the regularization u_n of u_n' is harmonic and that $\bar{H}_{A_n} u = \bar{H}_{A_{n-1}} u + u_n$ everwhere on S. Also we have

$$\bar{H}_{A_n} u_n = \bar{H}_{A_n} \bar{H}_{A_n} u - \bar{H}_{A_n} \bar{H}_{A_{n-1}} u = \bar{H}_{A_n} u - \bar{H}_{A_{n-1}} u = u_n \quad \text{on } S^u$$

and hence $\bar{H}_{A_n} u_n = u_n$ everywhere on S. Therefore there is a measure μ_n on A_n such that

$$u_n = \int_{A_n} \kappa(\cdot, \eta)\mu_n(d\eta), \qquad \mu_n(A_n) = \int r(dx)u_n(x).$$

Define a measure μ on A by $\sum \mu_n$, where each μ_n is regarded as a measure on A with the convention $\mu_n(A - A_n) = 0$. Then

$$\bar{H}_A u = \sum u_n = \int_A \kappa(\cdot, \eta) \{\sum \mu_n (d\eta)\} = \int_A \kappa(\cdot, \eta)\mu(d\eta),$$

$$\mu(A) = \sum \mu_n(A_n) = \int r(dx) \{\sum u_n(x)\} = \int r(dx)\bar{H}_A u,$$

which completes the proof of the proposition.

An excessive function u is said to be *extreme* (or *minimal*) if, whenever $u = u_1 + u_2$ with u_1 and u_2 both excessive, each u_i is a constant multiple of u. By the uniqueness of the measure determining a potential it follows that if y is in S_P, $G(\cdot, y)$ is extreme.[20] Therefore if y is a point of S_1, $\kappa(\cdot, y)$ is an extreme excessive function, not identically zero.

Let η be a point of S' and u, an excessive function of $\mathbf{L}(r)$. By Proposition 11.2, we have

$$\bar{H}_{\{\eta\}} u = \left\{\int r(dx)\bar{H}_{\{\eta\}} u(x)\right\} \times \kappa(\cdot, \eta), \tag{11.2}$$

$$\bar{H}_{\{\eta\}} u = \bar{H}_{\{\eta\}} \bar{H}_{\{\eta\}} u = \left\{\int r(dx)\bar{H}_{\{\eta\}} u(x)\right\} \times \bar{H}_{\{\eta\}} \kappa(\cdot, \eta). \tag{11.3}$$

Therefore if η is a point of S_1', $\kappa(\cdot, \eta)$ is not identically zero and

$$\kappa(\cdot, \eta) = \bar{H}_{\{\eta\}}\kappa(\cdot, \eta). \tag{11.4}$$

If $\bar{H}_{\{\eta\}} \kappa(\cdot, \eta)$ is not identically zero, then

$$\int r(dx)\bar{H}_{\{\eta\}} \kappa(\cdot, \eta) = 1, \tag{11.5}$$

[20] Actually, $G(\cdot, y)$ is extreme even for $y \in S - S_P$. See Proposition 12.7.

which follows from (11.3). Conversely, if (11.5) is satisfied, it is obvious that the point η belongs to S_1'. Moreover, as a result of (11.2) and (11.4), it follows that $\kappa(\cdot, \eta)$ is extreme if $\eta \in S_1'$ (see [11, p. 155], [13, p. 89]). Thus we have proved

PROPOSITION 11.3.

(i)
$$\int r\,(dx) \bar{H}_{\{\eta\}}\, \kappa(x, \eta) = 1 \quad \textit{if} \quad \eta \in S_1'$$
$$= 0 \quad \textit{if} \quad \eta \in S' - S_1'.$$

(ii) *If* $\eta \in S_1'$, *then* (11.4) *holds. Moreover*, $\kappa(\cdot, \eta)$ *is extreme and harmonic.*

The proofs of the following propositions are quite similar to those for the classical cases and will be omitted.

PROPOSITION 11.4 ([11, §4, Lemma 1], [13, Lemma 4.4]). *Let* u *be extreme and* μ, *a finite measure on a Borel subset* A *of* $S + S'$ *satisfying*

$$\mu(A) = \int r\,(dx) u(x) > 0.$$ [21]

If u *is expressed by*

$$u = \int_A \kappa(\cdot, \eta) \mu\,(d\eta),$$

then the total mass of μ *concentrates on a point of* A.

PROPOSITION 11.5 ([11, §4, Lemma 5], [13, Lemma 4.7)]. *Let* η *be a p*... *of* $S_1 + S_1'$ *and* A, *a Borel subset of* S'. *Then*

$$\bar{H}_A\, \kappa(\cdot, \eta) = 0 \quad \textit{if} \quad \eta \notin A.$$

PROPOSITION 11.6 ([11, §4, Theorem II], [13, Theorem 4.5]). *The set* $S' - S_1'$ *is an* F_σ*-set.*

PROPOSITION 11.7 ([11, §4, Lemma 2], [13, Lemma 3.9]). *If* u *is an excessive function of* $\mathbf{L}(r)$, *then* $\bar{H}_{S'-S_1'}\, u = 0$.

Using the above-obtained propositions, Theorem 4 is proved in the same way as for the classical cases ([11, §4, Theorem III], [13, pp. 92–93)]. In particular, if A is a Borel subset of S', then

$$\bar{H}_A u = \int_{S_1+S_1'} \{\bar{H}_A\, \kappa(\cdot, \eta)\} \mu\,(d\eta) = \int_{A \cap S_1'} \kappa(\cdot, \eta) \mu\,(d\eta), \tag{11.6}$$

$$\mu(A \cap S_1') = \int r\,(dx) \bar{H}_A\, u(x), \tag{11.7}$$

[21] Since $\int r(dx)\kappa(x, \eta) \leqq 1$ for each η of $S + S'$, this condition means that
$$\int r(dx)\kappa(x, \eta) = 1$$
on a set of (μ) measure 0. This fact is used in the proof.

which shows how the canonical measure is determined explicitly by u for the sets on the boundary.

Note. One can give a characterization of S_1' (not involving the reduced operator $\bar{H}_{\{\eta\}}$) as follows. S_1' is the set of points η of S' such that $\kappa(\cdot, \eta)$ is extreme and harmonic and such that $\int r(dx)\kappa(\cdot, \eta) = 1$. This fact follows from Theorem 4 and Proposition 11.4.

12. Terminal distributions of h-path processes

Let h be an excessive function and let $S^h = \{x; 0 < h(x) < \infty\}$. Set

$$(12.1)\qquad \begin{aligned} H_t^h(x, A) &= \frac{1}{h(x)} E_x\{h(x_t); x_t \in A\} \quad \text{if} \quad x \in S^h \\ &= \delta(x, A)e^{-t} \qquad\qquad\qquad \text{if} \quad x \in \tilde{S}^h, \end{aligned}$$

where $\delta(x, \cdot)$ denotes the unit distribution at the point x. In [10] we proved there is a standard process X^h (called the *h-path process*) such that

$$P_x^h\{x_t \in A\} = H_t^h(x, A).$$

Moreover for each x of S^h, $(\mathfrak{F})$ stopping time τ and $\Lambda \in \mathfrak{F}_{\tau+}$ we have

$$(12.2)\qquad P_x^h(\Lambda) = \frac{1}{h(x)} E_x\{h(x_\tau); \Lambda\}.$$

If $h = 1$, then $X^h = X$. For this reason the original process is sometimes called the 1-*path process.* If $h = \kappa(\cdot, \eta)$, $\eta \in S + S'$, then we will use the superfix η for the superfix h such as X^η and the word η-*path process* for the $\kappa(\cdot, \eta)$-path process.[22] It should be noted that the h-path transform may not preserve hypotheses (A_6) and (A_7) in general.

Suppose h is an excessive function of $\mathbf{L}(r)$. Then by Theorem 4, h has the canonical representation

$$(12.3)\qquad h = \int_{S+S'} \kappa(\cdot, \eta)\mu^h(d\eta),$$

where the canonical measure μ^h is regarded as a measure over $S + S'$ with the usual convention that μ^h has no mass outside of $S_1 + S_1'$.

For each w, let $l(w)$ be the set of limit points (in $S + S'$) of $x_t(w)$ from the left at the life time ζ. Hypothesis (A_6) is equivalent to the statement that $l(w)$ is either a point of S or a subset of S', a.e. (P_x) for each $x \in S$. For each w such that $l(w)$ is a point of $S + S'$, we will write

$$x_{\zeta-}(w) = \lim_{t \uparrow \zeta} x_t(w).$$

We will say the excessive function h of $\mathbf{L}(r)$ *has the property* (D) if $x_{\zeta-}$

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[22] We also make the same convention for $h = G(\cdot, y)$, $y \in S$.

exists a.e. (P_x^h) and if

$$P_x^h\{x_{\zeta-} \in A\} = \frac{1}{h(x)} \int_A \kappa(x, \eta)\mu^h(d\eta), \tag{12.4}$$

for each x of S^h and for each Borel subset A of $S + S'$. $P_x^h\{x_{\zeta-} \in A\}$ is said to be the *terminal distribution of the h-paths*. Since, for each η of $S_1 + S_1'$, $\int r(dx)\kappa(x, \eta) = 1$, the canonical measure μ^h of the excessive function h having the property (D) is given by

$$\mu^h(A) = \int r(dx)h(x)P_x^h\{x_{\zeta-} \in A\}.$$

Doob [3] discovered that, in the case of Brownian motion, every excessive function of $\mathbf{L}(r)$ has the property (D). The same result for Markov chains was proved in [7], [8]. In the following we will generalize the theorem of Doob (see Theorem 5) and also obtain several related results.

PROPOSITION 12.1. *For each x of S^h and for each Λ of $\mathfrak{F}$, we have*

$$P_x^h(\Lambda) = \frac{1}{h(x)} \int_{S+S'} \kappa(x, \eta)P_x^\eta(\Lambda)\mu^h(d\eta). \tag{12.5}$$

The value of $P_x^\eta(\Lambda)$ at $x \in \tilde{S}^\eta$ is irrelevant to the value of $P_x^h(\Lambda)$ on the left side.

For each Λ of $\mathfrak{F}_t$ and x of S^h, we have formally

$$\begin{aligned} P_x^h(\Lambda) &= \frac{1}{h(x)} E_x\{h(x_t); \Lambda\} \\ &= \frac{1}{h(x)} \int_{S+S'} E_x\{\kappa(x_t, \eta); \Lambda\}\mu^h(d\eta) \\ &= \frac{1}{h(x)} \int_{S+S'} \kappa(x, \eta) \cdot \frac{E_x\{\kappa(x_t, \eta); \Lambda\}}{\kappa(x, \eta)} \mu^h(d\eta) \\ &= \frac{1}{h(x)} \int_{S+S'} \kappa(x, \eta)P_x^\eta(\Lambda)\mu^h(d\eta). \end{aligned}$$

The above evaluation is justified by the following observations. If $\kappa(x, \eta) = 0$, then $E_x\{\kappa(x_t, \eta); \Lambda\} = 0$ and $\kappa(x, \eta)P_x^\eta(\Lambda) = 0$. Since $x \in S^h$,

$$\mu^h\{\eta; \kappa(x, \eta) = \infty\} = 0.$$

Therefore $E_x\{\kappa(x_t, \eta); \Lambda\} = \kappa(x, \eta)P_x^\eta(\Lambda)$ for almost all η relative to μ^h. Hence (12.5) is true for each Λ of $\mathfrak{F}_t$. But since the both sides of (12.5) are probability measures over $\mathfrak{F}$, (12.5) is true over $\mathfrak{F}$.

PROPOSITION 12.2. *Let A and B be open sets of S whose closures are disjoint.*

Then if h is a potential, we have

$$\lim_{n\to\infty} \underbrace{H_B H_A \cdots H_B H_A}_{n} h(x) = 0 \tag{12.6}$$

at each point x of finiteness of h.

Denote by h_∞ the left side of (12.6). By Proposition 7.5 and 7.9, reg. h_∞ is a potential of a measure vanishing outside of $\bar{A}$ (= closure of A in S). In the same way, reg. h_∞ is also a potential of a measure vanishing outside of $\bar{B}$. Hence reg. $h_\infty = 0$ by the uniqueness of the potential measure, so that $h_\infty = 0$ a.e. (m). Now let Λ denote the set of w's such that the path $x_t(w)$ intersects with both A and B infinitely often. Obviously $\Lambda \in \mathfrak{F}$ and

$$\{w;\, \theta_t w \in \Lambda\} = \Lambda \cap \{t < \zeta\} \uparrow \Lambda \qquad (t \downarrow 0).$$

Therefore

$$E_x^h\{P_{x_t}^h(\Lambda)\} = P_x^h\{w;\, \theta_t w \in \Lambda\} \uparrow P_x^h(\Lambda),$$

that is, $P_x^h(\Lambda)$ is excessive relative to X^h. On the other hand an evaluation based on (12.2) leads to the fact that $P_x^h(\Lambda) = h_\infty(x)/h(x)$ for each x of S^h. Hence, for each x of S^h,

$$\begin{aligned} P_x^h(\Lambda) &= \lim_{\alpha\to\infty} \alpha G_\alpha^h P_x^h(\Lambda) \\ &= \lim_{\alpha\to\infty} \frac{\alpha}{h(x)} G_\alpha \left\{ h \cdot \frac{h_\infty}{h} \right\} (x) \\ &= 0. \end{aligned}$$

If $h(x) = 0$, obviously $h_\infty(x) = 0$.

As a result of the preceding proposition and its proof we have

PROPOSITION 12.3. *If h is a potential, then X^h is a Hunt process.*

PROPOSITION 12.4. *If h is a potential in the wide sense, then ζ is finite* a.e. (P_x^h). *In particular, X^h is transient.*

Let $h = G\mu$. If $x \in \tilde{S}^h$, the assertion is trivial by definition (see (12.1)). For each x of S^h we have

$$\begin{aligned} E_x^h(e^{-\alpha\zeta}) &= 1 - \alpha G_\alpha^h(x, S) \\ &= \frac{1}{h(x)} \left[h(x) - \alpha \int G_\alpha(x, z) h(z) m\,(dz) \right] \\ &= \frac{\int G_\alpha(x, y)\mu\,(dy)}{\int G(x, y)\mu\,(dy)} \to 1 \qquad (\alpha \to 0), \end{aligned}$$

which proves that $P_x^h\{\zeta < \infty\} = 1$.

PROPOSITION 12.5. *If* $y \in S_P$, *then* $x_{\zeta-} = y$ a.e.[23] (P_x^y) *for each* x *of* S^y.

Let A be a compact set of S. By (12.2) we have

$$P_x^y\{\tau(\tilde{A}) < \zeta\} = \frac{H_{\tilde{A}}\, G(x, y)}{G(x, y)} \to 0 \qquad (A \uparrow S),$$

whence $l(w)$ is included in S a.e. (P_x^y). Since X^y is a transient Hunt process, $x_{\zeta-}$ exists a.e. (P_x^y). Let A be an open neighborhood of y and B, an open set of positive distance with A. Define the sequence of $(\mathfrak{F})$ stopping times by

$$\tau_1 = \text{the hitting time for } A$$

$$\sigma_1 = \text{the first time hitting } B \text{ after } \tau_1$$

$$\tau_2 = \text{the first time hitting } A \text{ after } \sigma_1 .$$

$\sigma_2, \tau_3, \sigma_3, \cdots$ are defined successively. Suppose the statement of the proposition is false. Then it follows that there is a pair of sets A, B such that, for some $n > 0$ and for some x of S^y, $P_x^y\{\sigma_n < \zeta \leqq \tau_{n+1}\} > 0$. But using the fact that

$$P_x^y\{\tau_1 < \zeta\} = \frac{H_A\, G(x, y)}{G(x, y)} = 1,$$

we have

$$\begin{aligned} P_x^y\{\sigma_n < \zeta,\ \tau_{n+1} < \zeta\} &= E_x^y\{P_{x_{\sigma_n}}^y(\tau_1 < \zeta);\ \sigma_n < \zeta\} \\ &= P_x^y\{\sigma_n < \zeta\}, \end{aligned}$$

which contradicts

$$0 < P_x^y\{\sigma_n < \zeta \leqq \tau_{n+1}\} = P_x^y\{\sigma_n < \zeta\} - P_x^y\{\sigma_n < \zeta,\ \tau_{n+1} < \zeta\}.$$

PROPOSITION 12.6. *If* h *is a potential of* $\mathbf{L}(r)$, h *has the property* (D).

In this case, the total mass of μ^h concentrates on S_1. Choose the set $\{w;\ x_{\zeta-} \text{ exists and } x_{\zeta-} \in A\}$ as Λ in (12.5). According to the preceding proposition, if $\eta \in S_1$, then

$$P_x^\eta\{x_{\zeta-} \text{ exists and } x_{\zeta-} \in A\} = \delta(\eta, A),$$

so that

$$P_x^h\{x_{\zeta-} \text{ exists and } x_{\zeta-} \in A\} = \frac{1}{h(x)} \int_A \kappa(x, \eta)\mu^h(d\eta).$$

In particular, setting $A = S + S'$, $P_x^h\{x_{\zeta-} \text{ exists}\} = 1$.

PROPOSITION 12.7. *If* $y \in S - S_P$, *then* $G(\cdot, y)$ *is extreme and harmonic.*

By Proposition 6.2, $\bar{H}_{\{y\}}\, G(\cdot, y) = G(\cdot, y)$. Moreover, as in the proof of Proposition 11.2 one can show that $\bar{H}_{\{y\}}\, u = \text{const.} \times G(\cdot, y)$ for each excessive function u of $\mathbf{L}_0(m)$. From these two formulas it follows that $G(\cdot, y)$

[23] See footnote 22.

is extreme. Since $G(\cdot, y)$ is not a potential by assumption, it must be harmonic by Proposition 8.1.

Let y be a point of $S_r - S_1$. According to the preceding proposition and the note of Section 11, there is a point η of S_1' such that $\kappa(\cdot, y) = \kappa(\cdot, \eta)$. Such correspondence is one-one, because the function κ separates S_r and S' respectively. We will write $\eta = \pi(y)$.

PROPOSITION 12.8. *$\pi(S_r - S_1)$ is an analytic set of S'.*

A subset of S is said to be a K_σ-set if it is a countable union of compact sets. A $K_{\sigma\delta}$-set is defined as a countable intersection of K_σ-sets. Similarly a $K_{\sigma\delta\sigma}$-set is defined as a countable union of $K_{\sigma\delta}$-sets. Let A_n be a sequence of open sets with compact closure increasing to S. That y is a point of $S - S_P$ is equivalent to that there are some f of $\mathbf{C}_0$ and some positive integer p such that

$$\int m\,(dx)f(x)H_{\tilde{A}_n}G(x, y) > 1/p \qquad \text{for every } n.$$

Noting that the left side is lower semicontinuous relative to y, we can conclude that $S - S_P$ is a $K_{\sigma\delta\sigma}$-set. Since S_r is open, $S_r - S_1 = S_r \cap (S - S_P)$ is also a $K_{\sigma\delta\sigma}$-set. On the other hand, π is a continuous mapping, because $y_n \to y$(in $S_r - S_1$) implies that

$$f\kappa\{\pi(y_n)\} = f\kappa(y_n) \to f\kappa(y) = f\kappa\{\pi(y)\}$$

and hence that $\pi(y_n) \to \pi(y)$ (in S'). Therefore the image $\pi(S_r - S_1)$ is an analytic set of S'.

PROPOSITION 12.9. *Let η be a point of S_1'.* (i) *If η is not a point of $\pi(S_r - S_1)$, then $\kappa(\cdot, \eta)$ has the property* (D) *and hence X^η is a transient Hunt process.* (ii) *If η is a point of $\pi(S_r - S_1)$, then X^η is still transient but not a Hunt process (and hence $\kappa(\cdot, \eta)$ does not have the property* (D)*). More precisely, $l(w)$ consists of the two points $\{\eta, \pi^{-1}(\eta)\}$* a.e. *(P_x^η) for each x of S^η.*

We will only give the proof of (ii): a similar argument is applicable to the proof of (i).

Let $y = \pi^{-1}(\eta)$ and let x be any point of S^η. Since $X^\eta = X^y$, X^η is transient by Proposition 12.4. Since $\kappa(\cdot, \eta)$ is harmonic, $P_x^\eta\{\tau(\tilde{A}) < \zeta\} = 1$ for each compact set A. Hence $l(w)$ intersects with S' a.e. (P_x^η). Let B be a closed set of S' not including η and C, an open neighborhood of B in $S + S'$. Then

$$P_x^\eta\{l(w) \cap B \neq \emptyset\} \leqq P_x^\eta\{\tau(C \cap S) < \zeta\} = \frac{H_{C \cap S}\,\kappa(x, \eta)}{\kappa(x, \eta)}$$

When $C \downarrow B$, $H_{C\cap S}\,\kappa(x, \eta) \to \bar{H}_B\,\kappa(x, \eta)$. Since $\bar{H}_B\,\kappa(x, \eta) = 0$ by Proposition 11.5, $l(w) \cap S' = \{\eta\}$ a.e. (P_x^η).

Next let A be an open set with compact closure and B, a compact neighborhood of A. If $y \in A$, $H_A \kappa(\cdot, y) = \kappa(\cdot, y)$. Also since $\kappa(\cdot, y)$ is harmonic, $H_{\tilde{B}} H_A \kappa(\cdot, y) = H_{\tilde{B}} \kappa(\cdot, y) = \kappa(\cdot, y)$. Hence we have

$$\lim_{n\to\infty} \underbrace{H_{\tilde{B}} H_A \cdots H_{\tilde{B}} H_A}_{n} \kappa(x, y) = \kappa(x, y),$$

which means that $y \in l(w)$ a.e. (P_x^η).

Finally we will prove that, for almost all w (P_x^η), $l(w)$ includes no points of $S - \{y\}$. It is enough to show that

$$(12.7) \qquad P_x^\eta\{l(w) \text{ intersects with both } S - \{y\} \text{ and } S'\} = 0.$$

Take a pair of sets A, B of the preceding paragraph and suppose the point y does not belong to $\bar{A}$, the closure of A. By Proposition 7.10 we have

$$\begin{aligned} H_A \kappa(\cdot, y) &= \int_{\bar{A}} G(\cdot, z)\mu(dz) \\ &= \int_{\bar{A}\cap S_P} G(\cdot, z)\mu(dz) + \int_{\bar{A}\cap(S-S_P)} G(\cdot, z)\mu(dz). \end{aligned}$$

Assume the integral over $\bar{A} \cap (S - S_P)$ in the above display, say u, is not identically zero. Then u is harmonic and dominated by an extreme harmonic function $\kappa(\cdot, y)$. Hence u is a constant $(\neq 0)$ multiple of $\kappa(\cdot, y)$, which contradicts the uniqueness of the measure of the potential in the wide sense because of $y \notin \bar{A}$. Therefore u must vanish, that is, we have proved that $H_A \kappa(\cdot, y)$ is a potential. Applying Proposition 12.2 to $h = H_A \kappa(\cdot, y)$ (also $A \to \tilde{B}$, $B \to A$), we have

$$\lim_{n\to\infty} \underbrace{H_A H_{\tilde{B}} \cdots H_A H_{\tilde{B}} H_A}_{n} \kappa(x, y) = 0,$$

which implies (12.7).

PROPOSITION 12.10. *For each excessive function h of $\mathbf{L}(r)$, X^h is transient.*

We already proved that X^η is transient for every η of $S_1 + S_1'$. Hence our assertion is easily derived from the formula (12.5).

THEOREM 5. *Each excessive function of $\mathbf{L}(r)$ has the property* (D) *if and only if $S_P \supset S_r$. In particular, if $S = S_P$, the above statement is true for any reference measure.*

Suppose $S_P \supset S_r$. Then $S_r = S_1$, so that for each η of $S_1 + S_1'$, $\kappa(\cdot, \eta)$ has the property (D). Hence, similarly to the proof of Proposition 12.6, it follows that each excessive function of $\mathbf{L}(r)$ has the property (D). Next suppose S_P does not include S_r. Then $S_r - S_1 \neq \emptyset$. When $y \in S_r - S_1$, $\kappa(\cdot, y) = \kappa(\cdot, \pi(y))$ does not have the property (D) by Proposition 12.9.

PROPOSITION 12.11. *Let h be an excessive function of $\mathbf{L}(r)$ such that X^h is a Hunt process. Then h has the property (D).*

Since $\pi(S_r - S_1)$ is an analytic set of S', it is measurable relative to the canonical measure μ^h of h. Hence it is enough to show that $\mu^h\{\pi(S_r - S_1)\} = 0$. Take

$$\Lambda = \{w;\, l(w) \text{ intersects with both } S \text{ and } S'\}.$$

Then by the assumption and Proposition 12.9, we have

$$0 = P_x^h(\Lambda) = \frac{1}{h(x)} \int_{\pi(S_r - S_1)} \kappa(x, \eta)\mu^h(d\eta),$$

whence

$$\mu^h\{\pi(S_r - S_1)\} = \int r(dx)h(x)P_x^h(\Lambda) = 0.$$

13. Conditions for $S = S_P$

Let h be an excessive function of $\mathbf{L}_0(m)$. Then there is a reference measure r such that h is (r) integrable. Hence, according to Proposition 12.10, $\mathbf{L}^h$ is transient. Now suppose $S = S_P$. Then h has the property (D) relative to the above introduced r and hence X^h is a Hunt process. Next suppose $S \neq S_P$. Take any point y_0 of $S - S_P$. We may assume $y_0 \in S_r$. According to Proposition 12.9, X^{y_0} is not a Hunt process. Hence we have proved

THEOREM 6. *$S = S_P$ if and only if, for each excessive function h of $Y_0(m)$, X^h is a Hunt process.*

PROPOSITION 13.1.[24] *A sufficient condition for $S = S_P$ is that the co-resolvent kernel $\hat{\mathbf{G}}$ is the resolvent kernel of a standard process $\hat{X}$.*

Let A be an open set of S and $\hat{H}_A(x, B)$, the harmonic measure to the set A relative to $\hat{X}$. Like [6, Part III], we have

$$\int H_A(x, dz)G(z, y) = \int G(x, z)\hat{H}_A(y, dz) \tag{13.1}$$

for all x, y of S. Suppose $S \neq S_P$. Then for a point y of $S - S_P$ and for each compact set A, $H_{\bar{A}}\, G(\cdot, y) = G(\cdot, y)$ by Proposition 12.7. Therefore we have for each f of $\mathbf{B}_0$

$$\int m(dx)f(x)G(x, y) = \iint m(dx)f(x)G(x, z)\hat{H}_{\bar{A}}(y, dz). \tag{13.2}$$

But the left side is equal to $\hat{E}_y\{\int_0^\infty f(x_t)\, dt\}$ and the right side, to $\hat{E}_y\{\int_{\tau(\bar{A})}^\infty f(x_t)\, dt\}$. Hence (13.2) is impossible if we choose an A containing y as an interior point and an f strictly positive on A.

[24] This is a revised form of an incorrect statement of the previous paper [9, footnote 4].

PROPOSITION 13.2. (i) *If h is excessive and bounded on each compact set, then X^h is a Hunt process.* (ii) *If every harmonic function of* $\mathbf{L}_0(m)$ *is bounded on each compact set, then* $S = S_P$.

Let A be an open set with compact closure and B, a compact neighborhood of A. Since h is bounded on A by the assumption, we have

$$H_{\tilde{B}} H_A h \leqq \text{const.} \times P_\bullet\{\tau(\tilde{B}) + \tau(A, \theta_{\tau(\tilde{B})} w) < \zeta\}.$$

When $B \uparrow S$, the right side goes to 0 by hypotheses (A_6), (A_7). Hence $H_A h$ is a potential. Applying Proposition 12.2 to $H_A h$ in place of h (also, $A \to \tilde{B}$, $B \to A$), we have

$$\lim_{n\to\infty} \underbrace{H_A H_{\tilde{B}} \cdots H_A H_{\tilde{B}} H_A}_{n} h = 0,$$

which implies that X^h is a Hunt process.

For the proof of (ii), suppose $S \neq S_P$. Then by Theorem 6, X^y, $y \in S - S_P$, is not a Hunt process. Hence the harmonic function $G(\cdot, y)$ cannot be bounded on compact sets.

14. Notes on hypotheses (A_6) and (A_7)

So far we have assumed that the basic process is a transient Hunt process satisfying hypothesis (B). In this section we will show that the phrase *transient Hunt* can be replaced by the word *standard*, namely, that hypotheses (A_6) and (A_7) can be removed. To see this we first note that hypotheses (A_6), (A_7) were used only in the proof of Proposition 3.6.[25] But as is shown easily Proposition 3.6 remains still valid under the following, a little weaker than (A_6) and (A_7), hypotheses:

(A_6)′ Under the condition $\zeta < \infty$, each path $x_t(w)$ except on a set of (P_x) measure 0 has at most one limit point in S from the left at $t = \zeta$. (In other words, the other possible limit point is Δ, that is, the point at infinity.)[26]

(A_7)′ For each compact set A, $\tau(\tilde{A}) < \infty$ a.e. (P_x).

For each w, let $l(w)$ be the set of limit points[27] (in $S + \{\Delta\}$) of $x_t(w)$ from the left at $t = \zeta$. Using this notation, (A_6)′ is equivalent to

$$P_x\{l(w) \cap S = \text{ at most one point, or } \zeta = \infty\} = 1 \qquad \text{for all } x \text{ of } S.$$

The purpose of this section is to prove

THEOREM 7. *If X is a standard process satisfying hypothesis* (B), *then hy-*

[25] We also used these hypotheses in Proposition 13.2. But since the proposition is independent of the other results, we ignore it.

[26] For instance, (A_6)′ is satisfied if X is a standard process such that the resolvent kernel $G_\alpha(x, A)$ maps $\mathbf{C}_0$ into $\mathbf{C}$. But this is not our case.

[27] This $l(w)$ is different from that of Section 12.

potheses $(A_6)'$ *and* $(A_7)'$ *are satisfied. Hence all the results*[28] *of the preceding sections remain till valid without hypotheses* (A_6) *and* (A_7).

$(A_7)'$ is obvious, for $E_x\{\tau(\tilde{A})\} \leqq G_0(x, A) < \infty$ by (B).

To prove $(A_6)'$ we will use a result on *reversed processes* (Proposition 14.1), omitting the proof.

Choose a reference probability measure r such that $r(dx) = g(x)m(dx)$ with g strictly positive everywhere on S and such that rG is finite everywhere (see Section 9 for the existence of such r). Define

$$P^{(r)}(\cdot) = \int r\,(dx)P_x(\cdot), \tag{14.1}$$

$$\hat{G}_\alpha^{(r)}f(y) = \frac{1}{rG(y)} \int m\,(dx)\{rG(x)\}f(x)G_\alpha(x, y). \tag{14.2}$$

Using hypothesis (B) and the properties of the above-chosen r, we can easily show that $\hat{G}_\alpha^{(r)}(y, A)$ is a regular substochastic resolvent kernel which maps $\mathbf{B}_0$ into $\mathbf{C}$. Therefore there is a countable subcollection $\{f_n\}$ of positive functions of $\mathbf{C}_0$ such that $\hat{G}_\alpha^{(r)}f_n$ separates points of S.

Next we define the reversed path $x_{\zeta-t}(w)$. If $0 < t \leqq \zeta < \infty$, $x_{\zeta-t}$ has the obvious meaning. If $t > \zeta$ or $\zeta = \infty$, $x_{\zeta-t} = \Delta$ by definition. Then we can prove

PROPOSITION 14.1.[29] $x_{\zeta-t}$, $0 < t < \infty$, *is a Markov process with stationary transition probabilities as a stochastic process defined over the probability space* $(W, \mathfrak{F}, P^{(r)})$. *Moreover the resolvent kernel of* $x_{\zeta-t}$ *is the kernel* $\hat{G}_\alpha^{(r)}(y, A)$ *defined by* (14.2). *That is*

$$\hat{G}_\alpha^{(r)}f(x_{\zeta-t}) = \int_0^\infty e^{-\alpha s}\, E^{(r)}\{x_{\zeta-s-t} \mid x_{\zeta-u}, u \leqq t\}\, ds$$

a.e. $(P^{(r)})$ *for each* $t > 0$.

As a result of the proposition it follows that the process $y_t = e^{-\alpha t}\hat{G}_\alpha^{(r)}f(x_{\zeta-t})$ is a supermartingale is f is positive. Therefore, for the previously defined f_n, $y_t^n = e^{-\alpha t}\hat{G}_\alpha^{(r)}f_n(x_{\zeta-t})$ gives a bounded and separable supermartingale and hence $\lim_{t\to 0} y_t^n(w)$ exists a.e. (P^r). Take any w such that $\zeta(w) < \infty$ and such that $\lim_{t\to 0} y_t^n(w)$ exists for all n. Suppose y, y' are points of $l(w) \cap S$. Since $G_\alpha^{(r)}f_n$ is continuous, we have

$$\begin{aligned}\lim_{t\to 0} y_t^n(w) &= \lim_{t\to 0} e^{-\alpha t}\hat{G}_\alpha^{(r)}f_n(x_{\zeta-t}) \\ &= \hat{G}_\alpha^{(r)}f_n(y) \\ &= \hat{G}_\alpha^{(r)}f_n(y').\end{aligned}$$

But since $\hat{G}_\alpha^{(r)}f_n$, $n = 1, 2, \cdots$, separates points of S, we have $y = y'$.

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[28] Strictly speaking, except Proposition 13.2. See footnote 25.

[29] The proof will be given, in a more general form, in the following paper by the authors: *On certain reversed processes and their applications to potential theory*, to apin J. Math. Mech., 1966 Also see M. Nagasawa, *Time reversions of Markov processes*, Nagoya Math. J., vol. 24 (1964), pp. 177–204.

Let $\Lambda = \{w; l(w) \cap S = \text{at most one point, or } \zeta = \infty\}$. We have proved $P^{(r)}(\Lambda) = 1$. Hence $P_x(\Lambda) = 1$ for almost all x relative to r. From the definition of r, $P_x(\Lambda) = 1$ a.a. x relative to m. But in the same way as in the proof of Proposition 12.2 it is shown that $P_x(\Lambda)$ is an excessive function. Therefore $P_x(\Lambda) = 1$ for all x of S.

References

1. R. M. Blumenthal, R. K. Getoor and H. P. McKean, Jr., *Markov processes with identical hitting distributions*, Illinois J. Math., vol. 6 (1962), pp. 402–420.
2. M. Brelot, *Le problème de Dirichlet. Axiomatique et frontière de Martin*, J. Math. Pures Appl., vol. 35 (1956), pp. 297–335.
3. J. L. Doob, *Conditional Brownian motion and the boundary limits of harmonic functions*, Bull. Soc. Math. France, vol. 85 (1957), pp. 431–458.
4. ———, *Discrete potential theory and boundaries*, J. Math. Mech., vol. 8 (1959), pp. 433–458.
5. E. B. Dynkin, *Theory of Markov processes*, (English translation), Engelwood Cliffs, N. J., Prentice-Hall, 1961.
6. G. A. Hunt, *Markoff processes and potentials I, III*, Illinois J. Math., vol. 1 (1957), pp. 44–93; vol. 2 (1958), pp. 151–213.
7. ———, *Markoff chains and Martin boundaries*, Illinois J. Math., vol. 4 (1960), pp. 313–340.
8. H. Kunita and T. Watanabe, *Markoff chains and Martin boundaries I*, (Japanese), Sûgaku, vol. 13 (1961), pp. 12–30.
9. ———, *Markov processes and Martin boundaries*, Bull. Amer. Math. Soc., vol. 69 (1963), pp. 386–391.
10. ———, *Notes on transformations of Markov processes connected with multiplicative functionals*, Mem. Fac. Sci. Kyushu Univ. Ser. A, vol. 17 (1963), pp. 181–191.
11. R. S. Martin, *Minimal positive harmonic functions*, Trans. Amer. Math. Soc., vol. 49 (1941), pp. 137–172.
12. D. Ray, *Resolvents, transition functions and strongly Markovian processes*, Ann. Math. (2), vol. 70 (1959), pp. 43–78.
13. T. Watanabe, *On the theory of Martin boundaries induced by countable Markov processes*, Mem. Coll. Sci. Univ. Kyoto, Ser. A, vol. 33 (1960), pp. 39–108.
14. ———, *A remark on the strong Markov property*, Mem. Fac. Sci. Kyushu Univ. Ser. A, vol. 17 (1963), pp. 176–180.

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